A Combinatorial Proof of a Tantalizing Symmetry on Catalan Objects

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We investigate a tantalizing symmetry on Catalan objects. In terms of Dyck paths, this symmetry can be interpreted in the following way: if $w_{n,k,m}$ is the number of Dyck paths of semilength n with k occurrences of UD and m occurrences of UUD, then $w_{2k+1,k,m} = w_{2k+1,k,k+1-m}$. We give a combinatorial proof of this symmetry which makes use of the cycle lemma and an alternative interpretation of the numbers $w_{n,k,m}$ involving plane trees. In particular, our combinatorial proof expresses the numbers $w_{2k+1,k,m}$ in terms of Narayana numbers, and we generalize this to a relationship between the numbers $w_{n,k,m}$ and a family of generalized Narayana numbers due to Callan.