An extension of the Lindström–Gessel–Viennot Theorem

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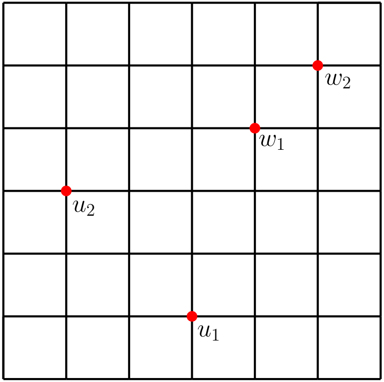
Outline

1. Motivating Example
2. The Lindström–Gessel–Viennot Theorem
3. New Results
4. Application: Tiling Problems

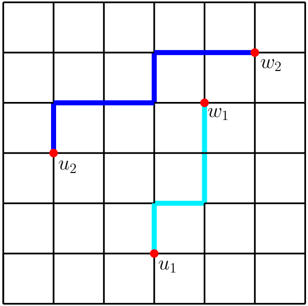
Motivating Example

Consider a square lattice with the horizontal edges oriented east, vertical edges oriented north. Let $U = \{u_1, u_2\}$ be the set of starting points and $W = \{w_1, w_2\}$ be the set of ending points. How many families of non-intersecting lattice paths from U to W do we have?

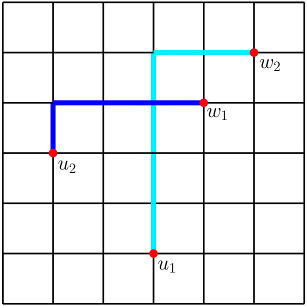


Motivating Example

- Two paths are non-intersecting if they do not pass through the same vertex.
- A family of paths is non-intersecting if any two of the paths are non-intersecting.



(a) Non-intersecting paths



(b) Intersecting paths

The LGV Theorem

Let G be a finite, connected, **directed acyclic graph** with a weight function $\mathrm{wt} : E(G) \mapsto R$, where R is a commutative ring.

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- $\mathscr{P}^{\pi}(U, V)$ is a set of n-tuples of paths $(p_1, \ldots, p_n)$, where $p_i \in \mathscr{P}(u_i, v_{\pi(i)})$. The permutation π is called the connection type.

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- $\mathscr{P}_0^{\pi}(U, V)$ is a set of n-tuples of non-intersecting paths of the connection type π.

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Theorem [Lindström '73], [Gessel–Viennot '85], [Stembridge '90]

Let G be a directed acyclic graph. Suppose that $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$ are two distinct sets of vertices of G. Then

$$\sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}(\pi) GF(\mathscr{P}_0^{\pi}(U, V)) = \det(M),$$

where the (i, j)-entry of the matrix M is given by $GF(\mathscr{P}(u_i, v_j))$.

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Two sets of the vertices $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$ are compatible if n-tuples of non-intersecting paths only consist of paths connecting u_i to v_i for $i = 1, \ldots, n$. That is, the connection type $\pi = \mathrm{id}$.

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Corollary

If $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$ are compatible, then we have

$$GF(\mathscr{P}_0^{\mathrm{id}}(U, V)) = \det(M),$$

where the (i, j)-entry of the matrix M is given by $GF(\mathscr{P}(u_i, v_j))$.

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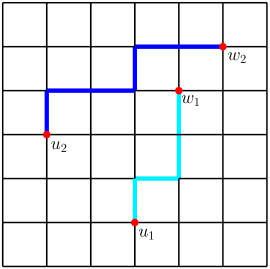
- enumeration of semi-standard Young tableaux.
- enumeration of various types of plane partitions.
- enumeration of lozenge and domino tilings.
- evaluation of the Hankel determinants.
- combinatorial proof of the determinant formulas.
  (e.g. the Cauchy–Binet formula)
- combinatorial proof of the Jacobi–Trudi type identities for Schur functions.
- ...

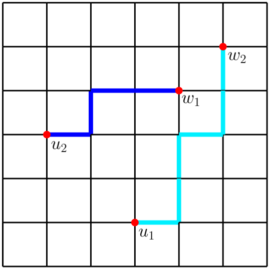
Survey paper: Christian Krattenthaler, Lattice path enumeration, 2017.
https://arxiv.org/abs/1503.05930

The LGV Theorem—Motivating Example

Go back to our motivating example:

- Both connection types are possible, the sets $U = \{u_1, u_2\}$ and $W = \{w_1, w_2\}$ are NOT compatible.





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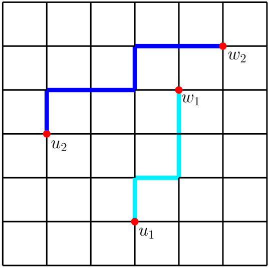
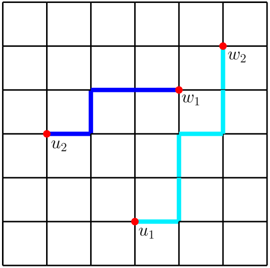
- We can still apply the LGV theorem and obtain the "signed" enumeration:
  $\mathrm{sgn}(\mathrm{id}) GF(\mathscr{P}_0^{\mathrm{id}}(U, W)) + \mathrm{sgn}((12)) GF(\mathscr{P}_0^{(12)}(U, W)) = \det M.$





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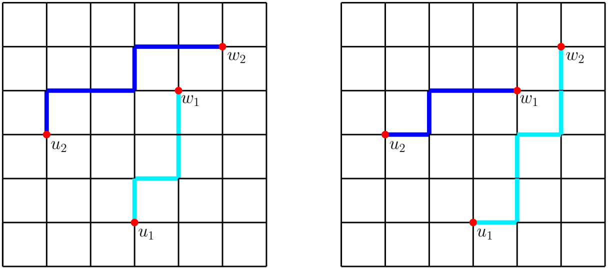
- How to find the total number of families of non-intersecting paths?



New Results—An Overview

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Setting:

- Let G be a directed acyclic graph with the special property*.
- Let $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$ be two sets of distinct vertices of G, not necessarily compatible.
- The new result is the "straight" enumeration

$$\sum_{\pi \in \mathfrak{S}_n} GF(\mathscr{P}_0^{\pi}(U, V)) = |\det M^*|,$$

where the (i, j)-entry of M^* depends only on the paths from u_i to v_j.

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Key ideas:

- We associate a sign (called the path sign) to each path from u_i to v_j.
- We form a new matrix M^*.
- These path signs in the det M^* cancel out the effect of the permutation signs on the LHS of the LGV theorem.

New Results—Special Property of G

Definition

An upward planar drawing of G is a drawing of G on the Euclidean plane such that

- each edge is drawn as a line segment that is either horizontal or up-pointing, and
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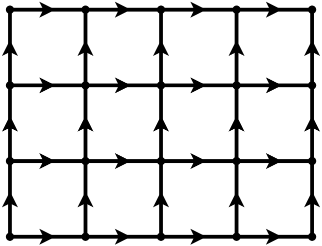
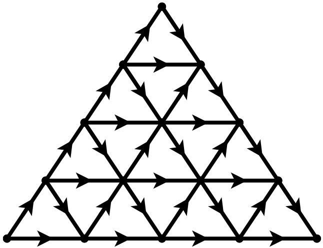
Examples:





New Results—Special Property of G

Definition

An *st-planar graph* is a planar, acyclic digraph with one source (a vertex with no incoming edges) and one sink (a vertex with no outgoing edges), so that these two special vertices lie on the outer face of the graph.

Theorem [Di Battista et al. '98]

A graph G has an upward planar drawing if and only if G is a subgraph of an *st*-planar graph $\widetilde{G}$ on the same vertex set.

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Now, we consider a directed acyclic graph G having an **upward planar drawing**, which is a subgraph of an *st*-planar graph $\widetilde{G}$.

New Results—Path Sign

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Definition

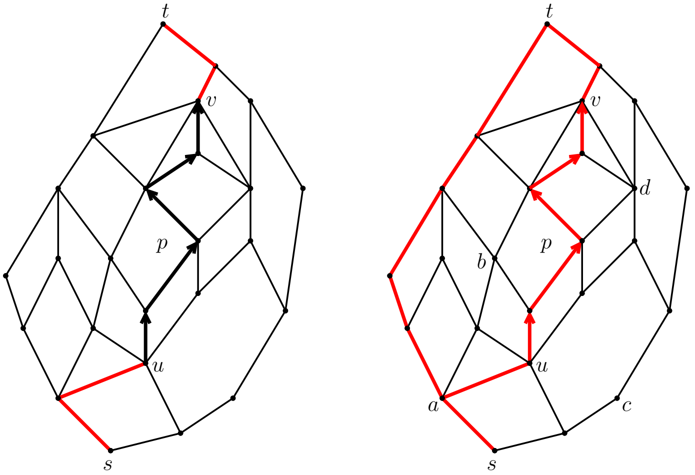
The left side of the path $p \in \mathscr{P}(u, v)$ is the closed region of the plane bounded by the following paths in $\widetilde{G}$:

- the leftmost path from s to u,
- the path p itself,
- the leftmost path from v to t, and
- the left boundary of $\widetilde{G}$ going from s to t.

Let $L(p)$ be the collection of the starting and ending points of $U \cup V$ which are on the left side of the path p (including u and v).

New Results—Path Sign

The left side of the path p is the region bounded by red edges.



New Results—Path Sign

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The path sign of a path $p \in \mathscr{P}(u, v)$ is defined to be

$$\operatorname{sgn}(p) = (-1)^{|L(p)|}.$$

In the previous graph, suppose $U \cup V = \{a, b, c, d, u, v\}$, then $L(p) = \{a, b, u, v\}$ and hence $\operatorname{sgn}(p) = (-1)^4 = 1$.

Main Theorem

Given an *st*-planar graph $\widetilde{G}$ and a subgraph G with the same vertex set. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be two sets of the distinct vertices of G.

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Then the total weight of families of non-intersecting paths connecting U to V is given by

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Our goal is to find the matrix $M^* = (m_{ij})$, where $1 \leq i, j \leq 2$.

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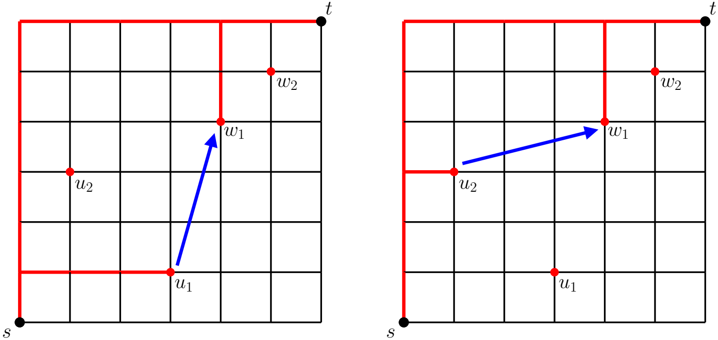
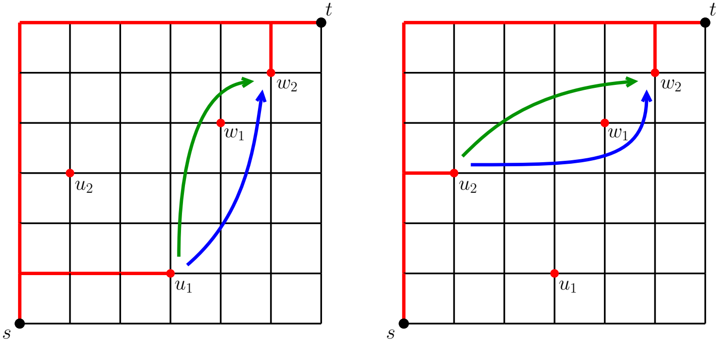
	$L(p)$	$\mathrm{sgn}(p)$	size	matrix
$p \in \mathscr{P}(u_1, w_1)$	$\{u_1, w_1, u_2\}$	$(-1)^3$	$\binom{4}{1}$	$m_{11} = -4$
$p \in \mathscr{P}(u_2, w_1)$	$\{u_2, w_1\}$	$(-1)^2$	$\binom{4}{3}$	$m_{21} = 4$



New Results—Motivating Example

	$L(p)$	$\mathrm{sgn}(p)$	size	matrix
$\mathbf{p} \in \mathscr{P}(u_1, w_2)$ (green)	$\{u_1, w_2, u_2\}$	$(-1)^3$	1	$m_{12} = -1 + 14$
$\mathbf{p} \in \mathscr{P}(u_1, w_2)$ (blue)	$\{u_1, u_2, w_1, w_2\}$	$(-1)^4$	$\binom{6}{2} - 1$	
$\mathbf{p} \in \mathscr{P}(u_2, w_2)$ (green)	$\{u_2, w_2\}$	$(-1)^2$	$\binom{4}{2}$	$m_{22} = 6 - 9$
$\mathbf{p} \in \mathscr{P}(u_2, w_2)$ (blue)	$\{u_2, w_1, w_2\}$	$(-1)^3$	$\binom{6}{2} - \binom{4}{2}$	



New Results—Motivating Example

We have $m_{11} = -4$, $m_{12} = 13$, $m_{21} = 4$, $m_{22} = -3$.
By the main theorem, the number of families of non-intersecting paths in the motivating example is given by

$$|\det M^*| = \left|\det \begin{pmatrix} -4 & 13 \\ 4 & -3 \end{pmatrix}\right| = 40.$$

Application—Tiling Problems

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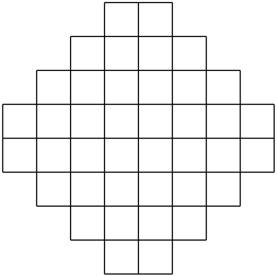
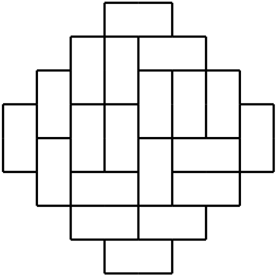
An example of a domino tiling (covered by 1×2 and 2×1 rectangles) of the Aztec diamond of order 4.

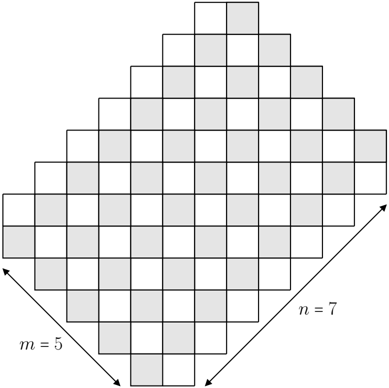


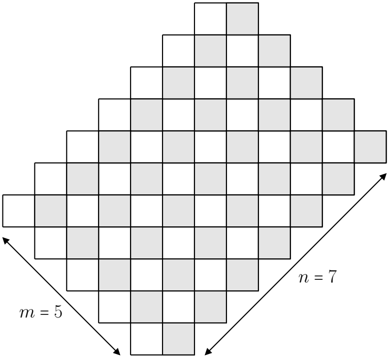


How many tilings do we have? Is there a "nice" formula?

Application—Mixed Aztec Rectangle

The Aztec rectangle (left) and the mixed Aztec rectangle (right) with the checkerboard coloring.

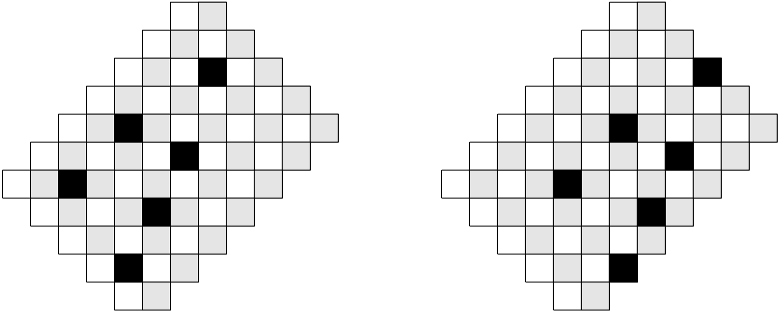




Application—Translation Invariant

Theorem [L. '22]

The number of domino tilings of the mixed Aztec rectangle with **arbitrary unit holes** is invariant under color-preserving translations of the set of holes, provided all the unit holes are still contained in the region.



Thank You