On the joins of regular graphs and applications to nonlinear dynamics

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Motivation

- Let G be a graph/network with adjacency matrix $A = (a_{ij})$. The Kuramoto model on G is described by the following differential equations

$$\dot{\theta}_i = \epsilon \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i)\,, \tag{0.1}$$

  where $\theta_i(t) \in [-\pi, \pi]$ is the state of oscillator $i \in [1, N]$ at time t and ϵ is the coupling strength.
- It is known that structure of G strongly influences the dynamics governed by the Kuramoto model.

A multilayer network

- Many real-world networks are multilayers; namely they are composed of several smaller "communities" joined together.

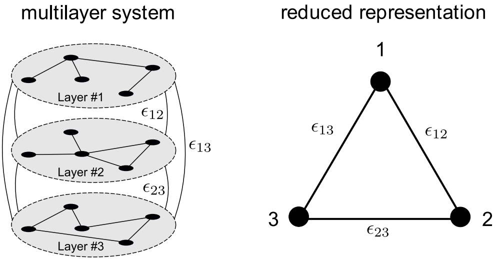

Figure 1: A multilayer network with three layers

- From both a theoretical and an applied perspective, it is interesting and important to study the spectra of these multilayer networks.

The joined union of graphs as a model for multilayer networks

Let G_1, G_2 be two graphs. The join of G_1 and G_2 is defined pictorially as follow.

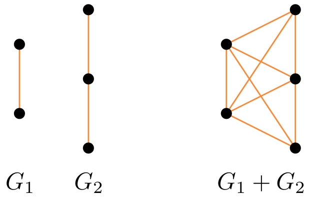

Figure 2: The join of two graphs

- More generally, suppose G is a (weighted) graph with d vertices $\{v_1, v_2, \ldots, v_d\}$. Let $G_1, G_2, \ldots, G_d$ be (weighted) graphs on $k_1, k_2, \ldots, k_d$ vertices. The **joined union** $G[G_1, G_2, \ldots, G_d]$ is obtained from the union of $G_1, \ldots, G_d$ by joining with an edge each pair of a vertex from G_i and a vertex from G_j whenever v_i and v_j are adjacent in G.
- The adjacency matrix of $G[G_1, G_2, \ldots, G_d]$ has the following form

$$A = \left[\begin{array}{c|c|c|c} A_{G_1} & a_{12}J_{k_1,k_2} & \cdots & a_{1d}J_{k_1,k_d} \\ \hline a_{21}J_{k_2,k_1} & A_{G_2} & \cdots & a_{2d}J_{k_2,k_d} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d1}J_{k_d,k_1} & a_{d2}J_{k_d,k_2} & \cdots & A_{G_d} \end{array}\right].$$

Here $J_{m,n}$ is the matrix of size $m \times n$ with all entries equal to 1, A_{G_i} is the adjacency matrix of G_i, and $A = (a_{ij})$ is the adjacency matrix of G.

Definition

- Let $A = (a_{ij})$ be an $n \times n$ matrix. We say that A is r_A-row regular if the sum of all entries in each row of A is equal to r_A, namely
$$\forall 1 \leq i \leq n, \sum_{j=1}^{n} a_{ij} = r_A.$$
Similarly, we say that A is c_A-column regular if the sum of all entries in each column of A is equal to c_A.
- We say that A is a **semimagic square** if it is both r_A-regular and c_A-regular and $r_A = c_A$.
- We say that A is normal if $AA^* = A^*A$. Here A^* is the conjugate transpose of A.

Let A be a normal semimagic square. Because A is a semimagic square, $v_1^A = \frac{1}{\sqrt{n}}\mathbb{1}_n = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^t \in \mathbb{C}^n$ is an eigenvector of A associated with the eigenvalue r_A. Furthermore, because A is normal, we have.

Proposition

Then there exists an orthonormal basis $\{v_1^A, v_2^A, \ldots, v_n^A\}$ of eigenvectors of A associated with the eigenvalues $\{\lambda_1^A, \lambda_2^A, \ldots, \lambda_n^A\}$ such that $v_1^A = \frac{1}{\sqrt{n}}\mathbb{1}_n = \frac{1}{\sqrt{n}}(1, \ldots, 1)^t \in \mathbb{C}^n$. In particular, $r_A = \lambda_1^A$ and, for $2 \leq k \leq n$, the standard inner product $\langle v_1^A, v_k^A \rangle = 0$.

Spectra of multilayer networks

- We will assume that $A_i = A_{G_i}$ is normal semimagic square of size $k_i \times k_i$ for $1 \leq i \leq d$.
- Let $\{v_1^{A_i}, v_2^{A_i}, \ldots, v_{k_i}^{A_i}\}$ and $\{\lambda_1^{A_i}, \lambda_2^{A_i}, \ldots, \lambda_{k_i}^{A_i}\}$ be the set of eigenvectors and eigenvalues of A_i as described previously.
- For two vectors $(x_1, \ldots, x_m)^T$ and $(y_1, \ldots, y_n)^T$ their concatenation is defined as

$$(x_1, \ldots, x_m)^T * (y_1, \ldots, y_n)^T = (x_1, \ldots, x_m, y_1, \ldots, y_n)^T.$$

Proposition

For each $1 \leq i \leq d$ and $2 \leq j \leq k_i$ let

$$w_{i,j} = \vec{0}_{k_1} * \ldots * \vec{0}_{k_{i-1}} * v_j^{A_i} * \vec{0}_{k_{i+1}} * \ldots * \vec{0}_{k_d}$$

Then $w_{i,j}$ is an eigenvector of A associated with the eigenvalue $\lambda_j^{A_i}$.

The reduced characteristic polynomial of A is defined as

$$\overline{p}_A(t) = \frac{p_A(t)}{\prod_{\substack{1\leq i\leq d,\\ 2\leq j\leq k_i}} (t-\lambda_j^{A_i})} = \frac{p_A(t)}{\prod_{i=1}^d \frac{p_{A_i}(t)}{t-r_{A_i}}}.$$

Theorem

The reduced characteristic polynomial of A coincides with the characteristic polynomial of $\overline{A}$: $\overline{p}_A(t) = p_{\overline{A}}(t)$. Here $\overline{A}$ the following matrix

$$\overline{A} = \begin{pmatrix} r_{A_1} & a_{12}k_2 & \cdots & a_{1n}k_d \\ a_{21}k_1 & r_{A_2} & \cdots & a_{2n}k_d \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1}k_1 & a_{d2}k_2 & \cdots & r_{A_d} \end{pmatrix}.$$

A corollary

Corollary

If G_1 is r_1-regular with k_1 vertices and G_2 is r_2-regular with k_2 vertices then the characteristic polynomial of the join $G_1 + G_2$ is given by

$$p_{G_1+G_2}(t) = \frac{p_{G_1}(t)p_{G_2}(t)}{(t-r_1)(t-r_2)}\left((t-r_1)(t-r_2)-k_1k_2\right).$$

Proof.

Let A_1, A_2 be the adjacency matrix of G_1, G_2 respectively. We have

$$\overline{A} = \begin{pmatrix} r_1 & k_2 \\ k_1 & r_2 \end{pmatrix}.$$

Hence

$$p_{\overline{A}}(t) = (t-r_1)(t-r_2) - k_1k_2.$$

$\square$

Let us consider the Kuramoto model on $\overline{A}$

$$\dot{\theta}_i = \epsilon \sum_{j=1}^{N} \overline{A}_{ij} \sin(\theta_j - \theta_i)\,, \tag{0.2}$$

where

$$\overline{A} = \begin{pmatrix} r_{A_1} & a_{12}k_2 & \cdots & a_{1n}k_d \\ a_{21}k_1 & r_{A_2} & \cdots & a_{2n}k_d \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1}k_1 & a_{d2}k_2 & \cdots & r_{A_d} \end{pmatrix} \simeq \begin{pmatrix} 0 & a_{12}k_2 & \cdots & a_{1n}k_d \\ a_{21}k_1 & 0 & \cdots & a_{2n}k_d \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1}k_1 & a_{d2}k_2 & \cdots & 0 \end{pmatrix}$$

Given a solution of the reduced Kuramoto model:

$$\bar{\boldsymbol{\theta}}^* = (\bar{\theta}_1^*, \bar{\theta}_2^*, \cdots, \bar{\theta}_M^*), \tag{0.3}$$

we define

$$\boldsymbol{\theta}^* = (\underbrace{\bar{\theta}_1^*, \bar{\theta}_1^*, \cdots, \bar{\theta}_1^*}_{1^{\mathrm{st}}\mathrm{layer}}, \underbrace{\bar{\theta}_2^*, \bar{\theta}_2^*, \cdots, \bar{\theta}_2^*}_{2^{\mathrm{nd}}\mathrm{layer}}, \cdots, \underbrace{\bar{\theta}_M^*, \bar{\theta}_M^*, \cdots, \bar{\theta}_M^*}_{M^{\mathrm{th}}\mathrm{layer}}), \tag{0.4}$$

Theorem

$\boldsymbol{\theta}^*$ is a solution of the Kuramoto model on A. It is called the broadcasted solution from $\overline{A}$.

Theorem

The fixed point $\bar{\boldsymbol{\theta}}^*$ for the reduced system is linearly stable if and only if the corresponding broadcasted fixed point for the multilayer system is linearly stable.

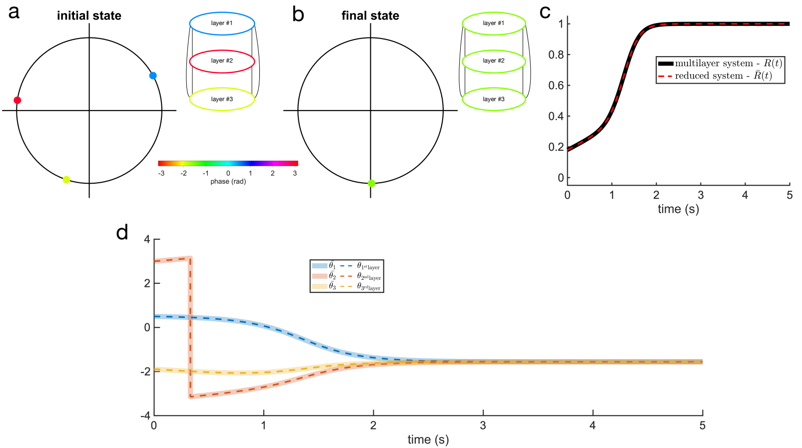

Figure 3: Dynamics on a 3 layers system and its reduced system with random initial conditions.

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