On Sets of Cardinality 2 of Nondecreasing Diameter

Adam O’Neal and Michael Schroeder

October 21st, 2022

Fundamental Definitions

Integer Coloring

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Example

Let's define $\Delta : [15] \to \{a, b\}$ as given below:

$$\Delta: \quad b \quad b \quad a \quad b \quad a \quad a \quad b \quad a \quad b \quad b \quad a \quad a \quad a \quad b \quad a$$

For example, $\Delta(1) = b$ and $\Delta(5) = a$.

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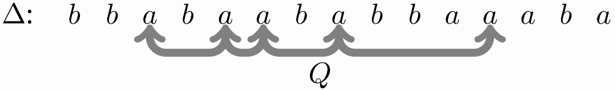

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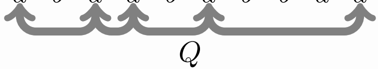

Notice that each element of Q are all colored with a, so the set Q is monochromatic.

Set Precedence

Definition

Let B and B' be subsets of $[n]$.

Then B **precedes** another set B' (denoted $B <_p B'$) if the biggest element in the first set ($\max(B)$) is smaller than the smallest element in the second set ($\min(B')$). That is $\max(B) < \min(B')$.

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Since $4 < 6$, B precedes B', or $B <_p B'$.

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So the diameter of C is $\mathrm{dm}(C) = 15 - 9 = 6$.

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- each set is monochromatic,
- all the sets have the same size (cardinality), m,
- the sets can be ordered by precedence, and
- in that order, the diameters of the sets are nondecreasing.

Example of (m, r, t)-perimissibity

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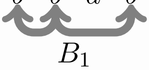

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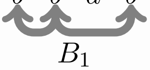

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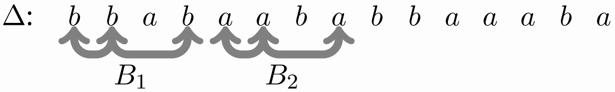

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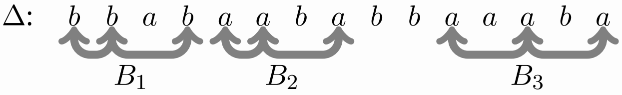

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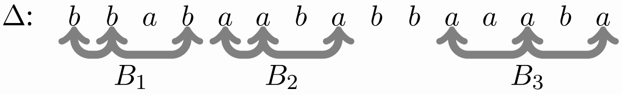

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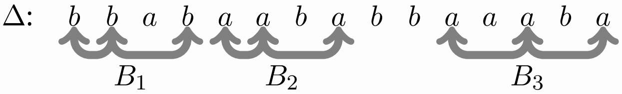

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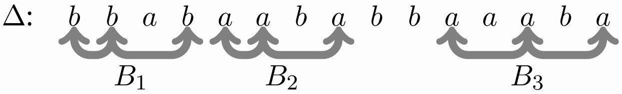

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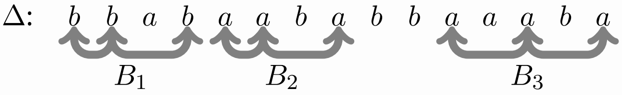

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So Δ is $(3, 2, 3)$-permissible.

The Function: $f(m, r, t)$

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- $m, r,$ and t are all positive integers.
- describes the least positive integer n such that every r-coloring of $[n]$ is (m, r, t)-permissible.
- $f(m, r, t)$ is well-defined. It follows as a consequence of van Der Waerden's Theorem (1927).

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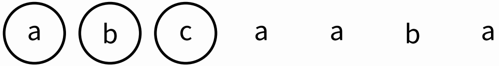

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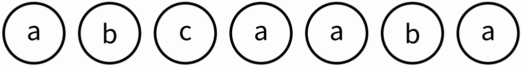

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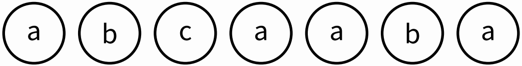

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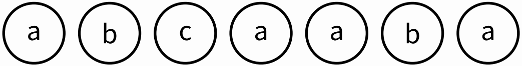

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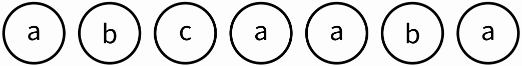

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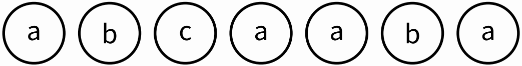

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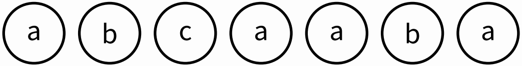

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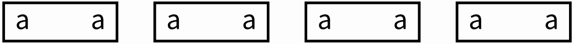

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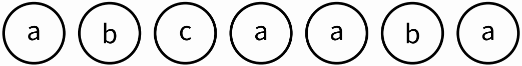

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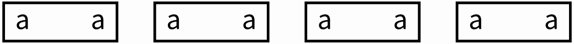

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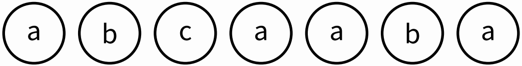

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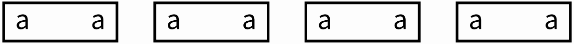

- $f(m,r,1) = (m-1)r + 1$
  = The length guaranteed to have m identical symbols.

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Theorem (Bialostocki, Erdös, and Lefmann, 1995)

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Theorem (Bernstein, Grynkiewicz, and Yerger, 2015)

If $m \geq 2$, $f(m, 2, 3)$ is known.

Our Work

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The first step towards our goal: establishing a lower bound!

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For $t = 2$... try it! $\Delta : a\ b\ a\ b\ a$

Lower Bound

Lemma (O., Schroeder)

If $t \geq 1$, then $f(2,2,t) > 5t - 5$.

Sketch of Proof

We did this through induction and contradiction using the coloring $\Delta = (ababa)^{t-1}$ as a counterexample.

For the base case, $t = 1$, this is trivial.

For $t = 2$... try it! $\Delta : a\ b\ a\ b\ a$

When $t \geq 3$, it can be shown that $\mathrm{dm}(B_2) \geq 3$ and that $\mathrm{dm}(B_t) = 2$.

Upper Bound

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We created a computer program to find the following values for $f(2,2,t)$:

t	1	2	3	4	5	6	7
$f(2,2,t)$	3	7	12	16	21	26	31

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From this data, it appeared that when $t \geq 4$, $f(2, 2, t) = 5t - 4$.

Theorem!

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Lemma (O., Schroeder)

If $t \geq 1$, then $f(2, 2, t) \leq 5t - 2$.

In fact, any 2-colored string of length $5t - 2$ has t permissible pairs with maximum diameter 2.

Sketch of Upper Bound Proof

Proof by Example

If $t = 4$, then a coloring $\Delta : [18] \to \{a, b\}$ is one of two types:

Δ: $b \quad b \quad a \quad b \quad b \quad a \quad b \quad a \quad b \quad a \quad a \quad a \quad b \quad a \quad b \quad b \quad a \quad b$

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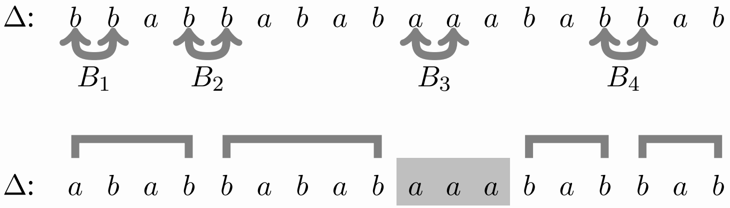

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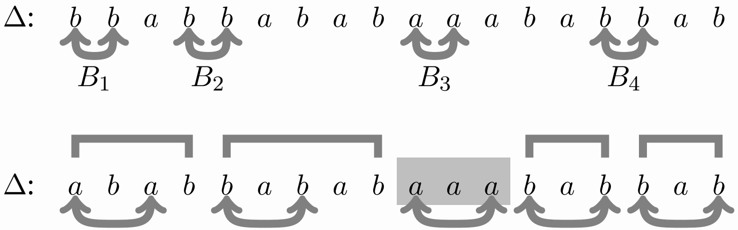

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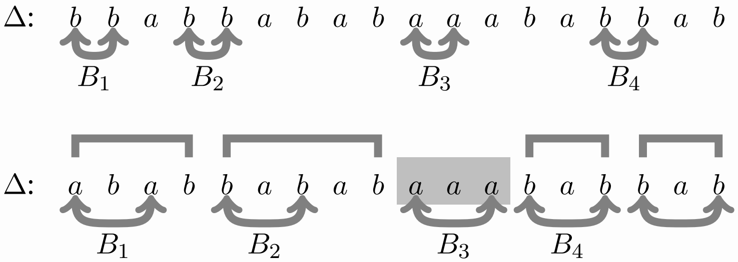

The Math

$$
\begin{aligned}
w + \sum_{i=1}^{v+w+1} \left\lfloor \frac{k_i}{3} \right\rfloor &= w + \sum_{i=1}^{v+w+1} \left\lceil \frac{k_i - 2}{3} \right\rceil \\
&\geq w + \left\lceil \frac{\sum_{i=1}^{v+w+1} (k_i - 2)}{3} \right\rceil \\
&= w + \left\lceil \frac{\sum_{i=1}^{v+w+1} k_i - 2(v+w+1)}{3} \right\rceil \\
&= w + \left\lceil \frac{|S'| - 2(v+w+1)}{3} \right\rceil \\
&= w + \left\lceil \frac{|S| - 3w - 2(v+w+1)}{3} \right\rceil \\
&= \left\lceil \frac{|S| - 2(v+w+1)}{3} \right\rceil \\
&= \left\lceil \frac{5t - 2 - 2(v+w+1)}{3} \right\rceil \\
&= t + \left\lceil \frac{2t - 2 - 2(v+w+1)}{3} \right\rceil \\
&\geq t + \left\lceil \frac{2t - 2 - 2(t)}{3} \right\rceil \\
&= t + \left\lceil \frac{-2}{3} \right\rceil \\
&= t.
\end{aligned}
$$

Proving the Theorem

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Suppose Δ is a 2-coloring of $[5t-4]$ and is NOT t-permissible.

We establish some properties of Δ:

- Δ contains t alternating substrings
- restrictions on the lengths of substrings
- cannot start or end with a triple
- at most, there is 1 triple
- conditions on the substrings around the triple
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Establish that the END of Δ falls into one of 12 cases:

$$\begin{array}{cccccc} (\tau,2) & (\tau,1,2) & (\overline{2},4,2) & (\overline{8},1) & (\overline{2},2) & (\overline{2},1,2) \\ (2,1) & (2,\tau,1) & (\overline{7},2) & (\tau,4,2) & (5,1) & (\overline{5},\tau,1) \end{array}$$

For example, ending with $(\overline{5},\tau,1)$: $\Delta = \cdots babab\, bbb\, b$.

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Show that in each case, Δ is actually t-permissible.

Short Sketch of Proof

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Use contradiction.

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Future Work

Incrementing m and r

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  Proof. Using the coloring $(ababababa)^{t-1}$.

There is a combined result for the two, but the proof has been elusive.

Thank You!