

# Critical Questions:

- 1. What constitutes a LT? **How stable and linear are LTs?**
- Do LTs developmental progressions represent “natural” development, informal experience, or instruction?
- Are there LT **transition points** in time or in *content*?
- May we expect to generate **multiple LTs in one domain**? Can a **LT be too extensive**, or **too focused** in *grain size*? How might a LT in one **domain relate to LTs** in other domains?
- 5. What kinds of **research data support construction or validation of a LT**? Do particular **assessment tasks indicate** a given level of a LT?

# Building and Validating a Learning Trajectory for Children's Measurement

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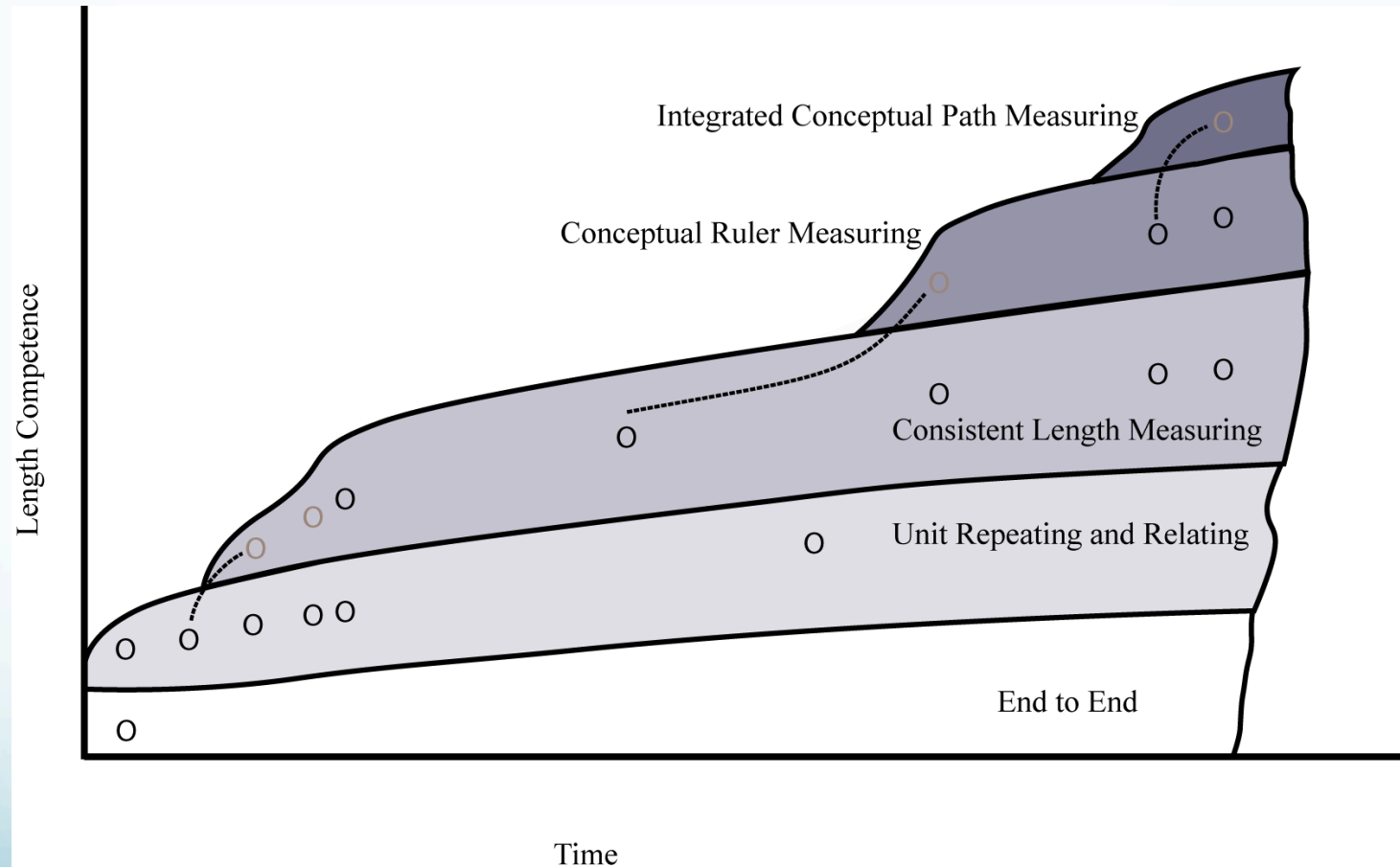
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# We use a teaching experiment to check for linearity and stability of LTs

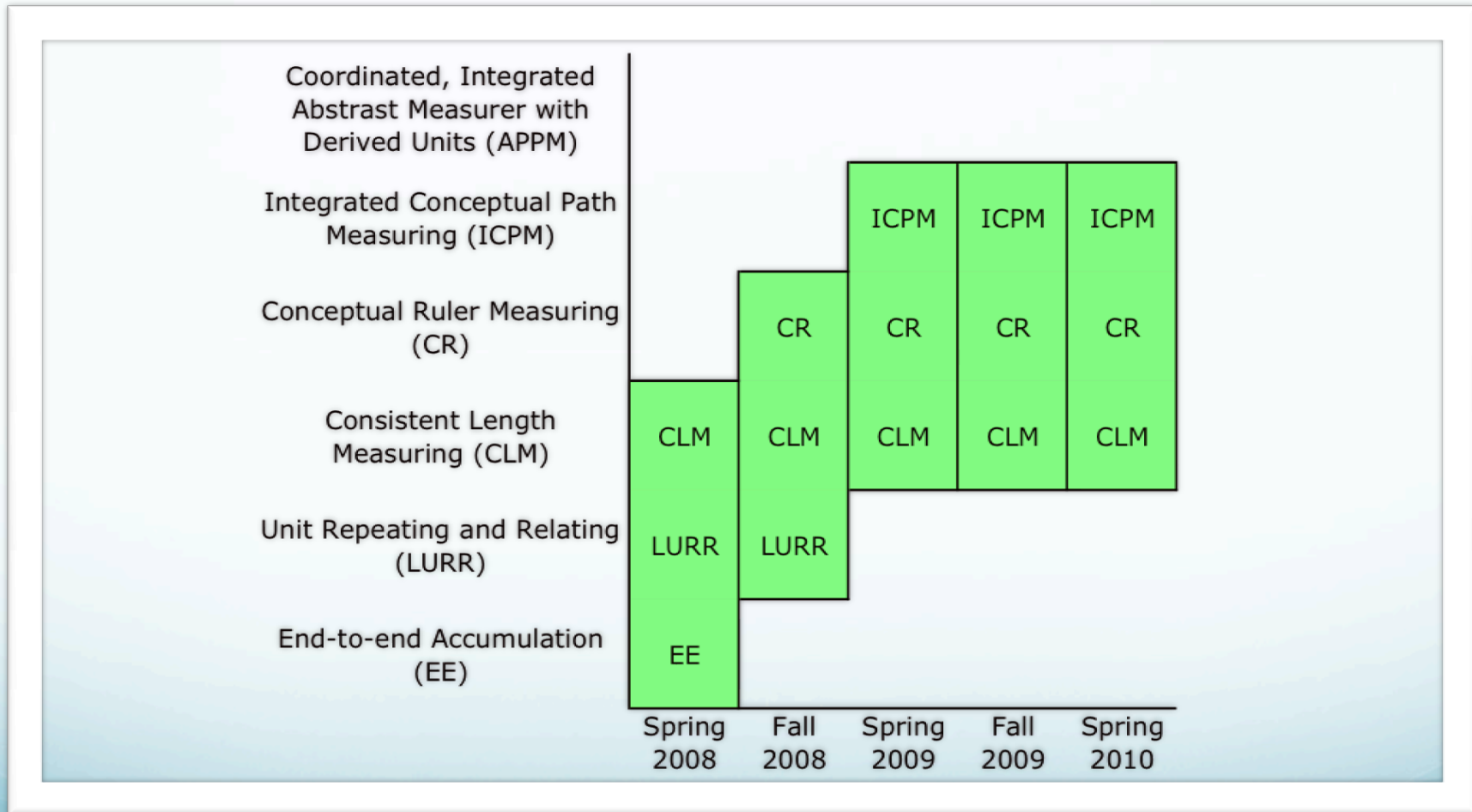
- We propose profiles for each case student, across multiple developmental shifts (every six months) occurring over four years.
- We group students based on these profiles to set instructional groups, and to check for the fit of each LT (length, area and volume).
- We note adaptations to the LT with each cycle of implementation at a given grade level.

# A Profile of one student's performance, showing data on Length LT levels: 20 months from Feb 2008 to Nov 2009



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# Complemented by a *composite profile* of a group of 6 students within a LT across 5 semesters



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## Project Focus and Design (2007-present)

- We use the LT to tie levels of development (first column), mental actions on objects (second column) and instructional interventions (third column) together in each row in an LT (see next slide).
- We are investigating LTs for Length, Area and Volume with 8 case students as they move through grades 2, 3, 4 and 5 in IL and with 8 case students in grades PreK, K, 1 and 2 in NY.
- We are using Teaching Experiments with individual, group, and classroom settings to form *plausible instructional sequences* to test and improve the LTs through design cycles.

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# What does a Learning Trajectory include (Sarama & Clements, 2009)?

- Three parts:
  - Description of level in observable characteristics
  - Description of mental actions on objects (cognition)
  - Description of instructional tasks to motivate growth from current level onward

Age 4: Length Direct comparer: Physically aligns two objects to determine the longer, or find equality.

A scheme for length as a linear extent from endpoint to endpoint (holistic). Shape can affect scheme use.

Set objects apart further, and then eventually compare objects that are distant and immovable, to prompt use of a longer, movable object to transfer.

Age 5: Indirect Length Comparer (ILC): Compares the length of two objects by representing them with a third object. Some guessing.

A mental image of a particular length can be built, maintained and manipulated.

*To shift toward End to End:* children should talk about numbers for lengths that they can compare indirectly. Use physical or drawn units along objects to compare.

(Serial Orderer to 6+) Orders sets of objects in ascending length, with lengths from 1 to 6 units.

Scheme is organized in a hierarchy, coordinating concept for an ordered series with direct comparison.

N/A

Age 6: End-to-End Length Measurer (EE): Lays units end-to-end. May not recognize the need for equal-length units. Needs a complete set of units to span.

An implicit concept that lengths can be composed as repetitions of shorter lengths underlies a scheme of laying lengths end to end. The scheme improves by attention to splitting and recomposing from parts.

Use "Length Riddles" providing only one unit per child to compare longer items. Have child plan and make a ruler from cardboard and mark it with ticks and numerals to match units (in or cm).

**Age 7: Length Unit Relater and Repeater (URR): Measures by repeated use of a unit (initially may be imprecise). Relates size and number of units explicitly, but may use units of varying lengths. Can add parts, and use rulers with slight errors.**

**Action schemes iterate a mental unit along a perceptually-available object. The image of each placement can be maintained while the physical unit is moved to the next iterative position. These action schemes allow counting-all addition schemes.**

**Pretend to gap or overlap units as they are repeated to challenge consistent measures. Have students draw objects beginning from a zero point and discuss the end-to-end measures coordination with intervals and numbers along rulers. Measure in different-sized units for the same object.**

Age 8: Consistent Length Measurer (CLM): Considers the length of a bent path as the sum of its parts (not the distance between the endpoints). Uses identical units, relates different units, partitions units, keeps zero point on rulers, accumulates distance. Begins to coordinate units and subunits.

The length scheme has hierarchical components, coordinating an object's length as a total extent and as a composition of units. This scheme adds constraints for equal-length units and, with rulers, on use of a zero point. Units themselves can be partitioned to increase precision.

Use a physical unit and a ruler to measure line segments and objects that require both an iteration and subdivision of the unit. Build sub-units to fourths and eighths. Discuss how to deal with leftover space, to count it as a whole unit or as part of a unit.

Age 9: Conceptual Ruler Measurer (CR): Possesses an "internal" measurement tool. Mentally moves along an object, segmenting it, and counting the segments. Operates arithmetically on measures ("connected lengths"). Estimates with accuracy.

Interiorization of the length scheme allows mental partitioning of a length into a given number of equal-length parts or the mental estimation of length by projecting an imaged unit onto present or imagined objects.

In "Missing Measures," students have to figure out the measures of figures using measures for a subset of sides. Prompt students to make explicit strategies for estimating lengths, including developing benchmarks for units and composite units.

Integrated, Coordinated Path Measurer (ICPM): coordinating length attributes, yet with further tendency and ability to relate multiple cases.

Can iterate units of units. Can also integrate whole sides with other sides and with entire perimeter within a figure; coordinates operations on measured objects between several figures in dynamic sequences, even for boundary cases.



# Unit Relater and Repeater

## Level in the LT for Length Measurement

### Age 7: **Length Unit Relater and Repeater (URR):**

Measures by repeated use of a unit (initially may be imprecise).

Relates size and number of units explicitly, but may use units of varying lengths.

*Can add parts, and use rulers with **slight errors**.*

Action schemes iterate a mental unit along a perceptually-available object.

The image of each placement can be maintained while the physical unit is moved to the next iterative position.

These action schemes allow counting-all addition schemes.

Pretend to gap or overlap units as they are repeated to challenge consistent measures.

Have students draw objects beginning from a zero point and discuss the end-to-end measures coordination with intervals and numbers along rulers.

Measure in different-sized units for the same object.

## What kinds of **research data support construction or validation of a LT?**

We focus closely on Problems in the Teaching of Length

- Growth from non-standard informal units like paperclips to the use of inches or centimeters with rulers (motions to sticks, or ruler marks?)
- Growth from counting items in a row, to counting hash-marks or points, to counting intervals or gaps to stand for units (discrete items or continuous space?).
- Growth from counting actions to reading the number labels at endpoints of tools/rulers (zero?)

# Problems coordinating indirect comparison with serial ordering and end to end collections

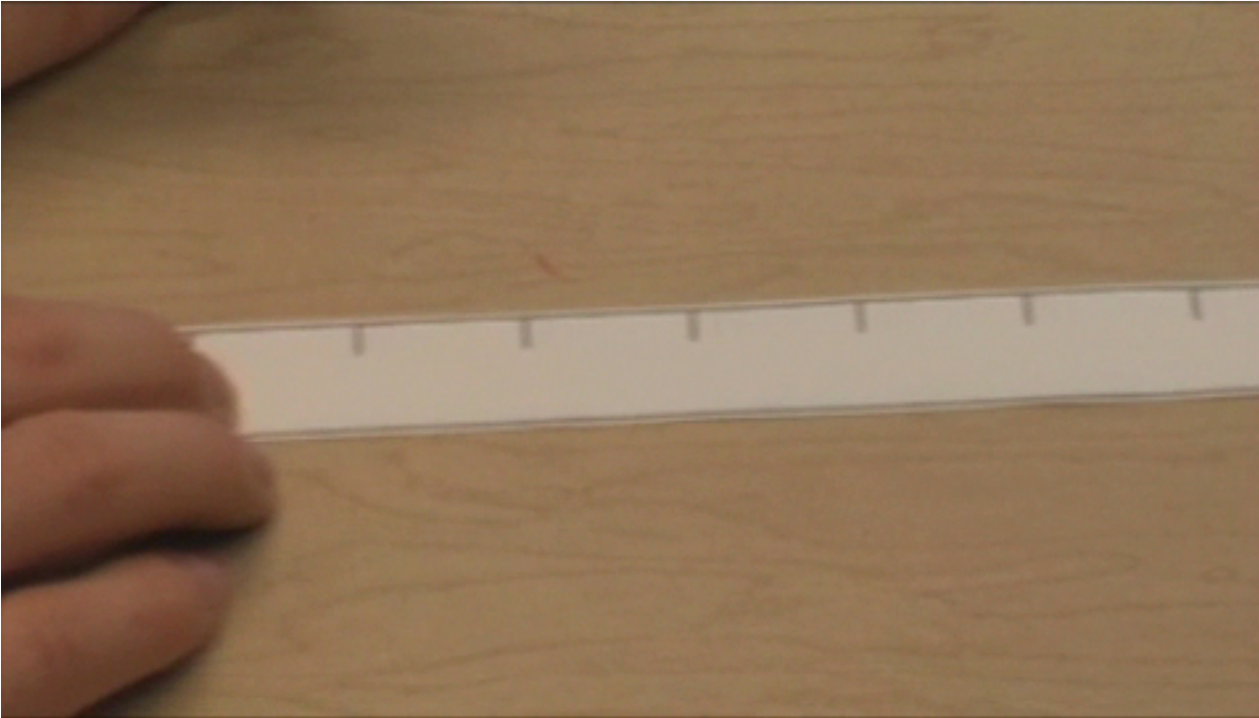
- See notes from UB about the early length sequence of levels (Julie and Doug share)

# Problems integrating the unit concept, representations and tool meaning

**In Grades 2 and 3, students required extensive support to integrate several aspects of linear quantity to surpass mere gap counting** (illustrate with 1 student):

- hash-mark counting (representing units) in February 2008 and April 2008,
- **Partly coordinated: count hash marks, sweeping gaps, & touching sticks (Apr 11 08).**
- reading numbers at endpoints, stick counting to show units, and re-assigning number labels (May 7, 2008)
- Coordinate with arithmetic operations on obscured ruler section and the ribbon (unit concept) (May 15, 2008)
- Re-established meaning for gaps as pairs of “lines” to show sequences of units (Sept 30, 2008).

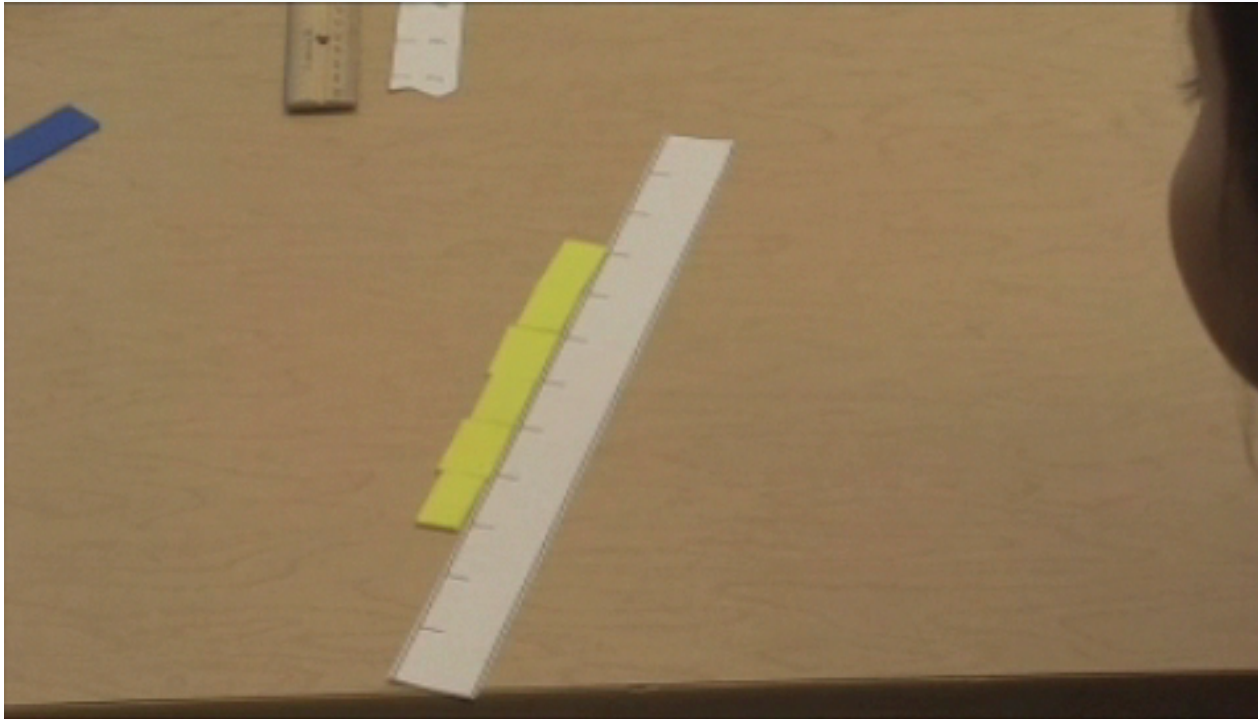
## Counting hash marks, Apr 11 Grade 2



Can you tell me how long it is?

Five. Five inches? Yes. Can you show me how you got that? I counted this as 1, 2, ... (touching the successive hash marks, getting five along a four-inch strip).

Partly coordinated: count hash marks, sweeping gaps, & touching sticks (Apr 11 08).



5 strips is 5 if aligned to end of ruler, but 6 if not aligned; Now we add another strip, and it is 6 in both cases, with a *new way* of coordinating the pointing to stick motion and moving through the ruler gaps between hash marks.



reading numbers at endpoints, stick counting  
to show units, and re-assigning number labels  
(May 7, 2008)



Tries to count gaps, but covered section prevents this approach. She is trying to get rid of the 8 at the front, and then says she can figure it out:  $8 + 2$  is 10, ... (she is treating 34 as  $8 + x1 + x2 + x3$ . She wants to subtract 8 from 34).

## Summary of shift from URR into CLM in the Learning Trajectory:

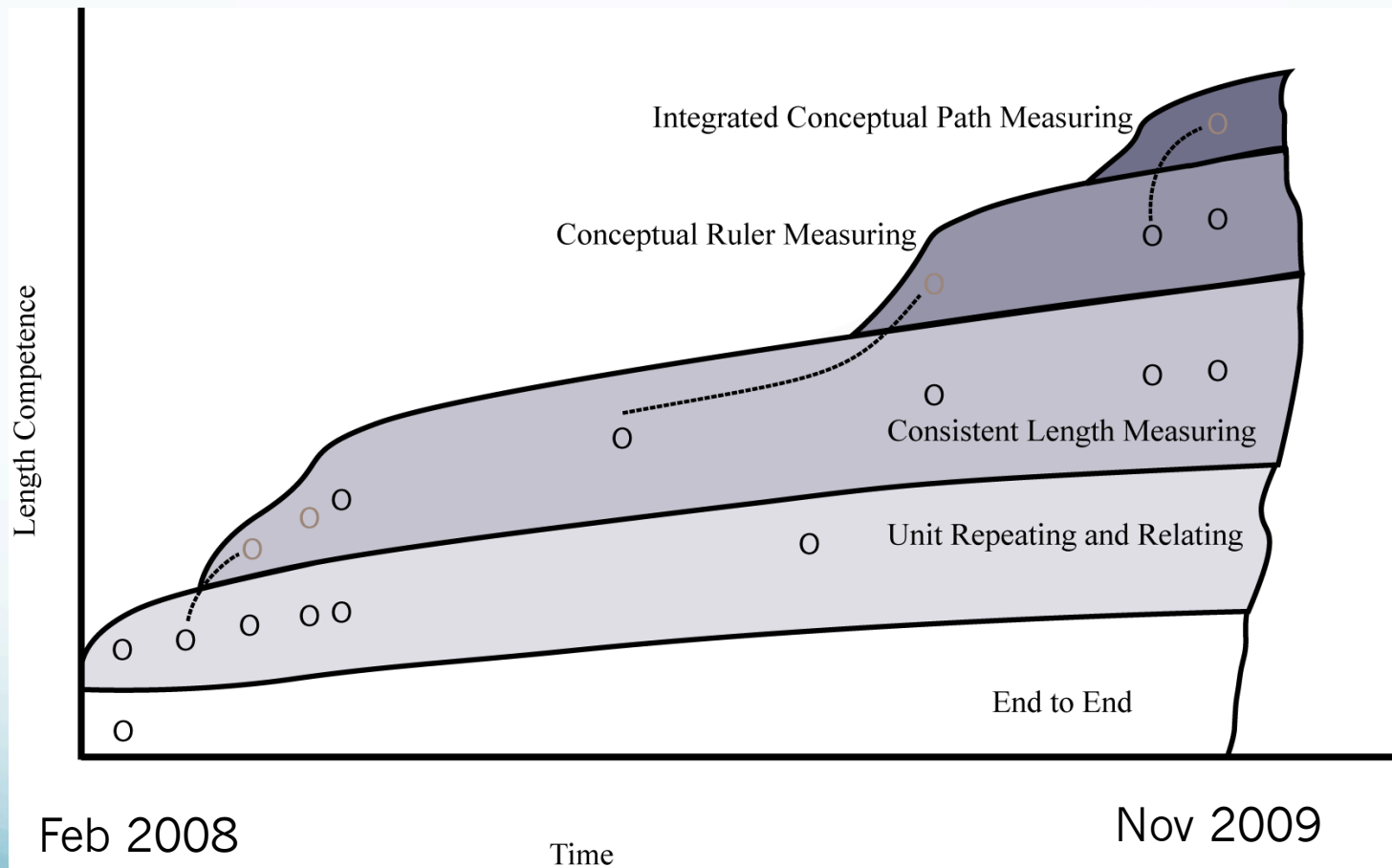
Students need to coordinate gaps, marks and numbers with arithmetic operations and unit iteration

- Counting gaps is inadequate for longer measures and for mixed units.
- Counting hash marks does not indentify unit portions of length directly, allowing errors.
- Using number labels on rulers at endpoints assumes alignment to zero and arithmetic.

**Thus, by integrating schemes for** reading number labels, arithmetic, and counting intervals or hash marks, students build flexible strategies and abstract linear quantity.

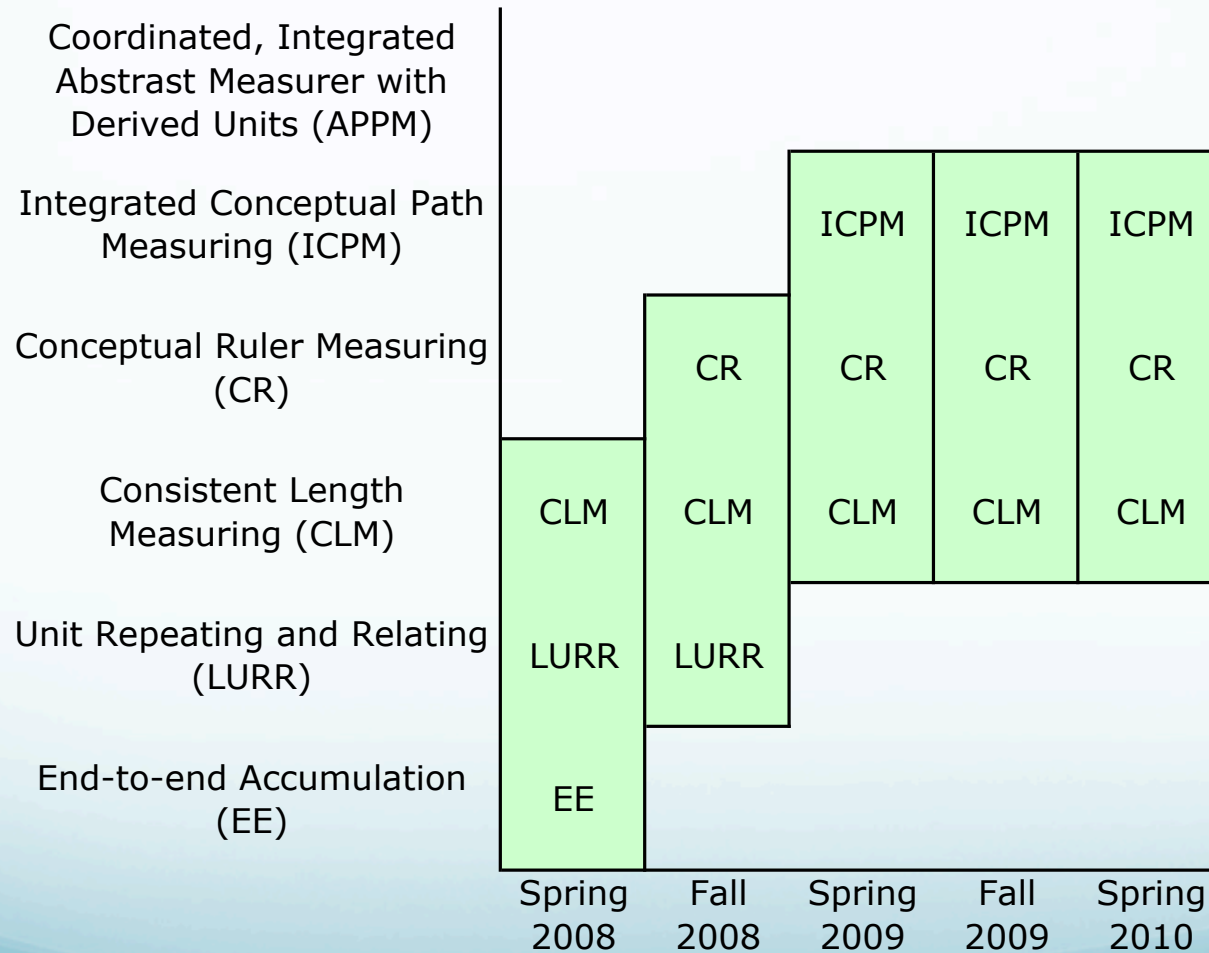


What kinds of **research data support construction or validation of a LT?** We argue for performance profiles of each student showing both regressive and progressive levels



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# Complemented by *composite profiles* of groups of students within a LT across substantive periods of development (6 months)



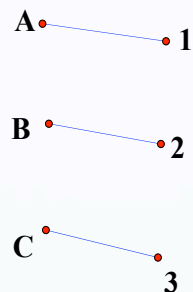
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# Linearity and stability

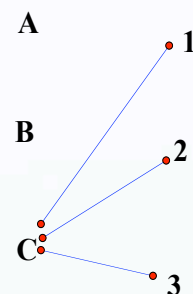
## 1. What constitutes a LT? **How stable and linear are LTs?**

- Some researchers argue that stability only occurs with high-performing students (Steedle & Shavelson, 2009), but “buggy” strategies are also unstable (cf. van Lehn, 1996).
- We argue that longer periods of development are stable even for low-performing students, requiring monthly or semi-annual steps.
- Also, we have begun using is to check for the fit between LT accounts of other research teams to establish more *generality*.
  - This requires a rubric for comparing different trajectory schemes.
  - We propose such a rubric (Barrett & Battista, forthcoming) in a pending report of the Consortium for Policy Research in Education report on Learning Progressions.

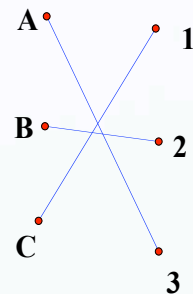
# Rubric for comparing similar LTs on a common domain



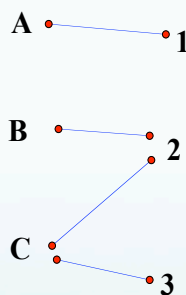
1. exact correspondence



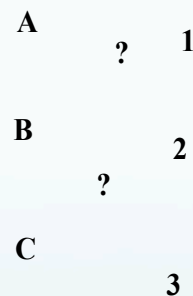
2. grain-size differs



3. similar levels,  
different order



4. differ slightly



5. misaligned, unrelated

# Thanks!

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