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THE HAUSDORFF DIMENSION OF GRAPHS OF DENSITY CONTINUOUS FUNCTIONS II

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ABSTRACT. In this paper we complete the proof of the fact that the Hausdorff dimensions of graphs of density continuous functions vary continuously between one and two. This result was announced in our previous paper, but the proof there contained a gap and the construction given there should also be slightly modified. This correction is done in this paper.

INTRODUCTION

B. Kirchheim [K] observed and pointed out to the authors that in the proof of Theorem 2 in [BO] there is a gap. That proof shows only that functions f, defined in Theorem 2 of [BO], are measure preserving. This property is not sufficient for density continuity. In fact, in Theorem 1 of this paper we show that functions f in Theorem 2 of [BO] are not necessarily density continuous. This will also illustrate that there are measure preserving but not density continuous functions. On the other hand, by changing slightly the definition of f one can obtain density continuous functions. This implies that the Hausdorff dimensions of graphs of density continuous functions $f: [0, 1] \to \mathbf{R}$ vary continuously between one and two.

In this paper we shall use the notation of [BO]. Recall that a function $f: \mathbf{R} \to \mathbf{R}$ is *density continuous* if it is continuous with respect to the density topology on both the domain and the range. Our work [BO] contains a construction of functions done in a manner resembling the way the first coordinate of the Peano area-filling curve is obtained. Those functions are claimed to be density continuous.

Theorem 1. *The functions f constructed in Theorem 2 of* [BO] *are measure preserving but not necessarily density continuous.*

Proof. Assume that we use the construction of Theorem 2 of [BO, pp. 1040–1041], with $n = 3$, $m = 5$, and $l = 2$. Put $k_0 = 0$, and choose a sequence of

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integers $k_j > 2$ for $j = 1, 2, \ldots$ such that

(1) $$\left(\frac{4}{5}\right)^{k_0+\cdots+k_j} \cdot 5^{k_0+\cdots+k_{j-1}} < \frac{1}{j}$$

holds for all $j = 1, 2, \ldots$. Put $s_1 = t_1 = \frac{1}{3}$. If s_{j-1} and t_{j-1} are given, let

$$s_j = s_{j-1} + \frac{1}{3}\frac{1}{(3 \cdot 5)^{k_0+\cdots+k_{j-1}}}$$

and

$$t_j = t_{j-1} + \frac{1}{3}\frac{1}{3^{k_0+\cdots+k_{j-1}}}.$$

Observe that $f(s_j) = t_j$. We define x and y in $(0, 1)$ as $x = \lim_{j\to\infty} s_j$ and $y = \lim_{j\to\infty} t_j$. It follows form the continuity of f that $y = f(x)$. We also put

$$I_j = \left(s_j - \frac{1}{3}\frac{1}{(3 \cdot 4 \cdot 5)^{k_0+\cdots+k_{j-1}} \cdot (3 \cdot 4)^{k_j}},\ s_j\right)$$

and

$$J_j = \left(t_j - \frac{5}{3}\frac{1}{(3 \cdot 4)^{k_0+\cdots+k_j}},\ t_j\right).$$

Observe that f is linear on

$$\left(s_j - \frac{1}{3} \cdot \frac{1}{(3 \cdot 5)^{k_0+\cdots+k_{j-1}+1}},\ s_j\right) \supset I_j$$

with slope $5^{k_0+\cdots+k_{j-1}+1}$. Hence it is easy to see that $f(I_j) = J_j$. Define

$$F = [0, 1] \setminus \bigcup_{j=1}^{\infty} J_j$$

and $E = f^{-1}(F)$. We have $y \in F$ and $x \in E$. Since $f(I_j) = J_j$, we also have

$$E \cap \bigcup_{j=1}^{\infty} I_j = \varnothing.$$

Then using the definition of the points t_j and the facts

$$\bigcup_{j=1}^{\infty} J_j = [0, 1] \setminus F, \qquad J_j = [t_{j-1}, t_j] \setminus F,$$

$$\frac{t_j - t_{j-1}}{|J_j|} > \frac{t_{j+1} - t_j}{|J_j|} = \frac{4^{k_0+\cdots+k_j}}{5},$$

and

$$\frac{(y - t_j)}{|J_j|} > \frac{t_{j+1} - t_j}{|J_j|} = \frac{4^{k_0+\cdots+k_j}}{5},$$

one can easily see that y is a density point of F. Plainly,

$$x - s_j = \sum_{l=j}^{\infty}(s_{l+1} - s_l) < 2(s_{j+1} - s_j).$$

Using $x \in f^{-1}(F)$ and (1) from

$$\frac{(x - s_j)}{|I_j|} < 2 \cdot \frac{s_{j+1} - s_j}{|I_j|} = 2 \cdot \left(\frac{4}{5}\right)^{k_0 + \cdots + k_j} \cdot 5^{k_0 + \cdots + k_{j-1}} < \frac{2}{j},$$

it follows that x is not a density point of E. This implies that f is not density continuous.

On p. 1041 of [BO] it is verified that for any measurable $E \subset [0, 1]$, $f^{-1}(E)$ has the same measure as E; that is, f is measure preserving. This concludes the proof of Theorem 1.

Theorem 2. *The set of Hausdorff dimensions of graphs of surjective density continuous functions* $f\colon [0, 1] \to [0, 1]$ *is dense in* $[1, 2]$.

Proof. Let $n \geq 3$ be arbitrary, $1 < l < n$, and $m \in \mathbf{N}$ be odd. Assume that the functions ϕ_{rm+j} are defined as in [BO, pp. 1040–1041]. Denote by e_1 the line segment connecting the points $(0, 0)$ and $(\frac{1}{n}, \frac{1}{n})$, and denote by e_2 the line segment connecting the points $(\frac{l}{n}, \frac{l}{n})$ and $(1, 1)$. Consider the invariant set of the affine functions system ϕ_{rm+j}, where $1 \leq r \leq l-1$ and $1 \leq j \leq m$. We remark that here we do not use the functions ϕ_k for $k = 1, \ldots, m$. This is the slight change in the construction of [BO]. The invariant set is not the graph of a function defined on [0, 1]; however, it will become one if we add to it the line segments e_1 and e_2, and the countable collection of their images under maps of the form $\phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_N}$, where i_j is an integer between $m+1$ and lm for each $j = 1, 2, \ldots, N$. Denote by f the function obtained above.

Denote $f_0(x) = x$ for $x \in [0, 1]$, and let $f_N\colon [0, 1] \to [0, 1]$, $N \in \mathbf{N}$, be the function whose graph is the union of the images of the graph of f_0 under the mappings $\phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_N}$, where i_j is an integer between $m+1$ and lm for each $j = 1, 2, \ldots, N$, and the union of e_1, e_2, and their images under the maps of the form $\phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_{N'}}$, where $1 \leq N' < N$ and i_j is an integer between $m + 1$ and lm for each $j = 2, \ldots, N'$. It is easy to see that the continuous functions f_N converge uniformly to f, and hence f is continuous.

Assume that $[a, b] \times [c, d] = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_N}([0, 1] \times [0, 1])$, where i_j is an integer between $m + 1$ and lm for each $j = 1, 2, \ldots, N$. Then it is easy to see that $b - a = 1/(mn)^N$ and $d - c = 1/n^N$. Put $\psi_1\colon [0, 1] \to [a, b]$, $\psi_1(x) = (b - a)x + a$ and $\psi_2\colon [0, 1] \to [c, d]$, $\psi_2(x) = (d - c)x + c$. Observe that $f|[a, b] = \psi_2 \circ f \circ \psi_1^{-1}$, where by $f|[a, b]$ we denoted the restriction of f onto $[a, b]$. An argument like the one presented in [BO, p. 1041] can show that f is measure preserving, i.e., for every measurable $E \subset [0, 1]$ we have $|f^{-1}(E)| = |E|$.

Assume now that $E \subset [c, d]$ is measurable. Then $|f^{-1}(E) \cap [a, b]| = |(f|[a, b])^{-1}(E)| = |\psi_1 \circ f^{-1} \psi_2^{-1}(E)|$. Plainly, $|\psi_2^{-1}(E)| = |E|/(d - c)$, $|f^{-1} \circ \psi_2^{-1}(E)| = |E|/(d - c)$, and finally $|\psi_1 \circ f^{-1} \circ \psi_2^{-1}(E)| = (b - a) \cdot |E|/(d - c) = |E|/m^N$. Therefore, we proved that

$$|f^{-1}(E) \cap [a, b]| = \frac{b - a}{d - c}|E|. \tag{2}$$

Denote by F_1 the invariant set of the affine functions system ϕ_{rm+j}, $1 \leq r \leq l - 1$, $1 \leq j \leq m$, and by F_2 the projection of F_1 onto the x-axis.

Assume that $E \subset [0, 1]$ is measurable, y_0 is a point of density of E, and $x_0 \in f^{-1}(y_0)$.

If $x_0 \notin F_2$, then it follows from the definition of f that there exits a $\delta > 0$ such that f is nonconstant and linear on the intervals $(x_0 - \delta, x_0]$ and $[x_0, x_0 + \delta)$. Therefore, x_0 is a point of density of $f^{-1}(E)$.

Assume now that $x_0 \in F_2$. Then there exists a sequence $i_1, i_2, \dots, i_N, \dots$ such that $(x_0, y_0) = \bigcap_{N=1}^{\infty} \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_N}([0, 1] \times [0, 1])$, and i_N is an integer between $m+1$ and lm for each nonnegative integer N. We also choose a_N, b_N, c_N, d_N such that

$$[a_N, b_N] \times [c_N, d_N] = \phi_{i-1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_N}([0, 1] \times [0, 1]).$$

Since $m < i_{N+1} \leq lm \leq (n-1)m$, we have

$$x_0 \in \left[a_N + \frac{b_N - a_N}{n}, b_N - \frac{b_N - a_N}{n}\right] \tag{3}$$

for $N = 1, 2, \dots$.

Let $\varepsilon > 0$. Since y_0 is a density point of E, we can choose an N_0 such that if $y_0 \in [q, w] \subset [c_{N_0}, d_{N_0}]$, then

$$|[q, w] \backslash E| < \varepsilon \cdot (w - q). \tag{4}$$

We obtain from (2) easily that

$$|[a_N, b_N] \backslash f^{-1}(E)| < \varepsilon (b_N - a_N)$$

holds for any $N \geq N_0$. From (3) it follows that x_0 is the open interval (a_N, b_N) for any N, and hence we can find a $\delta_0 > 0$ such that $(x_0 - \delta_0, x_0 + \delta_0) \subset (a_{N_0}, b_{N_0})$.

Assume that $0 < \delta < \delta_0$. Choose $N \geq N_0$ such that $(x_0 - \delta, x_0 + \delta) \subset [a_N, b_N]$ and $(x_0 - \delta, x_0 + \delta) \not\subset [a_{N+1}, b_{N+1}]$. Using (3) with $N+1$ we obtain

$$x_0 \in \left[a_{N+1} + \frac{b_{N+1} - a_{N+1}}{n}, b_{N+1} - \frac{b_{N+1} - a_{N+1}}{n}\right].$$

This and $(x_0 - \delta, x_0 + \delta) \not\subset [a_{N+1}, b_{N+1}]$ imply $\delta > (b_{N+1} - a_{N+1})/n$, that is, $b_{N+1} - a_{N+1} < \delta_n$, and hence

$$\begin{aligned} |(x_0 - \delta, x_0 + \delta) \backslash f^{-1}(E)| &\leq |[a_N, b_N] \backslash f^{-1}(E)| < \varepsilon (b_N - a_N) \\ &= \varepsilon nm (b_{N+1} - a_{N+1}) < \varepsilon nm \cdot n\delta. \end{aligned}$$

Thus for every $\varepsilon > 0$ there exists δ_0 such that for every $0 < \delta < \delta_0$ we have

$$\frac{|(x_0 - \delta, x_0 + \delta) \backslash f^{-1}(E)|}{2\delta} < \varepsilon \frac{n^2 m}{2}.$$

Therefore, x_0 is a point of density of $f^{-1}(E)$. This concludes the proof of the fact that f is density continuous.

The calculation of the Hausdorff dimension of f is similar to the one presented in [BO, p. 1042]. One obtains that the Hausdorff dimension of f equals

$$\log_n \sum_{j=1}^{l-1} m^{\log_{nm} n} = \frac{\ln(l-1)}{\ln n} + \frac{\ln m}{\ln m + \ln n}.$$

It is also clear that

$$\left\{\frac{\ln(l-1)}{\ln n} : n \in \mathbf{N},\ n \geq 3,\ 1 < l < n,\ l \in \mathbf{N}\right\}$$

is dense in $[0, 1]$. This implies (cf. [BO]) that the set of Hausdorff dimensions of graphs of the functions f is dense in [1, 2]. This completes the proof of Theorem 2.

Note that Theorem 2 of this paper together with Corollary 2 of [BO] imply that the set of the Hausdorff dimensions of graphs of density continuous functions equals the entire interval [1, 2].

References

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