Face-magic calendula graphs

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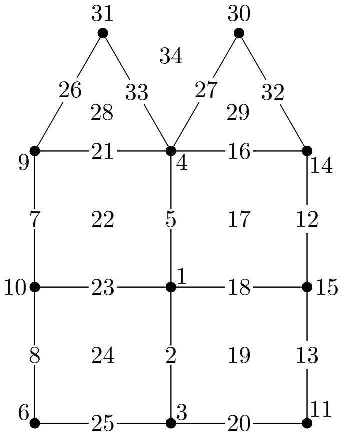
Type (a, b, c) Labeling

Definition (Lih, 1983)

For a planar graph $G = (V, E, F)$ and $a, b, c \in \{0, 1\}$, a **labeling of type (a, b, c)** is an assignment of the integers $\{1, 2, \ldots, a|V| + b|E| + c|F|\}$ to the elements of $V \cup E \cup F$ so that each vertex, edge, and face, receives exactly a, b, and c labels, respectively.



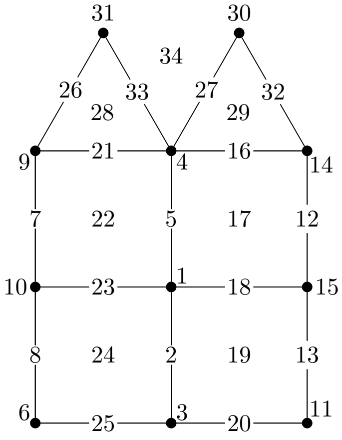
Type (a, b, c) Face-magic Labeling

Definition (Lih, 1983)

The **weight** of a face $f \in F$ is the sum of the label of f (if present) and the labels of the vertices and edges surrounding f.

Definition (Lih, 1983)

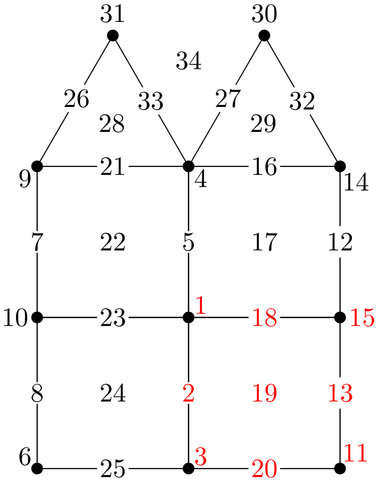
If the weights of all n-sided faces are equal to a fixed constant $\mu(n)$ for every possible n, then the labeling is **face-magic**.



Type $(1, 1, 1)$

Weights

$1+2+3+18+19+20+11+13+15 = 102$



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$1+2+3+18+19+20+11+13+15 = 102$
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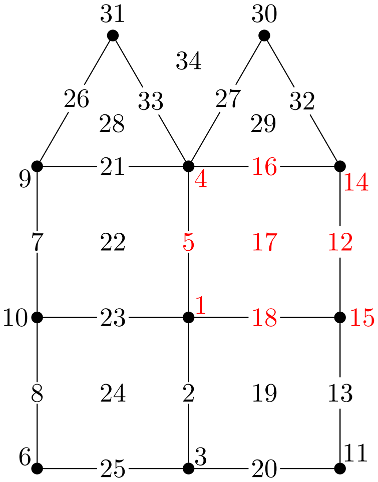
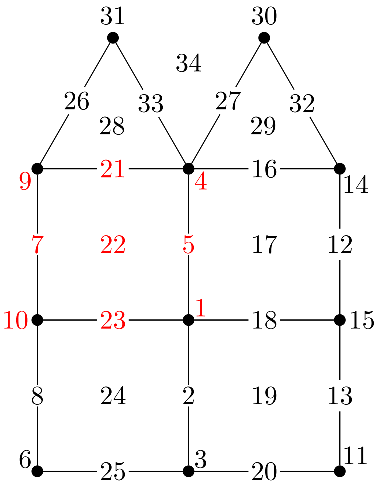


Weights

$1+2+3+18+19+20+11+13+15 = 102$
$1+4+5+18+17+16+12+14+15 = 102$
$9+21+4+7+22+5+10+23+1 = 102$



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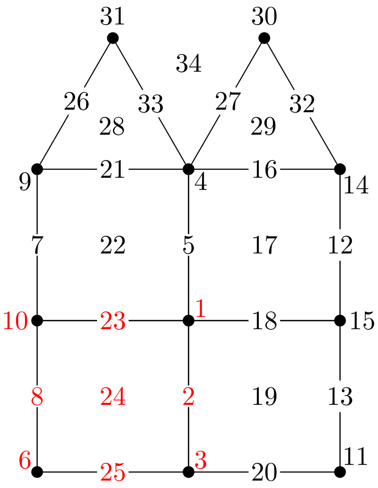

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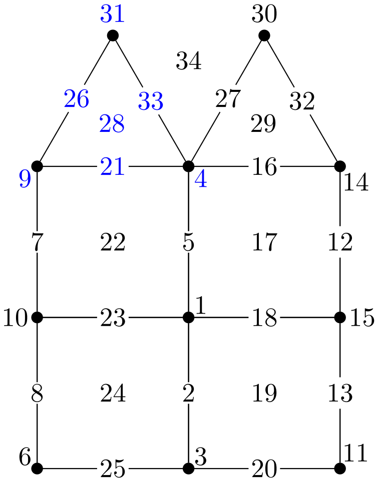
$9 + 21 + 4 + 26 + 28 + 33 + 31 = 152$



It's Magic!

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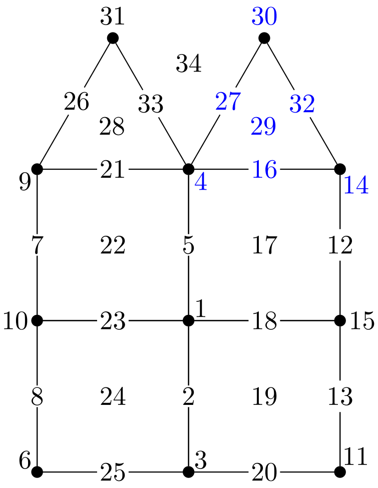
$9 + 21 + 4 + 26 + 28 + 33 + 31 = 152$
$4 + 16 + 14 + 27 + 29 + 32 + 30 = 152$



Known Results in Graph Labeling

See Gallian, J. *A dynamic survey of graph labeling*.

- 24^{th} edition (December 2021)
- $3,000+$ references!
- 25 is coming soon

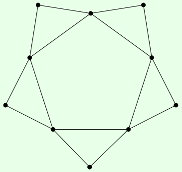
Calendula Graphs

Definition

Let $m, n \geq 3$. The calendula graph $Cal(m, n)$ consists of a central m-cycle and m peripheral n-cycles such that each peripheral cycle shares a unique edge with the central cycle.

Example

$Cal(5, 3)$



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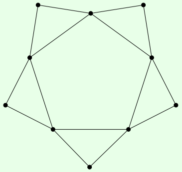


$$
\begin{aligned}
|V| &= m(n-1) \\
|E| &= mn \\
|F| &= m+2
\end{aligned}
$$

Known Results

Theorem (Pradipta & Salman, 2018)

The calendula graph $Cal(m, n)$ *admits a type* $(1, 1, 0)$ *face-magic labeling for* $m, n \geq 3$.

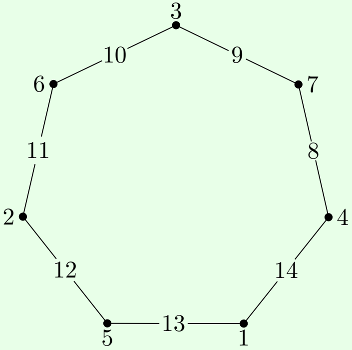
Case 1 ($m \neq n$). A Nice Tool.

Definition (Kotzig & Rosa, 1970)

Let $f : V \cup E \to \{1, 2, \dots, |V| + |E|\}$ be a bijection. If there exists an integer k such that $f(u) + f(uv) + f(v) = k$ for every edge $uv \in E$, then f is an **edge-magic total labeling** (EMT) of G.

Example

EMT labeling of C_7, $k = 19$



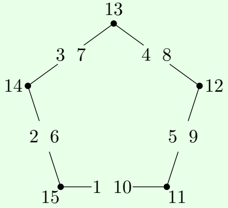
Case 1 ($m \neq n$). A Nice Tool.

Definition (F, 2021)

Let $\alpha, \beta \geq 0$ and $f : V \cup E \rightarrow \{1, 2, \ldots, \alpha|V| + \beta|E|\}$ be an assignment such that every vertex, edge receives exactly α, β labels, respectively and no label is repeated. If there exists an integer k such that $f(u) + f(uv) + f(v) = k$ for every edge $uv \in E$, then f is an **edge-magic total labeling of type** (α, β) ($EMT(\alpha, \beta)$) of G.

Example

$EMT(1, 2)$ labeling of C_5, $k = 37$



Known Results

Theorem (F, 2021)

If $n \geq 3$ then C_n admits an edge-magic total labeling of type $(0, \beta)$ if and only if β is even or $\beta \geq 3$ and n are both odd.

Theorem (F, 2021)

If $n \geq 3$ and $\alpha, \beta \geq 1$, then C_n admits an edge-magic total labeling of type (α, β).

Case 1. $m \neq n$

Notice the weight of the inner face is irrelevant in this case.

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Theorem (F, Jensen, Scott, 2022+)

If $m, n \geq 3$, $m \neq n$, then $Cal(m, n)$ admits a face-magic labeling of type (a, b, c) except for the following cases.

- $(a, b, c) = (0, 0, 1)$
- $(a, b, c) = (0, 1, 0)$, *m is even, and n is odd*

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Notice the weight of the inner face is irrelevant in this case.

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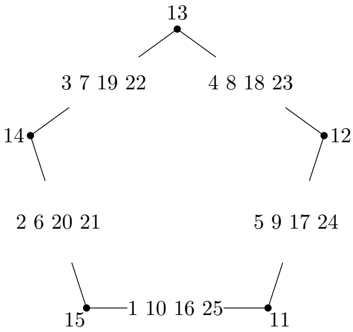
- $(a, b, c) = (0, 1, 0)$, *m is even, and n is odd*

Sketch of proof.

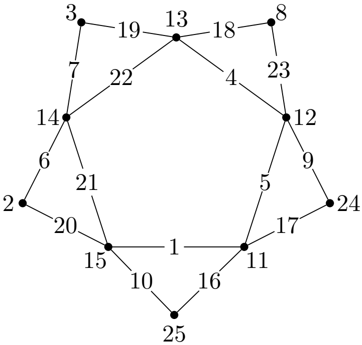
- $a = 1$ by an $EMT(a, a(n-2) + b(n-1) + c)$ labeling of C_m.
- $(0, 1, 0)$ iff there exists a constant sum partition of $\{1, 2, \ldots, mn\}$ into m subsets of size n.
- $(0, 1, 1)$ iff there exists a constant sum partition of $\{1, 2, \ldots, mn + m + 2\} \setminus \{\lambda_1, \lambda_2\}$ into m subsets of size $n + 1$ for some $\lambda_1, \lambda_2 \in \{1, 2, \ldots, mn + m + 2\}$.

□

Case 1. $m \neq n$. $EMT(\alpha, \beta) \to$ Face-magic.



$EMT(1,4)$ labeling of C_5



Type $(1,1,0)$ face-magic labeling of $Cal(5,3)$

Case 2.1. $m = n$ and $(a, b, c) = (1, 0, 0)$

$Cal(m, m)$ has $m^2 - m$ vertices.

Lemma

If $Cal(m, m)$ admits a face-magic labeling of type $(1, 0, 0)$, then m is even.

Proof.

Add the weights of all the faces in two different ways:

$$\begin{aligned} \mu(m+1) &= \textstyle\sum_{v \in V} f(v) + 2\mu \\ \implies \mu(m-1) &= \tfrac{m(m-1)(m^2-m+1)}{2} \\ \implies \mu &= \tfrac{m(m^2-m+1)}{2} \\ \implies m &\text{ is even.} \end{aligned}$$

□

Case 2.1. $m = n$ and $(a, b, c) = (1, 0, 0)$

$Cal(m, m)$ has $m^2 - m$ vertices.

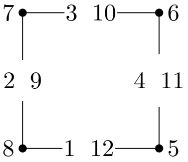
Theorem (F, Jensen, Scott, 2022+)

If $m \geq 4$ is even, then $Cal(m, m)$ admits a face-magic labeling of type $(1, 0, 0)$.

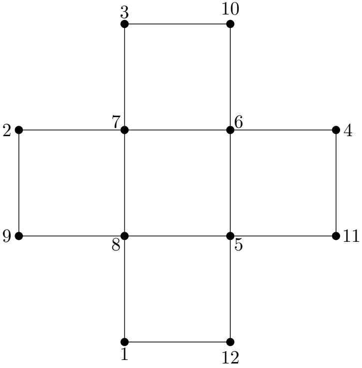
Proof.

There exists a type $(1, m - 2)$ EMT labeling of C_m with magic constant k with the property that the sum of the m vertex labels is also k. □

Case 2.1. $m = n$ and $(a, b, c) = (1, 0, 0)$



Type $(1, 2)$ EMT labeling of C_4



Type $(1, 0, 0)$ face-magic labeling of $Cal(4, 4)$

Case 2.2. $m = n$ and $(a, b, c) = (0, 1, 0)$. Magic squares!

$Cal(m, m)$ has m^2 edges.

Theorem (F, Jensen, Scott, 2022+)

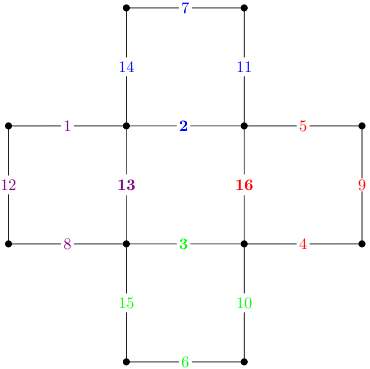
If $m \geq 3$, then $Cal(m, m)$ admits a face-magic labeling of type $(0, 1, 0)$.

Proof.

There exists a magic square of order m. □

Magic Squares $\Longrightarrow$ Face-magics

Magic square of order 4



Type $(0, 1, 0)$ face-magic labeling of $Cal(4, 4)$

Case 2.3. $m = n$ and $(a, b, c) = (0, 1, 1)$. Quasi-magic rectangles!

Definition (Froncek, Paramasivam, and Prajeesh, 2021)

An $a \times b$ quasi-magic rectangle, $QMR(a, b; \varsigma)$ is an a by b array with elements from the set of integers $\{1, 2, \ldots, ab + 1\} \setminus \{\varsigma\}$ such that no integer is repeated, the sum of integers in each row is equal to some fixed σ, and the sum of integers in each column is equal to some fixed σ.

Example

$QMR(2, 3; 4)$

1	6	5	12
7	2	3	12
8	8	8	

Case 2.3. $m = n$ and $(a, b, c) = (0, 1, 1)$. Quasi-magic rectangles!

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Example

$QMR(2, 3; 4)$

1	6	5	12
7	2	3	12
8	8	8	

Theorem (Froncek, et al., 2021)

A quasi-magic rectangle $QMR(a, 2t; at + 1)$ exists for all odd $a \geq 1$ and $t \geq 1$ except when $t = 1$ and $a \equiv 1 \pmod 4$.

Case 2.3. $m = n$ and $(a, b, c) = (0, 1, 1)$.

$Cal(m, m)$ has m^2 edges and $m + 2$ faces.

Theorem (F, Jensen, Scott, 2022+)

If $m \geq 3$, then $Cal(m, m)$ admits a face-magic labeling of type $(0, 1, 1)$.

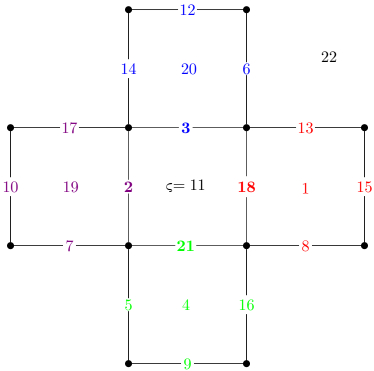
Proof.

There exists an $m \times m + 1$ quasi-magic rectangle with row sum ρ, column sum σ, and left out integer ς such that $\rho = \sigma + \varsigma$. □

Case 2.3. $m = n$ and $(a, b, c) = (0, 1, 1)$

3	**18**	**21**	**2**	44
14	8	5	17	44
20	1	4	19	44
6	15	16	7	44
12	13	9	10	44
55	55	55	55	

QMR(5,4;11)



Type $(0, 1, 1)$ face-magic labeling of $Cal(4, 4)$

Case 2.4. $m = n$ and $(a, b, c) = (1, 0, 1)$.

$Cal(m, m)$ has $m^2 - m$ vertices and $m + 2$ faces.

Theorem (F, Jensen, Scott, 2022+)

If $m \geq 3$, then $Cal(m, m)$ admits a face-magic labeling of type $(1, 0, 1)$.

Sketch of proof.

- If m is odd, we use an $EMT(1, 2)$ labeling on the inner cycle, giving the edge labels to the peripheral cycles.
- If m is even, we use an EMT labeling of the inner cycle, applying the edge labels to the outer faces.
- In both cases, we lift these labels and carefully pair up the remaining labels into constant sum pairs.

□

Case 2 Solved!

The remaining cases are:

- type $(1, 1, 0)$ (Pradipta & Salman)
- type $(1, 1, 1)$ (Case 2.2, Case 2.4)

Case 2 Solved!

The remaining cases are:

- type $(1, 1, 0)$ (Pradipta & Salman)
- type $(1, 1, 1)$ (Case 2.2, Case 2.4)

Theorem (F, Jensen, Scott, 2022+)

If $m \geq 3$ then $Cal(m, m)$ admits a face-magic labeling of type (a, b, c) unless

- $(a, b, c) = (0, 0, 1)$ *or*
- $(a, b, c) = (1, 0, 0)$ *and m is odd.*

Face-magic Classification of Calendulas

Let $m, n \geq 3$. Then $Cal(m, n)$ admits a face-magic labeling of type

- $(0, 0, 0)$ for all m, n (trivially).
- $(0, 0, 1)$ never.
- $(0, 1, 0)$ unless m is even and n is odd.
- $(0, 1, 1)$ for all m, n.
- $(1, 0, 0)$ unless $m = n$ is odd.
- $(1, 0, 1)$ for all m, n.
- $(1, 1, 0)$ for all m, n.
- $(1, 1, 1)$ for all m, n.

Gratitude

Thank you for supporting me and my students!

- Organizers of this conference
- University of Minnesota Office of Undergraduate Research

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