DRAFT Paper: Characterizing Children's Developing Understanding of Linear Measurement with a Learning Trajectory: A Tool for Integrated Assessment and Instruction

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Abstract: *Learning trajectories* provide an integrative tool for mathematics education research and curriculum development, enabling researchers to pose and check theories of learning and teaching, formulate formative and summative assessments in a domain, and develop standards-focused curricula (e.g., Brown, Blondel, Simon, & Black, 1995; Confrey, 2009). Here we propose and examine the veracity and coherence of a learning trajectory for children's length measurement. Linear measurement forms a foundation for quantitative reasoning including reasoning about ratio, proportion, and relations among variables (cf. V. V. E. Davydov, 1991; J. Smith & Thompson, 2007; Sowder et al., 1998; A. G. Thompson & Thompson, 1996; Patrick W. Thompson, 1993); all of these aspects of reasoning anticipate the formal study of algebra in upper school. Measurement also provides a window into the coordination of schemes for number and space, a critical developmental achievement in mathematical reasoning during the school years (cf. Lakoff & Nunez, 2000). Using a teaching experiment approach, our findings support the hypothesized learning trajectory for length measurement on three adjacent levels spanning strategies for end to end arrangements of unit objects up through coordinated iterations of units among students in grades 2 and 3. Our trajectory specifies a content-specific sequence of developmental levels with instructional tasks linked closely to each successive level.

Introduction

Measurement knowledge and strategies play broadly and deeply into children's understanding of both science and mathematics, making measurement a critical component of Pre-K through Grade 8 curricula. Geometric measurement bridges the mathematical domains of number and geometry (Clements & Sarama, 2007). Measurement experience provides images of variables as quantities in algebra (P. W. Thompson & Thompson, 1994). The *National Science Education Standards* (National Research Council, 1996) indicate that mathematical measurement plays an essential role in all aspects of scientific inquiry. Smith, Wiser, Anderson and Krajcik (2006) point out the integrative role of measurement for learning science conceptually: "Given the centrality of measurement in science and the ways measurement can contribute to conceptual understandings, it is important to start early in developing a rich understanding of the measurement of important physical quantities." (p. 33)

Measurement is a central aspect of spatial thinking; a quantitative understanding of space is often critical for posing questions, for developing explanations (Simon, 2006; J. Smith & Thompson, 2007), and for communicating results (National Academies Press (U.S.), 2006; Presmeg & Barrett, 2003). Within technological design for spatial problems, students need to choose suitable tools and techniques, and work with appropriate measurement methods to ensure precision and accuracy. The *Atlas of Science Literacy* (American Association for the Advancement of Science, 2001) describes measurement concepts as a critical foundation for establishing evidence and as a basis for modeling processes and systems (cf. Lehrer & Schauble, 2002; Lehrer & Schauble, 2004; Petrosino, Lehrer, & Schauble, 2003).

Recently, the National Council of Teachers of Mathematics (NCTM) published *Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics: A Quest for Coherence (2006)* (CFP) to promote a focused set of curriculum topics for each grade level, PreK-8. It offers guidelines for age-appropriate learning activities, clustered about central themes in mathematics. The CFP portrays measurement as a *critical topic* (cf. NCTM, 2000), underscoring the important role measurement instruction plays, especially prior to middle school. We believe the CFP highlights a need for developmental accounts of children's concept and strategy growth from Pre-K through Grade 8 to strengthen the empirical basis for continuing work to focus the elementary school curriculum in this country. Furthermore, naturalistic, longitudinal accounts are needed to sequence assessment goals across grades, inform curricular development programs and strengthen teacher development by describing proto-typical narratives of children's ways of gaining conceptual and strategic competence on critical measurement topics relevant to science and mathematics.

Problems with Measurement Instruction and Potential Improvements

Measuring is a non-trivial aspect of children's developing mathematical and scientific thinking based in the exercise of comparative judgment. Children often set out to compare magnitudes to make decisions or to interpret their surroundings. To make the comparisons, they observe patterns of change over time or patterns of variation among several cases, eventually quantifying their experiences and observations. We see a close connection between the genesis of quantity and the intention to compare within a 1D, 2D or 3D space. Sometimes these comparisons involve sets of objects and yet they often begin by comparisons along continuous dimensions involving length, area or volume. Thus, our characterization of measurement knowledge and strategies links directly to children's quantitative reasoning.

However, elementary mathematics instruction on measurement rarely focuses toward this kind of comparative judgment; instead, teachers often emphasize procedural competence with the ruler as a tool for finding a number to report without a purpose or context. Measurement lessons developed close to textbooks are often narrow in scope, with few mathematical discussions or connections; understanding is less developed than procedural competence (Jack Smith, Sisman, Dietiker et al., 2008; Jack Smith, Sisman, Figueras et al., 2008). Consequently, profound gaps in students understanding persist (e.g., Blume, Galindo, & Walcott, 2007; Chappell & Thompson, 1999). In addition to these conceptual struggles with measurement, and equally concerning from our standpoint, is the challenge of integrating instruction on ruler use with number line applications of number operations (Carraher, Schliemann, Brizuela, & Earnest, 2006; Saxe, Shaughnessy, Shannon, Chinn, & Gearhart, 2007; J. Smith & Thompson, 2007). In summary, most curriculum and our teaching practices tend to focus on developing successful measuring procedures without associating this activity with the understanding of continuity along number lines (cf. C. Smith, Solomon, & Carey, 2005; Carol Smith, Wiser, & Carey, 1985); thus, we see a great need to focus on comparative analysis of quantities displayed as distances or lengths, or the integration of unit, sub-units and super-units.

We believe children need extensive opportunities to develop quantity based on measures of linear space out of their initially figural and intuitive knowledge gained through qualitative comparisons with linear dimensions of objects in their world. This shift requires the integration of their counting number sequence and their knowledge of continuous spatial objects and images. Children gradually differentiate between counting and describing by continuous, extensive labels (Huntley-Fenner, 2001). Through comparisons, Pre-K aged children imitate the use of tools for measures. As children in K and Grade 1 extend their comparison they gain patterns allowing them to achieve simple measures (Lehrer, Jenkins and Osana, 1998). By reflecting on direct comparisons, children anticipate taking one object as a unit.

Concurrently, through experience with a ruler (a tool presenting iterated units and units of units), children in K and Grade 1 begin to expect a relation between units as a quantity and spatial extent (Hiebert, 1981, 1984; Kamii & Clark, 1997; Miller & Baillargeon, 1990; Szilagyi, 2007). During Grades 2 and into 3, children tend to avoid gapping or overlap (Lehrer, Jacobson, Kemney, & Strom, 1999; Lehrer, Jenkins, & Osana, 1998; Nunes, Light, & Mason, 1993). As children reflect on their own operations with collections of units to measure using ever-smaller units they make goal-directed decisions about ways to curtail their motion along objects (Clements, Battista, Sarama, & Swaminathan, 1997). Children gradually expect a *collection of units* to be conserved as well (Barrett, Clements, Klanderman, Pennisi, & Polaki, 2006, p. 208); they develop what Steffe calls a conceptual ruler (1991) as early as Grade 3. As children in Grades 4 and 5 reflect on their activities with path length problems, they integrate motions, drawings, and verbal narratives to act on an internal image of a nested sequence of units (Barrett & Clements, 2003; Chiu, 1996). This paper contributes to the literature by investigating these claims through extensive sequences of teaching episodes with children in grades 2 and 3.

Operationalizing a learning trajectory for Length measurement

We believe that a workable example of a learning trajectory can be useful for improving the teaching and learning of linear measurement concepts (cf. Clements, 1999; Kamii, 2006; Kamii & Clark, 1997; Lehrer, Jaslow, & Curtis, 2003; Jack Smith, Sisman, Figueras et al., 2008). *A learning trajectory* is suitable tool for conducting research into children's thinking and strategies on a specified mathematical learning goal (Clements & Sarama, 2004, 2007).

This paper presents our findings regarding a portion of a learning trajectory for length (c.f., Barrett & Clements, 2003; Barrett et al., 2006; Lehrer, Jacobson et al., 1998; C. L. Smith et al., 2006; Szilagyi, 2007). As such, it is an initial description of a set of prototypical, longitudinal stories of second and third-grade children's ways of understanding the mathematical ideas implicated by measurement activities. This report is part of a larger project describing children's measurement from grades PreK up through Grade 5 through longitudinal and cross-sectional approaches.[1] We present a set of stories based on our cyclical development of models for children learning to measure these dimensions, through a teaching experiment during the first year of our four-year project; we predicted and then checked how children would respond to increasingly complex tasks by relating each child to a model from the trajectory. The resulting set of instructional task sequences may serve to guide teachers toward key concepts relevant to their particular grade level, in keeping with the CFP (2006). We expect this account of a learning trajectory on measurement will help to improve formative assessment by teachers if they can recognize opportunities to engage their students in conceptual change. We also expect the learning trajectory will support more formal assessment efforts, the writing of state or national standards documents for mathematics and research on curriculum development (Clements, 2007).

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Based on our hypothesized Length Measurement Learning Trajectory (LMLT) (Clements, Sarama & Barrett, 2009), we designed a teaching experiment focusing on eight case study children in their second grade year of schooling and continuing into the fall of their third grade year (See Table 1 below).

Table 1. Length Measurement Learning Trajectory (integrated progression and instructional tasks)

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Developmental Progression	*Conceptual Structures and Strategies*	*Instructional Tasks*
Age 5: **Indirect Length Comparer (ILC)**: Compares the length of two objects by representing them with a third object. When asked to measure, may assign a length by guessing or moving along a length while counting (without equal length units). May be able to measure with a ruler, but often lacks understanding or skill (e.g., ignores starting point)	A mental image of a particular length can be built, maintained, and (to a simple degree) manipulated. With the immediate perceptual support of some of the objects, such images can be compared. If asked to measure, a counting scheme operates on an intuitive unit of spatial extent or amount of movement, directing physical or eye movement along a length while counting.	
Age 6: **End-to-End** Length Measuring (**EE):** Lays units end-to-end. May not recognize the need for equal-length units. The ability to apply resulting measures to comparison situations develops later in this level. (This develops in parallel with "**Serial Ordering** to 6+"). Lays 9 inch cubes in a line beside a book to measure how long it is.	An implicit concept that lengths can be composed as repetitions of shorter lengths underlies a scheme of laying lengths end to end. This initially only applied to small numbers of length parts. Starting with few restrictions, the scheme is enhanced by the growing conception of length measuring as covering distance or composing a length with parts.	Use "Length Riddles" providing only one unit per child to compare longer items. Provide incomplete sets of linear objects to span the length of an object to measure.
Age 7: **Length Unit Repeating and Relating (URR)**: Measures by repeated use of a unit	Action schemes include the ability to iterate a mental unit along a perceptually-available	Given a drawing of a 5 unit segment, ask students to draw a 3

Developmental Progression	*Conceptual Structures and Strategies*	*Instructional Tasks*
(initially may not be imprecise). Relates size and number of units explicitly (but may not appreciate the need for identical units). Relates size and number of units explicitly. Can add up two lengths to obtain the length of a whole. Iterates a single unit to measure. Uses rulers with minimal guidance.	object. The image of each placement can be maintained while the physical unit is moved to the next iterative position. With the support of a perceptual context, scheme can predict that fewer larger units will be required to measure an object's length. These action schemes allow the application of counting-all addition schemes to be applied to measures.	unit length line segment. Have students create units of units, such as a “footstrip”. Measure in different-sized units. Coordinate strips of units with un-numbered ruler tools with tick marks at unit intervals only.
Age 8: **Length Measuring (LM)**: Considers the length of a bent path as the sum of its parts (not the distance between the endpoints). Measures, knowing need for identical units, relationship between different units, partitions of unit, zero point on rulers, and accumulation of distance. Begins to estimate. “I used a meter stick three times, then there was a little left over. So, I lined it up from 0 and found 14 centimeters. So, it's 3 meters, 14 centimeters in all.”	The length scheme has additional hierarchical components, including the ability to simultaneously image and conceive of an object's length as a total extent and a composition of units. This scheme adds constraints on the use of equal-length units and, with rulers, on use of a zero point. Units themselves can be partitioned, allowing the accurate use of units and subordinate units.	Use a physical unit and a ruler to measure line segments and objects that require both an iteration and subdivision of the unit. Fold a unit in halves, mark the fold as a half, and then continue to do so, to build fourths and eighths. Discuss how to deal with leftover space, to count it as a whole unit or as part of a unit.
Age 9: **Conceptual Ruler Measuring (CM)**: Possesses an “internal” measurement tool. Mentally moves along an object, segmenting it, and counting the segments. Operates arithmetically on measures ("connected lengths"). Estimates with accuracy. “I imagine one meter stick after another along the edge of the room. That’s how I estimated the room’s length is 9 meters.”	Interiorization of the length scheme allows mental partitioning of a length into a given number of equal-length parts or the mental estimation of length by projecting an imaged until onto present or imagined objects.	In "Missing Measures," students have to figure out the measures of figures using measures for a subset of sides. Children employ explicit strategies for estimating lengths, including developing benchmarks for units (e.g., an inch-long piece of gum) and composite units (e.g., a 6-inch dollar bill) and mentally iterating those units.
Age 10: **Integrated Conceptual**	This action scheme anticipates	Compute perimeters,

Developmental Progression	*Conceptual Structures and Strategies*	*Instructional Tasks*
Path Measuring: Processes perimeter as an integration of sets of units and as a flexibly wrapped single collection of units (the entire perimeter. Computes perimeter of a polygon, including complex cases and determines sides from perimeter.	and monitors sets of related cases of a figure under an operation (like a change of shape, with a constant perimeter). The labels along a ruler are understood as symbols showing cumulative distance from an arbitrary point and iterative quantity. The system operates on mostly static internal images, relating parts of paths to the entire path.	including units and divisions of units, even when measures of sides are non-integers and some measures are missing but can be computed.
Age 12 +: **Coordinated, Integrated Abstract Measures with Derived Units** (Level of argumentation and verification of claims). Coordinates and operates on collections of units, collections of units of units, and on collections of entire paths (complex, bent paths). Students may address rates (derived units) as support for arguments about complex interactions.	This integrated action scheme engages dynamic imagery to coordinate and operate internally on collections of units, collections of units of units, and on collections of entire paths (complex, bent paths). Uses derived units as a dimension, without requiring decomposition to the involved sub-units or dimensions. Still attends to the dimensions directly, as an integrative relation.	

This paper narrates our implementation of the LMLT to provide differentiated instruction through cycles of formative assessment and appropriately targeted task sequences. We also describe the process of task design for formative assessment, commenting on the varying types of tasks and the roles they played in the developmental history of these case study children. We attribute the growth of children's understanding of quantity among the case study children to engagement with conceptual elements of the LMLT.

Theoretical Framework for Describing Mathematics Learning and Development

We view the development of children's understanding of geometric measurement through a theoretical lens termed *hierarchic interactionalism* (Clements & Sarama, in press, p.

464). Our view addresses the influence and interaction of global and local (domain specific) cognitive levels and the interactions of innate competencies, internal resources and experience (including instruction and available tools from the culture). In general, we expect children to progress through levels of understanding for measurement of length or distance in ways that can be characterized by specific mental objects and actions (i.e., both concept and process) with the most visible progress through levels for domain-specific topics. For example, researchers have identified levels of increasing awareness and competency for partitioning objects into same-size segments to be used as unit objects (Cannon, 1992; Steffe & Olive, 1996).

We expect that a critical mass of ideas would need to emerge within any given cognitive level before thinking characteristic of the next higher level would become evident in the child's thinking and behavior (Clements, Battista, & Sarama, 2001). This often involves "fallback" to prior levels of thinking under increasingly complex demands as typified in our findings about children's developing sophistication of units and units of units for measuring perimeter (Barrett et al., 2006). Most often, progressions follow a predictable pattern of learning activity: first, there are sensory-concrete levels wherein perceptual concrete supports are necessary, and reasoning proceeds through limited cases; second, verbally based generalizations follow as the child abstracts ideas beyond the immediate sense; third, there are integrated-concrete understandings that rely on internalized mental representations that serve the child as mental models for operations or abstractions (Clements & Sarama, in press, p. 465). We also see concepts and skills proceeding in tandem (Siegler, 2003, p. 223), with a benefit accruing where concepts are established to give context and foundation to the skills.

Hart (1981, pp. 232-233) posed questions regarding the veracity of hierarchical levels of growth. These issues are captured well by Steffe and Cobb (1988) in four essential questions: (1) are the tasks that relate to each level consistent? (2) are successive levels incorporating prior levels completely? (3) is the order of the levels invariant? and (4) are the aspects of each level coherent? We address these four questions through the analysis of our learning trajectory for children's length measurement.

In particular, we focus on the following research questions:

(1) How do students develop coherent knowledge and integrated strategies for measurement from preKindergarten age through grade 5? In this paper, we focus attention on grades 2 and 3.

(2) How does the Length Measurement Learning Trajectory support formative assessment and structured learning sequences?

Method

Student Selection

For the purposes of this study we selected 16 second-grade students from two separate classes from a small elementary school in the Midwest. Each of the 45 second-grade students from two different classes were administered a pretest consisting of 29 measurement tasks. Eight students were selected from each class, two from the top third, four from the middle third and two from the bottom third based on their performance.

Teaching Episodes

Once the students had been selected for the study we conducted a teaching experiment with these students (Steffe & Thompson, 2000). There were four main purposes. First we wanted to verify the LMLT. Secondly we expected to increase the students' knowledge of measurement using the LMLT. Thirdly we wanted to describe the process of using the LMLT to guide our instruction. And finally we wanted to modify the LMLT based on working with the students.

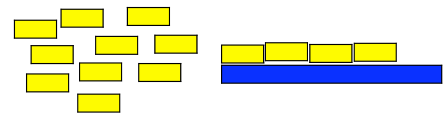
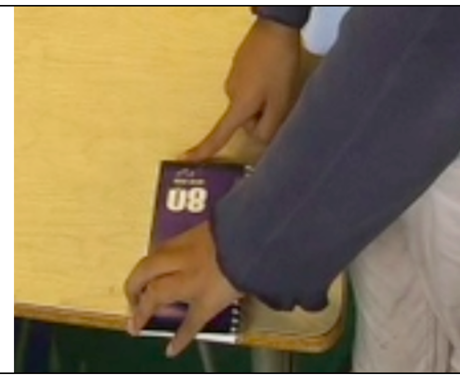
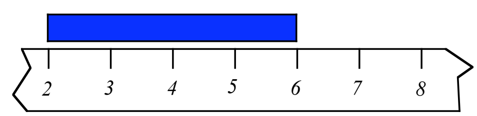
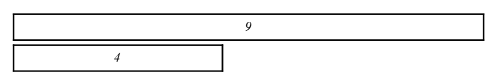
Prior to beginning the first teaching episode we conducted a follow-up interview to clarify student responses on the pretest. The eight students selected for case study were put into groups based on their placement in the LMLT. Teaching episodes were conducted with individual students, and were video taped for further analysis. Each student's responses during the teaching episode were used to reevaluate his or her placement in a level and change the placement if needed. This process was then repeated; students would be grouped based on their level, the next teaching episode would be planned for each group, the teaching episodes would be conducted and again the students level would be reevaluated. Periodically we tried to summarize our findings into a succinct set of tasks that we would present to the background students. This process allowed us to modify and improve our tasks based on reflection and discussion that emerged from the individual teaching episodes.

Our analysis addressed emergent themes from our ongoing and post hoc review of the teaching experiment work: primarily the concepts of unit, of rational number as comparisons of quantity, and operations on variable quantities that included rational numbers. We investigated the children's growth along the learning trajectory; we asked how children responded to sustained challenges to coordinate their number scheme and spatial schemes for linear objects, in relation to measurement tools and diagram use within a concept-focused learning environment (a clinical teaching experiment) addressing length measurement.

Tasks Related to our Trajectory Through the Teaching Experiment

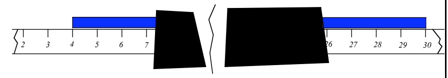
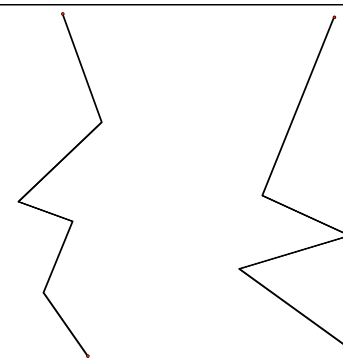
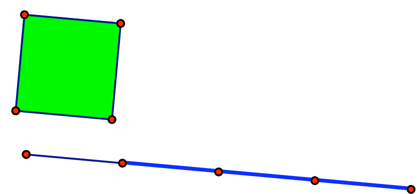
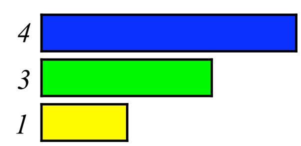
We developed a set of tasks through the interactions with three distinct groups of students to support the differential instruction indicated by the learning trajectory. The tasks were designed in part by referring to the existing task notes in our initial LMLT, in part by referring to the cognitive accounts of thinking at levels exhibited by the students in this study, and in part as a response to the thinking and strategies demonstrated by the students we were engaging in the learning trajectory. The next section of the paper narrates this complex interaction between students exhibited actions and expressions of thinking processes while working on particular tasks, the researchers' ways of selecting and using tasks to fit hypothesized models of the student's levels, and the development of further tasks to elaborate on thinking characteristic of a given level. The following table displays the major tasks used in our teaching experiment during the initial year of our work with the case study students (See Figure 1a and Figure 1b).

Figure 1a. Researcher-developed Tasks for Length Measurement

Task	Purpose	Description	Image
a. Yellow-Strip Tool*	To determine if the student was capable of displaying EE level thinking and to help the student identify units of length. Use: assess and instruct.	The student was given one-inch yellow-strips and asked to measure longer strips that were cut to whole number lengths. Next, use a taped strip of one-inch strips to measure.	
b. Compare length of the table and width of the door	To motivate URR strategies and precise measure by providing a real measurement task without a procedure. Judge whether the student was at least an EE measurer. Use: assess and instruct.	The interviewer measured in a way that was obviously wrong because of the gaps and overlaps to see how the student would react to the measuring. Then the student was given the chance to offer a technique for measuring more accurately.	
c. Broken Ruler*	To determine if the student is beyond the EE level and can correctly identify the units of length from the tool or can re-zero the object. Use: assess and instruct.	The student is given a ruler with the ends broken off. In some instances the student was not allowed to slide the ruler along the object to re-align it.	
d. Wooden Rods	To determine if the student has the arithmetic ability to answer the broken ruler tasks. Use: assess only.	Using twelve different unmarked and unlabeled wooden rods one rod was set below another and the student was asked which rod would be added to shorter rod to match the length of the longer rod.	

* These tasks were most helpful in assessing shifts from one level to the next.

Figure 1b. Researcher-developed Tasks for Length Measurement

e. Ribbon partially covering part of the standard ruler*	To determine if students can coordinate arithmetic and measurement when units are not visually available. This ability occurs after URR level. Use: assess and instruct.	A ribbon covered some of the standard ruler that was needed for the student to measure a strip that did not start at the zero point.	
f. Bent-Path Analysis	To determine if the student understood bent paths and if they differentiated between distance traveled and displacement. We were also interested in observing how the students dealt with accumulation of length around turns. Use: assess only.	The student was shown two different paths and asked to predict which was longer. Then the student compared the paths using a standard ruler to see which was actually longer.	
g. Compare a pipe cleaner and the perimeter of a set of tiles*	To observe how the students dealt with the coordination of units and super units in a context that they have not experienced previously. Use: assess and instruct.	The student was shown that a four-inch pipe cleaner was the same length as the length around one square inch, then asked to compare the length of the pipe cleaner to the path around the set of tiles.	
h. Draw a set of line segments that are decreasing in length*	To help a student who is counting tick marks when measuring that this technique will result in assigning a length of one inch to something that has essentially no length. This task relies on the importance of zero and one to force a student to develop a consistent understanding of measurement Use: instruct	The student is asked to draw or measure a set of objects with decreasing lengths starting with three inches down to one inch.	

Differentiated Instruction Using the Trajectory

The table below summarizes our findings for the eight students who were interviewed by the research team multiple times each semester beginning in February 2008.

Table 2. Number of Students at a Given Level of Thinking Over Time

Learning Trajectory Level	Number of Students (April 2008-Gr. 2)	Number of Students (October 2008-Gr. 3)
End-to-End (EE)	1	0
Unit Relater/Repeater (URR)	7	2
Length Measurer (LM)	0	6

Viewed from another perspective, here are three different groups of students, based upon their learning trajectory level in April 2008 and in October 2008. These groupings are important in that they were used to help determine appropriate tasks and interview protocols for subsequent teaching episodes with the research team.

Table 3. Number of Students Exhibiting Specific Transitions from April to October 2008

Learning Trajectory Level (April 2008)	Learning Trajectory Level (October 2008)	#Students
End-to-End (EE)	Unit Relater/Repeater (URR)	1
(weak) Unit Relater/Repeater (URR)	Unit Relater/Repeater (URR)	1
(clear) Unit Relater/Repeater (URR)	Length Measurer (LM)	6

Most specifically, the children's growth along the trajectory proceeded smoothly as described in Table 4.

Table 4. Level Assignments for Each Student Over Time

Group	*Child's name*	Initial assessment	Follow -up	TE 1: April 2008	TE 2 May	TE 3 May	TE 4 Fall 2008
1	*Sara*	ILC	ILC	EE	EE	EE	URR
2	*Anselm*	EE	URR	URR	URR	URR	URR
3	*David*	EE	URR	(LM)	URR	URR	LM
3	*Abby*	URR	URR	URR	(LM)	LM	LM
3	*Owen*	URR	URR	URR	URR	(LM)	LM
3	*Ryan*	URR	URR	LM	LM	LM	LM
3	*Drew*	URR	URR	LM	LM	(LM)	LM
3	*Arielle*	URR	URR	LM	LM	LM	(CM)

The data shown in Table 4 argue for even, stepwise growth along the trajectory for all eight students, with two exceptions. David reverted to URR in TE 2 and 3 after showing strong evidence of strategies consistent with a Length Measurer level during TE 1. Not until TE 4 did David make a sustained shift up to the Length Measurer strategies. Drew also appeared to falter during TE 3, reverting to a less clear level of Length Measurer. We understand these two exceptions as evidence of growth patterns that are uneven yet consistently pressing up along the trajectory. Until a child sustains evidence of a given level for two or more observations, we hesitate to confirm their change of level (Clements & Sarama, 2007, pp. 464-466).

Next, we describe the patterns of growth displayed by each case student. We provide detailed narratives discriminating between the first two students and the third group of six students, though we limit our detailed narrative of the group of six to a single student. These first two students (grouping 1 and grouping 2) typified other students in the two classes who performed comparably on the initial assessment and who evinced similar strategies in classroom sessions (data we discuss in a forthcoming paper).

Grouping 1: Movement from End-to-End to Unit Relater/Repeater

Focus Narrative: Sara

On the initial assessment, Sara consistently overlapped a short object along an object she was asked to measure, rather than iterating it to find a direct comparison of its unit length to the object length. On a similar task, she reported different numbers to describe the length of the same object measured repeatedly. In the follow-up interview (April 2008), the student was asked to use a single yellow-strip to measure a five-inch strip (see Figure 1a). When using a one-inch strip to measure a five-inch long blue strip, Sara reported ten, eleven, and fourteen on three different attempts to measure. She iterated the yellow-strip inconsistently without attention to precise iterative movement along the strip. However, Sara correctly reported the length of various strips ranging in length from 4 to 7 inches when given enough one-inch long yellow-strips to cover the strip she wanted to measure. Thus, we classified her thinking at the EE level in the learning trajectory.

To check her ability at the ILC level (the prior level), the interviewer asked her to compare two towers made out of blocks during the next episode (TE 1). It was predicted that the student would want to move the items close to each other and to the same level because of her established ability at the Direct Length Compare (DLC) level. Alternatively, we wondered if she might directly count the blocks. When asked to compare the towers, Sara attempted to use a broomstick as an indirect object. She held it against one tower and started counting where the broomstick hit the base. She slid her finger up at inconsistent intervals and counted to six for the upper tower and counted to five for the tower on the floor. Sara suggested that if she moved the towers together she could get an answer easier, a characteristic of students at the DLC level.

Sara exhibited strategies at the EE level in the first teaching episode. We predicted she would not lay sticky-notes end to end to measure a length, based on our review of her actions during the initial assessment (March 2008). Actually, Sara was able to correctly use sticky-notes to measure the length of tape and explain important aspects of measuring. For example she talked about beginning at the end of the item to be measured, making sure all of the tape is covered, and lining the strips up “like a choo-choo train”. Though she was successful with the EE task, when pushed to a Length Unit Relater and Repeater (URR) situation by having all but one strip taken from her, she was not able to measure a different object.

To account for awkwardness using the broom as a tool in the previous TE, the research team decided to give her a similar task during TE 2 (May 2008). The team wanted to see if she could use a third object, a string that was longer than two separate objects. It was predicted that she would count her own hand motion along the string instead of marking the height of one object and comparing that point of the string to the other object's height. However, before being given a string, the student predicted, "If I put them together, the white board is skinnier so it's taller." Even when given the string, she struggled to use it to compare two objects not next to each other. We note that she was apparently comfortable with her visual scan of the two distant objects, leading her to conclude the white board must be taller. This appears to be an adequate fall-back level of thinking on this task (from the student's standpoint it was an adequate measure). Thus, we saw a DLC strategy where we had expected to prompt a higher level strategy. This emphasizes what we have come to see as the need for persistence in checking with different tasks to assess a *potential level*.

To help motivate precise measurements at a higher level on the trajectory (URR), Sara was asked to compare the length of the table and the width of the door only using a small notebook as a tool (Task b in Figure 1a). We predicted Sara would notice and object to the interviewer's exaggerated errors allowing gapped or overlapped positions of the notebook. The intent of the interviewer was to focus Sara's attention on the need to imitate a set of notebooks laid end to end. When Sara initially tried to do a better job herself, she was not precise in her placements of the notebook. The interviewer decided to model careful use of her finger to see if Sara would see the relation of iterated units and end to end collections of units she might count to find a length. Subsequently, when pressed about how she knew where to put the notebook each time, Sara said she could use her finger, but she did not consistently place her finger to mark the endpoint of the notebook as she slid it from one position to the next.

During TE 3 (May 2008), Sara showed some evidence of URR strategies, but was not consistently iterating to find lengths. Yet, she always succeeded when given enough units to fill the object in an EE strategy. For this reason, we kept our EE level label. Sara was not yet able to correctly answer “broken ruler” tasks (see Figure 1a); she reported one length as five inches using one tool and six inches using another tool.

The limitations observed during spring influenced our plans for fall and TE 4 (October 2008). Because Sara was lower on the learning trajectory than the other case students, we created a different sequence of tasks. We created tasks that we thought would prompt transitions in her use of units similar to EE and URR levels. We predicted Sara would still struggle with broken ruler tasks based on TE 3. The team also thought she would count the tick-marks without attributing a zero point first. Next, we designed a set of tasks in which Sara would use large units to build her own ruler tool. Our prediction was that using big units would engage a different, more precise action scheme with unit iteration, since it would be closer to the enactment of a foot path, a scale of activity more closely related to the everyday actions of young elementary students. This differing scheme for unit iteration at a larger scale was hypothesized in keeping with work by Reynolds and Wheatley (1997; 1996). They proposed that changes in scale are significant and provide ways for children to integrate and generalize ideas about their activity at one scale level that closely maps to their own motion along foot paths either up to larger scale settings or down to more precise settings with small or tiny units.

We expected that this transition would provide Sara material to reflect on at a more workable scale (the big scale) and promote generalization of the scheme into her activities with small scales on her desktop using cm, in or even smaller units (Lehrer et al., 1999; Lehrer et al., 2003; Stephan, Bowers, Cobb, & Gravemeijer, 2004).

As expected, during the initial part of TE 4, Sara read the number at the end of the strip to find length. However, she was successful when she could use the yellow-strip tool, a tool we created by taping 1-inch strips together (consistent again with EE level).

Next, during TE 4, one pen-length was drawn on a large sheet of paper. The interviewer described the length of the line segment she had drawn as “a one-pen-length path”. The student seemed to understand the idea, and was asked to make her own path that was six pen-lengths long. When she made her paths however, the lengths of the pens were not uniform. The interviewer demonstrated a very obviously incorrect measurement of the table by gapping widely and also overlapping; this exaggerated gapping and overlapping sparked the student into accurately marking a path with 6-pen lengths. Sara was able to talk about the six pen lengths and draw a different path. When she labeled the length of the paths, she wrote “6 lins” (six lines). We believe Sara understood the idea of a unit when the unit was very large. She correctly iterated the pen length and could identify the unit of length for the first time.

Because students at the URR level require close attention to perceptual objects to reflect on the correspondence of motion and distance with counting all schemes (LMLT), we wanted to create a task where measuring would be associated with motion in a different way than the motion of a pen-length. The research team predicted that the student would be able to complete a task using her foot as the unit especially after she completed the question involving the pen-lengths. Because using feet as a unit is a low URR task and at the time she was bridging the gap between the levels EE and URR, the research team predicted that the physical iteration motions with her feet would prompt her see that the earlier part of the path had been measured. She did not correctly iterate the unit at first. In response, the interviewer exaggerated the overlapping use of steps in relation to the counting sequence as a way of prompting Sara to notice and reflect on the iterative action scheme for sliding a single unit along to measure. Now that she could reflect on having seen the interviewer move her foot too little for this stepping motion while still counting as if they were new steps, Sara objected. Sara was successfully iterating and monitoring the process of unit iteration as carried out by another, suggesting the establishment of a unit repeater scheme (although she was unable to correctly relate changes in unit size to predict changes in the quantity value during this episode).

For TE 5 (October 2008), the team put together additional tasks that incorporated large units and motion to help Sara identify and iterate the unit more carefully. The tasks were created because she seemed to have success in the TE 4 with the pen-lengths. The student recognized a poorly done 5 pen-length line segment and was able to give a strategy (using fingers) for making sure that the ends of the pencil line up. The interviewer showed Sara that the tick marks work like the finger marker. She made a line segment on a long strip of paper that was 10 pencil-lengths long and could show where one pen-length and all ten pencil-lengths were. She seemed to use a sweeping motion to indicate where a pen length was. She wanted to label the pencil-lengths at the tick marks and did not want to label the spaces. She was asked again later in TE to label the spaces but did not accept the idea. It was a positive step for Sara that she did not need to label the spaces because she could explain that the number at the tick mark represented the prior pen length. She was building a clear correspondence between tick marks and units. She reasoned that you could not label in the space because that was not a whole pencil length yet.

During this episode, Sara was able to correctly measure strips on the tool, even when the strip was not at the zero-point. She was solving broken ruler tasks correctly, something that she had not done confidently or correctly before. When the strip went from 3 to 6, she could identify its length as “3 pencil lengths”. When asked why it was not 6, she could show the 3 pencil lengths. Lastly in TE 5, the student was asked to measure the table using her new tool and she was not hesitant about starting at the 10 and working backwards. However, Sara got answers of 7, 8 and 9 on three attempts at the same length, struggling to maintain the new strategy. She started sweeping, and then transitioned to counting tick marks without attaching the marks to pencil lengths directly. She appeared to falter when she stopped sweeping thru “lengths”.

Sara apparently gained proficiency at consistent iteration of units as the interviewer modeled poor strategies in an exaggerated way, prompted Sara to use large units, and emphasized the need for a sweeping motion as a record of unit lengths between ticks.

Grouping 2: Consistent use of Unit Relater/Repeater Throughout

Focus Narrative: Anselm

In the follow-up interview to the initial assessment (April 2008), Anselm was asked to measure the length of an object using a measurement tool that had the end broken off between the one and two (see task c, Figure 1a). During the initial assessment it appeared that Anselm slid the object to the left until the right end of the object was in line with the next lowest integer marking on the tool. When given a chance to explain his thinking on this problem Anselm revealed that he had used a sophisticated strategy to assure that he had translated exactly one inch. This strategy required Anselm to identify a unit as well as copy, or iterate it. This strategy, although incomplete, demonstrated Anselm's ability to iterate a unit of length which is beyond the level of an End-to-End thinker. This evidence caused us to change his placement to Unit Relater and Repeater as well as plan his next interview based on this placement.

Throughout TE 1 (April 2008), Anselm struggled with unit identification. In many instances he was counting both the initial and final tick mark causing him to report one extra inch than the actual measure. In other instances, when the object was aligned with the end of the measuring tool, Anselm was able to arrive at the correct measure by counting the tick marks, but not counting the initial mark. We feel that these two struggles are typical of a student who is at the length Unit Relater and Repeater level of strategic development. Certainly Anselm consistently demonstrated his ability to use the tool made from a set of ten 1-inch strips taped in a line indicating that he is at least at the End-to-End measurer level. We found that Anselm's readiness to iterate unit length objects to establish length was clear from his responses to the spaghetti and ribbon task in the initial assessment.

In planning Anselm's TE 2 (May 2008), the research team focused on developing and presenting tasks that would help Anselm to identify a unit consistently across any of the tools discussed in teaching episode 1. The main task selected to meet this goal was to have Anselm draw a 3-inch line segment, then a 2-inch line segment, and finally a 1-inch line segment. As this sequence continues it was predicted that Anselm would have to deal with the realization that what he would call a 1-inch segment would have no length.

TE 2 opened with Anselm transitioning from incorrectly counting the tick marks to measure the length of objects to correctly counting the spaces, as he called them. The first teaching episode had ended without Anselm having resolved this issue and the second began without any remnants of this confusion. From the outset of TE 2, Anselm consistently identified units correctly on each task, without constant recourse to the yellow set of 1-inch strips tool. After 20 minutes of apparent success with URR level strategy, the interviewer asked Anselm to recall his old technique of counting the tick marks to count units; this prompted Anselm to revert to this prior strategy of counting tick marks rather than counting spaces. His strategy was less firmly established than we had thought. With just a slight suggestion of his old strategy Anselm slid back into counting tick marks and counting the initial mark as one rather than zero. Due to this unstable understanding of the zero-point we placed Anselm in the URR level again.

During TE2, seeing Anselm revert to counting tick marks, the interviewer asked him to draw successively shorter line segments toward a length of 1 (task h, Figure 1b). Anselm began by drawing a 3-inch line segment but was unsure if it would measure three or four inches. Next he moved over on his paper and drew a 2-inch segment but called it three. In response, the interviewer asked Anselm to draw another segment that was one inch shorter than the last (that he called three inches). Anselm drew a 1-inch segment and said that it was one inch. The interviewer restated the measures for Anselm's drawings; this one is three inches (pointing to a 2-inch segment) and this one is one inch (pointing to a 1-inch segment)? Anselm measured his segments again to verify and said, "that's really weird!" At this moment Anselm continued by drawing several other 1-inch and 2-inch segments but when he was asked to measure them he counted the tick marks and reported one extra inch for the length of each. At one moment, Anselm appeared extremely confident in drawing a 1-inch segment, but when asked to measure the segment he lined up the measurement tool saying "oh come on!" Anselm decided that he wanted to try drawing a 1-inch segment one more time; he started to try again, stopped and said "this is so confusing!"

Task 5 (TE2)

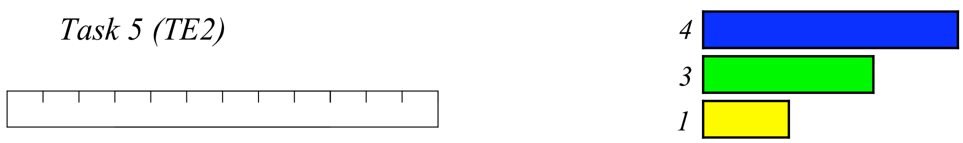


Figure 2. The set of 3, 2 and 1 inch strips as Anselm labeled them during TE 2.

Next the interviewer transitioned into the same basic task, but this time Anselm was given a 1-inch, 2-inch, and 3-inch strip to measure rather than draw. Anselm was asked to measure the 3-inch strip with the tool shown in figure 2. Anselm immediately explained that the this strip measured three inches by visual inspection, paused, and then aligned the unnumbered ruler and reported four inches. Here Anselm said that he was confusing himself by counting tick marks instead of spaces, but later concluded that he should have been counting tick marks.

Anselm was shown a 2-inch strip and asked how long it must be. Anselm seemed confident that the strip was two inches by looking at it (which we believe engaged his internal schema for scanning inch long gaps) but then compared it to the white un-numbered tool with tick marks at inch intervals (the measurement tool here) and reported a length of three. When Anselm was shown a 1-inch strip he predicted that it would measure one inch and measured it to be one inch. This action set Anselm's schemes into an initial coordination between figural images and his tick counting to indicate length for these short objects at least; he looked back and forth between several of the strips as well as the measurement tool.

At this point Anselm had measured all three strips and reported the 1-inch strip to be one inch, the 2-inch strip to be three inches, and the 3-inch strip to be four inches. These three strips are set in front of Anselm as shown in figure 2. The interviewer asked Anselm how many of the 1-inch strips would fit along the 3-inch strip and Anselm reported that two would fit. Next the interviewer recapped Anselm's measures of one, three, and four, and was about to ask Anselm what a 2-inch strip would look like. But Anselm blurted out "actually you should count the spaces", demonstrating that two of the 1-inch strips would fit along the 2-inch strip. When asked the length of the 3-inch piece, he iterated the 1-inch strip along it and reported the strip to be three inches. Anselm also measured the 3-inch strip with the unnumbered, unbroken ruler and reported the length to be three inches "because one entire space is one inch."

In summary, TE 2 focused on prompting the coordination of a variety of tools and drawings to show units and measures. During this session he correctly identified and counted units for the first time. Although the research team felt that it was important for Anselm to count spaces to find units we also felt that this strategy should be extended to coordinate with meaningful interpretation of tick mark values. Motivating a new more efficient strategy for measuring longer objects was the impetus for teaching episode 3.

TE 3 (May 2008) was centered on two main tasks. These two tasks were developed to see how students would respond to measuring lengths when the unit indicators were no longer perceptible. The first set of tasks involved an 8-foot measuring stick with tick marks labeled at one inch intervals from one to 95. The second set of tasks involved a set of wooden sticks ranging from 1-inch to 12-inches. The sticks were approximately ½ inch wide and ¼ inch thick, had no markings on them and the set consisted of only whole number lengths (see tasks d and e, Figure 1).

The start of TE 3 took this author by surprise as Anselm reverted to counting tick marks rather than counting spaces again. The realization that Anselm verbalized regarding spaces as units had seemed so clear and strong in TE 2, yet in just over a week Anselm reverted to tick counting. This fragile and implicit understanding of zero-point on a ruler is evidence that Anselm was not yet at the length measurer level although he had begun to show some abilities indicative of this level.

For example Anselm was able to connect the measurement world with the arithmetic world on the task with the different length sticks. When asked how long of a stick would be needed to pair with a 7-inch stick to match the length of a 12-inch stick Anselm responded that he needed a 5-inch stick because 7+5=12. This skill alone would be indicative of a student at the length unit relater and repeater level. However, the interviewer also identified the blank 5-inch space and asked Anselm how long this space would be if he measured it with a ruler. Anselm seemed to estimate at first stating it would be five or six, but then after the interviewer restated the set-up Anselm paused for a long time and finally responded that the space would be five inches and that he knew that "just by counting on regular numbers, it would still be the same." This statement is evidence that Anselm may be close to moving into the Length Measurer level.

TE 4 (September 2008) was the first of the year after summer vacation; therefore we wanted to check Anselm's abilities on the broken ruler task. Additionally we wanted to ask some questions that would provide Anselm with the opportunity to display his ability to answer some questions at the Length Measurer level, in this case the comparison of two bent paths. The final part of the interview attempted to explore the student's ability to work with composite units and required the student to coordinate two measurement units, one being four times the length of the other.

Anselm's difficulties with identifying units of length persisted as he continued to count tick marks, incorrectly, in some instances and count spaces correctly in others. Although Anselm seemed capable of dealing with the bent path task, he did measure two separate portions of the second path incorrectly with the yellow strip tool. These difficulties continue to place Anselm in the Unit Relater and Repeater level and motivated later teaching episodes, which focused on eliminating his confusion with zero and one, as well as unit identification processes.

What seemed to help Anselm learn about unit identification was the transition from using a collections of individual 1-inch yellow strips to using a taped-together strip made of the yellow strips, the sweeping motion while drawing segments of varying lengths, and measuring a set of objects with decreasing lengths down to one inch. We believe Anselm's limited number fluency has constrained his progress into the subsequent level that involves quantitative operations.

Grouping 3: Movement from Unit Relater/Repeater to Length Measurer

Focus Narrative: Arielle

In our initial assessment of Arielle (February 2008), we classified her measurement strategies as Unit Relater Repeater (URR). In part, this was based upon the observation that Arielle seemed to line up a 5-inch strip from 2 to 7 on a broken ruler (see task c, Figure 1a), but she reports a length of 6 inches. We conjecture that she was likely counting tick marks, but this strategy can be identified more precisely in the follow-up interview (April 2008). Indeed, in that later interview, Arielle counts tick marks, including the initial mark, to obtain one extra for total length. Arielle's apparent confusion regarding tick marks and length is most clearly demonstrated when she correctly identifies the length of a 1-inch strip but then proceeds to count both tick marks bracketing this inch, concluding it must be 2 inches.

From the follow-up interview to the first teaching experiment (TE 1) (April 2008), we predicted Arielle would obtain different measures for standard and broken rulers because the former has an actual zero (that she will presumably skip) but the latter does not. During TE 1, Arielle initially measured a 4-inch strip with both a broken ruler and a standard ruler before reporting its length as 6 inches. After a lengthy pause, she changed her answer to 5 inches, presumably counting tick marks including the initial mark. When asked for an explanation, she said she counted each inch. However, when she demonstrates this strategy, she changed her answer a third time, now correctly identifying the length as 4 inches. Without further prompting, she drew a picture to represent both the strip and a ruler. It is at this point, that we document Arielle's apparent transition from URR to length measurer (LM), span counting length intervals.

As we prepared for TE 2 (May 2008), we planned to have Arielle measure ribbons of different lengths, hoping that she would create a way of building a composite unit to avoid tedious repetitive counting. On the first task, Arielle initially counted each unit, using tick marks correctly to find 24 inches. Later, when asked to measure a longer ribbon (26 inches long), she started to count the first few inches but was interrupted by the interviewer asking how the length of the ribbon without allowing a recount. Arielle stated that it would be 26 inches and explained why she did not count all of the inches. She said, "Since this one's 24 inches and it stops right here and then 2 more inches equals 26." Thus, she confirmed our prediction that she could use known lengths to produce new lengths.

A second task for TE 2 related to the use of a ribbon to completely cover a measuring stick. During the interview, it appears that the starting number influenced Arielle's notion of length. When a ribbon spanned from 4 to 30 inches on the measuring stick (see task e, Figure 1b), Arielle started her count with 5 (rather than 1). To probe this issue further, the interviewer asked her to measure a 3-inch strip starting at four. Arielle identified its length correctly, saying "I counted the spaces." Asked to show this, she illustrated her reasoning with a sweeping motion across the first 3 inches with her finger. Arielle displayed some URR (tick-based) strategies and some LM (span count intervals and coordinating units) strategies.

During TE 3 (May 2008), we asked Arielle to measure a 26-inch ribbon, spanning from 8 inches to 34 inches on the measuring stick. However, we covered a portion of stick with a second ribbon. We predicted that Arielle would more completely make the transition to LM by connecting length measure with arithmetic. In this case, our predictions proved accurate. Asked to explain her correct answer of 26 inches, Arielle said “I am adding 10 to 8, and I am doing it until I get to 34.” A written record of her work shows Arielle first added two groups of 10 and later groups of 2 and 4 to produce a final sum of 34 inches. She collected these values to state a total length of 26 inches for the ribbon. In a related task, we expected Arielle to measure a 51-inch ribbon, placed between 13 inches and 64 inches, using the missing addend model for subtraction. Arielle initially repeated her earlier strategy of “adding on” yielding a total of five 10s and one 1 to reach as far as 64 inches. In a subsequent problem, however, Arielle used the missing addend model for subtraction to compute the length of a long ribbon.

A second task in TE 3 involved a series of wooden rods with integral lengths varying from 1 inch to 12 inches (see task d, Figure 1a). We predicted that Arielle would be able to decompose a 10-inch rod into two or more shorter rods. Once again, Arielle demonstrated that she could successfully complete this task. In fact, she produced multiple correct decompositions, including an interesting case where an 11-inch rod could be shortened using a 1-inch rod to produce a net length of 10 inches. This episode provides evidence that Arielle was once again working as a length measurer (LM). This time she demonstrated her ability to partition and to decompose length units.

In the Fall of 2008, we resumed our interviews with Arielle, now a 3rd grader. To assess whether any reliance on tick marks remained, we planned to return to the broken ruler task (see task c, Figure 1a) during TE 4 (September 2008). Arielle correctly identified the length of a 5-inch strip, placed from 2 inches to 7 inches, but now she focused on the end tick marks of each inch-long gap. On a related task, she ignored the first tick mark "because that is where it starts." Despite additional probing, Arielle was unable to elaborate on her reasoning for omitting the initial tick mark.

Later during TE 4, Arielle is presented with two bent paths (see task f, Figure 1b). We predicted that she would be able to coordinate lengths from the various paths to produce the total length. She was successful, working with fractional inches to produce the correct total lengths of 9 ¼ and 11 ½ inches, respectively. This is our first, albeit tentative, evidence, that Arielle started a transition to conceptual measure (CM) as she coordinated units in a bent path and worked successfully with fractional units. However, her behavior during this interview most often fit into the length measurer (LM) category.

A final task during TE 4 required Arielle to compare the length of a 4-inch (unmarked) pipe cleaner with the perimeter of a square with 1-inch sides (see task j, Figure 1b). We anticipated that she would be able to identify the perimeter of the square correctly. Arielle eventually provided the correct ratio of pipe cleaners to the perimeter of rectangles composed of first three and later two squares. Intriguingly, she refined her correct answer of 1 ½ pipe cleaners for the perimeter of two joined 1-inch squares in terms of ½ pipe cleaners, eventually claiming "3 half-pipe cleaners goes around once." This final episode provides evidence that Arielle was able to iterate units of units with the entire perimeter, a behavior associated with the conceptual measurer (CM) strategy. Overall, we classified Arielle as length measure (LM) but showing an initial transition to conceptual measurer (CM) during this interview.

Additional Brief Overviews: Drew

Drew was classified as URR based upon his work during the initial interview (February 2008) and follow-up interview (April 2008). This decision was based upon his apparent recognition of the need to translate a 5-inch strip past the broken end of the ruler in the broken ruler task (see task c, Figure 1a). He also showed the ability to estimate an inch during the same interview. Beginning with TE 1 (April 2008), Drew made the transition to LM. At this point, he correctly used the broken ruler to measure a strip by spanning along the intervals and counting, showing that he understood that the ruler was a record of an interated unit. This LM behavior was also noted in subsequent interviews. In TE 4 (October 2008), Drew appears to begin the transition to CM when he successfully used both unit and subunits during the task with the pipe cleaner and the red squares (see task g, Figure 1b). At one point, he used the phrase "half of a quarter" to describe a subunit that indicates a successful transition between fourths and eighths.

Additional Brief Overviews: Ryan

Ryan exhibited URR strategies during the follow-up to the initial interview (April 2008). He was able to find the total length of a 4-inch strip and a 7-inch strip placed end to end. However, he was unable to decompose that length into its basic units. Ryan makes the transition to LM during TE 1 (April 2008). At this point, he consistently counted gaps rather than tick marks, and he no longer needed to start at the beginning of the ruler because he viewed the numbers on the ruler as arbitrary. In subsequent interviews, Ryan continues to operate at the LM level.

Additional Brief Overviews: David

Based upon evidence from the follow-up interview (April 2008) of the initial assessment (January 2008), we classified David as URR. Evidence for this conclusion surfaced during the broken ruler task (see task c, Figure 1a), in which David attempted to answer this question by identifying a length of one inch on the object and then translating the object to the left one unit to compensate for the fact that the ruler did not start at one. This strategy also highlights his confusion with the zero value on the ruler, even as he does consistently attend to and iterate a unit feature. David appears to begin his transition to LM during TE 2 (May 2008). During this interview, he demonstrates the ability to connect measurement with number, such as when he computes the length of a ribbon by adding onto the lefthand endpoint (4) until reaching the righthand endpoint (30). The transition to LM became more complete in TE 3 (September 2008). In the bent paths task (see Figure 1b), David showed no confusion when dealing with the turns in the path and clearly paid attention to the length of the path rather than the displacement between the initial and final positions.

Additional Brief Overviews: Abby

During the initial assessment (February 2008), follow-up interview (March 2008), and TE 1 (April 2008), Abby's strategies for measuring length were classified as URR. In part, this classification was based on her use of units in repeated actions when assigning length, but her consistent inability to coordinate unit iterations symbolized by the tick marks along a ruler in relation to the gaps between the tick marks if the object to be measured was unaligned with the end of the ruled tool. During TE 2 (May 2008), Abby's thinking and strategies reflect a shift from URR to LM. She consistently assigned linear objects values based on her coordination of pointing along the sequence of tick marks or end points of unitary segments and the whole numbers (i.e. 0, 1, 2, …). She set this counting sequence into relation with the numbered values along the ruler tool. For example, to measure an object with one end at the 3 mark and the other at the 7 mark, she explained aloud, "3 is 0, and then 4 is 1, and 5 is 2, 6 is 3, and 7 is 4, so it is 4 long." This pattern continued with TE 4 and the additional interviews conducted during fall 2008.

Additional Brief Overviews: Owen

Owen demonstrated the URR strategy during both the initial assessment (February 2008) and the follow-up interview (April 2008). During the latter session, he worked successfully with several broken ruler tasks. However, he seemed confused by fractional units when they were associated with a ruler. He labeled the tick before the 1.5 mark 1 and the tick mark after the 1.5 mark 2. When asked how long it was from 1.5 to 2, Owen said ½ inch. He explained how the same interview could be one inch on one ruler and half an inch on another by the fact that the ruler with the fraction intervals was "not real." Later in TE 3 (May 2008), Owen began to exhibit early signs of LM strategies. In particular, he was able to make super units made up of every four small steps shown on a task on paper. When asked if he could make "half bit steps, like two small ones," Owen correctly doubled his prior answer to determine the total number of steps using this new unit size. In a subsequent interview, Owen continued to operate at the LM level.

In summary, for this group of students, it was helpful to coordinate multiple representations across tools, to connect the arithmetic of the number line to the hidden collection of unitary spaces within the ribbon task, and to coordinate units and super units within the pipe cleaner tasks.

Conclusions

We have demonstrated predictive and descriptive properties of the Length Measurement Learning Trajectory for research and development. We used the LMLT as a model-building tool to guide the conduct of our teaching experiment; it enabled us to build predictive models of student's thinking and acting given instructional tasks close to their level in the model; it also enabled us to generate or select such tasks that would serve to assess the current level or prompt growth into the succeeding level. Each row of the LMLT can be used to build a model for a particular student.

For the three different groups identified in our sample, the trajectory enabled us to characterize students' current level, including apparent struggles, offered insight into their thinking, and provided tasks to prompt transition to the subsequent level. In particular, we were able to use the Length Measurement Learning Trajectory to identify students' level of thinking along three of the levels on the trajectory: End to End, Unit Relater and Repeater, and Length Measurer. We used it to predict significant conceptual and procedural struggles, to gain insight about students' cognitive processes necessary to select or create instructional tasks. Further, the learning trajectory was helpful in structuring our work with groups of students to identify the next set of appropriate tasks.

First, we identified Sara's thinking as End to End; the student displayed the ability to measure an object by laying out shorter objects end to end and counting them. Second, the trajectory led our research team to examine her ability to coordinate her counting sequence with movement of smaller objects along an object being measured. Since the trajectory predicts progress into URR as a student reflects on a record of an iterated object in relation to a record of a complete collection of objects laid end to end, we posed tasks to constrain her end-to-end scheme by providing fewer objects than needed to match the length of an object. Based on our work with Sara, we expect other students at the end to end level will benefit if they are challenged in three ways: to notice and evaluate the use of poor iteration strategies including exaggerated gapping and overlapping; to use relatively large objects as units, and to trace over each unit with a sweeping motion.

We identified Anselm's thinking at the Unit Relater and Repeater level; the student displayed the ability to measure an object by iterating a single unit and keeping track of iterations by counting. Next, the trajectory led our research team to examine his ability to coordinate his counting sequence with tick marks, spaces and units. Since the trajectory predicts progress into Length Measurer as a student reflects on the consequences of assigning quantity to varying lengths of objects, we posed tasks to prompt the student to forge consistent systems for assigning whole numbers to objects. Based on our work with Anselm, we expect other students at the unit repeater and relater level will benefit from transition between using a set of 1-inch objects and a collection of 1-inch objects joined end to end (recall the taped set of yellow strips), by carrying out sweeping motions to identify units along objects, especially by drawing and attributing quantity to each segment drawn. We believe that students who are unable to find distances along rulers or number lines where several units are obscured from view may struggle to advance to Length Measurer; similarly students who do not maintain a careful correspondence of counting numbers with arithmetic operations will struggle to proceed to the level of Length Measurer.

We identified the thinking of the third group (including Arielle) at the Unit Relater and Repeater level; the students displayed the ability to measure an object by iterating a single unit and keeping track of iterations by counting. Thus we examined their ability to coordinate the counting sequence with tick marks, spaces and units. Since the trajectory predicts progress into Length Measurer follows from a coordination of figural (perceptual) images and internal images of an object's length and by part-whole integrations of unit collections, we posed tasks that would prompt the students to find the length of objects by number labels along tools with the constraint that the object was not aligned at the zero mark along the tool. (Recall the work of Arielle on the ribbon task). Based on our work with students in the third group, we expect other students at the unit repeater and relater level will benefit by carrying out sweeping motions to identify units along objects, especially by drawing and attributing quantity to each segment drawn. Also, it would be helpful to coordinate multiple representations across tools, to connect the arithmetic of the number line to compute a length for an object with a ruler that is partially obscured, and to coordinate subunits, units and super units.

We have revised the LMLT at the E to E, and URR levels. Although we found that it guided our interventions and that it was productive in the specification of intervening tasks to pull students toward higher levels of sophistication, our work with the first group, (exemplified by Sara) indicated a need to elaborate the transition from End to End up to Unit Relater Repeater. We also revised the LMLT based on our work with Anselm and with the third group of students as we elaborated on the set of tasks directed at the transition from Unit Relater Repeater up to Length Measurer. We conclude with a summary account of a revised LMLT based on modifications indicated by our analysis of the work with the case study students. We report our changes to the LMLT for two model rows using italics to show additions or strikethrough font to show deletions.

Age 6: **End-to-End** Length Measuring: Lays units end-to-end. May not recognize the need for equal-length units. The ability to apply resulting measures to comparison situations develops later in this level. (This develops in parallel with "**Serial Ordering** to 6+" *and **Length Relation** of unit size to quantity).* Lays 9 inch cubes in a line beside a book to measure how long it is.	An implicit concept that lengths can be composed as repetitions of shorter lengths underlies a scheme of laying lengths end to end. This initially only applied to small numbers of length parts. Starting with few restrictions, the scheme is enhanced by the growing conception of length measuring as covering distance *(sweeping) coordinated with* composing a length with parts *(unit sticks).*	Use "Length Riddles" providing only one unit per child to compare longer items. *a. Yellow-Strip Tool Create rulers with a few large units to iterate and record*[2] *b. Compare door and desk width*
Age 7: **Length Unit Repeater ~~and Relating~~**: Measures by repeated use of a unit (initially may not be imprecise). Relates size and number of units explicitly (but may not appreciate the need for identical units). Relates size and number of units explicitly. Can add up two lengths to obtain the length of a whole. Iterates a single unit to measure. Uses rulers with minimal guidance.	Action schemes include the ability to iterate a mental unit along a perceptually-available object. The image of each placement can be maintained while the physical unit is moved to the next iterative position. With the support of a perceptual context, scheme can predict that fewer larger units will be required to measure an object's length. These action schemes allow the application of counting-all addition schemes to be applied to measures.	Given a drawing of a 5 unit segment, ask students to draw a 3 unit length line segment (Cannon, 1992). Have students create units of units, such as a "footstrip". *Repeat measures using several different-sized units and relate the units.* *c. Broken ruler task.* *e. ribbon covered ruler section* *g. Compare wire around tile perimeter with tile edge as units.* *h. Ask students to draw and measure decreasing sequences of segments, using only one unit object or using a ruler.*

We adapted the method of the *teaching experiment* to treat six students as a coherent group, while following them through extended individual yet separate sequences of interviews provided a promising avenue for research related to differentiated instruction.

[2] (note: letters of tasks in this column of the table refer to task descriptions in Figure 1a and 1b)

We agree with many educators who suggest that any group of students may be treated broadly by subdividing to three or four strategy levels on a given concept (given a common age level in the class group). Our grouping of these six students allowed us adequate predictive power to address struggles and predict effective interventions while allowing an efficiency of formative assessment and analysis necessary for classroom instruction. This adaptation may well extend to other research situations and populations.

Finally, we found it challenging to integrate the imagistic accounts of linear quantity based in iteration and in collections of units to number lines and operations along number lines. The work with the obscured ruler (with the ribbon) indicated that children need very clear connections between diagrams if they are expected to coordinate their thinking about numbers and their distribution along number lines with length measure tasks.

Implications of our Study

The LMLT may be useful generally as a tool for assessment and formative instruction on linear measurement. We think teachers can gain by focusing their instruction more closely to major themes recommended by NCTM (National Council of Teachers of Mathematics, 2006), especially concepts of unit and unit iteration as ways to support the growth of quantitative reasoning (e.g., V. V. Davydov, 1975; Fernandez & Cannon, 2005; Nagasaki & Becker, 1993; Sophian, 2003). The LMLT should adapt well to action research by groups of teachers in a process such as Lesson Study (Lewis, 2000).

We believe that the writing of standards and the development of state assessments will be improved by employing the learning trajectory accounts of connective work engaging assessment, instruction and descriptions of student's thinking. This particular learning trajectory describes an important sequence of knowledge about quantity, based on a ratio between a unit and the measured object, and other measured lengths as ratios. Inasmuch as these become scalars, they are available for arithmetic operations, precursor to operations with fractions, proportional reasoning, and accounts of variable quantities. These are all precursors to variable quantity appropriate to the curriculum for algebra.

It would be helpful to check for broader collections of tasks beyond those used here for both assessment and instruction, and continue to classify tasks for assessment, instruction, or for both. We plan to relate these longitudinal accounts of individual learning with whole class lesson sequences to provide further validation and triangulation of empirical sources. We also expect to check the development of ideas of unit and measuring structure with other dimensions of mass, time, volume, temperature. Further work is needed to elaborate at more advanced and at earlier levels. More work is needed to look at contexts including the study of science or engineering.

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