TRANSVERSAL COSET PARTITIONS OF GROUPS

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Abstract. The Herzog-Schönheim Conjecture states that there is no partition of any group into finitely many cosets where the corresponding subgroups have distinct indices (repeated indices may come from the same subgroup). In a different approach from most existing proofs of the conjecture for families of groups, we study the inner structure of coset partitions and propose a stronger conjecture for certain cases: namely, that whenever a group is partitioned by finitely many cosets of mutually commuting subgroups, at least one subgroup must contribute more than one coset to the partition. This conjecture holds for the cases of two and three subgroups, and with some restrictions, of four subgroups. In addition, we prove that for $2 \leq r \leq 7$ distinct subgroups' cosets, Herzog-Schönheim holds; for $5 \leq r \leq 7$, our proof is computational. In terms of partition structure, we show that for any group G and either two arbitrary subgroups H_1, H_2 or three mutually commuting subgroups H_1, H_2, H_3 of G, the only possible partitions of G are obtained via successive decompositions of cosets of products of H_i's into cosets of smaller products, until single cosets are reached (a "standard" construction). However, we prove that non-standard constructions exist for $r \geq 4$ subgroups. Finally, we explore applications to exact multi-covers of groups by cosets and to subspace partitions.

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1. Introduction and Main Results

Let G be any group and $H_1, \ldots, H_r$ be a list of subgroups of G, not necessarily distinct. We will call a set partition of G into left cosets $g_1H_1, \ldots, g_rH_r$ (where $r \geq 2$ and $g_1, \ldots, g_r \in G$) a **coset partition** of G. It is known [25] that when covering a group by the union of finitely many cosets, any cosets of subgroups of infinite index can be omitted. Hence, for a coset partition (a.k.a. an "exact cover" or "disjoint cover" by cosets), all H_i involved must be of finite index. Coset partitions have been widely studied with the aim of settling the Herzog-Schönheim Conjecture [22]:

> *In any coset partition of a group G as above, at least two of the indices $[G\colon H_i]$ must be equal.*

Note that the statement does not distinguish between repeated indices of distinct subgroups $H_i \neq H_j$ and of the same subgroup H_i. The conjecture has been confirmed for various cases, including the group $\mathbb{Z}$ of integers [18, 23], finite nilpotent groups [7], (finite) pyramidal groups [6], finite groups of small order [24], and finite groups admitting a certain kind of prime factorization [21]. Also see [10]-[13] for recent work on the conjecture. In due course, we will (i) point out that the Herzog-Schönheim Conjecture holds for all nilpotent groups,

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including abelian groups, and (ii) prove that it also holds unconditionally for any group with a coset partition involving cosets of up to 7 distinct subgroups. However, our main focus in this article will be to state and support a variant of the Herzog-Schönheim Conjecture when the subgroups involved **mutually commute**, i.e., satisfy $H_iH_j = H_jH_i$ for all i, j (Conjecture 1 below), and to study the structure of coset partitions of G subject to this mutual commutativity condition.

Henceforth, we let $H_1, \dots, H_r$ denote *distinct* subgroups of a group G and consider coset partitions $\mathcal{P}$ of G such that for each i, $1 \le i \le r$, there exists at least one H_i-coset in $\mathcal{P}$. We will refer to such a coset partition of G as an $\{H_1, \dots, H_r\}$**-transversal coset partition**, or simply, a **transversal coset partition**, if the subgroups $H_1, \dots, H_r$ are clear from the context. We will further say that an $\{H_1, \dots, H_r\}$-transversal coset partition $\mathcal{P}$ is **pure** if it contains exactly one H_i-coset for $1 \le i \le r$. Here is our conjecture:

Conjecture 1. *Let G be a group and $r \ge 2$. If $H_1, \dots, H_r$ are mutually commuting distinct proper subgroups of G, with possibly repeating finite indices in G, then there exists no pure $\{H_1, \dots, H_r\}$-transversal coset partition of G.*

Our main results are as follows, with longer proofs and applications postponed to later sections.

Theorem 2. *Let G be any group and H, K be distinct proper subgroups of G. Then there exists an $\{H, K\}$-transversal coset partition of G if and only if $G \neq \langle H, K\rangle$. If such a partition exists, then the left H-cosets (resp., left K-cosets) in the partition are obtained by fully decomposing some left $\langle H, K\rangle$-cosets, and Conjecture 1 holds even if $HK \neq KH$.*

Note that Theorem 2 does not require commuting subgroups. Here $\langle H, K\rangle$ denotes the join of the subgroups H and K, which comprises all strings of products of elements of $H \cup K$.

Theorem 3. *Let G be a group and H, K, L be distinct, proper, and mutually commuting subgroups of G. Then there exists an $\{H, K, L\}$-transversal coset partition of G if and only if $G \neq HKL$. If such a partition exists, then it can only be achieved by first decomposing G fully into left HKL-cosets, then each HKL-coset fully into one of left HK-, HL-, or KL-cosets, and finally decomposing each of the latter fully into one of H-, K-, or L-cosets; consequently, Conjecture 1 holds. In particular, if $G \neq HKL$ and none of the three subgroups is contained in the product of the other two, then every $\{H, K, L\}$-transversal coset partition of G is the result of a standard construction.*

Theorem 3, like Theorem 2, is mainly about the structure of partitions of a group into cosets of three mutually commuting subgroups (which is, for example, valuable in understanding constructions of subspace partitions). A **standard construction** of an $\{H_1, \dots, H_r\}$-transversal coset partition is one that consists of successive decompositions of cosets of products of the H_i's into cosets of smaller products of the H_i's as described in the third sentence of the theorem above; the conditions we impose on the subgroups for this definition are (i) mutual commutativity of the subgroups H_i (so that their products are also subgroups) and (ii) that no subgroup is contained in the product of the remaining $r - 1$ subgroups (so that products $\prod_{i \in I} H_i$ and $\prod_{j \in J} H_j$ are equal if and only if $I = J$, and an inclusion $H_i \le \prod_{j \in J} H_j$ happens if and only if $i \in J$).

We also prove Conjecture 1 for certain three-subgroup cases without the commutation restriction:

Theorem 4. *Let G be any group and H, K, L be distinct proper subgroups such that all three do not have the common index of 3 in G. Then there exists no pure $\{H, K, L\}$-transversal coset partition of G, and Conjecture 1 holds even if the three subgroups do not commute mutually.*

The omission of equal indices is necessary. We will provide examples of pure $\{H, \dots, H_r\}$-transversal coset partitions with $[G\colon H_1] = \cdots = [G\colon H_r] = r$ in Section 2. Next, we prove the following result:

Theorem 5. *For any group G with four mutually commuting distinct proper subgroups, one of which has index 2 in G, Conjecture 1 holds.*

We also show independently that the Herzog-Schönheim Conjecture holds when there are cosets of at most 7 distinct subgroups:

Theorem 6. *Let G be any group and $H_1, \dots, H_r$ be distinct proper subgroups of G, with $2 \leq r \leq 7$. Then any $\{H_1, \dots, H_r\}$-transversal coset partition of G must contain at least two cosets whose subgroups have the same index.*

Finally, we revisit the concept of standard construction for four or more distinct, mutually commuting subgroups.

Theorem 7. *Let $r \geq 4$ and $G = H_1 \times \cdots \times H_r$ be a finite group that is a direct product of r nontrivial subgroups, $H_1, \dots, H_r$. Then there exists an $\{H_1, \dots, H_r\}$-transversal coset partition of G, necessarily unobtainable by a standard construction.*

After proving these results, we will highlight two main applications: parallelism of transversal coset partitions (a special case of "exact m-cover" of a group by cosets) and constructions of subspace partitions using coset partitions of the multiplicative group of a finite field (Subsections 7.1 and 7.2 respectively).

2. Background, Motivation, and Examples

2.1. **More about coset partitions and the Herzog-Schönheim Conjecture.** Studying the representations of 1 as a sum of unit fractions $1/d$ (where $d \geq 2$) is a useful tool for the elimination of unlikely combinations of subgroup indices $d_1 = [G\colon H_1], \dots, d_r = [G\colon H_r]$ in G that may appear in a transversal coset partition of G. [1] If each subgroup H_i contributes n_i cosets, then we will denote the list of indices belonging to the partition by $[d_1^{n_1}, \dots, d_r^{n_r}]$ for short. It is known that the reciprocals of all indices that appear in any partition must add up to 1 [23]:

$$\underbrace{\frac{1}{d_1} + \cdots + \frac{1}{d_1}}_{n_1 \text{ times}} + \cdots + \underbrace{\frac{1}{d_r} + \cdots + \frac{1}{d_r}}_{n_r \text{ times}} = 1. \tag{1}$$

(The proof for $G = \mathbb{Z}$ was given in [26].) Hence, if the indices of all r subgroups are equal in a pure transversal coset partition, then the common index must be r, the number of the subgroups. Yet another condition on the index list is that any pair (d_i, d_j) cannot be relatively prime [23].

[1] We thank our anonymous referee for bringing [20] to our attention, where index methods–some of them same as or similar to ours–were widely utilized in proofs.

We also note that there are two important mappings between coset partitions of finite and infinite groups, which reduce the proof of the Herzog-Schönheim Conjecture to the finite case for several classes of groups. First, we state the infinite-to-finite principle in [23].

Lemma 8. *Every infinite group with a coset partition has a finite quotient group that exhibits an induced coset partition with the same list of indices.*

This result follows from the fact that given any subgroup H of a group G with finite index $d = [G\colon H]$, there exists a normal subgroup N of G contained in H such that $[G\colon N] \leq d^d$. Setting $H = H_1 \cap \cdots \cap H_r$ in our customary notation, we obtain the desired quotient group G/N. We remark that distinct subgroups of the infinite group are mapped to distinct subgroups of the finite quotient group.

Conversely, we observe that new coset partitions can be constructed using direct products, and state the finite-to-infinite principle:

Lemma 9. *Let G_1, G_2 be groups and H_1, H_2 subgroups with finite indices in G_1 and G_2 respectively. Then we have*

$$[G_1 \times G_2\colon H_1 \times H_2] = [G_1\colon H_1][G_2\colon H_2].$$

Furthermore, if $\mathcal{P}_1, \mathcal{P}_2$ are coset partitions of G_1 and G_2 respectively, then

$$\mathcal{P}_1 \times \mathcal{P}_2 \stackrel{\text{def}}{=} \{(x_i, y_j)(H_i \times K_j)\colon x_iH_i \in \mathcal{P}_1,\ y_jK_j \in \mathcal{P}_2\}$$

is a coset partition of $G_1 \times G_2$. In particular, any coset partition $\mathcal{P}_1$ of a finite group G_1 can be extended to a coset partition of an infinite group $G_1 \times G_2$ with the same list of indices via the partition $\mathcal{P}_2 = \{G_2\}$ of any infinite group G_2.

Combining the two principles, we obtain the following corollary, parts of which were stated in [23].

Corollary 10. *Let $D = [d_1^{n_1}, \ldots, d_r^{n_r}]$ denote a possible list of finite indices representing $n_1, \ldots, n_r$ cosets of r distinct subgroups $H_1, \ldots, H_r$ respectively that may occur in a coset partition of a group G. Then*

(i) *The index list D can be realized for a coset partition of some infinite group if and only if it can be realized for a coset partition of some finite group.*
(ii) *Attributes such as nilpotent, solvable, abelian, etc., which are preserved by quotients and finite direct products, can be attached to the infinite and finite groups in part (i) as well.*
(iii) *In particular, all nilpotent (hence abelian) groups must satisfy the Herzog-Schönheim Conjecture (originally the Erdös Conjecture for $\mathbb{Z}$) by* [7].

It is now easy to prove the index-list condition (1), as we only need to verify it for a finite group. For an index list $D = [d_1^{n_1}, \ldots, d_r^{n_r}]$ of a partition, with our usual notation, we must have

$$\sum_{i=1}^{r} n_i|H_i| = |G| \Rightarrow \sum_{i=1}^{r} \frac{n_i}{d_i} = 1.$$

As observed in [20], any group whose order is a "deficient number" (i.e., a positive integer strictly greater than the sum of its proper divisors) must satisfy Herzog-Schönheim.

2.2. **The mutual commutativity condition and Conjecture 1.** Apart from the Herzog-Schönheim Conjecture, coset partitions have been utilized in the construction of subspace partitions, where G is the multiplicative group of a finite field (see Section 7.2 for further information on this application). Hence, it was natural for us to ask how coset partitions of abelian groups could be constructed, but a more general condition on the subgroups proved sufficient. Incidentally, we asked, could classifying possible construction methods help reframe the Herzog-Schönheim Conjecture? Recall that for subgroups H, K of any group G, the set product $HK = \{hk \colon h \in H, k \in K\}$ is also a subgroup if and only if $HK = KH$. For our application purposes, it suffices to assume that the given subgroups H_i of G commute pairwise, so that products of these subgroups are also subgroups of G. Note that any result proven under this assumption will hold more generally for abelian groups, Dedekind groups (for which all subgroups are normal), and Iwasawa groups (for which all pairs of subgroups commute, or are "permutable") in particular; see [4] for background. Moreover, since mutual commutativity of subgroups is a property that is preserved under quotients (by a normal subgroup that is contained in these subgroups) as well as under direct products, we may extend Corollary 10 to

Corollary 11. *Let $D = [d_1^{n_1}, \dots, d_r^{n_r}]$ denote a possible list of finite indices representing $n_1, \dots, n_r$ cosets of r distinct subgroups $H_1, \dots, H_r$ respectively that may occur in a coset partition of a group G. Then D can be realized for a transversal coset partition of an infinite group with r distinct, mutually commuting subgroups if and only if it can be realized for a transversal coset partition of a finite group with r distinct, mutually commuting subgroups. In particular, Conjecture 1 holds (i.e., $n_i > 1$ for some i) for infinite groups if and only if it holds for finite groups.*

Naturally, if the Herzog-Schönheim Conjecture holds, then there is no pure $\{H_1, \dots, H_r\}$-transversal coset partition when the subgroups H_i have distinct indices. However, some families of examples we have studied suggest that more may be true: that is, it is possible that at least one subgroup must contribute more than one coset to any transversal partition, whether all indices are distinct or not, when the subgroups involved mutually commute (Conjecture 1). For $r = 2, 3, 4$, see Theorems 2–5 above. There are other families of examples. For $G = \mathbb{Z}$ and $H_i = n_i\mathbb{Z}$, where the n_i's are pairwise distinct for $1 \leq i \leq r$, Mirsky, Newman, Davenport, and R. Rado proved (unpublished) a conjecture of Erdös (see [17]) stating that there is no pure $\{H_1, \dots, H_r\}$-transversal. Of course, it is not possible to have distinct subgroups with the same index in this case. Finally, we deduce from Lemma 23 below that if G is any finite group that contains a direct product $H_1 \times \cdots \times H_r$ of nontrivial subgroups $H_1, \dots, H_r$ with $r \geq 3$, then we have

$$|H_1| + \cdots + |H_r| < |H_1| \cdots |H_r| \leq |G|,$$

and a disjoint union of single cosets of these r subgroups cannot cover G.

We also examined some naturally occurring examples of pure transversal coset partitions: if such partitions involved mutually commuting subgroups, then they would have provided counterexamples to Conjecture 1. Let us start with a generally-known result.

Lemma 12. *Let G be a group and H be a subgroup of index r, with right coset representatives $g_1, \dots, g_r$. Then*

$$G = Hg_1 \sqcup \cdots \sqcup Hg_r = g_1(g_1^{-1}Hg_1) \sqcup \cdots \sqcup g_r(g_r^{-1}Hg_r).$$

That is, G has a coset partition via left cosets of conjugates of H, with the same list of coset representatives.

When the normalizer $N_G(H)$ of H in G is equal to H, all these conjugates are distinct, and we obtain a pure transversal coset partition of G. Hence, if we could find a proper subgroup H of finite index in some group G where $N_G(H) = H$ and H commutes with all its $[G\colon N_G(H)]$ conjugates, that would have been a counterexample for Conjecture 1. Subgroups that commute with their conjugates are called "conjugate-permutable subgroups". The fact that conjugate-permutable subgroups of finite groups are subnormal [19] again favors Conjecture 1; such subgroups must be either equal to the whole group (not proper) or normal in a strictly larger subgroup (not self-normalizing). We will investigate pure transversal coset partitions further in another paper.

Pure transversal coset partitions constructed in the manner of Lemma 12 constitute a good introduction to our applications in Subsection 7.1. Carter [9] proved that every finite solvable group G has a nilpotent subgroup H that is self-normalizing, and that all such subgroups of G (which are now called "Carter subgroups") are conjugates. When G is nilpotent as well, it must be the unique Carter subgroup of itself that satisfies the two properties; indeed, no proper subgroup of a finite nilpotent group can be self-normalizing. Conrad [16] points out that finite non-solvable groups may not have similar subgroups, but when they do exist, those subgroups must also be conjugates [29]. Let us briefly examine two families of groups with proper Carter subgroups, and hence, with pure transversal coset partitions.

Example 13. *Let $r \geq 3$ be odd,*

$$D_n = \langle a, b\colon a^n = b^2 = 1,\, ba = a^{-1}b \rangle = \{1, a, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b\}$$

be the dihedral group of order $2n$, where $n = 2^k r$ and $k \geq 0$, $H = \langle a^r, b \rangle \leq D_n$, and $\gamma = a^{2^k}$. Note that D_n is solvable and H has $[D_n\colon H] = r$ conjugates (see [15] *for structural properties of D_n). By Lemma 12, the right cosets*

$$H,\, H\gamma, \ldots, H\gamma^{r-1}$$

of H form at the same time a pure $\{H_1, \ldots, H_r\}$-transversal left-coset partition of D_n with respect to the conjugates H_i of H in D_n. We will add on to this construction in Example 32. When $n = 2^k$, D_n is also nilpotent and has no proper Carter subgroups.

Example 14 (Also see [30])**.** *Let $r \geq 3$. Although the symmetric group S_r is not solvable for $r \geq 5$, any S_r with $r \geq 3$ still exhibits a pure transversal coset partition by r conjugate subgroups of index r. Namely, if $S_{[j]}$ denotes the stabilizer of j for $1 \leq j \leq r$, then the collection of cosets below is a pure partition of S_r:*

$$S_{[1]},\, (12)S_{[2]}, \ldots, (1r)S_{[r]}.$$

We will develop this construction further for S_4 in Example 33.

2.3. **Structure of coset partitions for mutually commuting subgroups.** Next, we consider the motivation of our article, namely, the problem of methods of construction of transversal coset partitions (for mutually commuting subgroups) from scratch. Here is one strategy that feels the most natural, and we ask whether it is the only one. When subgroups $H_1, \ldots, H_r$ of G commute among themselves, every transversal coset partition of G must partition the left $H_1 \cdots H_r$-cosets of G as well: each coset gH_i in G is contained in the coset $gH_1 \cdots H_r$. Therefore, it is instructive to study coset partitions when G is equal to the product $H_1 \cdots H_r$. Now, it is always possible to decompose the group G into $H_1 \cdots H_r$-cosets, then fully decompose each such coset into cosets of the product of any $r-1$ subgroups, etc., and arrive at many different coset partitions of G in this manner. If there are enough cosets of product groups in G, then we can obtain various $\{H_1, \ldots, H_r\}$-transversal partitions of G. We have called constructions of this form "standard" with a caveat.

When there are no additional restrictions on $G, H_1, \ldots, H_r$ apart from the pairwise commutativity, some issues that detract from the classification problem may arise. For example, if one of the subgroups is equal to G, then there are no transversal partitions. Or, if $H_1 \leq H_2H_3$, then $H_1H_2H_3$-cosets are the same as H_2H_3-cosets, and an H_2H_3-coset can be fully decomposed into H_1-cosets instead of only H_2- or H_3-cosets. Also, if $H_1 \leq H_2$, then a number of H_1-cosets will be completely contained in any H_2-coset in a partition, as opposed to "linking" this coset to other H_2- or H_3-cosets, etc. Thus, when desirable, we will impose the condition

$$H_i \not\leq H_1 \cdots \hat{H}_i \cdots H_r, \ \ 1 \leq i \leq r \tag{2}$$

on our mutually commuting subgroups. As is customary, a caret over a symbol in a list denotes the absence of that symbol from the list. We will now properly define a **standard construction** of an $\{H_1, \ldots, H_r\}$-transversal coset partition of G:

> Let G be a group with mutually commuting subgroups $H_1, \ldots, H_r$, where the subgroups satisfy Eq. (2). We say that a transversal coset partition of G is obtained by a **standard construction** if, in parallel, every left coset of a product of s subgroups H_j is completely decomposed into left cosets of a product of $s-1$ of those subgroups for $s = r, r-1, \ldots, 2$ to obtain the partition.

If the standard construction were the only way to obtain coset partitions for mutually commuting subgroups subject to the restriction in Eq. (2), then we could deduce that (i) there does not exist an $\{H_1, \ldots, H_r\}$-transversal coset partition of $G = H_1 \cdots H_r$, and (ii) Conjecture 1 and the Herzog-Schönheim Conjecture hold for such groups and subgroups, because the standard construction will generate multiple cosets of the same subgroups at each stage. However, we have found that the standard construction is the only possibility exactly when $r = 2$ or $r = 3$; see Theorems 2 and 3 above respectively. In fact, Theorem 2 holds for all groups and subgroups, where the concept of "standard" is necessarily broader. The following example exhibits a non-standard construction with $r = 4$.

Example 15. *Let $G = H \times K \times L \times M$ be the direct product of subgroups $H = \langle a \rangle$, $K = \langle b \rangle$, $L = \langle c \rangle$, and $M = \langle d \rangle$ of order two. Note that the four subgroups naturally commute and satisfy the non-inclusion condition in Eq. (2) as well. Then we have the following $\{H, K, L, M\}$-transversal coset partition of G:*

$$\mathcal{P} = \{cH,\, bdH,\, K,\, aK,\, dL,\, adL,\, bcM,\, abcM\}.$$

Further details of the construction of this transversal coset partition are in Example 24.

Also see an example in [27] for a non-standard partition of $\mathbb{Z}$ into cosets of four distinct subgroups.

3. Proof of Theorem 2 and More

Due to Lemma 16 below, one might think that the condition $G \neq HK$ is sufficient for the existence of an $\{H, K\}$-transversal coset partition of an arbitrary group G with distinct proper subgroups H, K. Indeed, there are no transversal partitions when $G = HK$, because any left cosets of H and K must intersect:

Lemma 16 (Sun [30] Lemma 2.1(i))**.** *Let H and K be subgroups of a group G. Then $G = HK$ if and only if $xH \cap yK \neq \emptyset$ for all $x, y \in G$.*

We include the proof of Sun for completeness:

Proof. Assume that $G = HK$ and $x, y \in G$ be given. Then there exists $h \in H$ and $k \in K$ such that $x^{-1}y = hk$. Hence, we have

$$xh = yk^{-1} \in xH \cap yK.$$

Conversely, assume that $xH \cap yK \neq \emptyset$ for all $x, y \in G$. Then for any $g \in G$, we can find $h \in H$ such that $h \in 1H \cap gK$. That is, $h = gk$ for some $k \in K$, and $g = hk^{-1} \in HK$ (as a set product). Therefore, $G = HK$. □

However, we have the following example, which shows that an $\{H, K\}$-transversal partition does not necessarily exist when $G \neq HK$:

Example 17. *Let $G = S_3$, $H = \langle(12)\rangle$, and $K = \langle(13)\rangle$. Then $G \neq HK$, but no $\{H, K\}$-transversal coset partition of G exists, as can be verified directly from the sets of left cosets of H and K respectively:*

$$H = \{(1), (12)\},\ (13)H = \{(13), (123)\},\ (23)H = \{(23), (132)\}$$

and

$$K = \{(1), (13)\},\ (12)K = \{(12), (132)\},\ (23)K = \{(23), (123)\}.$$

A more suitable way to interpret the condition $G \neq HK$ is that there exists more than one double (H, K)-coset in G. We find that the double coset

$$HK = \{(1), (12), (13), (132)\}$$

can be decomposed into the left cosets $(1)K$ and $(132)K$, while the remaining double coset

$$H(23)K = \{(23), (123)\}$$

is the right coset $H(23)$, giving rise to a mixed partition of G into both right cosets of H and left cosets of K. Switching the roles of the two double cosets, we also obtain the mixed coset partition

$$H(1),\ H(13),\ (23)K.$$

We formally summarize our findings for all groups G and subgroups H, K, not necessarily of finite index.

Lemma 18. *Let G be a group, H, K be subgroups of G, and $x, y \in G$. Then*

$$Hx \cap yK \neq \emptyset \Leftrightarrow HxK = HyK.$$

Proof. We have

$$hx = yk \text{ for some } h \in H,\ k \in K \Leftrightarrow x \in HyK \Leftrightarrow HxK = HyK.$$

□

Proposition 19. *Let G be a group and H, K be distinct subgroups of G. Then there exists a mixed partition of G into both right cosets of H and left cosets of K if and only if $G \neq HK$.*

Proof. If $G = HK = H1K$, then there exists exactly one double coset of G and every right coset of H intersects every left coset of K by Lemma 18. Hence, no partition into both right cosets of H and left cosets of K is possible. But if $G \neq HK$, then there are at least two double (H, K)-cosets, and some can be decomposed into right H-cosets, while the rest can be decomposed into left K-cosets. $\square$

The correct condition for a partition of an arbitrary group G into left H- and K-cosets to exist is then given by Theorem 2, namely, if and only if $G \neq \langle H, K\rangle$. (If $HK = KH$, then each double coset becomes a left coset of HK, and a transversal partition exists if and only if $G \neq HK$.) Recall that if G is a group and H is a subgroup of G, then the rule $g \cdot h = gh^{-1}$ defines a right action of H on G. We will say that any subset of G that is invariant under this action of H is a "right H-set". Then the left cosets of H are the orbits of this action in G, and any right H-set in G is a union of left cosets of H.

Proof Theorem 2. Set $U = \langle H, K\rangle$. First, suppose that there exists a partition $\mathcal{P}$ of G into left cosets of both H and K, and let X, Y be the unions of the left H-cosets and the left K-cosets in the partition respectively. Then X is a right H-set and Y is a right K-set, both nonempty. Since X and Y are complements in G, we conclude that both X and Y are right H-sets and K-sets, hence, right U-sets. This shows that (i) there are at least two left U-cosets, so that $G \neq U$, and (ii) any transversal partition of G is obtainable only by decomposing each left U-coset fully into left H- or K-cosets, where U is possibly equal to H or K if $K \leq H$ or $H \leq K$ resp. Conversely, if there exist at least two left U-cosets in G, which are necessarily invariant under the right actions of $H, K \leq U$, then some U-cosets can be decomposed into left H-cosets and the rest into left K-cosets. $\square$

Continuing to use the notation of Theorem 2, we may impose the restrictions $H \not\leq K$ and $K \not\leq H$ on H and K. Then each decomposition of U-cosets becomes a proper decomposition, with $U \neq H, K$. Hence, if $HK = KH$ holds as well, we can further conclude that the only possible transversal coset partitions of G are the standard ones.

4. Proof of Theorem 3

The following lemmas are straightforward.

Lemma 20. *Let G be a group and H, K be subgroups. Then the intersection of two left cosets aH and bK in G is either empty or a coset of the subgroup $H \cap K$. In particular, an inclusion $aH \subseteq bK$ implies that $H \subseteq K$.*

Lemma 21. *Let H, K be subgroups of a group G such that $HK = KH$. Then every left H-coset within HK intersects every left K-coset within HK and no left K-coset outside HK. The same holds when HK is replaced by any left coset of HK in G. Equivalently, we have*

$$xH \cap yK \neq \emptyset \Leftrightarrow xHK = yHK,$$

and

$$xH \cap yK = \emptyset \Leftrightarrow xHK \cap yHK = \emptyset.$$

Proof. We have already observed that the subgroup $HK = \langle H, K\rangle$ of G is a disjoint union of left H-cosets in G as well as a disjoint union of left K-cosets in G. Hence, Lemma 16 applies to the group $G' = HK$. Clearly, the relationships of the H- and K-cosets within do not change when they are left-translated together with HK by an element of G. $\square$

We will informally call the set of all K-cosets inside a coset xHK a **stack of K-pancakes**, with parallel threads of full H-cosets binding them together.

For the rest of this preamble to the proof of Theorem 3, let $r \geq 3$, G be any group, $H_1, \ldots, H_r$ be mutually commuting subgroups of G, and $\mathcal{P}$ be an $\{H_1, \ldots, H_r\}$-transversal coset partition of G. We distinguish any one of the subgroups, say H_1, from the rest and set aside the collection $\mathcal{P}_1$ of left H_1-cosets in $\mathcal{P}$. Let us also denote the complement of $\mathcal{P}_1$ in $\mathcal{P}$ by $\mathcal{P}'$. Then $\mathcal{P}'$ is a collection of mutually disjoint left cosets of $H_2, \ldots, H_r$, and the union of the cosets in $\mathcal{P}'$ forms a right H_1-set. All left H_1-cosets not in $\mathcal{P}$ (equivalently, not in $\mathcal{P}_1$) occur in this union, intersecting some cosets in $\mathcal{P}'$. We would like to describe subsets of $\mathcal{P}'$ that are "connected" via threads of left H_1-cosets. First, we will say that two cosets aH_i, bH_j in $\mathcal{P}'$ are **H_1-linked** (or just "linked") if both intersect the same left H_1-coset, which we will call a **link** between aH_i and bH_j. This is equivalent to having an element of aH_i mapped to an element of bH_j by some right action of H_1 within the linking H_1-coset (and vice versa). Hence, we may also denote this linkage by either of the conditions $aH_iH_1 \cap bH_j \neq \emptyset$ and $aH_i \cap bH_jH_1 \neq \emptyset$. Second, we define an equivalence relation $\sim$ on $\mathcal{P}'$ by declaring $aH_i \sim bH_j$ if and only if there exists a list of cosets

$$C_1 = aH_i, C_2, \ldots, C_{s-1}, C_s = bH_j$$

in $\mathcal{P}'$, with $s \geq 2$, such that each C_i is H_1-linked to C_{i+1} for all $i = 1, \ldots, s-1$. We will refer to the equivalence classes of this relation as **blobs**. The union of cosets belonging to $\mathcal{P}'$ in each blob is then a right H_1-set. One possible type of blob consists only of some H_i-cosets in $\mathcal{P}'$:

Lemma 22. *Suppose that a blob in $\mathcal{P}'$ consists entirely of left H_i-cosets in $\mathcal{P}'$. Then the H_i-cosets in the blob form a complete stack of pancakes, i.e., their union is an H_1H_i-coset.*

Proof. The underlying set of the blob is closed under the right action of H_1 by construction, because it is a disjoint union of links. Also, the underlying set is a disjoint union of left H_i-cosets, which makes it invariant under the right action of H_i. Therefore, this set is invariant under the right action of the group H_1H_i. Moreover, we can move within each H_i-coset of the blob with the right action of H_i, and within any link, with the right action of H_1. This makes the set transitive under the H_1H_i-action and a left H_1H_i-coset. $\square$

See examples of blobs that are stacks of pancakes in Figure 1, page 14.

We will describe the union of cosets in a blob as the underlying **blob-set** to distinguish this equivalence class of cosets in $\mathcal{P}'$ from their union in G. Hence, the subset of G covered by the cosets in $\mathcal{P}'$ is a disjoint union of blob-sets. In the next section, we will prove that if G has a coset partition consisting of cosets of exactly three distinct subgroups, then each blob-set of $\mathcal{P}'$ can only be one of three objects: a left H_1H_2-coset, a left H_1H_3-coset, or a left $H_1H_2H_3$-coset. Interestingly, this structure may break down as soon as we have $r \geq 4$: a blob-set containing mutually H_1-linked H_2-, H_3-, and H_4-cosets in $\mathcal{P}'$ must be contained in the same left $H_1H_2H_3H_4$-coset by Lemma 21, but it is not necessarily equal to the whole coset. That is why the case $r = 3$ is special. A problematic blob for $r = 4$ is pictured in Figure 2, page 18.

Proof of Theorem 3. Let $r = 3$, and the set-up be as in Section 4, with $H_1 = H$, $H_2 = K$, and $H_3 = L$. Here is our plan for the proof: (1) In any blob of $\mathcal{P}'$ that contains both left K-cosets and left L-cosets of $\mathcal{P}'$, every H-link must intersect both K- and L-cosets of the blob. (2) For the same blob as in part (1), the underlying blob-set is then a right HKL-set.

(3) Every such blob-set is also transitive under the right HKL-action, and must be a left HKL-coset. (4) Finally, we describe the construction that leads to the partition $\mathcal{P}$.

(1) Let us fix a blob in $\mathcal{P}'$ that contains both left K-cosets and left L-cosets of $\mathcal{P}'$ for the first three parts of the proof. Assume that a link aH intersects only some distinct cosets $b_1K, \dots, b_sK$ in this blob, and hence, lies completely within the union of these K-cosets (the blob-set must contain the whole link). Then all $s+1$ cosets $aH, b_1K, \dots, b_sK$ must lie in the same left coset aHK of HK in G by Lemma 21. Since aH cannot intersect any other K-coset than the listed ones outside of aHK by Lemma 21, or inside of aHK because of its inclusion in the union of $b_1K, \dots, b_sK$, we must conclude that aHK is the disjoint union of $b_1K, \dots, b_sK$, where the latter list must form a pancake-type blob. This contradicts the fact that the K-cosets $b_1K, \dots, b_sK$ are part of a strictly larger blob (an equivalence class).

(2) We have established that each link aH in our blob-set intersects at least one K-coset bK in the blob and at least one L-coset cL in the blob. It suffices to show that elements of aH are sent to other elements of the same blob-set under the right actions of both K and L, so that the blob-set is invariant as a whole under the right actions of H, K, L. Indeed, note that $aH \cap bK \neq \emptyset$ implies that $aHK = bHK = bKH$ by Lemma 21, where bK is in our blob-set, and so is bKH, because the blob-set is closed under the right action of H by construction. Similarly, we have $aHL = cHL = cLH$, also staying in the blob-set. This proves that the subgroup HKL acts on the blob-set.

(3) We observe that the HKL-action on the blob-set is transitive. Indeed, we can move within links with the right H-action, within K-cosets of the blob with the right K-action, and within the L-cosets of the blob with the right L-action to go from any point in the blob-set to any other point. Therefore, the blob-set is an orbit of the right HKL-action by translations, which is a left HKL-coset.

(4) By Lemma 22 and parts (1)–(3) above, we know that blob-sets can only be HK-cosets, HL-cosets, or HKL-cosets. Also, if aK and bL are in $\mathcal{P}'$, then $aK \cap bL = \emptyset$, and we must have $aKL \cap bKL = \emptyset$ by Lemma 21. We start by decomposing any blob-set that is an HKL-coset in G (if any) completely into KL-cosets. If any KL-coset in it contains at least one K-coset (resp., L-coset) of the blob, then we decompose it fully into K-cosets (resp., L-cosets). This construction should capture all pairs aK, bL in the blob as they never reside in the same KL-coset. Therefore, it must also *exhaust* all the K- and L-cosets in the blob.

For blob-sets of type aHK (if any), a complete decomposition into K-cosets gives us all the K-cosets in the blob. Similarly, a blob-set of type bHL (if any) can be decomposed into the collection of the L-cosets of the blob. Can two such pancake-type blob-sets, say aHK and bHL, be in the same HKL-coset? No, because distinct blobs are disjoint equivalence classes, and we must have $aHKL \cap bHKL = \emptyset$ again by Lemma 21.

To recap:

- Decompose G into HKL-cosets.
- For any blob-set of the form $aHKL$, do the decomposition into first KL- then K- or L-cosets as described above. Consider the remaining HKL-cosets.
- If an HKL-coset contains at least one blob-set of type aHK, then completely decompose it into HK-cosets. Set aside those HK-cosets that are blob-sets and decompose them into K-cosets of $\mathcal{P}'$. Decompose the rest of the HK-cosets into H-cosets, because no L-pancake can be in the same HKL-coset simultaneously.
- Similarly treat HKL-cosets containing at least one blob-set of type bHL.
- If there are any leftover HKL-cosets, then decompose them completely into H-cosets.

This construction accounts for all cosets in $\mathcal{P}'$, which means that the complementary H-set (the union of all H-cosets in $\mathcal{P}_1$) must be the union of the leftover H-cosets in our construction. When $H \nleq KL$, $K \nleq HL$, and $L \nleq HK$, this is exactly the clean standard construction of $\mathcal{P}$. □

5. Proofs of Theorems 4, 5, and 6

Proof of Theorem 4. Let G be a group and H, K, L be distinct proper subgroups. There are precisely three unit-fraction decompositions of 1 into three parts, given by

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}, \quad \frac{1}{2}+\frac{1}{4}+\frac{1}{4}, \quad \frac{1}{3}+\frac{1}{3}+\frac{1}{3}.$$

Then any pure $\{H, K, L\}$-transversal coset partition of G must have index list $[2, 3, 6]$, or $[2, 4, 4]$, or $[3, 3, 3]$. If the last one is not allowed, then one subgroup (without loss of generality, say H) has index 2. The coset of H that is not in the partition must have been partitioned into the cosets of K and L that are in the partition. If necessary, we can translate this coset of H by the inverse of an element to make it H itself, without changing the fact that it is partitioned into one coset each of K and L. We then conclude that (i) $K, L \subseteq H$ (Lemma 20), and hence, (ii) H has a pure $\{K, L\}$-transversal coset partition, which is known not to exist for any group (Theorem 2). □

Proof of Theorem 5. Just as in the last proof, this case can be reduced to $r = 3$ for mutually commuting subgroups, which is covered by Theorem 3. □

Proof of Theorem 6. The case $r = 2$ is a side product of Theorem 2. Independently, we can argue as follows: since the only unit-fraction decomposition of 1 into two terms is $1/2 + 1/2$, the only possible index list is $[2, 2]$. (Even this list cannot be realized, which proves Conjecture 1 directly but is not necessary to show for Herzog-Schönheim.)

When $r = 3$ and there are repeated indices, we are done. Otherwise, suppose that a pure transversal partition with three cosets exists. Recall from the proof of Theorem 4 that the only numerically possible list of distinct indices of length three is $[2, 3, 6]$. However, the integers 2 and 3 are relatively prime and cannot be part of the index list of a coset partition by [23].

Suppose that a group G is partitioned into cosets of $r = 4$ distinct proper subgroups. If the partition exhibits repeated indices, then we are done. If it does not, then the only realizable index list of length 4 and consisting of distinct entries is $[2, 4, 6, 12]$ (Appendix A), as the remaining five lists contain pairs of coprime indices (ruled out in [23]). Once again, we conclude that three of the subgroups are contained in the one of index 2, and their cosets partition that largest subgroup, with indices $[2, 3, 6]$. This is not viable for the same reason as before.

Now assume that we have a coset partition of a group into cosets of $r = 5$ subgroups; if an index is repeated, we are done. Otherwise, we check the list of all possible decompositions of unity into a sum of five unit fractions (Appendix A). We find that all 147 decompositions are ineligible or corroborating Herzog-Schönheim as sums of reciprocal indices: each falls under at least one of three categories. That is, (i) some pairs of index candidates in the list are relatively prime; or (ii) the list has 2 as one index, which reduces the case to $r = 4$; or (iii) the list already has repetitions. This is a reasonable point to switch to a computer-assisted search.

Running a Haskell program by D. Akman, we verified all cases with $2 \leq r \leq 7$. Starting with lists of r distinct unit fractions adding up to 1, and eliminating all lists with a 2 in the denominator (induction) as well as lists with pairs of coprime denominators, we were able to cross out all lists and did not find any violations of Herzog-Schönheim. $\square$

We stopped at $r = 7$ since the computational time for $r = 8$ exceeded our limits. In any case, this method cannot eliminate all candidates for index lists alone, as [20] exhibits an index list for $r = 15$ that excludes all criteria (i)–(iii) we mention in the proof above:

$$[4, 6, 8, 10, 12, 16, 20, 24, 30, 40, 48, 60, 80, 120, 240].$$

D. Akman has also supplied us with a shorter list with $r = 13$ distinct indices, not necessarily of minimal length, with a least common multiple of 1080:

$$[3, 6, 9, 12, 15, 18, 24, 27, 30, 45, 54, 60, 72].$$

6. Proof of Theorem 7

We will first establish the following result:

Lemma 23. *Let $n, x_1, \ldots, x_n$ be integers such that $n, x_1, \ldots, x_n \geq 2$. Then we have*

$$x_1 + \cdots + x_n \leq x_1 \cdots x_n,$$

with equality holding if and only if $n = x_1 = x_2 = 2$.

Proof. Let us first do induction on the positive integer $x_1 + x_2$. When $x_1 = x_2 = 2$, we have $4 = 4$. Now assume that the inequality holds for some $x_1 + x_2 \geq 4$, that is, $x_1 + x_2 \leq x_1x_2$, and prove that the strict inequality is true for $(x_1 + 1) + x_2$. We have

$$(x_1 + 1) + x_2 = (x_1 + x_2) + 1 \leq x_1x_2 + 1 < x_1x_2 + x_2 = (x_1 + 1)x_2.$$

Similarly, we fix $n \geq 3$ and do induction on the sum $x_1 + \cdots + x_n$. For the base case $x_1 = \cdots = x_n = 2$, we would like to have $2n < 2^n$, or, $n < 2^{n-1}$ for any $n \geq 3$, which is true (calculus). Assume that the strict inequality holds for some $x_1 + \cdots + x_n$ and prove it for $(x_1 + 1) + x_2 + \cdots + x_n$:

$$(x_1+1)+x_2+\cdots+x_n = (x_1+x_2+\cdots+x_n)+1 < x_1x_2\cdots x_n+x_2\cdots x_n = (x_1+1)x_2\cdots x_n.$$

$\square$

We also provide a construction for the special case of Theorem 7, which we exhibited as our first non-standard case in Example 15:

Example 24. *Let $G = H \times K \times L \times M$ be the direct product of four subgroups of order two. Then G has a coset partition into cosets of all four subgroups. Let $H = \langle a \rangle$, $K = \langle b \rangle$, $L = \langle c \rangle$, and $M = \langle d \rangle$. We fix any HK-coset, say*

$$HK = \{1, a, b, ab\}.$$

We then pick any L-coset that does not intersect HK, say, $dL = \{d, cd\}$. Then dHL will not intersect HK either by Lemma 21:

$$dHL = \{d, ad, cd, acd\}.$$

Now pick an M-coset that does not intersect HK or dHL, say, $abcM = \{abc, abcd\}$. Then HM does not intersect HK and dHL, again by Lemma 21. Indeed, we have

$$abcHM = \{abc, bc, abcd, bcd\}.$$

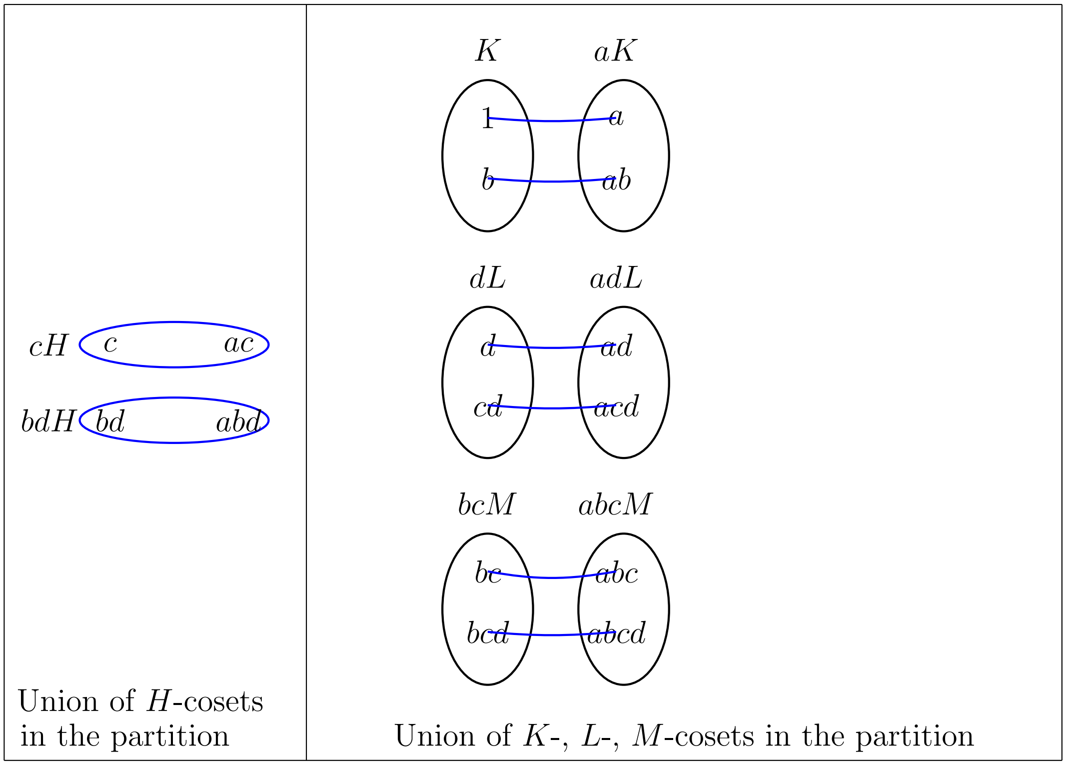

FIGURE 1. Blobs that are stacks of pancakes

The remaining elements of G can be compiled into two H-cosets:

$$cH = \{c, ac\} \text{ and } bdH = \{bd, abd\}.$$

By decomposing the HK-, HL-, and HM-cosets into K-, L-, and M-cosets as well, we obtain a coset partition of G into cosets of all four subgroups:

$$K,\ aK,\ dL,\ adL,\ bcM,\ abcM,\ cH,\ bdH.$$

The blob structure of this partition is depicted in Figure 1.

Remark 25. *These choices were random and the construction would have worked for any others as long as the new cosets are chosen to have no intersection with the previous ones. First, any choice of the first HK-coset is fine, as we may translate any coset partition of G by any element of G and still obtain a coset partition of G with the same distribution of coset types. Therefore, we may assume that the first HK-coset is HK itself. Second, the coset HK can (and does) intersect exactly 4 L-cosets by Lemma 20: we have $|HK \cap L| = 1$. There remain $8 - 4 = 4$ L-cosets in G to choose from. When we pick one (dL), the number of M-cosets that intersect $HK \sqcup dHL$ should be at most 8, but that would immediately exhaust all available M-cosets. However, $HK \sqcup dHL$ contains two complete M-cosets, M and aM (equivalently, the coset HM), which gives us room to pick from 2 more unrelated M-cosets. This would have occurred regardless of our choices. There are exactly two HL-cosets that do not intersect HK: dHL and $bdHL$. In the first case, $HM \subset HK \sqcup dHL$; in the second case, $bHM \subset HK \sqcup bdHL$. For the general proof of Theorem 7, we can be more generous with our* *estimates, assuming that every element of the union of mutually disjoint cosets constructed so far might lie in a different coset of the next subgroup.*

Finally, we can finish the proof of Theorem 7:

Proof. If $r = 4$ and all $|H_i| = 2$, then Example 24 applies. Suppose not, so that without loss of generality we have $|H_1| > 2$ or $r > 4$. We will show the existence of mutually disjoint cosets

$$a_1H_1H_r, \ldots, a_{r-1}H_{r-1}H_r$$

in G, whose union is an H_r-set. We will also show that this union is not all of G, so that the remaining elements of G can be written as a union of H_r-cosets. It then remains to decompose each H_iH_r, where $1 \le i \le r-1$, fully into H_i-cosets. Let us prove the first assertion: we begin by picking any coset $a_1H_1H_r$. There must be some coset a_2H_2 that does not intersect this coset, because we have

$$|a_1H_1H_r| = |H_1||H_r| < |H_1||H_3| \cdots |H_r| = [G\colon H_2].$$

In other words, even if every element in the coset $a_1H_1H_r$ were in a different H_2-coset, we would still have H_2-cosets left over. Say a_2H_2 does not intersect $a_1H_1H_r$. This condition is equivalent to

$$a_1H_1H_r \cap a_2H_2H_r = \emptyset.$$

We contend that there exists a coset a_3H_3 such that

$$a_1H_1H_r \cap a_3H_3 = \emptyset \text{ and } a_2H_2H_r \cap a_3H_3 = \emptyset.$$

Once again, we have

$$\begin{aligned} & |a_1H_1H_r \sqcup a_2H_2H_r| \\ = \; & |H_1||H_r| + |H_2||H_r| \\ = \; & (|H_1| + |H_2|)|H_r| \\ \le \; & |H_1||H_2||H_r| \\ \le \; & [G\colon H_3], \end{aligned}$$

where the overall inequality must be strict (Lemma 23 with $x_1 = |H_1| > 2$ or else $|H_1| = 2$ but $r > 4$). This is equivalent to saying there exists a coset $a_3H_3H_r$ such that

$$a_1H_1H_r \cap a_3H_3H_r = \emptyset \text{ and } a_2H_2H_r \cap a_3H_3H_r = \emptyset.$$

If $r = 4$, then we stop here. Otherwise, we continue until we reach a coset of H_{r-1}. That is, when $3 \le j \le r-2$ and we can safely use the Lemma with a strict inequality, suppose we already have $a_iH_iH_r$, with $1 \le i \le j$, where these cosets are mutually disjoint. Then

$$\begin{aligned} & |a_1H_1H_r \sqcup \cdots \sqcup a_jH_jH_r| \\ = \; & (|H_1| + \cdots + |H_j|)|H_r| \\ < \; & |H_1| \cdots |H_j||H_r| \\ \le \; & [G\colon H_{j+1}], \end{aligned}$$

and there exists some $a_{j+1}H_{j+1}$ (hence, also $a_{j+1}H_{j+1}H_r$) disjoint from all these. When $j+1 = r-1$ (or $j = r-2$), we stop and consider the disjoint union of all double cosets $a_1H_1H_r, \ldots, a_{r-1}H_{r-1}H_r$ constructed so far; this is an H_r-set. We can decompose these into

H_1-,...,H_{r-1}-cosets respectively. If there is anything left over, then it can be decomposed into H_r-cosets, finishing the proof. We have

$$\begin{aligned} & |a_1H_1H_r \sqcup \cdots \sqcup a_{r-1}H_{r-1}H_r| \\ = \ & (|H_1| + \cdots + |H_{r-1}|)|H_r| \\ < \ & |H_1| \cdots |H_{r-1}||H_r| = |G| \quad \text{(Lemma 23 with } r-1 \geq 3), \end{aligned}$$

indicating that there are indeed some H_r-cosets left over. □

Remark 26. *Note that this construction also applies to any collection of subgroups $H_{T_1}, \ldots, H_{T_m}$ of G, with*

$$\emptyset \neq T_1, \ldots, T_m \subset \{1, \ldots, r\},\ m \geq 4,\ T_1 \sqcup \cdots \sqcup T_m = \{1, \ldots, r\},$$

instead of the subgroups $H_1, \ldots, H_r$ of G. Consequently, many other constructions of transversal coset partitions of G than the one described in the proof of the theorem are possible, with various iterations of standard and non-standard constructions.

7. Extensions and Applications

7.1. **Transversal Coset Parallelisms.** The ability to cover an object X with a collection of smaller objects Y_j in such a way that every element, or special subset, of X is covered by the same number of Y_j's is a recurring theme in combinatorics and design theory. Let G be a group and $H_1, \ldots, H_r$ be distinct proper subgroups of G of finite index. An **exact** m**-cover** (see [30]) of a group G by cosets is a collection of cosets of G such that each element of G belongs to exactly m cosets in the collection. We will call those exact m-covers with no proper exact m'-covers (with $1 \leq m' < m$) **irreducible** (a.k.a. minimal); otherwise, the cover will be **reducible**. Note that any exact m-cover must be a disjoint union of irreducible exact covers of G, as extracting an m'-cover would leave behind an $(m-m')$-cover. We refine the concept of an exact m-cover in steps as follows.

We say that a collection $\{\mathcal{P}_1, \ldots, \mathcal{P}_m\}$ of sets of cosets in G is an $\{H_1, \ldots, H_r\}$**-coset parallelism** of G if each $\mathcal{P}_i$ is a coset partition (not necessarily transversal) of G, and each coset in $\{gH_j : g \in G \text{ and } 1 \leq j \leq r\}$ belongs to exactly one $\mathcal{P}_i$. If the subgroups $H_1, \ldots, H_r$ are clear from the context, then we simply call it a "coset parallelism". A trivial $\{H_1, \ldots, H_r\}$-coset parallelism can be obtained by combining all left H_i-coset partitions of G for $1 \leq i \leq r$. On the other hand, we may have an $\{H_1, \ldots, H_r\}$-coset parallelism composed of $\{H_1, \ldots, H_r\}$-transversal coset partitions. In this case, we say that G admits an $\{H_1, \ldots, H_r\}$**-transversal coset parallelism**, or simply a transversal coset parallelism, if the set $\{H_1, \ldots, H_r\}$ of subgroups is clear from the context. When each partition is also pure, the expression **pure transversal coset parallelism** will be used.

The problem of determining when transversal coset parallelisms exist for a group G is hard to tackle in its full generality, but we can ask when it is possible to construct such a coset parallelism using only left translations of a partition of G. Clearly, if $\mathcal{P}$ is any transversal coset partition and $g \in G$, then $g\mathcal{P} = \{gC \colon C \in \mathcal{P}\}$ is another transversal partition of G with the same numbers of each type of coset. There are some numerical limitations for parallelism by translations: if there are $m_1, \ldots, m_r$ cosets of $H_1, \ldots, H_r$ respectively in any one partition, and we want a total of m partitions that cover each coset of each subgroup exactly once, then we must have

$$m_1|H_1| + \cdots + m_r|H_r| = |G| \tag{3}$$

as well as

$$mm_i = [G\colon H_i] \Rightarrow mm_i|H_j| = |G| \text{ for all } i, \tag{4}$$

implying that

$$\frac{|G|}{m} + \cdots + \frac{|G|}{m} = \frac{r|G|}{m} = |G| \Rightarrow m = r. \tag{5}$$

We find that the number of subgroups must divide the order of G, among other things. Note that Eq. (3) is equivalent to Eq. (1). The following proposition highlights a common thread in the examples of parallelism that we have in this article:

Proposition 27. *Let G be a group, $H_1, \dots, H_r$ be distinct proper subgroups of G, and $\mathcal{P}$ be a transversal coset partition of G, where each subgroup H_i has m_i cosets. We can express this partition as*

$$\mathcal{P} = \{a_{ij}H_i \colon 1 \le i \le r,\ 1 \le j \le m_i\}.$$

Suppose that $[G\colon H_i] = rm_i$ for all i as well, and that there exist a subgroup Γ of G satisfying the following properties:

(a) *$\Gamma H_i = H_i\Gamma$ for $1 \le i \le r$;*
(b) *$\Gamma \cap H_i = \{1\}$ for $1 \le i \le r$;*
(c) *$|\Gamma| = r$, with $\Gamma = \{\gamma_1, \dots, \gamma_r\}$; and*
(d) *$a_{ij}\Gamma H_i \cap a_{il}H_i = \emptyset$ if $j \ne l$ for $1 \le i \le r$.*

Then the collection of cosets

$$\mathcal{P}\Gamma \stackrel{\text{def}}{=} \{a_{ij}\gamma_k H_i \colon 1 \le i \le r,\ 1 \le j \le m_i,\ 1 \le k \le r\}$$

forms an exact r-covering of G that is also a coset parallelism: every coset of every H_i appears exactly once. If, in addition, the condition

(e) *Elements of Γ commute with the coset representatives a_{ij} for all i, j with $1 \le i \le r$ and $1 \le j \le m_i$*

is satisfied, then we obtain an $\{H_1, \dots, H_r\}$-transversal coset parallelism $\{\mathcal{P}_1, \dots, \mathcal{P}_r\}$, with

$$\mathcal{P}_k = \gamma_k\mathcal{P} = \{a_{ij}\gamma_k H_i \colon 1 \le i \le r,\ 1 \le j \le m_i\},\ \ 1 \le k \le r.$$

Remark 28. *The construction of parallelisms via conditions (a)–(e) shows that irreducible components of an exact r-cover are not necessarily unique: we can decompose the same cover into the collections G/H_i of all left cosets of each H_i as well as into the transversal coset partitions $\lambda_k\mathcal{P}$. This is a demonstration of the aptness of the descriptor "transversal", as the $\lambda_k\mathcal{P}$'s cut through all single-subgroup coset collections G/H_i.*

Proof. Assume that conditions (a)–(d) are satisfied. Condition (d) (which is equivalent to $a_{ij}\Gamma H_i \cap a_{il}\Gamma H_i = \emptyset$ by Lemma 21) tells us that the a_{ij}'s form nonequivalent coset representatives for the group ΓH_i as well. Since

$$rm_i = [G\colon H_i] = [G\colon \Gamma H_i][\Gamma H_i \colon H_i] = [G\colon \Gamma H_i][\Gamma\colon \Gamma \cap H_i] = [G\colon \Gamma H_i]r,$$

we conclude that the set $\{a_{i1}, \dots, a_{im_i}\}$ is a full set of coset representatives for ΓH_i, which can in turn be decomposed into left cosets of H_i. Thus, the collection $\mathcal{P}\Gamma$ exhibits a coset parallelism where all cosets of all H_i's are present: the left-coset representatives for H_i are

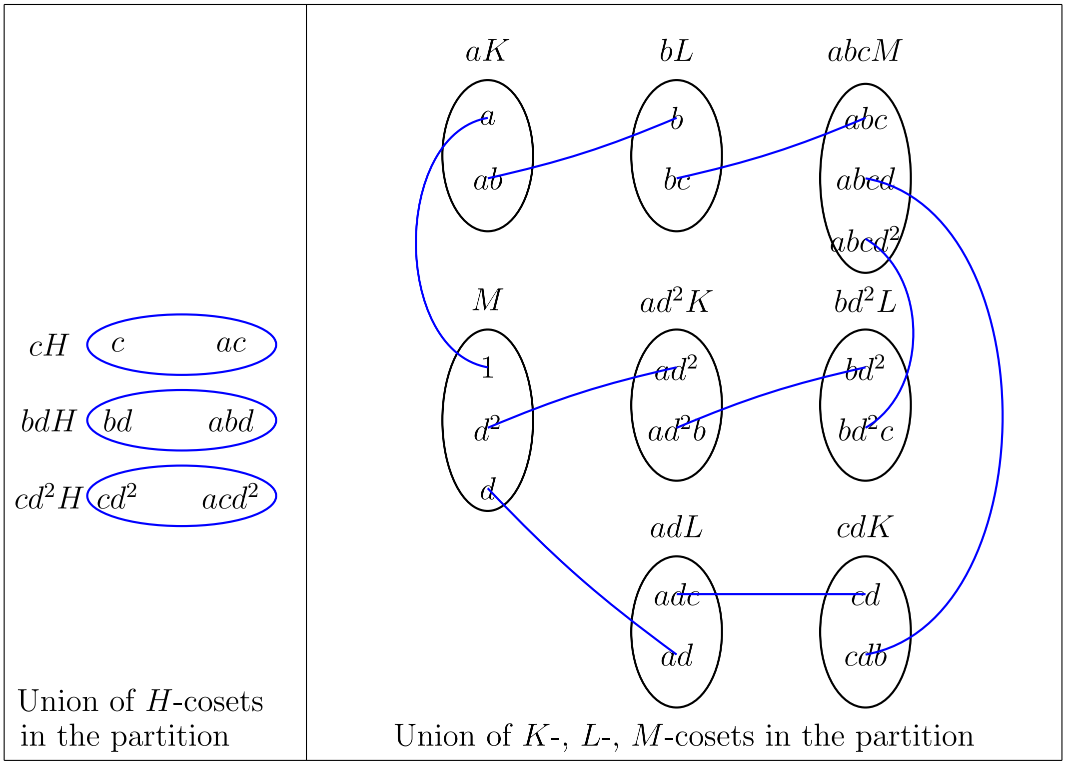

FIGURE 2. A blob that is not an $HKLM$-coset

$\{a_{ij}\gamma_k \colon 1 \le j \le m_i,\ 1 \le k \le r\}$. Now assume further that condition (e) holds. Then each $\mathcal{P}_k = \gamma_k\mathcal{P}$ is another transversal coset partition of G, which is true for all left translations of left-coset partitions by a group element. □

Example 29. *Let $H = \langle a\rangle$, $K = \langle b\rangle$, and $L = \langle c\rangle$ be cyclic groups of order 2, and $M = \langle d\rangle$ be a cyclic group of order 3. Consider the direct product $G = H \times K \times L \times M = \{a^{t_1}b^{t_2}c^{t_3}d^{t_4} : t_1, t_2, t_3 \in \{0, 1\},\ t_4 \in \{0, 1, 2\}\}$. Then we have the following transversal coset parallelism of G (thus, each $\mathcal{P}_i$ is an $\{H, K, L, M\}$-transversal coset partition of G):*

1. $\mathcal{P}_1 = \{cH,\ bdH,\ cd^2H,\ aK,\ ad^2K,\ cdK,\ bL,\ adL,\ bd^2L,\ M,\ abcM\}$. *(We note that this transversal coset partition was obtained greedily and not via the construction outlined in Theorem 7.)*
2. $\mathcal{P}_2 = ab\mathcal{P}_1 = \{abC : C \in \mathcal{P}_1\}$
3. $\mathcal{P}_3 = ac\mathcal{P}_1 = \{acC : C \in \mathcal{P}_1\}$
4. $\mathcal{P}_4 = bc\mathcal{P}_1 = \{bcC : C \in \mathcal{P}_1\}$

In other words, the transversal coset parallelism is obtained from $\mathcal{P}_1$ via translations by the Klein 4-group $\Gamma = \{1, ab, ac, bc\}$ of G. The blob structure of the partition $\mathcal{P}_1$ is depicted in Figure 2.

Another abelian example is the following:

Example 30. *In Examples 15 and 24, Γ can be chosen to be the following Klein 4-group:*

$$\Gamma = \{1, cd, abc, abd\}.$$

The hypotheses of Proposition 27 are easily satisfied in the two examples above, where G is abelian and a product of the subgroups. Another special case is applicable to groups G with $r \geq 3$ conjugate subgroups H_i of index r, if there exists a **complementary** abelian subgroup Γ to all H_i's (which necessarily commutes with them). That is, we would like to have $G = \Gamma H_i$ and $|\Gamma \cap H_i| = 1$ for all i.

Corollary 31. *Let G be a group, H be a subgroup of G of index r such that $N_G(H)=H$, and $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$ be an abelian subgroup of G of order r that is complementary to H. Then*

(1) *The set Γ is a full set of right coset representatives for H;*
(2) *The subgroups $H_i = \gamma_i^{-1} H \gamma_i$ for $1 \leq i \leq r$ are distinct and also complemented by Γ;*
(3) *The collection $\mathcal{P} = \{\gamma_1 H_1, \ldots, \gamma_r H_r\}$ is a pure transversal coset partition of H; and*
(4) *The partitions $\{\mathcal{P}_1, \ldots, \mathcal{P}_r\} = \{\gamma_1 \mathcal{P}, \ldots, \gamma_r \mathcal{P}\}$ form a transversal coset parallelism of G.*

Proof. The result follows from Lemma 12 and Proposition 27; note that $m_i = 1$ for all i and the elements of Γ commute among themselves. We additionally verify that

$$\Gamma \cap H_i = \Gamma \cap \gamma_i^{-1} H \gamma_i = \gamma_i^{-1} (\Gamma \cap H) \gamma_i = \langle 1 \rangle$$

and

$$\Gamma H_i = \Gamma \gamma_i^{-1} H \gamma_i = \Gamma H \gamma_i = G \gamma_i = G.$$ □

Example 32. *Let $r \geq 3$ be odd. We have indicated in Example 13 that D_n, the dihedral group of order $2n$, where $n = 2^k r$ and $k \geq 0$, has a pure transversal coset partition for the conjugates of the subgroup $H = \langle a^r, b \rangle$ of order 2^{k+1} and index r. These are the r distinct Sylow 2-subgroups of D_n. We also have*

$$D_n = \Gamma \langle a^r, b \rangle ,$$

where $\gamma = a^{2^k}$ and $\Gamma = \langle \gamma \rangle$ is abelian, is complementary to H, and comprises a set of right-coset representatives for H. Applying Corollary 31, we obtain a pure transversal coset partition $\mathcal{P}$ of D_n via single cosets of r subgroups of index r. Finally, we extend this to a pure transversal coset parallelism of D_n:

$$\mathcal{P}, \gamma \mathcal{P}, \ldots, \gamma^{r-1} \mathcal{P}.$$

Example 33. *In Example 14 we considered the symmetric group S_r for $r \geq 3$ and employed Lemma 12 for*

$$H = S_{[1]} \text{ and } g_1 = 1,\ g_i = (1i) \text{ for } 2 \leq i \leq r.$$

Since $g_i^{-1} S_{[1]} g_i = S_{[i]}$, we are able to say that

$$\mathcal{P} = \{S_{[1]},\ (12)S_{[2]}, \ldots, (1r)S_{[r]}\}$$

is a pure transversal coset partition of S_r. When $r = 4$, we are also able to find an abelian complement Γ for $S_{[1]}$, namely, the normal Klein 4-subgroup of A_4: we have

$$\Gamma = \{(1),\ (12)(34),\ (13)(24),\ (14)(23)\},\ S_4 = \Gamma S_{[1]},\ \text{and } \Gamma \cap S_{[1]} = \langle (1) \rangle .$$

By Corollary 31, the following is a pure transversal coset parallelism of S_4:

$$\mathcal{P},\ (12)(34)\mathcal{P},\ (13)(24)\mathcal{P},\ (14)(23)\mathcal{P}.$$

Remark 34. *By the Schur-Zassenhaus Theorem, any normal subgroup Γ of a finite group G whose order and index are relatively prime must have a complement H in G, and all complements are conjugates of one another. Hence, we obtain a transversal coset partition of G into $|\Gamma|$ left cosets of the conjugates of H, whether the conjugates are distinct or repeated. This translates into an exact $|\Gamma|$-cover of G that is a coset parallelism. If Γ is further abelian, then we obtain a transversal coset parallelism, not necessarily pure.*

7.2. **Application to Subspace Partitions.** Our interest in coset partitions of finite abelian groups stems from subspace partitions of finite vector spaces. Let V be a vector space of dimension n over the Galois field $\mathbb{F}_q$, where q is a prime power. Then any collection of nonzero subspaces of V that cover V, in which distinct pairs of subspaces have trivial intersections, is called a **subspace partition** (also, a vector space partition) of V. Now, V can be identified with the field $\mathbb{F}_{q^n}$, and the multiplicative group $\mathbb{F}_{q^n}^*$ is cyclic. If a is a positive divisor of n, then the unique cyclic subgroup H of $\mathbb{F}_{q^n}^*$ of order q^a, together with the zero vector, is always a subspace of V. Hence, H-cosets, appended with zero, form a subspace partition of V [3, 28]. Similarly, if a, b are distinct divisors of n, then double cosets of the subgroups H, K of $\mathbb{F}_{q^n}^*$ of orders $q^a - 1$ and $q^b - 1$ respectively can be decomposed into a coset partition of $\mathbb{F}_{q^n}^*$ into H- and K-cosets (necessarily by a standard construction). This in turn induces a subspace partition of V into subspaces of dimensions a and b [2, 5, 8]. More generally, if $a_1, \ldots, a_r$ are mutually relatively prime divisors of n with corresponding subgroups $H_1, \ldots, H_r$ of orders $q^{a_1}, \ldots, q^{a_r}$ respectively, then $H_1, \ldots, H_r$-transversal coset partitions of $\mathbb{F}_{q^n}^*$ together with the zero vector give us subspace partitions of V with subspace dimensions $a_1, \ldots, a_r$. Thus, the non-standard constructions described above become possible when n is divisible by at least four distinct primes. This fact significantly increases the construction possibilities of subspace partitions via cosets of $\mathbb{F}_{q^n}^*$.

Appendix A. Unit-Fraction Decompositions of 1

This Appendix contains our program outputs for $r = 3, 4, 5$ used in the main paper (Theorem 6). These are the lists of distinct indices such that the sum of their reciprocals is 1. Note any such list includes 2, or it contains two entries that are relatively prime. The full Haskell codes for $2 \leq r \leq 7$ can be found[2] in [1].

[6, 3, 2], [42, 7, 3, 2], [24, 8, 3, 2], [18, 9, 3, 2], [15, 10, 3, 2], [20, 5, 4, 2], [12, 6, 4, 2],

[1806, 43, 7, 3, 2], [924, 44, 7, 3, 2], [630, 45, 7, 3, 2], [483, 46, 7, 3, 2], [336, 48, 7, 3, 2],

[294, 49, 7, 3, 2], [238, 51, 7, 3, 2], [189, 54, 7, 3, 2], [168, 56, 7, 3, 2], [140, 60, 7, 3, 2],

[126, 63, 7, 3, 2], [105, 70, 7, 3, 2], [91, 78, 7, 3, 2], [600, 25, 8, 3, 2], [312, 26, 8, 3, 2],

[216, 27, 8, 3, 2], [168, 28, 8, 3, 2], [120, 30, 8, 3, 2], [96, 32, 8, 3, 2], [88, 33, 8, 3, 2],

[72, 36, 8, 3, 2], [60, 40, 8, 3, 2], [56, 42, 8, 3, 2], [342, 19, 9, 3, 2], [180, 20, 9, 3, 2],

[126, 21, 9, 3, 2], [99, 22, 9, 3, 2], [72, 24, 9, 3, 2], [54, 27, 9, 3, 2], [45, 30, 9, 3, 2],

[240, 16, 10, 3, 2], [90, 18, 10, 3, 2], [60, 20, 10, 3, 2], [40, 24, 10, 3, 2], [231, 14, 11, 3, 2],

[110, 15, 11, 3, 2], [33, 22, 11, 3, 2], [156, 13, 12, 3, 2], [84, 14, 12, 3, 2], [60, 15, 12, 3, 2],

[2] For $r \geq 8$, the program takes too long and we do not have an output yet.

$[48, 16, 12, 3, 2], [36, 18, 12, 3, 2], [30, 20, 12, 3, 2], [28, 21, 12, 3, 2], [35, 15, 14, 3, 2],$
$[420, 21, 5, 4, 2], [220, 22, 5, 4, 2], [120, 24, 5, 4, 2], [100, 25, 5, 4, 2], [70, 28, 5, 4, 2],$
$[60, 30, 5, 4, 2], [45, 36, 5, 4, 2], [156, 13, 6, 4, 2], [84, 14, 6, 4, 2], [60, 15, 6, 4, 2],$
$[48, 16, 6, 4, 2], [36, 18, 6, 4, 2], [30, 20, 6, 4, 2], [28, 21, 6, 4, 2], [140, 10, 7, 4, 2],$
$[42, 12, 7, 4, 2], [28, 14, 7, 4, 2], [72, 9, 8, 4, 2], [40, 10, 8, 4, 2], [24, 12, 8, 4, 2],$
$[18, 12, 9, 4, 2], [15, 12, 10, 4, 2], [120, 8, 6, 5, 2], [45, 9, 6, 5, 2], [30, 10, 6, 5, 2],$
$[20, 12, 6, 5, 2], [20, 6, 5, 4, 3].$

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