Chip-Firing on Signed Graphs

A Combinatorial Game

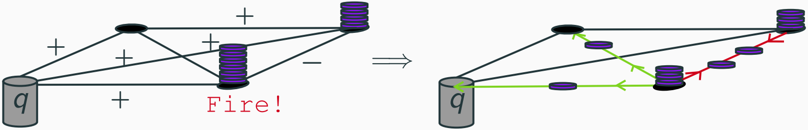
Matthew Cho, Ryota Inagaki, Dylan Snustad, and Bailee Zacovic
Drs. Anton Dochtermann and Suho Oh

22 October 2022

Texas State University NSF-REU 2022

Chip-Firing: A Disperson Process

- A discrete dynamical system examining the exchange of resources
- Combinatorial aspects dictated by various notion of *stability*
- The critical group of a graph is a subtle invariant, whose cardinality is given by the number of spanning trees
- We extend these notions in [GK16] to the setting of *signed* graphs, which distinguish neighbor relations

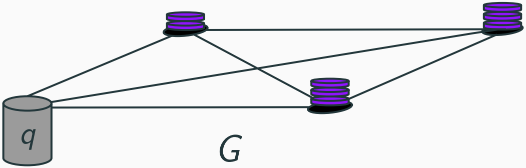


Basic Definitions & Properties

Graphs and Chip Configurations

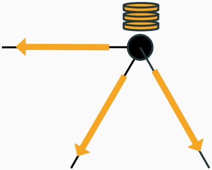
- Consider a **graph** $G = (V, E)$ with vertices $V = \{v_1, v_2, \ldots, v_n, q\}$, where q is the **sink.**
- Place c_i chips on vertex v_i, and encode this **chip configuration** in the column vector,

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$



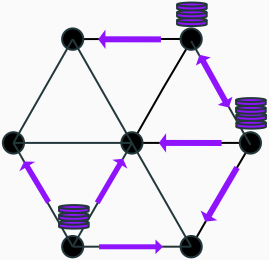
Vertex Firing

We **single fire** a vertex v_i if it has at least as many chips as its degree, $c_i \geq \deg(v_i)$, passing one chip to each neighbor.



- The resultant configuration is $c' = c - Le_i$.
- A configuration from which you cannot single fire is **stable**.

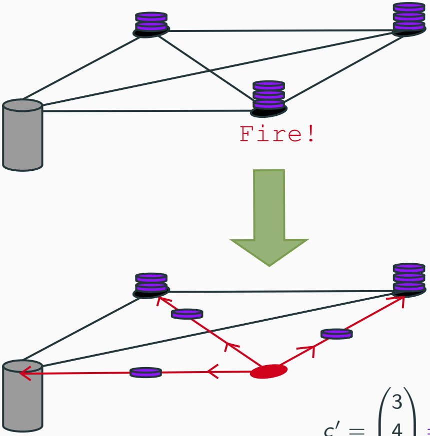
We **multi-fire** a collection of vertices $\{v_{i_1}, \ldots, v_{i_k}\}$.



- The resultant configuration is $c' = c - L\left(\sum_{j=1}^{k} z_i e_i\right)$.
- A configuration from which you cannot multi-fire is **superstable**

**Let $L = D - A$ denote the Laplacian Matrix*

Chip Firing Illustrated



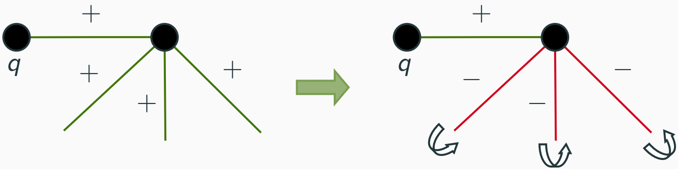
$$c = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

$$c' = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}}_{c} - \underbrace{\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}}_{\text{Laplacian Matrix}} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{Firing Script}}$$

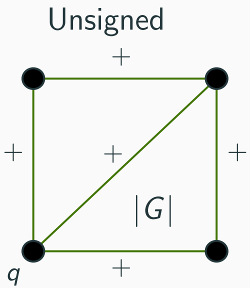
Signed Graphs

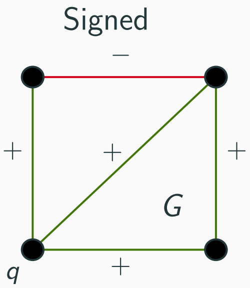
- A **signed graph** G consists of a graph $G = (V, E)$ equipped with a signature, $\sigma : E \to \{+, -\}$.
- An **unsigned graph** is a specific case of a signed graph: A graph $|G| = (V, E)$ with positive sign assigned to all edges.
- If a vertex fires to a neighbor via a negative edge, *both* vertices lose a chip.

- We can **switch** a vertex, flipping the sign of all its incident edges non-adjacent to q. We call two graphs **switching equivalent** if one can be obtained from the other by switching vertices.



Signed Graphs: Illustrated

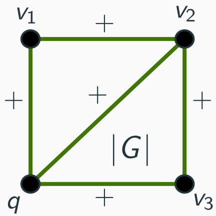


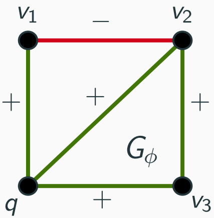


Signed Graphs: Laplacian and Allowable Configurations

We employ the Guzman-Klivans theory of ‘chip firing on invertible matrices’ in [GK16].

For signed graph G, we define its Laplacian matrix L_G to be the Laplacian of its underlying unsigned graph $L_{|G|}$ but with entries corresponding to negative edges to be 1 instead of -1.





$$L(|G|) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \qquad L(G_\phi) = \begin{pmatrix} 2 & +1 & 0 \\ +1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

In the signed case, the set of **'allowable'** configurations of chips are those in

$$S^+ := \{L_G L_{|G|}^{-1}\vec{v} : \vec{v} \in \mathbb{R}^n_{\geq 0} \text{ and } L_G L_{|G|}^{-1}\vec{v} \in \mathbb{Z}^n\}$$

[GK16; Moo21].

Given configuration $\vec{c} \in G$, a vertex $v_i \in V(G)$ can **fire** if

$$\vec{d} = \vec{c} - L_G e_i \in S^+.$$

A multiset of vertices $\{v_{i_1}, \ldots, v_{i_k}\}$ can **multi-fire** if

$$\vec{d} = \vec{d} - L_G \left(\sum_{j=1}^{k} e_{i_j} \right) \in S^+$$

is allowable.

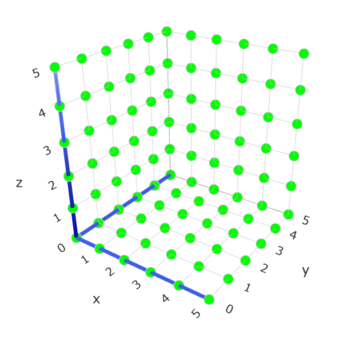


Figure 1: Allowable Configurations of $|G|$

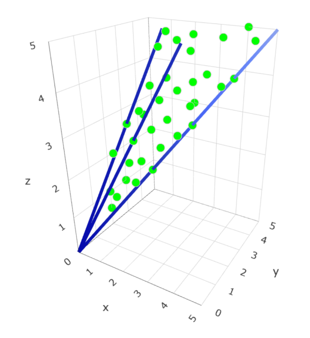


Figure 2: Allowable Configurations of G

Proposition

The allowable configurations for a signed graph sit in a nonnegative cone.

Signed Graphs: Special Configurations

- Chip-firing defines an equivalence relation on integer configurations in $\mathbb{Z}^n$, with $\det(L)$ equivalence classes.
- These classes have special representatives:
  - A configuration is z-**superstable** if you cannot multi-fire.
  - A configuration is **critical** if it's *stable* and *reachable*.
- The critical configurations form the **critical group** of G, denoted $\mathcal{K}(G)$. Note that $\mathcal{K}(G) \cong \mathbb{Z}^n/\text{im}L$.

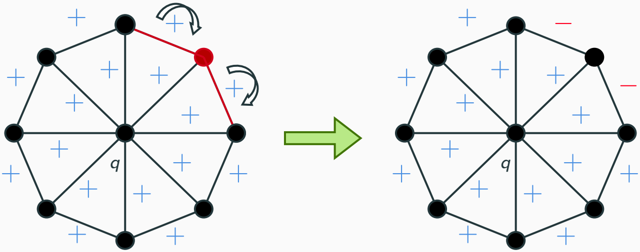
Main Results

Overview of Main Results

- Vertex switching preserves the critical group, $\mathcal{K}(G)$.
- Finding superstable configurations of signed graph G by using those of underlying unsigned graph.
- Calculation of the identity of $\mathcal{K}(G)$ in the signed case can be obtained from the identity element of $\mathcal{K}(|G|)$ in the unsigned case.
- Test for criticality of a configuration chips in signed graphs.

Vertex Switching and Critical Group

Lemma
The critical groups of two switching equivalent graphs are isomorphic.



Classifying Superstable Configurations

Theorem
Let G be a signed graph with sink q and n nonsink vertices, and denote $|G|$ as the underlying unsigned. Suppose $\vec{v}$ is a vector such that $\vec{v}$ is coordinate-wise less than some superstable configuration of $|G|$. Then we have that if $\vec{c} = L_G L_{|G|}^{-1} \vec{v}$ is allowable then $\vec{c}$ is z-superstable.

Upshot: We find an intuitive interplay/"semi-duality" between superstable configurations of signed graphs and underlying unsigned graphs!

The Identity Element

Theorem
Let G be a connected signed graph with sink. Given the configuration $I_{\mathcal{K}(|G|)}$ serving as the identity of the critical group of $|G|$,the identity of the critical group of G is $I_{\mathcal{K}(G)} = L_G L_{|G|}^{-1} I_{\mathcal{K}(|G|)}$

.

A Test For Criticality

Theorem
Consider a connected signed graph G with sink q, n nonsink vertices, and underlying unsigned graph $|G|$. Let $\vec{a}_q$ be the vector associated with firing the sink once, component-wise defined by

$$(\vec{a}_q)_i := \begin{cases} 1 & \text{, nonsink vertex } v_i \in G \text{ is adjacent to } q \\ 0 & \text{, otherwise} \end{cases}.$$

If $\vec{a}_q$ is an allowable configuration, then there exists positive integer N_0 such that allowable configuration $\vec{c}$ is critical if and only if $(\vec{c} + N_0(\vec{a}_q))$ stabilizes to $\vec{c}$.

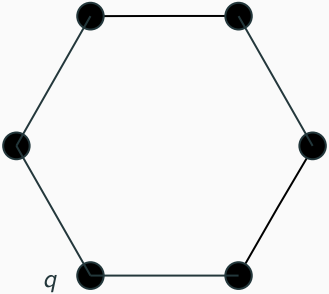
A Test For Criticality (ctd)

Corollary
Let G be a connected signed graph with sink q . Suppose $G - q$ is regular, q is adjacent to each nonsink vertex, and each nonsink vertex is incident with m_- negative edges not incident to q. Then allowable configuration $\vec{c}$ is critical if and only if $\vec{c}$ is the stabilization of $(\vec{c} + (1 + 2m_-)\vec{1})$.

Upshot: This extends the fact that for unsigned graph $|G|$ with sink q, configuration $\vec{c}$ is critical iff stabilizing the configuration after firing the sink once yields $\vec{c}$ (See Chapter 2.6 of *The Mathematics of Chip-Firing* [Kli19]).

A Few Case Studies

The Cycle Graph, C_n



Consider any signed cycle graph C_n of n vertices including the sink.
Then, $\mathcal{K}(C_n) \cong \mathbb{Z}/n\mathbb{Z}$ regardless of graph signature

Why? As in [Kli19], the critical group of the unsigned cycle graph on n-vertices with sink is $\mathbb{Z}/n\mathbb{Z}$. Use that result along with the fact that all signed cycle graphs of are switching equivalent to the unsigned cycle.

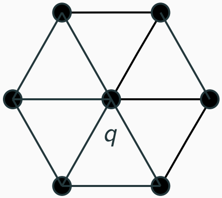
Interesting Duality between z-superstables and Criticals

Proposition
For fixed integer $m \geq 2$, such that $m \equiv 1, 3 \pmod 4$ the set map

$$f : \{z\text{-superstable configurations}\} \to \mathcal{K}(-C_{2n-1})$$

given by $w \mapsto I_{C_m^-} + w$ is a bijection.

The Wheel Graph, W_n



Consider signed wheel graph W_n, whose underlying graph is constructed by length n cycle and sink q adjacent to all other vertices.

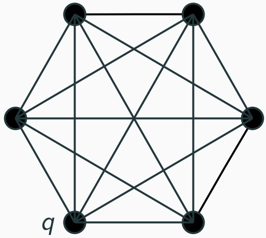
$$\mathcal{K}((W_n)_\sigma) \cong \begin{cases} \mathbb{Z}_{f_n} \oplus \mathbb{Z}_{5f_n} & n \text{ is even and } (W_n)_\sigma \text{ is equivalent to } |W_n| \\ \mathbb{Z}_{\ell_n} \oplus \mathbb{Z}_{\ell_n} & n \text{ is even and } (W_n)_\sigma \text{ is NOT equivalent to } |W_n| \\ \mathbb{Z}_{f_n} \oplus \mathbb{Z}_{5f_n} & n \text{ is odd and } (W_n)_\sigma \text{ is NOT equivalent to } |W_n| \\ \mathbb{Z}_{\ell_n} \oplus \mathbb{Z}_{l_n} & n \text{ is odd and } (W_n)_\sigma \text{ is equivalent to } |W_n| \end{cases}$$

where ℓ_n, f_n denotes the nth Lucas and Fibonacci numbers, respectively.

The Wheel Graph, W_n (ctd)

Why? The critical groups for $(W_n)_\sigma$ where are switching equivalent to $|W_n|$ can be found in [Big99]. For all other groups, we find the Smith Normal Form of the Laplacian of $(W_n)_\sigma$.

The Complete Graph, K_n



$$\mathcal{K}((K_n)_\sigma) \cong \begin{cases} \mathbb{Z}_n^{n-2} & \text{when } (K_n)_\sigma \text{ is equivalent to } |K_n| \\ \mathbb{Z}_{(n-2)(2n-3)} \oplus \mathbb{Z}_{n-2}^{n-3} & \text{when } (K_n)_\sigma \text{ is equivalent to } -K_n \end{cases}$$

Why? We utilize a technique from the proof of Corollary 7.6. of [RT14], in which it is used that $L_{(K_n)_\sigma}$ is a multiple of the identity matrix subtracted by the all ones matrix.

Thank You!

References

[Big99] N. L. Biggs. “Chip-firing and the critical group of a graph”. In: *J. Algebraic Combin.* 9.1 (1999), pp. 25–45.

[GK16] Johnny Guzmán and Caroline Klivans. “Chip firing on general invertible matrices”. In: *SIAM J. Discrete Math.* 30.2 (2016), pp. 1115–1127. ISSN: 0895-4801. DOI: 10.1137/15M1034404. URL: https://doi.org/10.1137/15M1034404.

[Kli19] Caroline J. Klivans. *The mathematics of chip-firing*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2019, pp. xii+295. ISBN: 978-1-138-63409-1.

[Moo21] Ram Melkote Moore. *Chip-Firing on Signed Graphs*. 2021.

[RT14] Victor Reiner and Dennis Tseng. "Critical groups of covering, voltage and signed graphs". In: *Discrete Math.* 318 (2014), pp. 10–40. ISSN: 0012-365X. DOI: 10.1016/j.disc.2013.11.008. URL: https://doi.org/10.1016/j.disc.2013.11.008.