

# Chip-Firing on Signed Graphs

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A *signed graph* consists of a graph  $G = (V, E)$  equipped with an edge signature  $\phi: E \rightarrow \{+, -\}$ . The combinatorial game of *chip firing* is a dynamical process by which a commodity, such as chips, are exchanged between vertices according to simple local rules governed by the signed Laplacian  $L_G$ . We employ the Guzmán-Klivans theory of chip-firing on general invertible matrices, where the allowable chip configurations are given by the lattice points of a certain cone determined by  $G$ . This theory gives rise to notions of *critical* as well as  *$z$ -superstable* chip configurations. We prove general results regarding the critical group and  $z$ -superstable configurations of the signed and unsigned graphs, including a duality. We then classify the critical groups of various classes of regular signed graphs.