Fast Algorithms for $(\Delta + 1)$-Edge-Coloring

Joint work with Anton Bernshteyn

Abhishek Dhawan

School of Mathematics
Georgia Institute of Technology

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Vizing's Theorem

Recall a proper k-**edge-coloring** is a function $\varphi : E(G) \to [k]$ such that $\varphi(e) \neq \varphi(f)$ when e and f share an endpoint.
The **chromatic index** of a graph (denoted $\chi'(G)$) is the minimum value k such that G admits a proper k-edge-coloring.

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The **chromatic index** of a graph (denoted $\chi'(G)$) is the minimum value k such that G admits a proper k-edge-coloring.

The following renowned result of Vizing provides a general bound on $\chi'(G)$.

Theorem (Vizing 1964)

For any simple graph G, $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.

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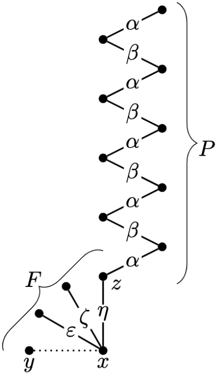
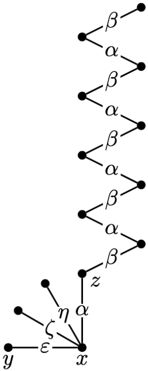


(a) Before shifting.



(b) After shifting.

Sequential Algorithms

Theorem

- *There is a deterministic* $(\Delta + 1)$*-edge-coloring algorithm that runs in* $O(mn)$ *time (Bollobás 1984; Rao, Dijkstra 1992; Misra, Gries 1992).*

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Locality of Vizing's Theorem

Definition (Augmenting Subgraphs)

Given a partial coloring φ, we say a subgraph $H \subseteq G$ is **augmenting** if there is at least one uncolored edge in H and there is a proper coloring ψ that can be obtained by modifying φ on the edges in H such that every edge in H is now colored.

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For an arbitrary n-vertex graph G of maximum degree Δ and partial $(\Delta + 1)$-edge-coloring φ, how small can H be?

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- *There exists G, φ such that $e(H) = \Omega(\Delta \log n)$ for every augmenting subgraph H (Chang, He, Li, Pettie, Uitto 2018).*

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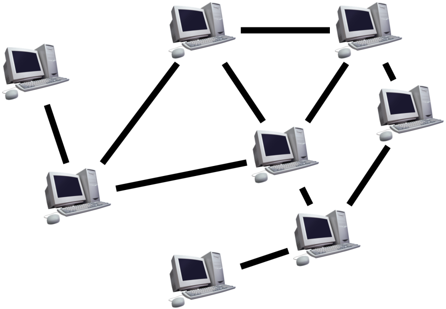
- *For every G, φ there exists an augmenting subgraph H such that $e(H) \leq \mathrm{poly}(\Delta)(\log n)^2$ (Bernshteyn 2021).*

Distributed Algorithms

A distributed algorithm in the LOCAL model occurs in rounds, where in each round, every vertex performs some local computation and then broadcasts its results to its neighbors in G.



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- *There is a randomised distributed* $(\Delta + \tilde{O}(\sqrt{\Delta}))$*-edge-coloring algorithm that runs in* $\mathrm{poly}(\Delta, \log\log n)$ *rounds (Chang, He, Li, Pettie, and Uitto 2018).*

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- *There is a deterministic distributed* $(\Delta + 1)$*-edge-coloring algorithm that runs in* $\mathrm{poly}(\Delta, \log\log n)\,(\log n)^{11}$ *rounds (Bernshteyn 2021).*

Our Results

Theorem (Bernshteyn, D. 2022+)

Let G be an n-vertex graph of maximum degree Δ and let φ be a partial $(\Delta + 1)$-edge-coloring. For any uncolored edge e, there is an augmenting subgraph H containing e such that $e(H) \leq \text{poly}(\Delta) \log n$.

Our Results

Theorem (Bernshteyn, D. 2022+)

There exists a randomized (resp. deterministic) distributed algorithm in the LOCAL model that computes a proper $(\Delta + 1)$*-edge-coloring of an* n*-vertex graph of maximum degree* Δ *in* $\mathrm{poly}(\Delta)(\log n)^3$ *rounds (resp.* $\mathrm{poly}(\Delta, \log\log n)(\log n)^6$*).*

Theorem (Bernshteyn, D. 2022+)

There exists a randomized sequential algorithm that computes a proper $(\Delta + 1)$*-edge-coloring of an* n*-vertex graph of maximum degree* Δ *in poly*$(\Delta)\, n$ *time with probability at least* $1 - 1/\Delta^n$*.*

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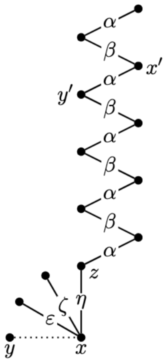
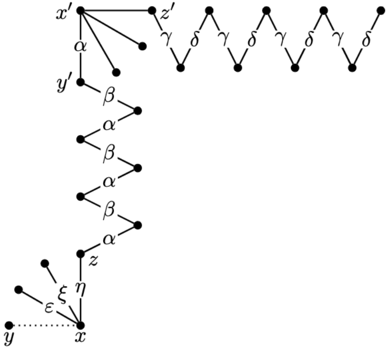
Non-Intersecting Vizing Chains

Idea (Grebík, Pikhurko 2020; Bernshteyn 2021)

- If P is too long, consider a random initial segment of P, shift and try again.





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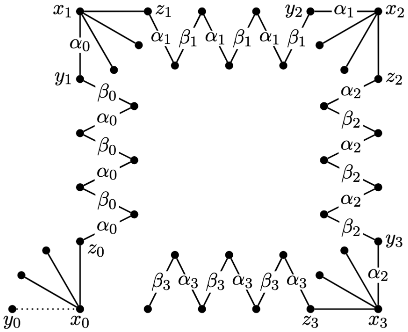
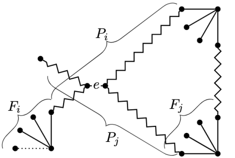


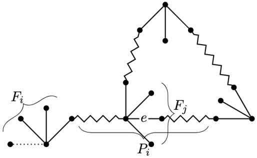
Non-Intersecting Vizing Chains

Idea (Grebík, Pikhurko 2020; Bernshteyn 2021)

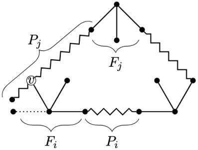
- If P is too long, consider a random initial segment of P, shift and try again.
- Repeat until the final path is short, while guaranteeing the chain is non-intersecting.



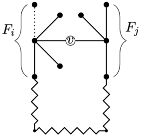
(a) Intersection between P_i, P_j at an edge e



(b) Intersection between P_i, F_j at an edge e



(c) Intersection between F_i, P_j at a vertex v.



(d) Intersection between F_i, F_j at a vertex v

The Multi-Step Vizing Algorithm

- Input: A graph G, a partial coloring φ, an uncolored edge $e = xy$, a vertex $x \in e$ and a parameter $\ell \in \mathbb{N}$.
- Output: A non-intersecting augmenting multi-step Vizing chain C such that $\mathsf{Start}(C) = xy$.

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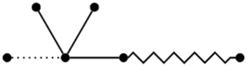
Step 1: First Chain

Find the first Vizing Chain $F + P$ on C.



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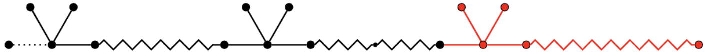
Step 2: Iterate

Let C (in black) and $F + P$ (in red) be the current chain and candidate chain at the start of the iteration.



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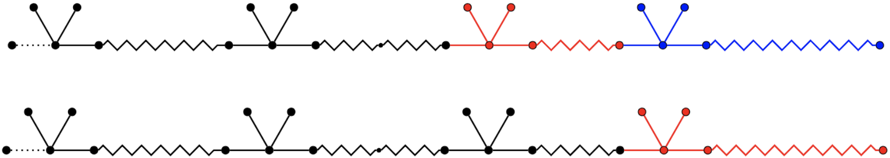
Let C (in black) and $F + P$ (in red) be the current chain and candidate chain after truncation. Let $\tilde{F} + \tilde{P}$ (in blue) be the next step chain.

Case 1: Non-intersecting.



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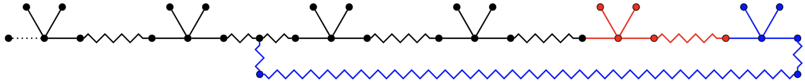
Case 2: Intersecting.



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Termination of the Multi-Step Vizing Algorithm

Theorem (Bernshteyn, D. 2022+)

For $\ell = \text{poly}(\Delta)$ *of large enough degree, the Multi-Step Vizing Algorithm terminates in* $O(\log n)$ *steps with probability at least* $1 - 1/\text{poly}(n)$*.*

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We utilise the entropy compression argument developed by Moser and Tardos in 2010.

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Some Notable Applications

- For every $n \geq 1$ and a sequence of sets $L_1, \ldots, L_n$ of size at least 4, there is a non-repetitive sequence chosen from $L_1, \ldots, L_n$ (Grytczuk, Kozik, Micek 2011).
- For every graph G of maximum degree Δ, there is a non-repetitive coloring using at most $(1 + o(1))\Delta^2$ colors (Dujmović, Joret, Kozik, Wood 2014; Rosenfeld 2020).
- The list chromatic number of triangle free graphs is at most $(1 + o(1))\Delta / \log \Delta$ asymptotically (Molloy 2017).

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- Suppose your process runs for t steps making random choices $\omega_1, \ldots, \omega_t$.

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- Define a record for each step $r_1, \dots, r_t$ such that given the record and the output $\mathcal{O}_t$ after t steps, we can recover $\omega_1, \dots, \omega_t$ as well as the outputs $\mathcal{O}_j$ for each $1 \leq j < t$.

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- We have

$$\mathbb{P}[\text{Failure in } t \text{ steps}] \leq \frac{\#(\text{record,output})\text{-pairs}}{\text{Amount of randomness generated}}$$

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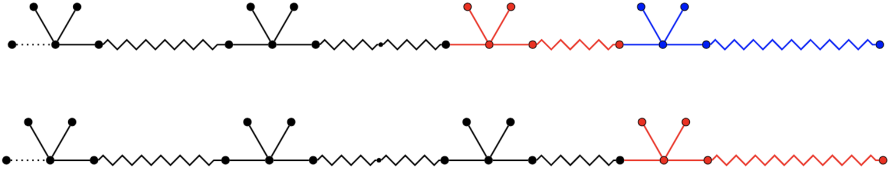

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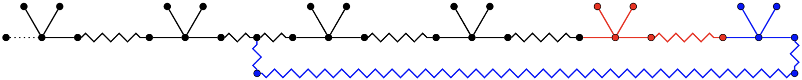
2. If the second step chain found intersects C, we let $d_i = -k$ and let k_i denote the $(k+1)$-step chain causing the truncation.



Termination of the Multi-Step Vizing Algorithm

A subtle counting argument shows that the number of k-step chains ending at a vertex x is at most $\mathrm{poly}(\Delta)^k$. This observation helps in bounding the number of (record, output)-pairs.

Lemma

Assuming the process doesn't terminate, there are at most $n(\mathrm{poly}(\Delta)\,\ell)^{t/2}$ (record, output)-pairs after t steps of the Multi-Step Vizing Algorithm.

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It follows that

$$\mathbb{P}[\mathrm{Failure}] \leq \frac{n(\mathrm{poly}(\Delta)\,\ell)^{t/2}}{\ell^t}.$$

For $\ell \sim \mathrm{poly}(\Delta)$ and $t \sim \log n$, the above is at most $1/\mathrm{poly}(n)$.

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3. The sequential coloring algorithm matches the lower bound asymptotically.
4. Moreover, the result implies that the average size of an augmenting subgraph found using the Multi-Step Vizing Algorithm is independent of n.
5. Unconventional use of the entropy compression argument.

Thank you!