Finiteness theorem of commutative rings with finite hyper-graph and finite commutative rings with $HG(R) = 2$

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Introduction

Consider $\mathbb{Z}_{12}$.
Elements $2, 3, 10$ have the following properties:

$$2 \cdot 3 \cdot 10 = 0$$

$$2 \cdot 3 \neq 0 \qquad 2 \cdot 10 \neq 0 \qquad 3 \cdot 10 \neq 0$$

Elements 2, 3 and 10 are examples of **3-zero divisors** and the set $\{2,\ 3,\ 10\}$ is an example of a **3-edge**.
There are four 3-zero divisors $\{2,\ 3,\ 9,\ 10\}$ and two 3-sdges, $\{2,\ 3,\ 10\}$and $\{2,\ 9,\ 10\}$.
Together, they form a **3 uniform hyper-graph** denoted by $H_3(\mathbb{Z}_{12})$.

3-Hyper-graph

The vertex set of $H_3(\mathbb{Z}_{12}$ is $Z(\mathbb{Z}_{12}, 3) = \{2, 3, 9, 10\}$. There are two 3-edges, $\{2, 3, 10\}$ and $\{2, 9, 10\}$ which are enclosed by an ellipse in the following hyper-graph:

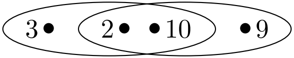

Figure: Hyper-graph $H_3(\mathbb{Z}_{12})$.

Definition

Let R be a commutative ring with identity. An element $a_1 \in R$ is called a k **zero divisor** if there exist $a_2, a_3, \ldots, a_k$ in R satisfying following three conditions:

1. All a_i are pairwise distinct.
2. The product $\prod_1^k a_i = 0$.
3. For any j, $\prod_{i \neq j} a_i \neq 0$.

Let $Z(R, k)$ denote the set of all k-zero divisors of R and $H_k(R)$ denote the k uniform hyper-graph on R. Ring with $Z(R, k) = \emptyset$ is called a k **domain**.

Hyper-graphic constant $HG(R)$

It is known that $Z(R,k)=\emptyset$ implies $Z(R,l)=\emptyset$ for all $l \geq k$.
Hence it is natural to find the largest integer g for which $Z(R,\, k) \neq \emptyset$.
We define the **hyper-graphic constant** $HG(R)$ to be the largest integer (if one exists) g such that $Z(R,\, g)$ is non-empty, but $Z(R,\, k)$ is empty for any $k > g$.
If R is an integral domain, we say that $HG(R)=0$. If R is not an integral domain, but $Z(R\,2)=\emptyset$, then we say that $HG(R)=1$.
If $HG(R)<\infty$, then $H_g(R)$ is called the **maximal hyoer-graph**.

Constant against degree

If $R_i = \mathbb{Z}_{p_i^{\alpha}{}_i}$ where p_i are odd primes, then $HG(R) = \sum_i \alpha_i$. But in general, HG is not additive. For example, $HG(\mathbb{Z}_8) = 2$, but $HG(\mathbb{Z}_8 \times \mathbb{Z}_8) = 6$.

History

- 1965-Ganesan: If R is not an integral domain, then R has finitely many zero-divisors if and only if R is finite.
- 1988 -Beck defined Zero divisor graphs and introduced coloring of commutative rings.
- 1990-2000 Anderson, Livingston and many more advanced this theory.
- 2007 - Ch. Eslahchi and A. M. Rahimi defined k uniform hyper-graphs as a generalization of zero divisor graphs.

Known results

1. If R is not a k integral domain and has two prime ideals P_1 and P_2 with $P_1 \cap P_2 = \{0\}$, then $H_k(R)$ is a bipartite hyper-graph.
2. If R is a product of n integral domains, then the chromatic number
   $\chi(H_n(R)) = 2$ and $\chi(H_{n+t}(R)) \leq 2 + t$ for any $t \geq 0$.
3. If $x^2 \neq 0$ for all $x \in Z(R, 3)$, then $H_3(R)$ is connected with diameter at most 4.
4. For any finite non-local ring R, the hyper-graph $H_3(R)$ is complete if and only if R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Recent development

1. Suppose $HG(\mathbb{Z}_n) = g$ then the maximal hyper-graph $H_g(\mathbb{Z}_n)$ and the group of automorphisms $Aut(H_g(\mathbb{Z}_n))$ can be completely described.
2. If $Z(R, k)$ is empty, then so is $Z(R, l)$ for any $l \geq k$. This result justifies the definition of the hyper-graphic constant $HG(R)$.
3. If $HG(R) = 1$, then R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/ < x^2 >$. .

Finiteness theorem

Im this paper, we claim the following results:
Suppose $k \geq 2$ and R is a ring with nonempty $Z(R, k)$. Then the following statements are equivalent;

1. The hyper-graph $H_k(R)$ contains a finite degree edge.
2. R is finite.
3. $Z(R, k)$ is finite.
4. All the vertices of $H_k(R)$ have finite degree.

Finite local rings

Suppose $(R, \mathfrak{m})$ is a finite local ring with $HG(R) = n$ for some $n \geq 3$. Then $\mathfrak{m}^n = 0$.

The above result has two corollaries:

- Suppose $R = \prod_1^n R_i$ is a finite ring where R_i is a local ring for each i such that $HG(R_i) \geq 4$. Then $HG(R) = \sum_1^n HG(R_i)$.
- Suppose R_i and R_2 are two finite rings with a property that the hyper-graphic constant of the localization $(R_i)_p$ is at least 4 for all prime ideals p. Then $HG(R_1 \times R_2) = HG(R_1) + HG(R_2)$.

Structure of finite rings

Structure theorem of abelian groups classifies finite abelian groups. But rings have two structures! Addition and multiplication (no guarentee for reciprocals).
The following rings are non-isomorphic!

$\mathbb{Z}_4$ $\qquad$ $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\qquad$ $\mathbb{Z}[x]/ < x^2 >$

Note that the following hold:

- Cardinality is 4.
- Hyper-graphic constant is 2.
- Additive structure of the first and the last rings are same.

Classification theorem

Suppose R is a finite ring with $HG(R) = 2$. Then R is isomorphic to one of the following rings:

$\mathbb{Z}_8$, $\mathbb{Z}_4[x]/ < x^2 + 2, 2x >$, $\mathbb{Z}_2[x]/ < x^3 >$.

$GF(p^\alpha, r)[x_1, x_2, \ldots x_n]/ < x_i \cdot x_j >$ where, $1 \leq i, j \leq n$, p is a prime, $r \geq 1$ and $GF(p, r)$ is a finite field other then $\mathbb{Z}_2$.

$\mathbb{Z}_2[x_1, x_2, \ldots x_n]/ < x_i \cdot x_j >$ where, $1 \leq i, j \leq n$ and $n > 1$.

$GR(p^2, r)[x_1, x_2, \ldots x_n]/ < x_i \cdot x_j, p \cdot x_j >$ where, $n \geq 0$, $1 \leq i, j \leq n$ and $GR(p^2, r)$ is a coefficient ring other than $\mathbb{Z}_4$.

$\mathbb{Z}_4[x_1, x_2, \ldots x_n]/ < x_i \cdot x_j, 2 \cdot x_i >$ where, $n \geq 1$, $1 \leq i, j \leq n$.

$GF(p^\alpha, r) \times GF(q^\beta, s)$, $GF(p^\alpha, r) \times \mathbb{Z}_4$ or $GF(p^\alpha, r) \times \mathbb{Z}_2[x]/ < x^2 >$ where p, q are primes.

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