The Tamari Block Lattice

An Order on Saturated Chains in the Tamari Lattice

Luke Nelson Kevin Treat*

Air Force Research Laboratory
United States Air Force Academy

October 22th, 2022

Outline

Scope Sequences

The set of scope sequences of length n, denoted $\mathcal{SS}_n$, is the set of all n-tuples, $(s_1, \ldots, s_n)$, such that for each $i \in [n]$,

1. $1 \leq s_i \leq n - i + 1$, and
2. $s_{i+r} \leq s_i - r$, for $0 < r < s_i$.

Let $s = (s_1, \ldots, s_n)$ be a scope sequence.

- The chamber of s with lead index $i \in [n]$ is the sequence $(s_i, s_{i+1}, \ldots, s_{i+s_i-1})$.
- A major chamber of a scope sequence is a chamber that is not properly contained by any other chamber.

Example

Consider $(2, 1, 4, 1, 2, 1, 3, 1, 1) \in \mathcal{SS}_9$

Pallo, Knuth, Björner & Wachs have used similar objects.

Alternative Definition of Scope Sequences

Let $\overline{\mathcal{SS}}_i$ be the multiset $\mathcal{SS}_{i-1}$ with i prepended to each element.

Definition

$\mathcal{SS}_n$ may be defined recursively as

$$\mathcal{SS}_0 = \{()\}, \qquad \mathcal{SS}_n = \sum_{i=1}^{n} \mathcal{SS}_{n-i}\overline{\mathcal{SS}}_i, \quad n \geq 1.$$

Example

$$\begin{aligned}\mathcal{SS}_3 &= \mathcal{SS}_2\overline{\mathcal{SS}}_1 + \mathcal{SS}_1\overline{\mathcal{SS}}_2 + \mathcal{SS}_0\overline{\mathcal{SS}}_3 \\ &= \{11, 21\}\{1\} + \{1\}\{21\} + \{()\}\{311, 321\} \\ &= \{111, 211, 121, 311, 321\}.\end{aligned}$$

The Tamari Lattice

The Tamari lattice, $\mathcal{T}_n$, may be represented on the set $\mathcal{SS}_n$, denoted $\mathcal{T}_{\mathcal{SS}_n}$, where $(s_1, \ldots, s_n) \leq (s'_1, \ldots, s'_n)$ if and only if $s_i \leq s'_i$, for each $i \in [n]$.

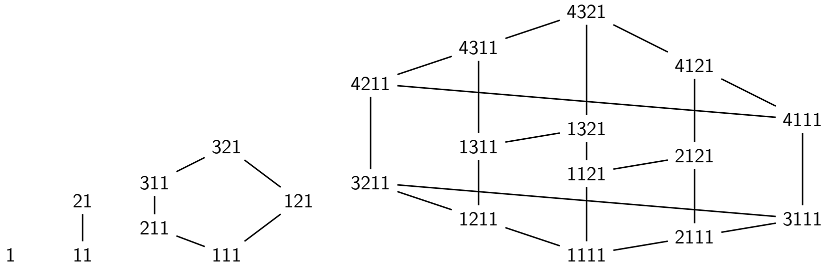

Figure: $\mathcal{T}_{\mathcal{SS}_n}$, for $n = 1, 2, 3, 4$.

Edges & Edge Sequences

For the cover relation $s \lessdot_{\mathcal{T}_{\mathcal{SS}_n}} s'$, identify its edge in the Hasse diagram with the ordered pair, (i, j), $i < j$, such that i and j are the lead indexes of the two chambers merged in s to obtain s'. An edge sequence in $\mathcal{T}_{\mathcal{SS}_n}$ is a sequence of edges obtained by traversing a saturated chain.

Example

The saturated chain in $\mathcal{T}_{\mathcal{SS}_6}$,

$$(4,1,1,1,2,1) \lessdot (4,2,1,1,2,1) \lessdot (6,2,1,1,2,1) \lessdot (6,2,1,3,2,1),$$

has the edge sequence, $((2,3),(1,5),(4,5))$.

A similar edge labeling is defined by Bjorner and Wachs.

Theorem

This is an EL-labeling on $\mathcal{T}_{\mathcal{SS}_n}$.

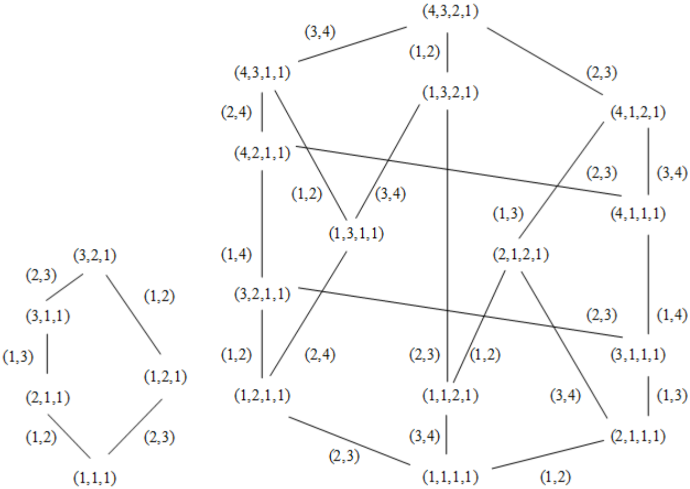

Figure: Hasse diagrams of $\mathcal{T}_{\mathcal{SS}_3}$ and $\mathcal{T}_{\mathcal{SS}_4}$ with the edge label encoding.

Preliminaries

(Devadoss & Read) The 2D faces of the associahedron (the 1-skeleton of $\mathcal{T}_{\mathcal{SS}_n}$) are either squares or pentagons.

Proposition

If $s \lessdot t$ and $s \lessdot u$, for distinct $s, t, u \in \mathcal{T}_{\mathcal{SS}_n}$, then the edges associated to these cover relations take one of the following forms:

1. *(i, j) and (k, l), i, j, k, l distinct, in which case $[s, t \vee u]$ is a square with sides,*
$$((i, j), (k, l)) \text{ and } ((k, l), (i, j)).$$
Moreover, assuming without loss of generality that $i < k$, then
$$i < k < l < j \quad \text{or} \quad i < j < k < l.$$
2. *(i, j) and (j, k), $i < j < k$, in which case $[s, t \vee u]$ is a pentagon with sides,*
$$((j, k), (i, j)) \text{ and } ((i, j), (i, k), (j, k)).$$

The Definition

Let I be an interval in the Tamari lattice.

- Define the set $\mathcal{M}_I = \{C \mid C \text{ a maximal chain of } I\}$.
- Suppose S (respectively, P) is a square (respectively, pentagon) in I, and suppose $C_1, C_2 \in \mathcal{M}_I$ agree except on S (respectively, P). A square move (respectively, pentagon move) on C_1 is the map that sends C_1 to C_2.
- There is an equivalence relation on $\mathcal{M}_I$ by $C_1 \sim C_2$ if and only if C_1 can be obtained from C_2 by making a set of square moves. We call the equivalence classes square blocks. Any two chains in the same square block are square equivalent.
- The square blocks of $\mathcal{M}_I$ form a poset which we call the Tamari Block Poset for I, $\mathcal{TB}_I$, with $\overline{S_1} \lessdot \overline{S_2}$ if and only if there is a $C_1 \in \overline{S_1}$ and a $C_2 \in \overline{S_2}$, such that C_2 can be obtained from C_1 by making an increasing pentagon move.

Cool Facts about $\mathcal{TB}_I$

Every chain is square equivalent to a chain that makes major chamber moves first.

Example

$$((3,1,1)(2,1)(4,1,1,1)(1)) \lessdot^{(7,8)} ((3,1,1)(2,1)(4,2,1,1)(1))$$
$$\lessdot^{(2,3)}((3,2,1)(2,1)(4,2,1,1)(1)) \lessdot^{(6,10)} ((3,2,1)(2,1)(5,2,1,1,1))$$
$$\lessdot^{(7,9)}((3,2,1)(2,1)(5,3,1,1,1)) \lessdot^{(1,4)} ((5,2,1,2,1)(5,3,1,1,1))$$
$$\lessdot^{(1,6)}((10,2,1,2,1,5,3,1,1,1))$$

Edge Sequence: $((7,8),(2,3),(6,10),(7,9),(1,4),(1,6))$

$$\equiv (((6,10),(1,4),(1,6),(7,8),(2,3),(7,9)))$$

Cool Facts about $\mathcal{TB}_I$

Proposition

For any interval I in $\mathcal{T}_{\mathcal{SS}_n}$, two maximal chains of I are in the same square block of $\mathcal{TB}_I$ if and only if they have the same edge set.

Proposition

Let $C, D \in \mathcal{TB}_I$. Then $C \leq D$ if and only if $\mathcal{ES}(C) \subseteq \mathcal{ES}(D)$.

Elements of $\mathcal{TB}_I$ can be characterized by:

- Square equivalent saturated chains
- Edge sets of chains
- The Super Chain

An Example

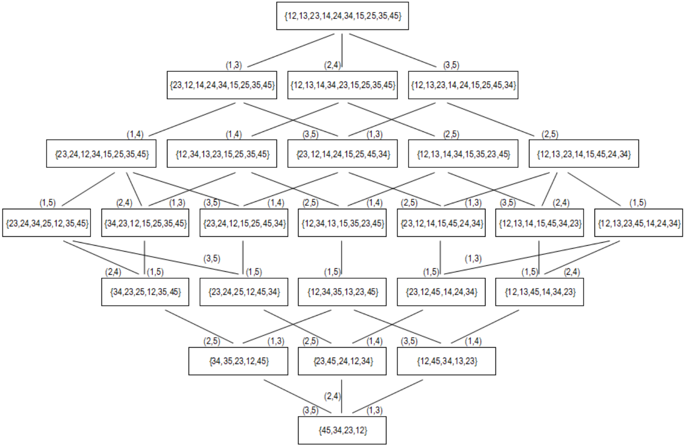

Figure: Hasse diagram of $\mathcal{TB}_5$ with each block identified by its edge set. Each edge of the diagram is labeled with the edge that augments the lesser block's edge set resulting in that of the greater block.

Important Properties of $\mathcal{TB}_I$

Theorem

Let I be an interval in the Tamari lattice, $\mathcal{T}_n$.

- *$\mathcal{TB}_I$ has a unique top and bottom element.*
- *$\mathcal{TB}_I$ is graded with rank function $\rho(\overline{S}) = \ell(\overline{S}) - \ell(\hat{0})$ (where $\overline{S}$ is an equivalence class of maximal chains in I, $\ell(\overline{S})$ is the length of any chain in the class, and $\ell(\hat{0})$ is the length of any chain in the bottom class).*
- *$\mathcal{TB}_I$ is a lattice.*

Theorem

$\mathcal{TB}_n$ is anti-isomorphic to $HST(n+2,3)$.

Higher Stasheff-Tamari Order

Kapranov & Voevodsky and later Edelman & Reiner introduced two orders on triangulations of the cyclic polytope, one defined locally, one globally.

Williams recently showed they are the same order.

Definition (Rambau & Reiner)

First define when a triangulation $\mathscr{T}'$ is obtained from a triangulation $\mathscr{T}$ by an upward flip: this means that there exists a $(d+2)$-subset of the vertices of $\mathbf{C}(n,d)$, whose convex hull gives a subpolytope $\mathbf{C}(d+2,d)$ of $\mathbf{C}(n,d)$ with the property that $\mathscr{T}$ and $\mathscr{T}'$ restrict to the lower and upper, respectively, triangulations of this $\mathbf{C}(d+2,d)$, and otherwise, $\mathscr{T}$ and $\mathscr{T}'$ agree on all of their other simplices not lying in this $\mathbf{C}(d+2,d)$. Then define $\mathrm{HST}(n,d)$ to be the transitive closure of the upward flip relation.

Case $d = 2$

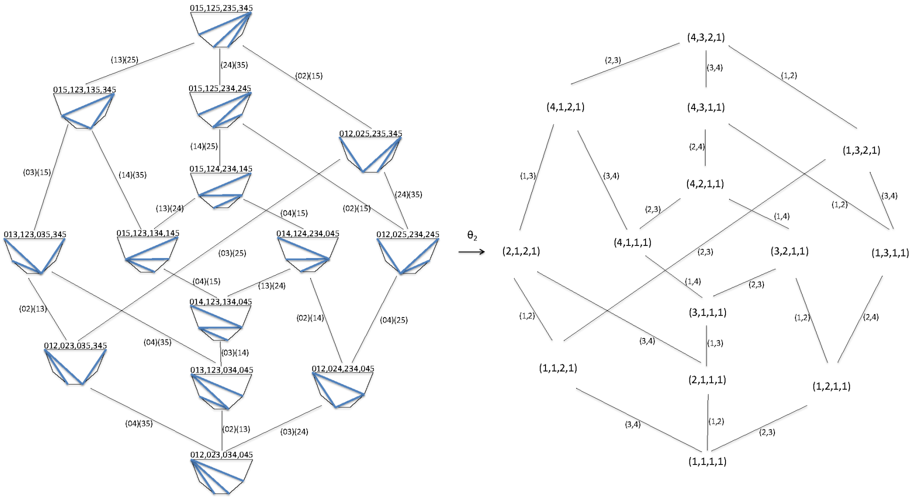

Figure: The maps θ_2 and θ_2^E take HST$(6,2)$ (left) to $\mathcal{T}_{\mathcal{S}S_4}$ (right).

The Edge Map

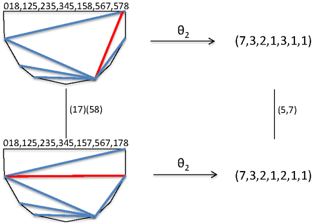

Figure: An example of θ_2 and θ_2^E for $n = 7$.

Case $d = 3$

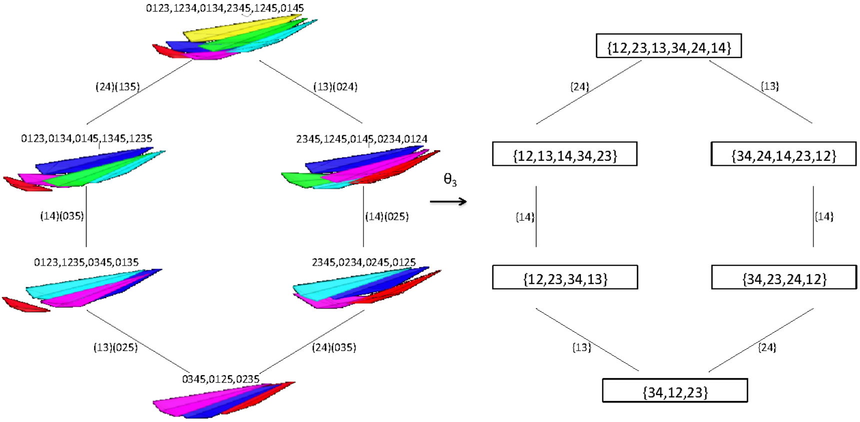

Figure: The poset on the left is the dual of HST$(6,3)$. θ_3 takes each element of this poset (a triangulation of $\mathbf{C}(6,3)$) to the respective edge set of $\mathcal{TB}_4$ on the right. Triangulation graphics are from Rambau & Reiner.

What's Next

Enumeration of $\mathcal{TB}_I$

- Working on some recursive formulas
- Trying to build a full characterization

QUESTIONS?

https://www.sciencedirect.com/science/article/pii/S0012365X22001571