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# Henstock integration in the plane

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## ABSTRACT

The paper deals with integration of abstract Henstock type.

Eleven derivation bases on the plane are investigated, those built with triangles, rectangles and regular rectangles, and the approximate bases. The relationships between the integration theories generated by them are found. Also the nonabsolute integrals of Perron, Kempisty, Mawhin, Pfeffer, and Chelidze-Dzhvarsheĭshvili are considered, and compared with the Henstock integrals.

Chapter 3 contains a generalized Fubini Theorem for the abstract Henstock integral. This theorem holds for any Henstock integral generated by a product base, in particular for the Lebesgue integral, Kurzweil integral, and the integral given by the approximate product base.

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## NOTATION

| | |
|---|---|
| $\mathbf{R}$ | The set of all real numbers. |
| $\mathbf{N}$ | The set of all positive integers. |
| $\underline{P}(A)$ | The class of all subsets of a set $A$. |
| $A^{\circ}$ | The interior of a set $A \subset \mathbf{R}^2$ (in the natural topology). |
| $A^{-}$ | The closure of $A \subset \mathbf{R}^2$. |
| $\partial A$ | The boundary of $A \subset \mathbf{R}^2$. |
| $d(A)$ | The diameter of a set $A \subset \mathbf{R}^2$. |
| $\lambda(A)$ | The outer Lebesgue measure of $A \subset \mathbf{R}^2$. |
| $\lambda_1(E)$ | The outer Lebesgue measure of a linear set $E$. |
| $D(x,r)$ | Disk on the plane, centered at $x$, with radius $r$. |
| $A \triangle B$ | The symmetric difference of $A$ and $B$, i.e., $(A \setminus B) \cup (B \setminus A)$. |
| sgn $x$ | The sign of a real number $x \neq 0$, i.e., $\frac{x}{\lvert x \rvert}$. |
| $\frac{\partial_{ap}}{\partial x}, \frac{\partial_{ap}}{\partial y}$ | The approximate partial derivatives. |
| $\prod_{s \in S} X_s$ | The Cartesian product of a class $\{X_s\}_{s \in S}$. |
| $a \cdot b$ | For vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ this is their scalar product $a_1 b_1 + a_2 b_2$. |

# INTRODUCTION

This work presents and compares various integration theories in the plane. It is our intention to put those theories in the framework based on the abstract Henstock integral, presented in chapter 1.

We introduce eleven derivation bases on the plane, and the Henstock integrals generated by them. We also consider nonabsolute integrals of Perron, Kempisty, Mawhin, Pfeffer, and Chelidze and Dzhvarshe̕ishvili. The relationships among them found are presented graphically in a diagram. In the diagram, integration theories are represented by the bases generating them, or by the names of their inventors. Arrows point to the more general theories. +(condition) means that the condition stated is necessary for the relationship. *CSS* denotes continuity in the sense of Saks, *SMC* — special assumption on majorants and minorants of theorem 4.5.4, and *SDC* — special decomposition condition of theorem 5.5.5.

The following is the list of references for the relationships presented in the diagram.

| | |
|---|---|
| Lebesgue $\to \Delta_1^*, \Delta_2^*, \Delta_3^*$; $\Delta_1^*, \Delta_2^*, \Delta_3^* \to$ Lebesgue | 2.5.5, |
| $\Delta_1^*, \Delta_2^*, \Delta_3^* \to \Delta_3$ | 2.3.4, 2.5.5, |
| $\Delta_3 \to \Delta_1$ | 2.3.4, 2.3.5, |
| $\Delta_1 \to \tilde{\Delta}_1$; $\tilde{\Delta}_1 \to \Delta_1$ | 2.3.4, 2.3.8, |
| $\Delta_1 \to$ Perron; Perron $\to \Delta_1$ | 2.1.1, |
| $\Delta_1 \to \Delta_2$ | 2.3.4, 3.4.1, |
| $\Delta_1 \to \Delta_4$ | 2.3.4, 2.3.5, |

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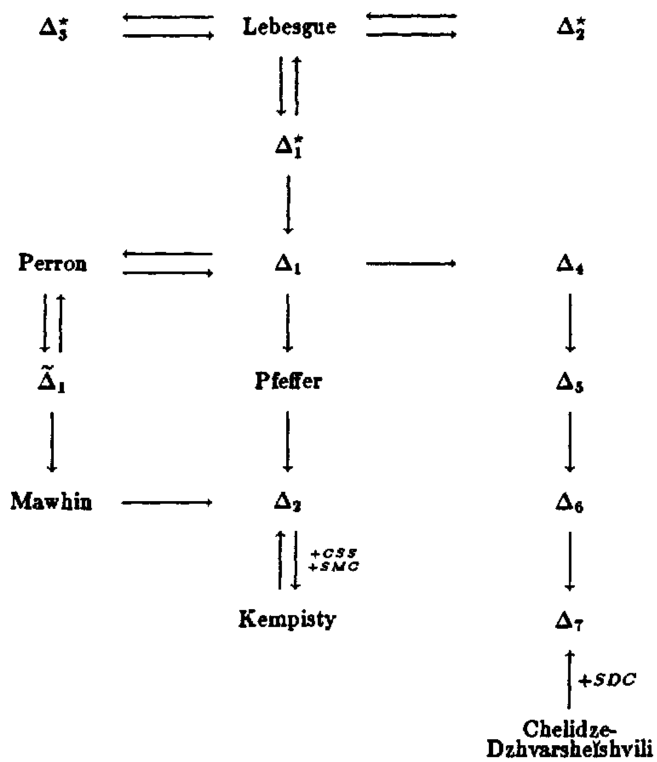
| | |
|---|---|
| $\Delta_4 \to \Delta_5$ | 5.3.7, |
| $\Delta_5 \to \Delta_6$; $\Delta_6 \to \Delta_7$ | 5.2.7, |
| $\Delta_1 \to$ Pfeffer; Pfeffer $\to \Delta_2$ | 4.7.9, 4.7.10, |
| $\Delta_1 \to$ Mawhin; Mawhin $\to \Delta_2$ | 4.7.2, |
| Kempisty $\to \Delta_2$ | 4.5.2, |
| $\Delta_2$ $(+CSS\ +SMC) \to$ Kempisty | 4.5.3, 4.5.4, |
| Chelidze-Dzhvarsheĭshvili $(+SDC) \to \Delta_7$ | 5.5.5. |



*Chapter 1*

# HENSTOCK INTEGRAL

In this chapter we introduce the notion of a derivation base, and then all the other notions fundamental for the whole work, such as: derivative, variation, (Henstock) integral, and the variational equivalence.

We define basic properties of certain bases, including possessing the partitioning property (which makes a base into an integration base), having local character, being additive.

Finally, five equivalent ways to define the Henstock integral are presented, including the classical one (a limit of Riemann sums), variational equivalence of the integral and the integrand, and the definition of the Perron-Ward type.

## 1.1. Derivation bases.

**1.1.1. Definition.** Let $X$ be a nonempty set and $\Psi$ a nonvoid class of its subsets. A nonempty class

$$\Delta \subset \underline{P}(X \times \Psi) \tag{1.1}$$

will be termed a ***derivation base*** on $X$. In specific cases, we will take $X$ to be $\mathbf{R}$ and $\Psi$ to be the class of all nontrivial closed intervals, or (more often) $X = \mathbf{R}^2$ and $\Psi$ — nondegenerate closed intervals, regular intervals, triangles, etc.

A more general setting is possible. In [1] an integration theory of Henstock type in a locally compact Hausdorff space is presented. A space $A$ equipped with a class $\{I\}$, as in [8] and [57], is also a possibility. We will, however, concentrate on $\mathbf{R}^2$.

The definition presented here is based on the one in [53], and is slightly different from that of [12]. The purpose of our choice is to get a definition closely related to the Henstock integral.

A base $\Delta$ is called *trivial* if $\emptyset \in \Delta$. Unless stated otherwise, all bases considered are nontrivial.

Elements of a base $\Delta$ will be denoted by small Greek letters $(\alpha, \beta, \gamma, \ldots)$. We will use the following notation:

$$\begin{aligned} \alpha[E] &= \{(x, I) \in \alpha : x \in E\}, \\ \alpha(E) &= \{(x, I) \in \alpha : I \subset E\} \end{aligned} \tag{1.2}$$

for $\alpha \in \Delta, E \subset X$. We will also write:

$$\begin{aligned} \Delta[E] &= \{\alpha[E] : \alpha \in \Delta\}, \\ \Delta(E) &= \{\alpha(E) : \alpha \in \Delta\}. \end{aligned} \tag{1.3}$$

**1.1.2. Remark.** We will assume that the class $\Psi$ has the following property: given $I_0, I_1, \ldots, I_n \in \Psi$, and $I_1, \ldots, I_n \subset I_0$,

$$I_0 \setminus (I_1 \cup I_2 \cup \ldots \cup I_n) = J_1 \cup J_2 \cup \ldots \cup J_m \tag{1.4}$$

where $J_1, J_2, \ldots, J_m$ are nonoverlapping elements of $\Psi$ (since we concentrate on $\mathbb{R}^2$ the meaning of "nonoverlapping" will be clear).

**1.1.3. Definition.** We say that a finite class $\mathcal{D}$ of elements of $\Psi$ is a *division* if its elements are nonoverlapping. If $\mathcal{D}$ is a division then

$$\sigma(\mathcal{D}) = \bigcup_{I \in \mathcal{D}} I. \tag{1.5}$$

A *partition* is a class $\pi \in \underline{P}(X \times \Psi)$ such that

$$\mathcal{D} = \{I \in \Psi : (x, I) \in \pi\} \tag{1.6}$$

has exactly as many elements as $\pi$ and is a division.

If $\pi$ is a partition, we will write

$$\sigma(\pi) = \bigcup_{(x,I) \in \pi} I. \tag{1.7}$$

Finally, $\mathcal{D}$ is a division of an element $I$ of $\Psi$ if $\sigma(\mathcal{D}) = I$. Similarly, $\pi$ is a partition of $I$ if $\sigma(\pi) = I$.

**1.1.4. Definition.** If $F : X \times \Psi \to \mathbf{R}$ and $\pi$ is a partition then we will write

$$F(\pi) = \sum_{(x,I)\in\pi} F(x,I). \tag{1.8}$$

If $H : \Psi \to \mathbf{R}$ and $\mathcal{D}$ is a division then

$$H(\mathcal{D}) = \sum_{I\in\mathcal{D}} H(I). \tag{1.9}$$

**1.1.5. Definition.** A base $\Delta$ is *filtering down* if for every $\alpha_1, \alpha_2 \in \Delta$ there exists an $\alpha \in \Delta$ such that $\alpha \subset \alpha_1 \cap \alpha_2$.

$\Delta$ has the *partitioning property* if for every $I \in \Psi$ and every $\alpha \in \Delta$ there exists a partition $\pi \subset \alpha$ of $I$.

**1.1.6. Definition.** A base $\Delta$ is *finer* than a base $\Delta'$ if for every $\alpha' \in \Delta'$ there exists an $\alpha \in \Delta$ such that $\alpha \subset \alpha'$.

If $\Delta$ is finer than $\Delta'$ then we write $\Delta \preceq \Delta'$.

If $\Delta \preceq \Delta'$ and $\Delta \succeq \Delta'$ then we will say that $\Delta$ and $\Delta'$ are *equivalent* and write $\Delta \simeq \Delta'$.

**1.1.7. Definition.** $\Delta$ has a *local character* if for every $\{\beta_x\} \in \prod_{x\in X} \Delta[\{x\}]$ there exists an $\alpha \in \Delta$ such that for every $x \in X$

$$\alpha[\{x\}] \subset \beta_x. \tag{1.10}$$

$\Delta$ has a *$\sigma$-local character* if for any sequence $\{X_n\}$ of disjoint subsets of $X$, and for every $\{\beta_n\} \in \prod_{n\in\mathbf{N}} \Delta[X_n]$ there exists an $\alpha \in \Delta$ such that for every $n \in \mathbf{N}$

$$\alpha[X_n] \subset \beta_n. \tag{1.11}$$

Any base that has a local character must also have a $\sigma$-local character.

**1.1.8. Definition.** $\Delta$ *ignores a point* $x \in X$ if there exists an $\alpha \in \Delta$ such that $\alpha[\{x\}] = \emptyset$.

## 1.2. Derivatives.

**1.2.1. Definition.** Let $F, G : X \times \Psi \to \mathbf{R}$. Define the *derivative* of $F$ with respect to $G$ (with respect to a derivation base $\Delta$) as a number $\mathrm{D}_\Delta F_G(x)$ such that for every $\varepsilon > 0$ there exists an $\alpha \in \Delta$ such that if $(x, I) \in \alpha\big[\{x\}\big]$ then

$$\left|\mathrm{D}_\Delta F_G(x) - \frac{F(x,I)}{G(x,I)}\right| < \varepsilon. \tag{1.12}$$

The *upper* and *lower derivatives* are defined as follows:

$$\begin{aligned} \overline{\mathrm{D}}_\Delta F_G(x) &= \inf_{\alpha\in\Delta}\sup_{(x,I)\in\alpha}\frac{F(x,I)}{G(x,I)}, \\ \underline{\mathrm{D}}_\Delta F_G(x) &= \sup_{\alpha\in\Delta}\inf_{(x,I)\in\alpha}\frac{F(x,I)}{G(x,I)}. \end{aligned} \tag{1.13}$$

Note that if $\Delta$ ignores a point $x$ then $\overline{\mathrm{D}}_\Delta F_G(x) = -\infty$ and $\underline{\mathrm{D}}_\Delta F_G(x) = +\infty$.

In the remainder of this section we take $F$, $G$ to be arbitrary functions from $X \times \Psi$ into $\mathbf{R}$, and $x \in X$. We have the following, easy to prove, statements.

**1.2.2. Proposition.** *If $\Delta$ does not ignore $x$ and is filtering down then*

$$\overline{\mathrm{D}}_\Delta F_G(x) \geq \underline{\mathrm{D}}_\Delta F_G(x). \tag{1.14}$$

**1.2.3. Proposition.** *(i) If $\Delta$ does not ignore $x$ and is filtering down, then the existence of $\mathrm{D}_\Delta F_G(x)$ implies the equality*

$$\underline{\mathrm{D}}_\Delta F_G(x) = \overline{\mathrm{D}}_\Delta F_G(x). \tag{1.15}$$

*$\mathrm{D}_\Delta F_G(x)$ is then equal to the extreme derivatives.*

*(ii) If $\Delta$ is filtering down then (1.15) implies the existence of $\mathrm{D}_\Delta F_G(x)$ which is then equal to the extreme derivatives.*

**1.2.4. Observation.** *If $\Delta \preceq \Delta'$ and $\Delta'$ ignores $x$, then $\Delta$ ignores $x$, as well.*

**1.2.5. Proposition.** *If $\Delta \preceq \Delta'$ then*

$$\underline{\mathrm{D}}_\Delta F_G(x) \geq \underline{\mathrm{D}}_{\Delta'} F_G(x), \tag{1.16}$$

$$\overline{\mathrm{D}}_\Delta F_G(x) \leq \overline{\mathrm{D}}_{\Delta'} F_G(x). \tag{1.17}$$

## 1.3. Henstock integral.

**1.3.1. Definition.** Let $I_0 \in \Psi$, $F : X \times \Psi \to \mathbf{R}$. We define the *Henstock integral* of $F$ with respect to $\Delta$ over $I_0$ as a number $(\Delta)\int_{I_0} F$ such that for every $\varepsilon > 0$ there exists an $\alpha \in \Delta$ such that for every partition $\pi \subset \alpha$ of $I_0$

$$\left|F(\pi) - (\Delta)\int_{I_0} F\right| \le \varepsilon. \tag{1.18}$$

The *upper* and *lower Henstock integrals* are defined as

$$(\Delta)\overline{\int_{I_0}} F = \inf_{\alpha\in\Delta}\sup_{\pi\subset\alpha} F(\pi), \tag{1.19}$$

$$(\Delta)\underline{\int_{I_0}} F = \sup_{\alpha\in\Delta}\inf_{\pi\subset\alpha} F(\pi). \tag{1.20}$$

Obviously, if $\Delta$ does not partition $I_0$, the definition becomes void, and

$$(\Delta)\underline{\int_{I_0}} F = +\infty, \qquad (\Delta)\overline{\int_{I_0}} F = -\infty. \tag{1.21}$$

**1.3.4. Proposition.** *If $\Delta$ has the partitioning property and is filtering down then*

$$(\Delta)\overline{\int_{I_0}} F \ge (\Delta)\underline{\int_{I_0}} F. \tag{1.22}$$

**Proof.** Let $\varepsilon > 0$. Find $\alpha_1, \alpha_2 \in \Delta$ such that

$$\begin{aligned} \sup_{\pi\subset\alpha_1} F(\pi) &\le (\Delta)\overline{\int_{I_0}} F + \varepsilon, \\ \inf_{\pi\subset\alpha_2} F(\pi) &\ge (\Delta)\underline{\int_{I_0}} F - \varepsilon. \end{aligned} \tag{1.23}$$

Let $\alpha \in \Delta$ be such that $\alpha \subset \alpha_1 \cap \alpha_2$. Then

$$\begin{aligned} (\Delta)\overline{\int_{I_0}} F + \varepsilon \ge \sup_{\pi\subset\alpha_1} F(\pi) \ge \sup_{\pi\subset\alpha} F(\pi) \ge \\ \inf_{\pi\subset\alpha} F(\pi) \ge \inf_{\pi\subset\alpha_2} F(\pi) \ge (\Delta)\underline{\int_{I_0}} F - \varepsilon. \end{aligned} \tag{1.24}$$

Since $\varepsilon$ was arbitrary, (1.22) follows.

**1.3.5. Proposition.** *(i) If $\Delta$ has the partitioning property and is filtering down then the existence of $(\Delta)\int_{I_0} F$ implies the equality*

$$(\Delta)\underline{\int}_{I_0} F = (\Delta)\overline{\int}_{I_0} F = (\Delta)\int_{I_0} F. \tag{1.25}$$

*(ii) If $\Delta$ is filtering down then the equality*

$$(\Delta)\underline{\int}_{I_0} F = (\Delta)\overline{\int}_{I_0} F \tag{1.26}$$

*implies the existence of $(\Delta)\int_{I_0} F$ which is then equal to the extreme integrals.*

**Proof.** (i) Let $\varepsilon$ be arbitrary. There exists an $\alpha \in \Delta$ such that for all $\pi \subset \alpha$, partitions of $I_0$

$$F(\pi) - \varepsilon \le (\Delta)\int_{I_0} F \le F(\pi) + \varepsilon. \tag{1.27}$$

This implies

$$(\Delta)\overline{\int}_{I_0} F - \varepsilon \le (\Delta)\int_{I_0} F \le (\Delta)\underline{\int}_{I_0} F + \varepsilon, \tag{1.28}$$

which together with 1.3.4 gives (1.25).

(ii) Let

$$A = (\Delta)\underline{\int}_{I_0} F = (\Delta)\overline{\int}_{I_0} F. \tag{1.29}$$

Let $\varepsilon > 0$ be arbitrary. Choose $\alpha_1, \alpha_2 \in \Delta$ so that

$$\sup_{\pi\subset\alpha_1} F(\pi) - \varepsilon \le A \le \inf_{\pi\subset\alpha_2} F(\pi) + \varepsilon. \tag{1.30}$$

Find an $\alpha \in \Delta$ which is contained in $\alpha_1 \cap \alpha_2$. For every $\pi \subset \alpha$, a partition of $\pi$, we have

$$F(\pi) - \varepsilon \le A \le F(\pi) + \varepsilon, \tag{1.31}$$

so that

$$(\Delta)\int_{I_0} F = A. \tag{1.32}$$

**1.3.6. Observation.** *If $\Delta \preceq \Delta'$ and $\Delta$ has the partitioning property then $\Delta'$ has the partitioning property.*

**1.3.7. Proposition.** *If $\Delta \preceq \Delta'$ then*

$$(\Delta) \underline{\int}_{I_0} F \geq (\Delta') \underline{\int}_{I_0} F \tag{1.33}$$

*and*

$$(\Delta) \overline{\int}_{I_0} F \leq (\Delta') \overline{\int}_{I_0} F. \tag{1.34}$$

**Proof.** We will prove only (1.33), as (1.34) may be shown in the same manner. By 1.3.6 we can assume that both $\Delta$ and $\Delta'$ have the partitioning property (otherwise the inequalities becomes trivial). Let $\varepsilon > 0$. There exists an $\alpha' \in \Delta'$ such that for every $\pi \subset \alpha'$, a partition of $I_0$

$$F(\pi) \leq (\Delta') \underline{\int}_{I_0} F + \varepsilon. \tag{1.35}$$

Let $\alpha \in \Delta$ be such that $\alpha \subset \alpha'$. Then, for any partition $\pi \subset \alpha$ of $I_0$, (1.35) is satisfied too, so that

$$(\Delta) \underline{\int}_{I_0} F \leq (\Delta') \underline{\int}_{I_0} F + \varepsilon. \tag{1.36}$$

This ends the proof.

## 1.4. Variation.

**1.4.1. Definition.** Let $\Delta$ be a derivation base and $F : X \times \Psi \to \mathbf{R}$. The *variation of $F$ over $\alpha \in \Delta$* is defined as

$$V(F, \alpha) = \sup_{\pi \subset \alpha} |F|(\pi), \tag{1.37}$$

where $\pi \subset \alpha$ are partitions.

The *variation of $F$ over $\Delta$* is

$$V(F, \Delta) = \inf_{\alpha \in \Delta} V(F, \alpha). \tag{1.38}$$

The *variational measure* of a set $E \subset X$ is defined as

$$F_\Delta(E) = V\left(F, \Delta[E]\right). \tag{1.39}$$

It is not necessarily a measure in the ordinary sense of the word (see [53], p. 161).

**1.4.2. Definition.** For any $F : X \times \Psi \to \mathbf{R}$ its variation generates a function from $\Psi$ into $\mathbf{R}$ which can be called a ***variation of*** $F$, namely

$$\Psi \ni I \longmapsto V\big(F, \Delta(I)\big). \tag{1.40}$$

We will write $V_F$ for that function.

**1.4.3.** Note that any function $H : \Psi \to \mathbf{R}$ may be viewed as $H : X \times \Psi \to \mathbf{R}$ by taking $H(x, I) = H(I)$ for any $x$.

**1.4.4. Definition.** A function $H : \Psi \to \mathbf{R}$ will be termed
(i) ***additive***,
(ii) ***subadditive***,
(iii) ***superadditive***,
if for any division $D \subset \Psi$ such that $\sigma(D) = I \in \Psi$ we have
(i) $H(D) = H(I)$,
(ii) $H(D) \geq H(I)$,
(iii) $H(D) \leq H(I)$,
respectively.

We will denote by $\Psi_+$ the class of unions of finite elements of $\underline{P}(\Psi)$.

A function $H : \Psi \to \mathbf{R}$ which is additive, extends naturally to an additive $H : \Psi_+ \to \mathbf{R}$.

**1.4.5. Proposition.** *Suppose $\Delta$ has a $\sigma$-local character and $F : X \times \Psi \to \mathbf{R}$. Then for any sequence $\{E_n\}$ of subsets of $X$ and $E_0 \subset \bigcup_{n\in\mathbf{N}} E_n$ we have*

$$F_\Delta(E_0) \leq \sum_{n=1}^{+\infty} F_\Delta(E_n). \tag{1.41}$$

*Consequently, for $\Delta$ with a $\sigma$-local character, $F_\Delta$ becomes a genuine outer measure.*

**Proof.** We can assume without loss of generality that the sets $E_n$ are disjoint. Let $\epsilon > 0$ be arbitrary. For every $n \in \mathbf{N}$ there exists an element $\alpha_n$ of the base

$\Delta$ such that

$$V(F, \alpha_n[E_n]) \leq F_\Delta(E_n) + \frac{\varepsilon}{2^n}. \tag{1.42}$$

Since $\Delta$ has a $\sigma$-local character and $E_n$'s are disjoint, there exists an $\alpha \in \Delta$ such that

$$\alpha[E_n] \subset \alpha_n \tag{1.43}$$

for every $n \in \mathbf{N}$.

Let $\pi \subset \alpha[E_0]$ be an arbitrary partition, set

$$\pi_n = \{(x, I) \in \pi : x \in E_n\}. \tag{1.44}$$

We have

$$|F|(\pi) = \sum_{n=1}^{+\infty} |F|(\pi_n) \leq \sum_{n=1}^{+\infty} V(F, \alpha_n[E_n]) \leq \sum_{n=1}^{+\infty}\left(F_\Delta(E_n) + \frac{\varepsilon}{2^n}\right) \tag{1.46}$$

so that

$$F_\Delta(E_0) \leq V(F, \alpha[E_0]) \leq \sum_{n=1}^{+\infty} F_\Delta(E_n) + \varepsilon. \tag{1.47}$$

Since $\varepsilon$ was arbitrary, this ends the proof.

## 1.5. Variational Integral.

**1.5.1. Definition.** Let us denote the class of additive $H : \Psi \to \mathbf{R}$ by $\mathcal{A}$. Furthermore, let $\underline{\mathcal{A}}$ stand for the class of all subadditive $H : \Psi \to \mathbf{R}$, and $\overline{\mathcal{A}}$ — for the collection of the superadditive ones.

If $\mathcal{K}$ is any of the above defined classes of functions, and $I_0 \in \Psi$, then $\mathcal{K}(I_0)$ will denote the set of all $H \in \mathcal{K}$ defined only for $I \subset I_0$.

$\mathcal{K}^+$ will stand for all those $H \in \mathcal{K}$ which are nonnegative.

**1.5.2. Definition.** If $F_1, F_2 : X \times \Psi \to \mathbf{R}$, we say that $F_1$ and $F_2$ are *variationally equivalent* (written as $F_1 \simeq F_2$) *on* $I_0 \in \Psi$ if for every $\varepsilon > 0$ there exists an $\alpha \in \Delta$ and $\Omega \in \overline{\mathcal{A}}^+$ (depending on $\alpha$) such that $\Omega(I_0) < \varepsilon$ and for every $(x, I) \in \alpha(I_0)$

$$|F_1(x,I) - F_2(x,I)| \leq \Omega(I). \tag{1.48}$$

We say that $F_1$ and $F_2$ are *variationally equivalent* if they are variationally equivalent on every $I_0 \in \Psi$.

It is easy to see that the variational equivalence is in fact an equivalence relation, because $\overline{\mathcal{A}}^+$ is an additive class.

**1.5.3. Lemma.** *Suppose $H_1, H_2 : \Psi \to \mathbf{R}$, $H_1, H_2 \in \mathcal{A}$, and $H_1 \simeq H_2$ with respect to a base $\Delta$ which possesses the partitioning property. Then $H_1 = H_2$.*

**Proof.** Let $\varepsilon > 0$ and $I_0 \in \Psi$. There are $\alpha \in \Delta$, $\Omega \in \overline{\mathcal{A}}^+$ such that $\Omega(I_0) \leq \varepsilon$ and for each $(x, I) \in \alpha(I_0)$

$$|H_1(I) - H_2(I)| \leq \Omega(I). \tag{1.49}$$

Let $\pi \subset \alpha$ be a partition of $I_0$. Then

$$\begin{aligned} |H_1(I_0) - H_2(I_0)| \leq \sum_{(x,I)\in\pi} |H_1(I) - H_2(I)| \leq \\ \sum_{(x,I)\in\pi} \Omega(I) \leq \Omega(I_0) \leq \varepsilon. \end{aligned} \tag{1.50}$$

Since $\varepsilon$ and $I_0$ are arbitrary, this ends the proof.

**1.5.4. Definition.** The *variational integral* of a function $F : X \times \Psi \to \mathbf{R}$ on $I_0 \in \Psi$ (with respect to a derivation base $\Delta$) is the number $H(I_0)$, where $H : \Psi \to \mathbf{R}$ is additive and variationally equivalent to $F$.

We will also refer to $H$ as the variational integral of $F$.

As long as $\Delta$ has the partitioning property, the variational integral is, by lemma 1.5.3, uniquely defined.

It might be interesting to note that, although the above definition originates from Henstock (see [13], [14], [19], and [22]), a similar approach is presented in the classical paper [28] of Kolmogorov. For a modern treatment of the Kolmogorov integral, see [11].

### 1.6. Various ways to define the Henstock integral.

**1.6.1. Theorem.** *Let $\Delta$ have the partitioning property and be filtering down. Let $F : X \times \Psi \to \mathbf{R}$, $I_0 \in \Psi$. The following are equivalent:*

*(i) $F$ is Henstock integrable on $I_0$;*

*(ii) For every $\varepsilon > 0$ there exists an $\alpha \in \Delta(I_0)$ such that for every $I \subset I_0$, $I \in \Psi$ and for every $\pi \subset \alpha$, a partition of $I$*

$$\left|(\Delta)\int_I F - F(\pi)\right| < \varepsilon; \tag{1.51}$$

*(iii) There exists an $H \in \mathcal{A}(I_0)$ such that $V(H - F, \Delta(I_0)) = 0$;*

*(iv) There exists an $H \in \mathcal{A}(I_0)$ such that $H \simeq F$ on $I_0$;*

*(v) For every $\varepsilon > 0$ there exist an $\alpha \in \Delta$, $A \in \overline{\mathcal{A}}$, and $B \in \underline{\mathcal{A}}$ such that*

$$A(I_0) - B(I_0) \le \varepsilon \tag{1.52}$$

*and for every $(x, I) \in \alpha(I_0)$*

$$A(I) \ge F(x, I) \ge B(I). \tag{1.53}$$

**Proof.** (ii)$\Rightarrow$(i) is obvious.

(i)$\Rightarrow$(iii). Let $\varepsilon > 0$. Choose $\alpha \in \Delta(I_0)$ so that for every partition $\pi \subset \alpha$ of $I_0$

$$\left|(\Delta)\int_{I_0} F - F(\pi)\right| \le \varepsilon. \tag{1.54}$$

Let $I_1 \subset I_0$, $I_1 \in \Psi_+$. Let $\pi'$, $\pi'' \subset \alpha$ be arbitrary partitions of $I_1$. By 1.1.2, $I_0 = I_1 \cup I_2 \cup \ldots \cup I_n$ for some $I_2, \ldots, I_n \in \Psi$, where $I_1, I_2, \ldots, I_n$ are nonoverlapping. Since $\Delta$ has the partitioning property, there are $\pi_1$, $\pi_2 \subset \alpha$, partitions of $I_0$ such that

$$\begin{aligned} \pi' &= \{(x, I) \in \pi_1 : I \subset I_1\}, \\ \pi'' &= \{(x, I) \in \pi_2 : I \subset I_1\}. \end{aligned} \tag{1.55}$$

We can assume that $\pi_1 \setminus \pi' = \pi_2 \setminus \pi''$. Then

$$|F(\pi') - F(\pi'')| = |F(\pi_1) - F(\pi_2)| \leq 2\varepsilon, \tag{1.56}$$

by (1.54).

Set

$$H(I) = (\Delta) \int_I F \tag{1.57}$$

for $I \subset I_0$, $I \in \Psi$. (1.56) implies

$$|H(I) - F(\pi)| \leq 4\varepsilon \tag{1.58}$$

whenever $\pi \subset \alpha$ is a partition of $I \subset I_0$, $I \in \Psi$. Now if $I'$ and $I''$ are nonoverlapping and contained in $I_0$, then by (1.58)

$$|H(I' \cup I'') - H(I') - H(I'')| \leq 12\varepsilon \tag{1.59}$$

Consequently — H is additive.

We will show now that

$$V(H - F, \Delta(I_0)) = 0. \tag{1.60}$$

Take $\alpha$ as before, and $\pi \subset \alpha$ — a partition of $I_0$. Write

$$\begin{aligned} \pi' &= \{(x, I) \in \pi : H(I) - F(x, I) \geq 0\}, \\ \pi'' &= \{(x, I) \in \pi : H(I) - F(x, I) < 0\}, \\ E' &= \sigma(\pi'), \\ E'' &= \sigma(\pi''). \end{aligned} \tag{1.61}$$

The estimates made before give

$$\begin{aligned} |H - F|(\pi) = &|H - F|(\pi') - |H - F|(\pi'') = \\ &|H(E') - F(\pi')| + |H(E'') - F(\pi'')| \leq 8\varepsilon \end{aligned} \tag{1.62}$$

so that

$$V(H - F, \alpha) \le 8\varepsilon \tag{1.63}$$

and consequently — (1.60).

(iii)⇒(iv). For each $\alpha \in \Delta$ set

$$\Omega(I) = V\big(H - F, \alpha(I)\big), \tag{1.64}$$

for $I \subset I_0$, $I \in \Psi$.

If $I_1$, $I_2$ are nonoverlapping, then

$$\begin{aligned}\Omega(I_1) + \Omega(I_2) = V\big(H - F, \alpha(I_1) \cup \alpha(I_2)\big) \le \\ V\big(H - F, \alpha(I_1 \cup I_2)\big) = \Omega(I_1 \cup I_2),\end{aligned} \tag{1.65}$$

so that $\Omega \in \overline{\mathcal{A}}$. Obviously, $\Omega$ is nonnegative, and

$$\big|H(I) - F(x, I)\big| \le \Omega(I). \tag{1.66}$$

We can make $\Omega(I_0) \le \varepsilon$, because $V\big(H - F, \Delta(I_0)\big) = 0$.

(iv)⇒(v). Let $\varepsilon > 0$. Choose an $\alpha \in \Delta(I_0)$ and $\Omega \in \overline{\mathcal{A}}^+$ so that for all $(x, I) \in \alpha$

$$\big|H(I) - F(x, I)\big| \le \Omega(I). \tag{1.67}$$

For $I \subset I_0$ set

$$\begin{aligned}A(I) &= \Omega(I) + H(I), \\ B(I) &= -\Omega(I) + H(I).\end{aligned} \tag{1.68}$$

We have $A \in \overline{\mathcal{A}}$ and $B \in \underline{\mathcal{A}}$. Moreover,

$$A(I_0) - B(I_0) \le 2\varepsilon. \tag{1.69}$$

Finally, if $(x, I) \in \alpha$ then by (1.67)

$$\begin{aligned}A(I) = \Omega(I) + H(I) \ge F(x, I) \ge \\ - \Omega(I) + H(I) = B(I).\end{aligned} \tag{1.70}$$

(v)⇒(iii) Set

$$H(I) = \inf_{A} A(I) = \sup_{B} B(I), \tag{1.71}$$

where inf and sup are taken over all possible $A$ and $B$ in (v). The function $H$ is well-defined, and it is easy to show that it is additive. If $\varepsilon > 0$ is arbitrary and $\alpha \in \Delta$ is chosen as in (v) (with $A$, $B$ chosen, too) then for $\pi \subset \alpha$, a partition of $I_0$, and

$$\begin{aligned} \pi' &= \{(x, I) \in \pi : H(I) \geq F(x, I)\}, \\ \pi'' &= \{(x, I) \in \pi : H(I) \leq F(x, I)\}, \end{aligned} \tag{1.72}$$

we have

$$\begin{aligned} |H - F|(\pi) = &(H - F)(\pi') - (F - H)(\pi'') \leq \\ &(A - F)(\pi') - (F - B)(\pi'') \leq \\ &(A - B)(\pi') - (A - B)(\pi'') = \\ &(A - B)(\pi) \leq \varepsilon. \end{aligned} \tag{1.73}$$

Thus

$$V(H - F, \Delta(I_0)) = 0. \tag{1.74}$$

(iii)⇒(ii) Let $\varepsilon > 0$. Choose an $\alpha \in \Delta(I_0)$ so that

$$V(H - F, \alpha) < \varepsilon. \tag{1.75}$$

Let $I \subset I_0$, and let $\pi \subset \alpha$ be a partition of $I$. Then by (1.75)

$$|H(I) - F(\pi)| = |(H - F)(\pi)| < \varepsilon. \tag{1.76}$$

This implies that $(\Delta) \int_I F$ exists and is equal to $H(I)$, and also that (ii) is satisfied.

The proof is ended.

Note that neither the statement nor the proof of (iii)⇔(iv)⇔(v) uses the partitioning property of $\Delta$.

**1.6.2. Lemma.** *Let $E_1, E_2 \subset X$ and $F : X \times \Psi \to \mathbf{R}$. Then*

$$F_\Delta(E_1 \cup E_2) \leq F_\Delta(E_1) + F_\Delta(E_2) \tag{1.77}$$

*for any base $\Delta$ which is filtering down.*

**Proof.** This can be proved just as 1.4.5 was — we do not need the assumption of $\Delta$ having a $\sigma$-local character, since there are only two sets considered.

**1.6.3. Lemma.** *Let $\Delta$ have a local character and the partitioning property. Let $\Lambda \in \mathcal{A}^+(I_0)$ for $I_0 \in \Psi$. For an $f : X \to \mathbb{R}$, $H \in \mathcal{A}(I_0)$, $F = f\Lambda$, and*

$$E = \{x \in I_0 : \mathrm{D}_\Delta H_\Lambda(x) = f(x)\} \tag{1.78}$$

*we have*

$$V(H - F, \Delta(I_0)[E]) = 0. \tag{1.79}$$

**Proof.** Let $\varepsilon > 0$. For every $x \in E$ there exists an $\alpha_x \in \Delta$ such that for $(x, I) \in \alpha_x[\{x\}]$

$$\left|\frac{H(I)}{\Lambda(I)} - f(x)\right| < \varepsilon, \tag{1.80}$$

i.e.,

$$|H(I) - F(x, I)| < \varepsilon\Lambda(I). \tag{1.81}$$

Let $\alpha \in \Delta$ be such that for every $x \in E$

$$\alpha[\{x\}] \subset \alpha_x[\{x\}]. \tag{1.82}$$

Let $\pi \subset \alpha[E]$ be a partition. Then from (1.81) and (1.82) we get

$$|H - F|(\pi) \leq \varepsilon\Lambda(\pi) \leq \varepsilon\Lambda(I_0). \tag{1.84}$$

Therefore

$$V(H - F, \alpha) \leq \varepsilon\Lambda(I_0). \tag{1.85}$$

Since $\varepsilon$ is arbitrary, this ends the proof.

**1.6.4. Proposition.** *Let $\Delta$ have a local character and the partitioning property, $I_0 \in \Psi$, $\Lambda \in \mathcal{A}^+(I_0)$, $f : X \to \mathbb{R}$, $H \in \mathcal{A}(I_0)$, and $F = f\Lambda$. Suppose for $E$ defined as is (1.78) we have*

$$V\Big(H - F, \Delta(I_0)[I_0 \setminus E]\Big) = 0. \tag{1.86}$$

*Then $F$ is Henstock integrable on $I_0$ and $H$ is its Henstock integral.*

**Proof.** This is a consequence of 1.6.3, 1.6.2, and 1.6.1.

### 1.7. Additive bases.

**1.7.1. Definition.** We will say that $\Delta$ is *additive* if $V_F \in \mathcal{A}$ for any $F : X \times \Psi \to \mathbf{R}$.

**1.7.2. Definition.** Consider the case when $X = \mathbf{R}^2$. We will say that $\Delta$ is *additive in the sense of Henstock* if for every $I \in \Psi$ and every $\beta = (\beta_1, \beta_2) \in \Delta(I) \times \Delta(X \setminus I^\circ)$ there exists an $\alpha \in \Delta$ such that

$$\alpha \subset \beta_1 \cup \beta_2. \tag{1.87}$$

**1.7.3. Proposition.** *If $\Delta$ is filtering down and additive in the sense of Henstock then it is additive.*

**Proof.** If $I_1$ and $I_2$ do not overlap, then for $\alpha \in \Delta$

$$V(F, \alpha(I_1) \cup \alpha(I_2)) = V(F, \alpha(I_1)) + V(F, \alpha(I_2)) \tag{1.88}$$

so that

$$V(F, \alpha(I_1 \cup I_2)) \geq V(F, \alpha(I_1)) + V(F, \alpha(I_2)). \tag{1.89}$$

Let $\varepsilon > 0$. Choose an $\alpha \in \Delta$ such that

$$V(F, \alpha(I_1 \cup I_2)) \leq V(F, \Delta(I_1 \cup I_2)) + \varepsilon. \tag{1.90}$$

Then

$$\begin{aligned} V(F, \Delta(I_1 \cup I_2)) + \varepsilon \geq & V(F, \alpha(I_1 \cup I_2)) \geq \\ & V(F, \alpha(I_1)) + V(F, \alpha(I_2)) \geq \\ & V(F, \Delta(I_1)) + V(F, \Delta(I_2)). \end{aligned} \tag{1.91}$$

Thus $V_F$ is superadditive (and this is true for any base, not only for one which is additive in the sense of Henstock).

For $\Delta$ additive in the sense of Henstock, we have also

$$V(F, \Delta(I_1 \cup I_2)) \leq V(F, \Delta(I_1)) + V(F, \Delta(I_2)). \tag{1.92}$$

To prove that, take an arbitrary $\varepsilon > 0$. Let $\alpha_1 \in \Delta$ be such that

$$V(F, \alpha_1(I_1)) \le V(F, \Delta(I_1)) + \varepsilon, \tag{1.93}$$

and $\alpha_2 \in \Delta$ such that

$$V(F, \alpha_2(I_2)) \le V(F, \Delta(I_2)) + \varepsilon. \tag{1.94}$$

Choose $\beta_i \in \Delta$, for $i = 1, 2$, such that

$$\beta_i \subset \alpha_i(I_i) \cup \alpha_i(X \setminus I_i^\circ). \tag{1.95}$$

Let $\alpha \in \Delta$ be such that

$$\alpha \subset \beta_1 \cup \beta_2. \tag{1.96}$$

Because of (1.95) and (1.96)

$$\alpha(I_1 \cup I_2) = \alpha(I_1) \cup \alpha(I_2). \tag{1.97}$$

Thus

$$\begin{aligned} V(F, \alpha(I_1 \cup I_2)) =& V(F, \alpha(I_1)) + V(F, \alpha(I_2)) \le \\ & V(F, \beta_1(I_1)) + V(F, \beta_2(I_2)) \le \\ & V(F, \alpha_1(I_1)) + V(F, \alpha_2(I_2)) \le \\ & V(F, \Delta(I_1)) + V(F, \Delta(I_2)) + 2\varepsilon, \end{aligned} \tag{1.98}$$

and this implies (1.92).

**1.7.3. Proposition.** *Suppose $\Delta$ is additive, has the partitioning property, and is filtering down. Let $F$ be integrable on two nonoverlapping $I_1$, $I_2 \in \mathfrak{I}$. Then $F$ is integrable on $I_1 \cup I_2$.*

**Proof.** Let $H$ be an additive function such that

$$V(H - F, \Delta(I_1)) = 0 \tag{1.99}$$

and

$$V(H - F, \Delta(I_2)) = 0 \tag{1.100}$$

— because of additivity of $H$ we may assume that it is the same function on $I_1$ and $I_2$. Since $\Delta$ is additive

$$V(H - F, \Delta(I_1 \cup I_2)) = V(H - F, \Delta(I_1)) + V(H - F, \Delta(I_2)) = 0, \tag{1.101}$$

so that $F$ is integrable on $I_1 \cup I_2$.

*Chapter 2*

# DERIVATION BASES ON THE PLANE

We start by proving that the Henstock integral on the plane is equivalent to the Perron integral.

Then we introduce seven bases on $\mathbf{R}^2$ — $\Delta_1$, $\tilde{\Delta}_1$, $\Delta_1^*$, $\Delta_2$, $\Delta_2^*$, $\Delta_3$, and $\Delta_3^*$, and the Henstock integration theories generated by them.

The star-bases yield the Lebesgue integral.

The bases $\Delta_1$ and $\tilde{\Delta}_1$ generate a theory which is equivalent to the classical Perron integral. The integration process generated by $\Delta_3$ is less general than the one of $\Delta_1$, but it is rotation-invariant ($\Delta_1$-integral is not). $\Delta_2$-integral is more general than $\Delta_1$-integral.

We also consider continuity of interval functions. We show that the common notion of continuity, used in classical monograph of Saks on the theory of integral, is less general than three, equivalent to each other, notions derived from derivation bases.

## 2.1. Perron Integral.

**2.1.1. Theorem.** *Let $X = \mathbf{R}^2$, $f : I_0 \to \mathbf{R}$, $F(x, I) = f(x)\lambda(I)$. Let $\Delta$ be a base on $X$ that is filtering down. The following are equivalent:*
*(i) For every $\varepsilon > 0$ there exist $\alpha \in \Delta$, and $A \in \overline{\mathcal{A}}$, $B \in \underline{\mathcal{A}}$ (depending on $\alpha$) such that*

$$A(I_0) - B(I_0) \le \varepsilon \tag{2.1}$$

*and*

$$A(I) \ge F(x, I) \ge B(I) \qquad \text{for every } (x, I) \in \alpha. \tag{2.2}$$

*(ii) For every $\varepsilon > 0$ there exist $A \in \overline{\mathcal{A}}$, $B \in \underline{\mathcal{A}}$ such that (2.1) holds and*

$$\underline{\mathrm{D}}_{\Delta} A_{\lambda}(x) \geq f(x) \geq \overline{\mathrm{D}}_{\Delta} B_{\lambda}(x) \qquad \text{for } x \in I_0. \tag{2.3}$$

**Proof.** (i)$\Rightarrow$(ii). Let $\alpha, A, B$ be given for $\varepsilon > 0$ as in (i). We have then from (2.2)

$$\inf_{(x,I)\in\alpha} \frac{A(I)}{\lambda(I)} \geq f(x) \geq \sup_{(x,I)\in\alpha} \frac{B(I)}{\lambda(I)}. \tag{2.4}$$

(2.4) implies

$$\sup_{\alpha\in\Delta} \inf_{(x,I)\in\alpha} \frac{A(I)}{\lambda(I)} \geq f(x) \geq \inf_{\alpha\in\Delta} \sup_{(x,I)\in\alpha} \frac{B(I)}{\lambda(I)}, \tag{2.5}$$

i.e., (2.3).

(ii)$\Rightarrow$(i). (2.3) implies the existence of $\alpha'$, $\alpha'' \in \Delta$ such that for all $(x', I') \in \alpha'$, $(x'', I'') \in \alpha''$

$$A(I') + \varepsilon\lambda(I') \geq f(x)\lambda(x) \quad \text{and} \quad B(I'') + \varepsilon\lambda(I'') \leq f(x)\lambda(I''). \tag{2.6}$$

Choose an $\alpha \in \Delta$ such that $\alpha \subset \alpha' \cap \alpha''$. Write

$$\begin{aligned} A_1(I) &= A(I) + \varepsilon\lambda(I), \\ B_1(I) &= B(I) - \varepsilon\lambda(I). \end{aligned} \tag{2.7}$$

Then $A_1 \in \overline{\mathcal{A}}$, $B_1 \in \underline{\mathcal{A}}$ because $\lambda \in \mathcal{A}$. We also have

$$A_1(I) \geq F(x, I) \geq B_1(I) \qquad \text{for } (x, I) \in \alpha, \tag{2.8}$$

$$A_1(I_0) - B_1(I_0) \leq \varepsilon\big(1 + \lambda(I_0)\big). \tag{2.9}$$

This completes the proof.

**2.1.2. Definition.** The functions $A$, $B$ of theorem 2.1.1 are called a *majorant* and a *minorant* (respectively) for $f$. A function $f$ which satisfies the condition (ii) of theorem 2.1.1 is called *Perron-integrable*.

**2.2. Specific bases on the plane.**

**2.2.1. Definition.** Let $\Phi$ stand for the class of all nondegenerate closed intervals in $\mathbf{R}^2$. Take $X = \mathbf{R}^2$ and $\Psi = \Phi$.

Let $\mathcal{P}$ be the class of all real-valued, positive functions on $\mathbf{R}^2$.

The *Kurzweil base* $\Delta_1$ consists of all $\alpha_p$, where $p \in \mathcal{P}$ and

$$\alpha_p = \Big\{(x, I) \in \mathbf{R}^2 \times \Phi : x \in I,\ I \subset D\big(x, p(x)\big)\Big\}. \tag{2.10}$$

If we drop the condition "$x \in I$" in (2.10), we get $\Delta_1^*$, which will be called the *weak Kurzweil base*.

If we replace "$x \in I$" in (2.10) by "$x$ is a vertex of $I$", then we get $\widetilde{\Delta}_1$ which will be called the *modified Kurzweil base*.

**2.2.2. Definition.** Let $I \in \Phi$. We define its *norm* $n(I)$ as the length of its longer side.

If $\{I_1, \ldots, I_n\}$ is a finite subclass of $\Phi$ then the *norm* $n(\{I_1, \ldots, I_n\})$ is defined to be the greatests of all $n(I)$ for $I \in \{I_1, \ldots, I_n\}$.

**2.2.3. Definition.** Let $I \in \Phi$. We define its *regularity* as the number

$$r(I) = \frac{\lambda(I)}{\big(n(I)\big)^2} \tag{2.11}$$

(see [52], p. 106).

It is easy to see that $0 < r(I) \leq 1$.

Let $\varrho \in (0,1)$. We will say that $I$ is $\varrho$-regular if $r(I) \geq \varrho$. And we will write $\Phi_\varrho$ for the class of all elements of $\Phi$ which are $\varrho$-regular.

**2.2.4. Definition.** Now let $\Psi = \Phi_\varrho$, and $X = \mathbf{R}^2$.

For $p \in \mathcal{P}$ let

$$\alpha_p^\varrho = \Big\{(x, I) \in \mathbf{R}^2 \times \Phi_\varrho : x \in I,\ I \subset D\big(x, p(x)\big)\Big\}, \tag{2.12}$$

and

$$\Delta_2^\varrho = \{\alpha_p^\varrho : p \in P\}. \tag{2.13}$$

We will call $\Delta_2^\varrho$ the *Kempisty $\varrho$-base.* If we drop the assumption "$x \in I$" in (2.12), we get the *weak Kempisty $\varrho$-base* $\Delta_2^{\varrho *}$.

We will usually fix a $\varrho \in (0,1)$ and write $\Delta_2$ and $\Delta_2^*$ instead of $\Delta_2^\varrho$ and $\Delta_2^{\varrho *}$.

**2.2.5. Definition.** If we replace intervals in the definition 2.2.1 by triangles (compare this with the work in [42], [43], [45], and [48]), the base so obtained will be called the *Pfeffer base* (*weak Pfeffer base*) and denoted by $\Delta_3$ ($\Delta_3^*$).

The class of all nondegenerate triangles in $\mathbf{R}^2$ will be denoted by $\mathbf{T}$.

## 2.3. Basic properties of the bases defined.

**2.3.1. Observation.** *All of the bases $\Delta_i$, $\Delta_i^*$ $(i = 1,2,3)$ and $\widetilde{\Delta}_1$ are filtering down, have a local character, and ignore no point.*

**2.3.2. Proposition.** *All of the bases $\Delta_i$ and $\Delta_i^*$ $(i = 1,2,3)$ and $\widetilde{\Delta}_1$ have the partitioning property.*

**Proof.** For the base $\Delta_3$, this is proved in [48]. And this implies the assertion for $\Delta_3^*$.

Let $\Delta$ stand for any of the bases $\Delta_1$, $\widetilde{\Delta}_1$, $\Delta_2$, $\Delta_1^*$, $\Delta_2^*$. Let $I_0$ be an element of the corresponding $\Psi$. Suppose that $I_0$ is such that $\Delta$ does not partition $I_0$, i.e., there is an $\alpha \in \Delta$ not partitioning $I_0$.

Divide $I_0$ into four subintervals by splitting its sides into halves. Then $\alpha$ does not partition at least one of so obtained subintervals — call that subinterval $I_1$.

Apply the same procedure to $I_1$, and obtain $I_2 \subset I_1$, etc.

If we stop after a finite number of steps, then $I_0$ will be partitioned.

Otherwise, we obtain a decreasing sequence of intervals $\{I_n\}$, such that neither of them can be partitioned by $\alpha$, and $\lambda(I_n) \to 0$, as $n \to \infty$.

Let $\bigcap_{n\in\mathbb{N}} I_n = \{x_0\}$. There is an $n \in \mathbb{N}$ such that $I_n$ is contained in $D(x_0, p(x_0))$, where $\alpha = \alpha_p$ and $p \in \mathcal{P}$. If $\Delta = \Delta_i$ or $\Delta = \Delta_i^*$ for $i = 1, 2$, then we already get a contradiction, since $\{(x_0, I_n)\}$ is a partition of $I_n$ which is contained in $\alpha$.

If $\Delta = \tilde{\Delta}_1$, then divide $I_n$ into at most four subintervals, each of which has a vertex at $x_0$. Let $\{J_j\}$ be those subintervals ($j$ can run through any of the following sets: $\{1\}$, $\{1,2\}$, $\{1,2,3,4\}$). Then $\{(x_0, J_j)\}$ is a partition of $I_n$ contained in $\alpha$, and that is a contradiction.

**2.3.3. Observation.** *None of the bases $\Delta_i$, $\Delta_i^*$ ($i = 1, 2, 3$) is additive in the sense of Henstock. $\tilde{\Delta}_1$ is additive in the sense of Henstock.*

**2.3.4. Observation.**

$$\begin{gathered} \Delta_i \preceq \Delta_i^* \qquad \text{for } i = 1,2,3, \\ \Delta_1 \succeq \Delta_2, \quad \Delta_1^* \succeq \Delta_2^*, \quad \Delta_1 \succeq \tilde{\Delta}_1. \end{gathered} \tag{2.14}$$

**2.3.5. Proposition.** *Let $f : I_0 \to \mathbb{R}$, $F(x, I) = f(x)\lambda(I)$, for $I_0 \in \Phi$. Then*

$$(\Delta_1)\underline{\int}_{I_0} F = (\tilde{\Delta}_1)\underline{\int}_{I_0} F, \tag{2.15}$$

$$(\Delta_1)\overline{\int}_{I_0} F = (\tilde{\Delta}_1)\overline{\int}_{I_0} F. \tag{2.16}$$

**Proof.** We will show only (2.15), as the proof of (2.16) is similar. Since $\tilde{\Delta}_1 \preceq \Delta_1$, we have

$$(\Delta_1)\underline{\int}_{I_0} F \le (\tilde{\Delta}_1)\underline{\int}_{I_0} F. \tag{2.17}$$

If

$$(\tilde{\Delta}_1)\underline{\int}_{I_0} F = -\infty \tag{2.18}$$

then we already get (2.15). Otherwise, for every $\varepsilon > 0$ there exists an $\tilde{\alpha} \in \tilde{\Delta}_1$ such that for every partition $\tilde{\pi} \subset \tilde{\alpha}$ of $I_0$ we have

$$F(\tilde{\pi}) \geq (\tilde{\Delta}_1) \underline{\int}_{I_0} F - \varepsilon. \tag{2.19}$$

Let $\tilde{\alpha} = \tilde{\alpha}_p$ for a certain $p \in \mathcal{P}$. Take $\alpha \in \Delta_1$ which is generated by the same $p \in \mathcal{P}$. Let $\pi \subset \alpha$. For every $(x, I) \in \pi$ divide $I$ into at most four subintervals having a vertex at $x$. For each of such subintervals $J$ make $(x, J)$ into a pair. By collecting all such pairs for all $(x, I) \in \pi$ we obtain a partition $\tilde{\pi}$ contained in $\tilde{\alpha}$. On the other hand, because $F(x, I) = f(x)\lambda(I)$ and $\lambda$ is additive, we have

$$F(\pi) = F(\tilde{\pi}). \tag{2.20}$$

Combining (2.20) and (2.19) we obtain

$$F(\pi) \geq (\tilde{\Delta}_1) \underline{\int}_{I_0} F - \varepsilon. \tag{2.21}$$

Therefore

$$(\Delta_1) \underline{\int}_{I_0} F \geq (\tilde{\Delta}_1) \underline{\int}_{I_0} F - \varepsilon. \tag{2.22}$$

This, together with (2.17), gives (2.15).

**2.3.6. Corollary.** *Under the same assumptions as in 2.3.5, for every $I \subset I_0$*

$$V\big(F, \Delta_1(I)\big) = V\big(F, \tilde{\Delta}_1(I)\big). \tag{2.23}$$

**Proof.** For $F_1(x, I) = |f(x)|\lambda(I)$ we have

$$\begin{aligned} V\big(F, \Delta_1(I)\big) &= (\Delta_1) \overline{\int}_I F_1, \\ V\big(F, \tilde{\Delta}_1(I)\big) &= (\tilde{\Delta}_1) \overline{\int}_I F_1, \end{aligned} \tag{2.24}$$

so that (2.23) is a consequence of (2.16).

**2.3.7. Corollary.** *$\Delta_1$ is additive.*

**Proof.** This follows from 1.7.3, 2.3.3, and 2.3.6.

**2.3.8. Corollary.** *The Henstock integration theories for functions of the kind presented in 2.3.5 generated by $\Delta_1$ and $\tilde{\Delta}_1$ are not only equivalent to each other, but also equivalent to the classical Perron integral of [34].*

**Proof.** The integral of [34] is defined by the condition (i) of the theorem 2.1.1.

**2.3.9. Definition.** We will say that a derivation base $\Delta$ on $\mathbf{R}^2$ is *compatible with the Euclidean topology* on $\mathbf{R}^2$ if for every set $G$, open in the Euclidean topology on $\mathbf{R}^2$, there exits and $\alpha \in \Delta$ such that $\alpha[G] \subset \alpha(G)$.

**2.3.10. Observation.** *Each of the bases $\Delta_i$, $\Delta_i^*$ $(i = 1, 2, 3)$, $\tilde{\Delta}_1$ is compatible with the Euclidean topology.*

**Proof.** This follows easily from the fact that if $G \subset \mathbf{R}^2$ is open, then for every $x \in G$ there exists a disk $D(x, p(x)) \subset G$.

**2.3.11. Proposition.** *Let $\Delta$ be any of the bases $\Delta_i$, $\Delta_i^*$ $(i = 1, 2, 3)$, or $\tilde{\Delta}_1$. Then for every $E \subset \mathbf{R}^2$*

$$\lambda_\Delta(E) = \lambda(E). \tag{2.25}$$

**Proof.** Let $E \subset \mathbf{R}^2$ be bounded. Since $\Delta$ has the partitioning property, for any $\alpha \in \Delta$ there exists a partition $\pi \subset \alpha[E]$ such that $E \subset \bigcup_{(x,I)\in\pi} I$. We have then

$$\sum_{(x,I)\in\pi} \lambda(I) \geq \lambda(E). \tag{2.26}$$

Consequently

$$\lambda_\Delta(E) \geq \lambda(E). \tag{2.27}$$

If $E$ is unbounded, (2.27) is true, as well, since both $\lambda$ and $\lambda_\Delta$ are outer measures.

Now assume again that $E$ is bounded. $\Delta$ is compatible with the Euclidean topology. If $G$ is an arbitrary open set containing $E$, then we can choose an

$\alpha \in \Delta$ such that

$$\alpha[G] \subset \alpha(G). \tag{2.28}$$

(2.28) implies that

$$\lambda_\Delta(G) \leq \lambda(G). \tag{2.29}$$

Since $G$ was an arbitrary open set containing $E$, (2.29) implies

$$\lambda_\Delta(E) \leq \inf_G \lambda_\Delta(G) \leq \inf_G \lambda(G) = \lambda(E). \tag{2.30}$$

The case when $E$ is unbounded may be handled as before.

The proof is ended.

### 2.4. Absolute Integration.

**2.4.1.** Let $I_0 \in \Phi$, $f : I_0 \to \mathbb{R}$, and $F(x,I) = f(x)\lambda(I)$. If $\Delta$ is a derivation base, and $(\Delta)\int_{I_0} F$ exists, we will write $(\Delta)\int_{I_0} f d\lambda$ for that integral. We will also say that $f$ is $\Delta$-integrable, if $F$ is so.

**2.4.2. Definition.** Let $\Delta$ be any of the bases $\Delta_i$, $\Delta_i^*$ $(i = 1,2,3)$, or $\tilde{\Delta}_1$. Let $\alpha \in \Delta$ and let $\pi_1$, $\pi_2 \subset \alpha$ be partitions of an interval $I_0$. Write $\mathcal{D}_1$, $\mathcal{D}_2$ for the corresponding divisions of $I_0$. Let $\mathcal{D}$ be a division that refines both $\mathcal{D}_1$ and $\mathcal{D}_2$ (we form it, of course, with the elements of the corresponding $\Psi$). Set

$$\begin{aligned} \pi^1 &= \{(x,I) : I \in \mathcal{D} \quad \text{and} \quad \exists\,(x_1,I_1) \in \pi_1 \text{ such that } x = x_1, I \subset I_1\}, \\ \pi^2 &= \{(x,I) : I \in \mathcal{D} \quad \text{and} \quad \exists\,(x_2,I_2) \in \pi_2 \text{ such that } x = x_2, I \subset I_2\}. \end{aligned} \tag{2.31}$$

Then $\pi^1$ will be called a $\pi_2$-*refinement* of $\pi_1$, and $\pi^2$ — a $\pi_1$-*refinement* of $\pi_2$.

The division $\mathcal{D}$ will be called a *refining division* for $\pi_1$ and $\pi_2$.

We are going to use the notions defined only for the bases listed. But the refinements may be defined for other bases as well, as long as a refining division exists.

**2.4.3. Definition.** We will say that a base $\Delta$ is *refining* if for any $\alpha \in \Delta$, $\pi_1$, $\pi_2 \subset \alpha$ — partitions of an $I_0 \in \Psi$, and $\pi^1$ — a $\pi_2$-refinement of $\pi_1$, $\pi^2$ — a $\pi_1$-refinement of $\pi_2$, we have $\pi^1 \subset \alpha$ and $\pi^2 \subset \alpha$.

**2.4.4. Lemma.** *If $I \in \Phi$ and $\varrho \in (0,1)$ then there exists a division $D$ of $I$ with elements of $\Phi_\varrho$.*

**Proof.** Let $I = [a_1, b_1] \times [a_2, b_2]$. Choose a $\xi > 0$ such that

$$\left(\frac{1-\xi}{1+\xi}\right)^2 > \varrho. \tag{2.32}$$

Then find integers $\varsigma_1$, $\varsigma_2$ such that

$$\left|\frac{b_2 - a_2}{b_1 - a_1}\frac{\varsigma_1}{\varsigma_2} - 1\right| < \xi. \tag{2.33}$$

Set

$$\begin{aligned} I_{ij} = &\left[a_1 + i\frac{b_1 - a_1}{\varsigma_1}, a_1 + (i+1)\frac{b_1 - a_1}{\varsigma_1}\right] \times \\ &\left[a_2 + j\frac{b_2 - a_2}{\varsigma_2}, a_2 + (j+1)\frac{b_2 - a_2}{\varsigma_2}\right] \end{aligned} \tag{2.34}$$

for $i \leq \varsigma_1 - 1$ and $j \leq \varsigma_2 - 1$.

Clearly, the intervals $I_{ij}$ form a division of $I$ and

$$\begin{aligned} r(I_{ij}) = &\frac{\frac{b_1-a_1}{\varsigma_1}\frac{b_2-a_2}{\varsigma_2}}{n(I_{ij})^2} \geq \\ &\frac{\left(\min\left(\frac{b_1-a_1}{\varsigma_1}, \frac{b_2-a_2}{\varsigma_2}\right)\right)^2}{\left(\max\left(\frac{b_1-a_1}{\varsigma_1}, \frac{b_2-a_2}{\varsigma_2}\right)\right)^2} \geq \left(\frac{1-\xi}{1+\xi}\right)^2 > \varrho. \end{aligned} \tag{2.35}$$

**2.4.5. Corollary.** *If $I \in \Phi$ and $\varrho \in (0,1)$, then for every $\varepsilon > 0$ there exists a division $D$ of $I$, $D \subset \Phi_\varrho$ such that $n(D) < \varepsilon$.*

**Proof.** First divide $I$ into subintervals of small norm, and then apply 2.4.4 to divide those with $\varrho$-regular intervals.

**2.4.6. Corollary.** *Refining divisions exist for all the bases on $\mathbf{R}^2$ defined in this chapter.*

**Proof.** The only nonobvious cases are $\Delta_2$ and $\Delta_2^*$ — but then 2.4.6 follows from 2.4.5.

**2.4.7. Observation.** *$\Delta_1^*$, $\Delta_2^*$, $\Delta_3^*$ are refining. $\Delta_1$, $\tilde{\Delta}_1$, $\Delta_2$, and $\Delta_3$ are not.*

**Proof.** It suffices to note that for $\alpha^* \in \Delta_1^*$, $\Delta_2^*$, or $\Delta_3^*$, $(x, I) \in \alpha^*$, and $I' \subset I$, we have $(x, I') \in \alpha^*$. And this does not hold for $\Delta_1$, $\tilde{\Delta}_1$, $\Delta_2$, and $\Delta_3$.

**2.4.8. Lemma.** *Let $f$ be a function as in 2.4.1. The following conditions are sufficient for $f$ to be $\Delta$-integrable on $I_0$:*

*(i) For every $\varepsilon > 0$ there exists an $\alpha \in \Delta$ such that if $\pi_1$, $\pi_2 \subset \alpha$ are partitions of $I_0$ and $D$ is their refining division then*

$$\left| \sum_{(x_1,I_1)\in\pi_1} \sum_{(x_2,I_2)\in\pi_2} \sum_{\substack{I\in D \\ I\subset I_1, I\subset I_2}} \Big(f(x_1) - f(x_2)\Big)\lambda(I) \right| < \varepsilon. \tag{2.36}$$

*(ii) For every $\varepsilon > 0$ there exists an $\alpha \in \Delta$ such that if $\pi_1$, $\pi_2 \subset \alpha$ are partitions of $I_0$ and $D$ is their refining division then*

$$\sum_{(x_1,I_1)\in\pi_1} \sum_{(x_2,I_2)\in\pi_2} \sum_{\substack{I\in D \\ I\subset I_1, I\subset I_2}} |f(x_1) - f(x_2)|\lambda(I) < \varepsilon. \tag{2.37}$$

**Proof.** (ii) implies (i), so it suffices to show that (i) implies integrability. In fact, (i) is equivalent to integrability, as by additivity of $\lambda$ we have

$$\begin{gathered} \sum_{(x_1,I_1)\in\pi_1} \sum_{(x_2,I_2)\in\pi_2} \sum_{\substack{I\in D \\ I\subset I_1, I\subset I_2}} (f(x_1) - f(x_2))\lambda(I) = \\ \sum_{(x_1,I_1)\in\pi_1} f(x_1)\lambda(I_1) - \sum_{(x_2,I_2)\in\pi_2} f(x_2)\lambda(I_2). \end{gathered} \tag{2.38}$$

**2.4.9. Lemma.** *If the condition (ii) of 2.4.8 holds then $|f|$ is also $\Delta$-integrable.*

**Proof.** This is obvious, because

$$\big||f(x_1)| - |f(x_2)|\big| \le |f(x_1) - f(x_2)|. \tag{2.39}$$

**2.4.10. Theorem.** *Let $\Delta$ be a refining base, and $f$ as in 2.4.1. In order for $f$ to be $\Delta$-integrable it is necessary and sufficient that the condition (ii) of lemma 2.4.8 is satisfied.*

**Proof.** It is enough to prove necessity, and in order to do that we will show that the condition (i) of lemma 2.4.8 implies the condition (ii) in this case. Let $\alpha$, $\pi_1$, and $\pi_2$ be given as in (i), and let $\pi^1$, $\pi^2$ be their refinements. Define

$$\begin{aligned} \sigma_1 = \{(x,I) : \exists (x_1, I_1) \in \pi^1 \quad \exists (x_2, I_2) \in \pi^2 \quad \text{such that} \quad I_1 = I_2 = I, \\ \text{and} \quad x = x_1 \quad \text{if} \quad f(x_1) \geq f(x_2), \\ x = x_2 \quad \text{otherwise}\}, \end{aligned} \tag{2.40}$$

$$\begin{aligned} \sigma_2 = \{(x,I) : \exists (x_1, I_1) \in \pi^1 \quad \exists (x_2, I_2) \in \pi^2 \quad \text{such that} \quad I_1 = I_2 = I, \\ \text{and} \quad x = x_2 \quad \text{if} \quad f(x_2) \geq f(x_1), \\ x = x_1 \quad \text{otherwise}\}. \end{aligned} \tag{2.41}$$

Because $\Delta$ is refining, $\sigma_1$ and $\sigma_2$ are contained in $\alpha$, so that for $F(x,I) = f(x)\lambda(I)$

$$|F(\sigma_1) - F(\sigma_2)| < \varepsilon. \tag{2.42}$$

On the other hand

$$\begin{aligned} |F(\sigma_1) - F(\sigma_2)| = \\ \sum_{(x,I)\in\pi_1} \sum_{(x,I)\in\pi_2} \sum_{\substack{I\in\mathcal{D} \\ I\subset I_1, I\subset I_2}} |f(x_1) - f(x_2)|\lambda(I), \end{aligned} \tag{2.43}$$

and this, together with (2.42), gives (ii).

**2.4.11. Definition.** Let $I_0 \in \Phi$ and $f : I_0 \to \mathbb{R}$. Write $f^+ = \frac{1}{2}(f + |f|)$ and $f^- = f^+ - f$.

**2.4.12. Corollary.** *If $\Delta$ is refining then $\Delta$-integrability of $f$ implies $\Delta$-integrability of $|f|$.*

**Proof.** This follows from 2.4.9 and 2.4.10.

**2.4.13. Corollary.** *If $\Delta$ is refining, then $f$ is $\Delta$-integrable if and only if both $f^+$ and $f^-$ are $\Delta$-integrable.*

**Proof.** This follows from 2.4.12, we just need to note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

**2.4.14. Corollary.** *For $i = 1, 2, 3$, $f$ is $\Delta_i^*$-integrable if and only if both $f^+$ and $f^-$ are $\Delta_i^*$-integrable.*

## 2.5. Lebesgue integral.

**2.5.1. Proposition.** *A function $f$ is $\Delta_1^*$-integrable on $I_0 \in \Phi$ if and only if it is Lebesgue-integrable on $I_0$.*

**Proof.** This is shown in [38] (theorem 13.6, p. 45).

**2.5.2. Lemma.** *Let $f$ be as in 2.4.1. We have:*

$$(\Delta_3^*) \overline{\int}_{I_0} f d\lambda \geq (\Delta_1^*) \overline{\int}_{I_0} f d\lambda \geq (\Delta_2^*) \overline{\int}_{I_0} f d\lambda \tag{2.44}$$

*and*

$$(\Delta_3^*) \underline{\int}_{I_0} f d\lambda \leq (\Delta_1^*) \underline{\int}_{I_0} f d\lambda \leq (\Delta_2^*) \underline{\int}_{I_0} f d\lambda. \tag{2.45}$$

**Proof.** It suffices to show (2.44). The inequality

$$(\Delta_1^*) \overline{\int}_{I_0} f d\lambda \geq (\Delta_2^*) \overline{\int}_{I_0} f d\lambda \tag{2.46}$$

is obvious, since $\Delta_2^* \preceq \Delta_1^*$.

We can assume that $(\Delta_1^*) \overline{\int}_{I_0} f d\lambda > -\infty$, because otherwise there is nothing to prove. To show

$$(\Delta_3^*) \overline{\int}_{I_0} f d\lambda \geq (\Delta_1^*) \overline{\int}_{I_0} f d\lambda, \tag{2.47}$$

take an $\varepsilon > 0$ and find an $\alpha_3 \in \Delta_3^*$ such that for every $\pi_3 \subset \alpha_3$, a partition of $I_0$, we have

$$(\Delta_3^*)\overline{\int}_{I_0} f d\lambda + \varepsilon \geq \sum_{(x,I)\in\pi_3} f(x)\lambda(I). \tag{2.48}$$

Let $\alpha_3$ be generated by a $p \in \mathcal{P}$. Take an $\alpha_1 \in \Delta_1^*$ which is given by the same $p \in \mathcal{P}$. Let $\pi_1 \subset \alpha_1$ be a partition of $I_0$. For every $(x, I) \in \pi_1$, $I$ can be divided into at most four triangles with a common vertex at $x$. By assigning $x$ to each of those triangles, we obtain a partition $\pi_3 \subset \alpha_3$. Therefore (2.48) holds for that partition. On the other hand, by additivity of $\lambda$

$$\sum_{(x,I)\in\pi_3} f(x)\lambda(I) = \sum_{(x,I)\in\pi_1} f(x)\lambda(I), \tag{2.49}$$

so that

$$(\Delta_3^*)\overline{\int}_{I_0} f d\lambda + \varepsilon \geq \sum_{(x,I)\in\pi_1} f(x)\lambda(I). \tag{2.50}$$

Since $\pi_1 \subset \alpha_1$ was arbitrary, we obtain

$$(\Delta_3^*)\overline{\int}_{I_0} f d\lambda + \varepsilon \geq (\Delta_1^*)\overline{\int}_{I_0} f d\lambda \tag{2.51}$$

which implies (2.47).

**2.5.3. Theorem.** *Under the hypotheses of 2.5.2 we have:*

$$(\Delta_1^*)\overline{\int}_{I_0} f d\lambda = (\Delta_3^*)\overline{\int}_{I_0} f d\lambda, \tag{2.52}$$

*and*

$$(\Delta_1^*)\underline{\int}_{I_0} f d\lambda = (\Delta_3^*)\underline{\int}_{I_0} f d\lambda. \tag{2.53}$$

**Proof.** It suffices to show that

$$(\Delta_1^*)\overline{\int}_{I_0} f d\lambda \leq (\Delta_3^*)\overline{\int}_{I_0} f d\lambda. \tag{2.54}$$

If

$$(\Delta_1^*)\overline{\int}_{I_0} f d\lambda = -\infty \tag{2.55}$$

then (2.54) is obvious. Otherwise, for an $\varepsilon > 0$ take an $\alpha_2 \in \Delta_2^*$ such that for every partition $\pi_2 \subset \alpha_2$ of $I_0$ we have

$$(\Delta_2^*) \overline{\int_{I_0}} f d\lambda + \varepsilon \geq \sum_{(x,I)\in\pi_2} f(x)\lambda(I). \tag{2.56}$$

Let $\alpha_2$ be generated by a $p \in \mathcal{P}$. Take an $\alpha_1 \in \Delta_1^*$ given by the same $p$. Let $\pi_1 \subset \alpha_1$ be an arbitrary partition of $I_0$. For $(x, I) \in \pi_1$, $I$ can be divided into a finite number of $\varrho$-regular intervals (see lemma 2.4.4). Let $J_1^x, \ldots, J_{k_x}^x$ be those intervals. Then

$$\pi_2 = \bigcup_{(x,I)\in\pi_1} \bigcup_{i=1}^{k_x} \{(x, J_i^x)\} \tag{2.57}$$

is a partition of $I_0$ contained in $\alpha_2$. Therefore (2.56) is satisfied. On the other hand

$$\sum_{(x,I)\in\pi_2} f(x)\lambda(I) = \sum_{(x,I)\in\pi_1} f(x)\lambda(I), \tag{2.58}$$

so that

$$(\Delta_2^*) \overline{\int_{I_0}} f d\lambda + \varepsilon \geq \sum_{(x,I)\in\pi_1} f(x)\lambda(I). \tag{2.59}$$

Since $\pi_1$ and $\varepsilon$ are arbitrary, this implies (2.54).

**2.5.4. Proposition.** *A function $f$ is $\Delta_1^*$-integrable if and only if it is $\Delta_3^*$-integrable, with both integrals equal when they exist.*

**Proof.** By corollary 2.4.14 we can assume that $f$ is nonnegative. And by lemma 2.5.2 it suffices to show that every $\Delta_1^*$-integrable function is also $\Delta_3^*$-integrable.

Let $f$ be $\Delta_1^*$-integrable. Let $\varepsilon > 0$. By théorem 1.6.1 we can choose an $\alpha_1 \in \Delta_1^*$ so that for every $I' \in \Phi_+$, $I' \subset I_0$, and $\pi_1 \subset \alpha_1$, a partition of $I'$, we have

$$\left| \sum_{(x,I)\in\pi_1} f(x)\lambda(I) - (\Delta_1^*) \int_{I'} f d\lambda \right| < \frac{1}{4}\varepsilon. \tag{2.60}$$

And by proposition 2.5.1 there exists an $\eta > 0$ such that if $I' \in \Phi_+$ and $\lambda(I') < \eta$ then

$$(\Delta_1^*) \int_{I'} f d\lambda < \frac{1}{4}\varepsilon. \tag{2.61}$$

Let $p \in \mathcal{P}$ be the function generating $\alpha_1$, and let $\alpha_3$ be an element of $\Delta_3^*$ which is given by the same $p$. Let $\pi_3 \subset \alpha_3$ be an arbitrary partition of $I_0$. Take an arbitrary $(x, I) \in \pi_3$. $I$ is a triangle. We can find a finite number of intervals $I_1^x, \dots, I_{k_x}^x$ contained in the interior of $I$ which are nonoverlapping and such that if $s$ is the number of elements of $\pi_3$, we have

$$f(x)\lambda\Big(I \setminus \bigcup_{i=1}^{k_x} I_i^x\Big) < \frac{\varepsilon}{4s} \tag{2.62}$$

and

$$\lambda\Big(I \setminus \bigcup_{i=1}^{k_x} I_i^x\Big) < \frac{\eta}{s}. \tag{2.63}$$

Let

$$\pi_1' = \bigcup_{(x,I)\in\pi_3} \bigcup_{i=1}^{k_x} \{(x, I_i^x)\}. \tag{2.64}$$

Then $\pi_1'$ is a partition of $\sigma(\pi_1') \in \Phi_+$ and $\pi_1' \subset \alpha_1$. Furthermore

$$\lambda\Big(I_0 \setminus \sigma(\pi_1')^\circ\Big) < \eta. \tag{2.65}$$

Choose $\pi_1''$, a partition of $I_0 \setminus \sigma(\pi_1')^\circ$ (an element of $\Phi_+$), $\pi_1'' \subset \alpha_1$. We have then

$$\Big|\sum_{(x,I)\in\pi_3} f(x)\lambda(I) - \sum_{(x,I)\in\pi_1'} f(x)\lambda(I)\Big| < \frac{1}{4}\varepsilon \qquad \text{by (2.62)}, \tag{2.66}$$

$$\Big|\sum_{(x,I)\in\pi_1''} f(x)\lambda(I) - (\Delta_3^*)\int_{I_0\setminus\sigma(\pi_1')^\circ} f\,d\lambda\Big| < \frac{1}{4}\varepsilon \qquad \text{by (2.60)}, \tag{2.67}$$

$$\int_{I_0\setminus\sigma(\pi_1')^\circ} f\,d\lambda < \frac{1}{4}\varepsilon \qquad \text{by (2.61) and (2.65)}, \tag{2.68}$$

$$\Big|(\Delta_1^*)\int_{I_0} f\,d\lambda - \sum_{(x,I)\in\pi_1'\cup\pi_1''} f(x)\lambda(I)\Big| < \frac{1}{4} \qquad \text{by (2.60)}. \tag{2.69}$$

Combining (2.66), (2.67), (2.68), and (2.69) we get

$$\Big|\sum_{(x,I)\in\pi_3} f(x)\lambda(I) - (\Delta_3^*)\int_{I_0} f\,d\lambda\Big| < \varepsilon, \tag{2.70}$$

so that $f$ is $\Delta_3^*$-integrable, as desired.

**2.5.5. Theorem.** *The following are equivalent:*

*(i) $f$ is Lebesgue-integrable on $I_0$;*

*(ii) $f$ is $\Delta_1^*$-integrable on $I_0$;*

*(iii) $f$ is $\Delta_2^*$-integrable on $I_0$;*

*(iv) $f$ is $\Delta_3^*$-integrable on $I_0$.*

*The integrals are equal if they exist.*

**Proof.** This follows from 2.5.1, 2.5.3, and 2.5.4.

## 2.6. Comparison of nonabsolute integrals.

**2.6.1. Observation.** *Let $f : I_0 \to \mathbb{R}$ for an $I_0 \in \Phi$. Then*

$$(\Delta_3)\overline{\int}_{I_0} f d\lambda \geq (\Delta_1)\overline{\int}_{I_0} f d\lambda \geq (\Delta_2)\overline{\int}_{I_0} f d\lambda \tag{2.71}$$

*and*

$$(\Delta_3)\underline{\int}_{I_0} f d\lambda \leq (\Delta_1)\underline{\int}_{I_0} f d\lambda \leq (\Delta_2)\underline{\int}_{I_0} f d\lambda \tag{2.72}$$

**Proof.** This can be shown in the manner similar to the one used in the proof of lemma 2.5.2.

**2.6.2. Definition.** We will say that a base $\Delta$ on $\mathbb{R}^2$ is ***rotation invariant*** if for every $\Delta$-integrable function $f$ and a rotation $R$ of the plane, $f \circ R$ is $\Delta$-integrable as well.

It is easy to see that the class $\Upsilon$ has the property: $R(\Upsilon) = \Upsilon$ for any rotation $R$.

Therefore, the $\Delta_3$- and $\Delta_3^*$-integrals are rotation-invariant. This way we also obtain a surprizing proof of the well-known fact that the Lebesgue integral is rotation-invariant. But the part which is more important now is the following.

**2.6.3. Observation.** *The $\Delta_3$-integral is rotation-invariant.*

**2.6.4. Example.** Let $I_0 = [0,1] \times [0,1]$,

$$a_n = 1 - \frac{1}{2^n} \qquad \text{for } n = 0, 1, 2, \ldots, \tag{2.73}$$

$$b_n = \frac{1}{n} \qquad \text{for } n = 1, 2, \ldots, \tag{2.74}$$

and for $n \in \mathbf{N}$

$$K_n = [a_{n-1}, a_n] \times [a_{n-1}, a_n], \tag{2.75}$$

$$L_n = \{(u,v) \in K_n : v \leq u\}. \tag{2.76}$$

For each $n \in \mathbf{N}$ construct a function $f_n : K_n \to \mathbf{R}$ such that

(i) $f_n$ is continuous on $K_n$, and $f_n = 0$ on $\partial K_n$;

(ii) $f_n \geq 0$ on $L_n$;

(iii) $f_n(u,v) = -f_n(v,u)$ for every $(u,v) \in K_n$;

(iv) $\int_{L_n} f_n(u,v)\,du\,dv = b_n$ (Lebesgue integral).

If we now define $f : I_0 \to \mathbf{R}$ by

$$f(u,v) = \begin{cases} f_n(u,v) & \text{if } (u,v) \in K_n, \quad n \in \mathbf{N}; \\ 0 & \text{otherwise} \end{cases} \tag{2.77}$$

then it is not hard to show that

(a) $f$ is not Lebesgue-integrable on $I_0$;

(b) $f$ is $\Delta_1$-integrable on $I_0$ (we apply theorem 8.9 from [30] when showing this);

(c) if a rotation by $\frac{\pi}{4}$ is applied, then $f$ fails to be $\Delta_1$-integrable.

**2.6.5. Corollary.** *The $\Delta_1$-integral is not rotation-invariant.*

**2.6.6. Corollary.** *$\Delta_3$-integral is strictly less general than $\Delta_1$-integral.*

**2.6.7. Remark.** $\Delta_1$-integral is less general than $\Delta_2$-integral. It will be shown in chapter 3 (example 3.4.1) that they are not the same.

## 2.7. Differentiation of integrals.

**2.7.1. Proposition.** *Let $f$ be $\Delta_2$-integrable on $I_0$, let*

$$H(I) = (\Delta_2)\int_I f d\lambda \tag{2.78}$$

*for $I \subset I_0$, $I \in \Phi$. Then*

$$D_{\Delta_2} H_\lambda(x) = f(x) \qquad \text{a.e. on } I_0. \tag{2.79}$$

**Proof.** Let

$$E_1 = \{x \in I_0 : D_{\Delta_2} H_\lambda(x) = f(x)\}, \tag{2.80}$$

and $E = I_0 \setminus E_1$.

If $x \in E$ then there exists an $\epsilon_x$ such that for every $\alpha \in \Delta_2$ there is an $(x, I) \in [\{x\}]$ such that

$$\left|\frac{H(I)}{\lambda(I)} - f(x)\right| \geq \epsilon_x. \tag{2.81}$$

For $n = 2, 3, 4, \ldots$ let

$$E_n = \left\{x \in E : \epsilon_x \geq \frac{1}{n}\right\}. \tag{2.82}$$

Choose an $\varepsilon \in \left(0, \frac{1}{n}\right)$ and find an $\alpha_0 \in \Delta_2$ such that for every $\pi \subset \alpha_0$, a partition of $I_0$,

$$\sum_{(x,I)\in\pi} |f(x)\lambda(I) - H(I)| < \frac{\varepsilon}{n}. \tag{2.83}$$

Let $\mathcal{R}$ be the family of all $\varrho$-regular intervals $I \subset I_0$ such that $(x_I, I) \in \alpha_0[E_n]$ for some $x_I$.

By (2.81), $\mathcal{R}$ covers $E_n$ in the sense of Vitali. By the Vitali Covering Theorem (see [52], theorem 4.3.1, p. 109) there exists a sequence $\{I_k\}$ of disjoint intervals such that

$$\lambda\Big(E_n \setminus \sum_{k\in\mathbb{N}} I_k\Big) = 0. \tag{2.84}$$

Since $\Delta_2$ has the partitioning property, every

$$\{(x_I, I_1), \ldots, (x_{I_k}, I_k)\} \tag{2.85}$$

is a subset of some partition $\pi \subset \alpha_0$ of $I_0$, so that by (2.81), (2.82), and (2.83)

$$\sum_{i=1}^{k} \lambda(I_i) \leq n \sum_{i=1}^{k} \left| f(x_{I_i})\lambda(I_i) - H(I_i) \right| < \varepsilon. \tag{2.86}$$

Since $\varepsilon$ was arbitrary, $\lambda(E_n) = 0$ for every $n \geq 2$, and consequently $\lambda(E) = 0$.

**2.7.2. Remark.** The analogue of 2.7.1 for the $\Delta_1$ base does not hold. In fact, we have the following classical "rarity" theorem of Saks (see [51]).

**Theorem.** Consider the class $\mathcal{L}$ of all Lebesgue-integrable functions $f$ on $[0,1] \times [0,1]$, with its natural topology (making it into a Banach space). For $f \in \mathcal{L}$, and $I \subset [0,1] \times [0,1]$, $I \in \Phi$ let

$$H_f(I) = \int_I f(u,v)dudv \qquad \text{(Lebesgue integral).} \tag{2.87}$$

Then the class $\mathcal{F}$ of all $f \in \mathcal{L}$ such that

$$\overline{D}_{\Delta_1} H_f(x) < +\infty \qquad \text{at some point } x \in [0,1] \times [0,1] \tag{2.88}$$

is of the first category in $\mathcal{L}$.

Proof of this theorem is given in [12] (theorem 4.2.1).

As we can see now, since $\Delta_1$ does not even differentiate the Lebesgue integral, it does not differentiate the more general $\Delta_1$-integral.

## 2.8. Continuity of interval functions.

**2.8.1. Definition.** If $\Delta$ is a derivation base and $H : \Psi \to \mathbf{R}$ then we say that $H$ is *$\Delta$-continuous at $x \in X$* if

$$V\Big(H, \Delta\big[\{x\}\big]\Big) = 0. \tag{2.89}$$

We say that $H$ is *$\Delta$-continuous on a set $E$* if it is $\Delta$-continuous at every point of $E$.

**2.8.2. Observation.** *Let $\Delta$ be a derivation base on $\mathbf{R}^2$ with the partitioning property and filtering down. Suppose that $F(x,I) = f(x)\lambda(I)$ is $\Delta$-integrable on $I_0 \in \Psi$ and $H$ is its integral. Then $H - f\lambda$ is $\Delta$-continuous on $I_0$.*

**Proof.** This is a consequence of the theorem 1.6.1 (part (iii)).

**2.8.3. Corollary.** *Let $\Delta$ be any of the bases $\Delta_i$, $\Delta_i^*$ $(i = 1,2,3)$, or $\widetilde{\Delta}_1$. If $f : I_0 \to \mathbf{R}$ and we define for $I \subset I_0$*

$$H(I) = (\Delta)\int_I f d\lambda \tag{2.90}$$

*then $H$ is $\Delta$-continuous on $I_0$.*

**Proof.** It is not hard to see that $f\lambda$ is $\Delta$-continuous, so that 2.8.3 follows from 2.8.2.

**2.8.4. Definition.** A function $H : \Phi \to \mathbf{R}$ is *continuous in the sense of Saks* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|H(I)| < \varepsilon$ whenever $\lambda(I) < \delta$.

If we consider $H$ as defined only for $I \subset I_0$ for a certain $I_0 \in \Phi$ then we call it *continuous in the sense of Saks in* $I_0$.

We will say that $H$ is *$\varrho$-regularly continuous in the sense of Saks* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|H(I)| < \varepsilon$ whenever $\lambda(I) < \delta$ and $I \in \Phi_\varrho$.

These definitions are based on the definition of continuity in [52] (p. 59).

**2.8.5. Definition.** A function $H : \Phi \to \mathbf{R}$ is *metrically continuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $I_1$, $I_2 \in \Phi$, and $\lambda(I_1 \Delta I_2) < \delta$, then $|H(I_1) - H(I_2)| < \varepsilon$.

Just as in 2.8.4, we can talk about functions which are *metrically continuous in* a certain $I_0 \in \Phi$.

**2.8.6. Definition.** $H : \Phi \to \mathbf{R}$ will be termed *$\Delta$-continuous in the sense of Burkill at $x$* if for every $\varepsilon > 0$ there exists an $\alpha \in \Delta$ such that if $(x, I) \in \alpha[\{x\}]$

then $|H(I)| < \varepsilon$. $H$ is *$\Delta$-continuous in the sense of Burkill on a set $E$* if it is $\Delta$-continuous in the sense of Burkill at every point of $E$.

This definition is based on the definition of continuity of interval functions in [4].

**2.8.7. Proposition.** *Let $H : \Phi \to \mathbb{R}$ be additive. It is continuous in the sense of Saks in $I_0$ if and only if it is metrically continuous.*

**Proof.** First suppose $H$ is metrically continuous. Let $\varepsilon > 0$ be arbitrary and $\delta$ be the number given by the metrical continuity. Take an $I$ such that $\lambda(I) < \delta$. Extend it in one direction so that to obtain an interval $I_1$ containing $I$ and such that

$$I_2 = (I_1 \setminus I)^- \tag{2.91}$$

is also an interval. Then we have

$$I_1 \Delta I_2 = I_1 \setminus I_2, \tag{2.92}$$

and thus

$$\lambda(I_1 \Delta I_2) < \delta. \tag{2.93}$$

Consequently

$$|H(I_1) - H(I_2)| < \varepsilon. \tag{2.94}$$

But

$$H(I_1) - H(I_2) = H(I) \tag{2.95}$$

by additivity of $H$. Therefore $H$ is continuous in the sense of Saks.

Now assume that $H$ is continuous in the sense of Saks. Let $\varepsilon > 0$, find a $\delta$ such that

$$H(I) < \frac{1}{6} \qquad \text{whenever} \quad \lambda(I) < \delta. \tag{2.95}$$

If we have two intervals $I_1$, $I_2$, then $I_1 \Delta I_2$ can be written as a union of at most six nonoverlapping intervals $J_i$. Now suppose $I_1$, $I_2 \in \Phi$ are chosen so that

$\lambda(I_1 \triangle I_2) < \delta$. Then for $I_1 \triangle I_2 = \bigcup_i J_i$, we have $\lambda(J_i) \le \lambda(I_1 \triangle I_2) < \delta$ for every $i$. Consequently

$$|H(I_1) - H(I_2)| = \Big|\sum_i \theta_i H(J_i) + H(I_1 \cap I_2) - H(I_1 \cap I_2)\Big| \le \sum_i |H(J_i)| < 6\frac{\varepsilon}{6} = \varepsilon \tag{2.96}$$

(where $\theta_i = \pm 1$). Thus $H$ is metrically continuous.

**2.8.8. Proposition.** *If $H : \Phi \to \mathbf{R}$ is additive then it is continuous in the sense of Saks in $I_0$ if and only if for every $\varepsilon > 0$ and every $x \in I_0$ there exists an $\eta(x) > 0$ such that whenever $I \in \Phi$, $x \in I$, and $n(I) < \eta(x)$ then $|H(I)| < \varepsilon$.*

**Proof.** This is shown in [4].

**2.8.9. Proposition.** *Let $H : \Phi \to \mathbf{R}$ be additive. The following are equivalent:*
*(i) $H$ is $\Delta_1(I_0)$-continuous.*
*(ii) $H$ is $\Delta_1(I_0)$-continuous in the sense of Burkill on $I_0$.*
*(iii) $H$ is $\Delta_1^*(I_0)$-continuous in the sense of Burkill on $I_0$.*

**Proof.** (i)⇒(ii). This is obvious.

(ii)⇒(iii). We wil show that $\Delta_1(I_0)$-continuity in the sense of Burkill at a point $x$ implies $\Delta_1^*(I_0)$-continuity at the same point. Let $\varepsilon > 0$. Choose $p(x) > 0$ so that if $I \subset D(x, p(x))$, and $x \in I$, then $|H(I)| < \frac{1}{4}\varepsilon$.

Take an arbitrary $I' \subset D(x, p(x))$ and if it does not contain $x$, consider $I$ — the smallest interval containing both $x$ and $I'$. It is easy to see that $I \subset D(x, p(x))$. We have either $I = I' \cup I_1$ for some $I_1$ containing $x$, nonoverlapping with $I'$, and contained in $D(x, p(x))$, or $I = I' \cup I_1 \cup I_2$ for some $I_1$, $I_2$ containing $x$, nonoverlapping with $I'$, and contained in $D(x, p(x))$.

In the first case

$$|H(I')| = |H(I) - H(I_1)| \le 2\frac{\varepsilon}{4} < \varepsilon. \tag{2.97}$$

And in the second case

$$|H(I')| = |H(I) - H(I_1) - H(I_2) + H(I_1 \cup I_2)| < 4\frac{1}{4}\varepsilon = \varepsilon, \tag{2.98}$$

so that $H$ is $\Delta_1^*$-continuous in the sense of Burkill.

(iii)$\Rightarrow$(i) Let $x \in I_0$. Choose an $\alpha^* \in \Delta_1^*$, $\alpha^*$ given by a certain $p \in \mathcal{P}$ such that if $(x, I) \in \alpha^*[\{x\}]$ then $|H(I)| < \frac{\varepsilon}{4}$. Let $\alpha \in \Delta_1$ the the one generated by the same $p \in \mathcal{P}$. If $\pi \subset \alpha[\{x\}]$ then $\pi$ consists of at most four elements, so that

$$\sum_{(x,I)\in\pi} |H(I)| < \varepsilon. \tag{2.99}$$

Consequently

$$V(H, \alpha) \leq \varepsilon. \tag{2.100}$$

Therefore $H$ is $\Delta_1(I_0)$-continuous.

**2.8.10. Proposition.** *Let $I_0 \in \Phi$. If $H : \Phi \to \mathbb{R}$ is additive and continuous in the sense of Saks then it is $\Delta_1(I_0)$-continuous.*

**Proof.** Let $\eta$ be chosen so that $\lambda(I) < \eta$ implies $|H(I)| < \frac{1}{4}\varepsilon$. Let $x \in I_0$, and choose a $p(x) > 0$ so that

$$\pi\big(p(x)\big)^2 < \eta \tag{2.101}$$

(to avoid confusion let us note that $\pi$ stands this time for the number $\pi$). The function

$$x \longmapsto p(x) \tag{2.102}$$

is an element of $\mathcal{P}$ and so it defines a certain element $\alpha \in \Delta_1$. If $\sigma \subset \alpha[\{x\}]$ is a partition then $\sigma$ consists of at most four elements (one point cannot belong to more that four nonoverlapping intervals) and for $(x, I) \in \sigma$

$$\lambda(I) < \pi\big(p(x)\big)^2 < \eta, \tag{2.103}$$

so that

$$\sum_{(x,I)\in\sigma} |H(I)| < 4\,\frac{\varepsilon}{4} = \varepsilon. \tag{2.104}$$

Consequently

$$V\Big(H, \Delta_1[\{x\}]\Big) = 0. \tag{2.105}$$

Since $x$ was arbitrary, this ends the proof.

**2.8.11. Proposition.** *If $H : \Phi \to \mathbb{R}$ is additive and $\Delta_1^*(I_0)$-continuous in the sense of Burkill on $I_0 \in \Phi$ then it is $\varrho$-regularly continuous in the sense of Saks in $I_0$.*

**Proof.** Let $\varepsilon > 0$ be arbitrary. For every $x \in I_0$ there exists a $p(x) > 0$ such that if $x \in I$, $I \subset D(x, p(x))$, then $|H(I)| < \varepsilon$. The class

$$C = \Big\{ D\big(x, p(x)\big) : x \in I_0 \Big\} \tag{2.106}$$

is an open cover of a compact metric space $I_0$. Applying the well-known Lebesgue Lemma (see [9], theorem 11.4.5, p. 234), we obtain a number $\eta > 0$ such that for every $x \in I_0$ there exists an $x'$ such that

$$D(x, \eta) \subset D\Big(x', p(x')\Big). \tag{2.107}$$

Let $\delta = \varrho\eta^2/\sqrt{2}$ and suppose $I \subset I_0$, $\lambda(I) < \delta$, and $I$ is $\varrho$-regular. Then for $x$ being the center of $I$ $d(I) < \eta$ so that $I \subset D(x, \eta)$. Therefore there exists an $x' \in I_0$ such that

$$I \subset D\big(x', p(x')\big) \tag{2.108}$$

and consequently — $|H(I)| < \varepsilon$. This shows that $H$ is continuous in the sense of Saks.

**2.8.12. Example.** Let $I_0 = [0,1] \times [0,1]$, and for $I = [a, b] \times [c, d] \subset I_0$,

$$H(I) = \begin{cases} d - c & \text{if } a = 0, \\ 0 & \text{if } a > 0. \end{cases} \tag{2.109}$$

Then $H$ is additive and $\Delta_1(I_0)$-continuous but it is not continuous in the sense of Saks.

**2.8.13. Example.** Let

$$a_n = 1 - \frac{1}{2^n}, \qquad n = 0, 1, 2, \dots, \tag{2.110}$$

$$b_n = \frac{1}{2}(a_{n-1} + a_n), \qquad n = 1, 2, \dots, \tag{2.111}$$

$$c_n = \frac{1}{n}, \qquad n = 1, 2, \dots, \tag{2.112}$$

$$I_0 = [0,1] \times [0,1], \tag{2.113}$$

$$I_n = [a_{n-1}, a_n] \times [a_{n-1}, a_n], \qquad n = 1, 2, \ldots, \tag{2.114}$$

$$M_n = [b_n, a_n] \times [a_{n-1}, b_n], \qquad n = 1, 2, \ldots, \tag{2.115}$$

$$N_n = [a_{n-1}, b_n] \times [b_n, a_n], \qquad n = 1, 2, \ldots. \tag{2.116}$$

For every $n \in \mathbf{N}$ let $f_n$ be a Lebesgue-integrable function on $I_n$ such that $\int_{M_n} f_n(x)dx = c_n$, $f_n(v,u) = -f_n(u,v)$, and $f_n = 0$ on $\partial M_n$ and off $M_n \cup N_n$. We can choose $f_n$ to be continuous.

Define

$$f(u,v) = \begin{cases} f_n(u,v) & \text{if } (u,v) \in I_n, \quad \text{for some } n \in \mathbf{N}, \\ 0 & \text{otherwise} \end{cases} \tag{2.117}$$

The function $f$ is very similar to the one in example 2.6.4. It is $\Delta_1$-integrable on $I_0$, but not $\Delta_1^*$-integrable. Let, for $I \subset I_0$

$$H(I) = (\Delta_1) \int_I f d\lambda. \tag{2.118}$$

Then $H$ is $\Delta_1$-continuous by 2.8.3. However, $H$ is not $\Delta_1^*$-continuous. No matter what disk around $(1,1)$ we take, all but finitely many (say, for $n \geq n_0$) of $M_n$'s will be in it, and for any $\varepsilon_0 > 0$

$$\sum_{i=n_0}^{m} |H(M_i)| \geq \varepsilon_0 \tag{2.119}$$

for sufficiently large $m$.

*Chapter 3*

# GENERALIZED FUBINI THEOREM

The following chapter contains the natural generalization of the Fubini Theorem for the Henstock integral. The theorem, however, holds only if the base considered is a product base — this notion is newly introduced here.

In particular, we get an alternative proof of the Fubini Theorem for the Lebesgue integral (in Euclidean spaces) and the $\Delta_1$-integral.

In chapter 5 another example of a product base, for which the theorem is true, will be given.

Since the Fubini Theorem does not hold for the $\Delta_2$-integral, we show that the $\Delta_2$-integral is strictly more general than the $\Delta_1$-integral.

## 3.1. Product bases.

**3.1.1. Definition.** Let $\Delta^1$ be a derivation base on $X$ and $\Delta^2$ — a base on $Y$. Assume that $\Delta^1$ and $\Delta^2$ have local character. Let $\Psi^1 \subset \underline{P}(X)$ and $\Psi^2 \subset \underline{P}(Y)$ be the corresponding classes of "intervals".

Set

$$\Psi = \{I \times J : I \in \Psi^1, J \in \Psi^2\} \quad \text{and} \quad Z = X \times Y. \tag{3.1}$$

$\Delta \subset \underline{P}(Z \times \Psi)$ will be termed the ***product base*** of $\Delta^1$ and $\Delta^2$ (written as $\Delta = \Delta^1 \times \Delta^2$) if for every $\alpha \in \Delta$ there exist functions

$$\begin{aligned} X \ni x \mapsto \alpha_x^2 \in \Delta^2, \\ Y \ni y \mapsto \alpha_y^1 \in \Delta^1 \end{aligned} \tag{3.2}$$

such that $(z, P) \in \alpha$ if and only if

$$z = (x, y) \quad \text{and} \quad P = I \times J \tag{3.3}$$

where

$$(x, I) \in \alpha^1_y \quad \text{and} \quad (y, J) \in \alpha^2_x. \tag{3.4}$$

We will use the $\alpha^2_y$, $\alpha^1_x$ as a standard notation for the functions in (3.2), if $\alpha$ is an element of a product base (i.e., if we have $\beta \in \Delta$, then we will also write $\beta^2_x$, $\beta^1_y$).

It is easy to see that if $\Psi^1$ and $\Psi^2$ satisfy the condition 1.1.2, then so does $\Psi$.

**3.1.2. Definition.** Let $P_1$ stand for the class of positive functions $p : \mathbf{R} \to \mathbf{R}$, and let $\Phi_1$ be the class of all closed nondegenerate subintervals of $\mathbf{R}$. Following [53] and [54] we will define bases on $\mathbf{R}$:

$$\mathrm{D} = \{\alpha_p\}_{p \in P_1} \tag{3.5}$$

where

$$\alpha_p = \Big\{(x, I) \in \mathbf{R} \times \Phi_1 : x \in I \subset \big(x - p(x), x + p(x)\big)\Big\}; \tag{3.6}$$

$$\mathrm{D}_0 = \{\alpha^0_p\}_{p \in P_1} \tag{3.7}$$

where

$$\alpha^0_p = \Big\{(x, I) \in \mathbf{R} \times \Phi_1 : x \text{ is an endpoint of } I, I \subset \big(x - p(x), x + p(x)\big)\Big\}; \tag{3.8}$$

$$\mathrm{D}^\# = \{\alpha^\#_p\}_{p \in P_1} \tag{3.9}$$

where

$$\alpha^\#_p = \Big\{(x, I) \in \mathbf{R} \times \Phi_1 : I \subset \big(x - p(x), x + p(x)\big)\Big\}. \tag{3.10}$$

Then we have

$$\Delta_1 \simeq \mathrm{D} \times \mathrm{D}, \tag{3.11}$$

$$\tilde{\Delta}_1 \simeq \mathrm{D}_0 \times \mathrm{D}_0, \tag{3.12}$$

$$\Delta^*_1 \simeq \mathrm{D}^\# \times \mathrm{D}^\#. \tag{3.13}$$

It is known that for the functions of the form $f(x)\lambda(I)$, D generates the Denjoy-Perron integral, and so does $\mathrm{D}_0$. $\mathrm{D}^\#$ generates the Lebesgue integral. For the complete discussion of this subject, see [53].

**3.1.3. Observation.** *Any product base has a local character. A product of two bases which are filtering down, is filtering down.*

**3.1.4. Definition.** Let $\pi^1 = \{(x_1, I_1), \ldots, (x_n, I_n)\}$ be a partition in $X$, and let, for each $i = 1, 2, \ldots, n$,

$$\pi^2_{x_i} = \{(y^i_1, J^i_1), \ldots, (y^i_{k_i}, J^i_{k_i})\} \tag{3.14}$$

be a partition in $Y$. Then

$$\begin{aligned} \pi = \bigcup_{(x,I)\in\pi^1} \bigcup_{(y,J)\in\pi^2_x} \Big\{\big((x, y), I \times J\big)\Big\} = \\ \Big\{\big((x_i, y^i_j), I_i \times J^i_j\big) : i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, k_i\Big\} \end{aligned} \tag{3.15}$$

is a partition of $X \times Y$, and such a partition will be called a ***compound partition***.

**3.1.5. Proposition.** *If $\Delta^1$ and $\Delta^2$ have the partitioning property and are filtering down, then $\Delta = \Delta^1 \times \Delta^2$ has the partitioning property.*

**Proof.** Let $\alpha \in \Delta$ and $I_0 \times J_0 \in \Psi$. Fix an $x \in I_0$. Since $\Delta^2$ has the partitioning property, there exists a partition $\pi^2_x \subset \alpha^2_x$ of $J_0$. Write

$$\pi^2_x = \{(y^x_1, J^x_1), \ldots, (y^x_{k_x}, J^x_{k_x})\}. \tag{3.16}$$

Since $\Delta^1$ is filtering down, for every $x \in I_0$ there exists an $\alpha^{1,x} \in \Delta^1$ such that

$$\alpha^{1,x} \subset \bigcap_{i=1}^{k_x} \alpha^1_{y^x_i}. \tag{3.17}$$

Since $\Delta^1$ has a local character, there is an $\alpha^1 \in \Delta^1$ such that

$$\alpha^1\big[\{x\}\big] \subset \alpha^{1,x}\big[\{x\}\big] \tag{3.18}$$

for every $x$. There exists a partition $\pi^1 \subset \alpha^1$ of $I_0$,

$$\pi^1 = \{(x_1, I_1), \ldots, (x_n, I_n)\}. \tag{3.19}$$

Construct now a compound partition

$$\pi = \Big\{\big((x_i, y^{x_i}_j), I_i \times J^{x_i}_j\big) : i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, k_{x_i}\Big\}. \tag{3.20}$$

It is easy to see that $\pi \subset \alpha$ so that the proof is ended.

**3.2. Fubini Theorem.**

**3.2.1. Theorem.** *Let $\Delta$ be a product $\Delta^1 \times \Delta^2$ of derivation bases on $X$ and $Y$ respectively, with $\Psi^1$ and $\Psi^2$ being the corresponding classes of subsets of $\underline{P}(X)$ and $\underline{P}(Y)$. Let $I_0 \in \Psi^1$, $J_0 \in \Psi^2$, and $U_1 : I_0 \times \Psi^1 \to \mathbf{R}$, $U_2 : I_0 \times (J_0 \times \Psi^2) \to \mathbf{R}$. Write $U = U_1 U_2$. Suppose $U$ is $\Delta$-integrable on $I_0 \times J_0$. Define*

$$T = \{x \in I_0 : U_2(x, \cdot, \cdot) \text{ is } \Delta^2\text{-integrable}\}. \tag{3.21}$$

*For $x \in T$ let*

$$g(x) = (\Delta^2) \int_{J_0} U_2(x, \cdot, \cdot). \tag{3.22}$$

*For $x \in J_0 \setminus T$ let $g(x)$ be chosen arbitrarily. Set*

$$W(x, I) = U_1(x, I) g(x) \tag{3.23}$$

*for $(x, I) \in I_0 \times \Psi^1$.*

*Then*

*(i) $V(U_1, \Delta^1[I_0 \setminus T]) = 0$;*

*(ii) $W$ is $\Delta^1$-integrable and*

$$(\Delta^1) \int_{I_0} W = (\Delta) \int_{I_0 \times J_0} U. \tag{3.24}$$

**Proof.** We will show (i) first. For every $n \in \mathbf{N}$ let

$$X_n = \Big\{ x \in I_0 : \forall \alpha^2 \in \Delta^2 \ \exists \text{ partitions } \pi^{2,1}, \pi^{2,2} \subset \alpha^2 \text{ of } J_0 \text{ such that} \\ \Big| \sum_{(y,J) \in \pi^{2,1}} U_2(x, y, J) - \sum_{(y,J) \in \pi^{2,2}} U_2(x, y, J) \Big| \geq \frac{1}{n} \Big\}. \tag{3.25}$$

We have then

$$I_0 \setminus T = \bigcup_{n \in \mathbf{N}} X_n \tag{3.26}$$

and is suffices to show that

$$V(U_1, \Delta^1[X_n]) = 0 \tag{3.27}$$

for every $n \in \mathbb{N}$.

Fix an $n \in \mathbb{N}$. Let $\varepsilon > 0$ be arbitrary. Take an $\alpha \in \Delta$ such that for every partition $\pi \subset \alpha$ of $I_0 \times J_0$

$$\left|(\Delta)\int_{I_0\times J_0} U - U(\pi)\right| \le \frac{1}{2}\varepsilon. \tag{3.28}$$

Let $x \in X_n$. Find partitions of $J_0$

$$\begin{aligned} \pi_x^2 &= \{(y_j^x, J_j^x) : j = 1, 2, \dots, m_x\} \\ \widehat{\pi}_x^2 &= \{(y_j^x, J_j^x) : j = 1, 2, \dots, \widehat{m}_x\} \end{aligned} \tag{3.29}$$

which are contained in $\alpha_x^2$, and such that

$$\left|\sum_{(y,J)\in\pi_x^2} U_2(x, y, J) - \sum_{(y,J)\in\widehat{\pi}_x^2} U_2(x, y, J)\right| \ge \frac{1}{n} \tag{3.30}$$

— this is possible by (3.25). Since $\Delta^1$ is filtering down and has a local character, there exists a $\beta^1 \in \Delta^1$ such that for every $x \in X_n$ and every $y$ such that $(y, J) \in \pi_x^2, \widehat{\pi}_x^2$

$$\beta^1[\{x\}] \subset \alpha_y^1[\{x\}]. \tag{3.31}$$

Now let $x \in I_0 \setminus X_n$ and

$$\begin{aligned} \pi_x^2 &= \{(y_j^x, J_j^x) : j = 1, 2, \dots, m_x\} \subset \alpha_x^2 \\ \widehat{\pi}_x^2 &= \pi_x^2. \end{aligned} \tag{3.32}$$

Choose $\gamma^1 \in \Delta^1$ such that for every $x \in I_0 \setminus X_n$ and every $y$ such that $(y, J) \in \pi_x^2, \widehat{\pi}_x^2$

$$\gamma^1[\{x\}] \subset \alpha_y^1[\{x\}]. \tag{3.33}$$

Finally, let $\phi^1 \in \Delta^1$ be such that

$$\phi^1 \subset \beta^1 \cap \gamma^1. \tag{3.34}$$

Let $\overline{\pi}^1$ be an arbitrary partition contained in $\phi^1[X_n]$. By our standard assumption 1.1.2, and because $\Delta^1$ has the partitioning property, there exists a partition $\pi^1 \subset \phi^1$ of $I_0$ such that

$$\overline{\pi}^1 = \{(x, I) \in \pi^1 : x \in X_n\}. \tag{3.35}$$

Write

$$\pi^1 = \{(x_1, I_1), \ldots, (x_k, I_k)\} \tag{3.36}$$

and define compound partitions

$$\begin{aligned} \rho &= \left\{\left((x_i, y_j^{x_i}), I_i \times J_j^{x_i}\right) : i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, m_{x_i}\right\}, \\ \widehat{\rho} &= \left\{\left((x_i, \widehat{y}_j^{\,x_i}), I_i \times \widehat{J}_j^{\,x_i}\right) : i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, \widehat{m}_{x_i}\right\}. \end{aligned} \tag{3.37}$$

Then we have $\rho, \widehat{\rho} \subset \alpha$, so that by (3.28)

$$\left|U(\rho) - U(\widehat{\rho})\right| \leq \varepsilon. \tag{3.38}$$

On the other hand

$$\begin{aligned} &\left|U(\rho) - U(\widehat{\rho})\right| = \\ &\left|\sum_{x_i \in X_n} U_1(x_i, I_i)\Big(\sum_{j=1}^{m_x} U_2(x_i, y_j^{x_i}, J_j^{x_i}) - \sum_{j=1}^{\widehat{m}_x} U_2(x_i, \widehat{y}_j^{\,x_i}, \widehat{J}_j^{\,x_i})\Big)\right|. \end{aligned} \tag{3.39}$$

We can assume without loss of generality that

$$\operatorname{sgn} U_1(x_i, I_i) = \operatorname{sgn}\left(\sum_{j=1}^{m_x} U_2(x_i, y_j^{x_i}, J_j^{x_i}) - \sum_{j=1}^{\widehat{m}_x} U_2(x_i, \widehat{y}_j^{\,x_i}, \widehat{J}_j^{\,x_i})\right) \tag{3.40}$$

whenever $x_i \in X_n$ and $U_1(x_i, I_i) \neq 0$. This is possible because if (3.40) does not hold, we can simply switch the roles of $\pi^2_{x_i}$ and $\widehat{\pi}^2_{x_i}$.

Therefore

$$\begin{aligned} &\left|U(\rho) - U(\widehat{\rho})\right| = \\ &\sum_{x_i \in X_n} \left|U_1(x_i, I_i)\right| \left|\sum_{j=1}^{m_x} U_2(x_i, y_j^{x_i}, J_j^{x_i}) - \sum_{j=1}^{\widehat{m}_x} U_2(x_i, \widehat{y}_j^{\,x_i}, \widehat{J}_j^{\,x_i})\right| \geq \\ &\qquad \frac{1}{n} \sum_{x_i \in X_n} \left|U_1(x_i, I_i)\right| \geq \\ &\qquad\qquad \frac{1}{n} \sum_{(x, I) \in \overline{\pi}^1} U_1(x, I). \end{aligned} \tag{3.41}$$

From (3.38) and (3.41) we conclude that

$$\sum_{(x, I) \in \overline{\pi}^1} \left|U_1(x, I)\right| \leq n\varepsilon, \tag{3.42}$$

and this shows that (3.27) holds for every $n \in \mathbb{N}$.

Now we turn to the proof of (ii).

We need to show that for every $\varepsilon$ there exists an $\alpha^1 \in \Delta^1$ such that for every partition $\pi^1 \subset \alpha^1$ of $I_0$

$$\left|(\Delta)\int_{I_0\times J_0} U - W(\pi^1)\right| \le \varepsilon. \tag{3.43}$$

Let $\varepsilon > 0$ be arbitrary. Find an $\alpha \in \Delta$ such that for every partition $\pi \subset \alpha$ of $I_0 \times J_0$

$$\left|(\Delta)\int_{I_0\times J_0} U - U(\pi)\right| \le \frac{1}{8}\varepsilon. \tag{3.44}$$

Note that (3.44) implies that whenever $\pi, \tilde{\pi} \subset \alpha$ are partitions of $I_0 \times J_0$ then

$$\left|U(\pi) - U(\tilde{\pi})\right| \le \frac{1}{4}\varepsilon. \tag{3.45}$$

Let $x \in I_0 \setminus T$. Choose a partition $\pi_x^2 \subset \alpha_x^2$ of $J_0$. Then find an $\alpha^1 \in \Delta^1$ such that for every $x \in I_0$

$$\alpha^1[\{x\}] \subset \bigcap_{(y,J)\in\pi_x^2} \alpha_y^1[\{x\}]. \tag{3.46}$$

Put

$$Q_1 = \left\{x \in I_0 \setminus T : |g(x)| + \Big|\sum_{(y,J)\in\pi_x^2} U_2(x,y,J)\Big| \le 1\right\}, \tag{3.47}$$

and for $r \in \mathbb{N}$, $n \ge 2$

$$Q_r = \left\{x \in I_0 \setminus T : r-1 < |g(x)| + \Big|\sum_{(y,J)\in\pi_x^2} U_2(x,y,J)\Big| \le r\right\}. \tag{3.48}$$

By (i) we have, for every $r \in \mathbb{N}$

$$V(U_1, \Delta^1[Q_r]) = 0. \tag{3.49}$$

Note that $Q_r$'s are pairwise disjoint.

By (3.49) for every $r \in \mathbb{N}$ there exists an $\alpha^{1,r} \in \Delta^1$ such that for every partition $\pi^{1,r} \subset \alpha^{1,r}[Q_r]$

$$|U_1|(\pi^{1,r}) \le \frac{\varepsilon}{r\,2^{r+2}}. \tag{3.50}$$

Put

$$\rho_x^2 = \tilde{\rho}_x^2 = \pi_x^2 \qquad \text{for } x \in I_0 \setminus T \tag{3.51}$$

and find a $\beta^1 \in \Delta^1$ such that for every $r \in \mathbb{N}$ and every $x \in Q_r$

$$\beta^1[\{x\}] \subset \alpha^1[\{x\}] \cap \alpha^{1,r}[\{x\}]. \tag{3.52}$$

Now consider an $x \in T$. Let $\pi_x^2$ be a partition of $J_0$ which is contained in $\alpha_x^2$. We can find a $\psi^2 \in \Delta^2$ such that for every partition $\pi^2 \subset \psi^2$ of $J_0$

$$\left| \sum_{(y,J)\in\pi^2} U_2(x,y,J) - g(x) \right| < \frac{1}{2} \left| \sum_{(y,J)\in\pi_x^2} U_2(x,y,J) - g(x) \right|. \tag{3.53}$$

— we can do that because the right-hand side is just a certain positive real number.

It should be noted here that the right-hand side of (3.53) may actually be equal to zero, but then we can skip the following estimates and go directly to (3.66), which remains true.

Let $\pi_x^{2,2}$ be a partition of $J_0$ which is contained in $\alpha_x^2 \cap \psi^2$. Then $\pi_x^2$ and $\pi_x^{2,2}$ are partitions of $J_0$, contained in $\alpha_x^2$, and such that

$$\left| \sum_{(y,J)\in\pi_x^{2,2}} U_2(x,y,J) - g(x) \right| < \frac{1}{2} \left| \sum_{(y,J)\in\pi_x^2} U_2(x,y,J) - g(x) \right|. \tag{3.55}$$

Choose $\alpha^1 \in \Delta^1$, $\alpha^1 \subset \beta^1$ such that for $x \in T$

$$\alpha^1[\{x\}] \subset \alpha_y^1[\{x\}] \tag{3.56}$$

for all $(y,J) \in \pi_x^2$ and $(y,J) \in \pi_x^{2,2}$.

Let $\pi^1 \subset \alpha^1$ be a partition of $I_0$. If $(x,I) \in \pi^1$ and $x \in T$, and

$$U_1(x,I)\Big(\Big(\sum_{(y,J)\in\pi_x^2} U_2(x,y,J)\Big) - g(x)\Big) > 0 \tag{3.57}$$

then put

$$\rho_x^2 = \pi_x^2 \qquad \text{and} \qquad \tilde{\rho}_x^2 = \pi_x^{2,2}. \tag{3.58}$$

Otherwise (still for $x \in T$) let

$$\rho_x^2 = \pi_x^{2,2} \qquad \text{and} \qquad \tilde{\rho}_x^2 = \pi_x^2. \tag{3.59}$$

Define compound partitions of $I_0 \times J_0$

$$\begin{aligned} \rho &= \bigcup_{(x,I)\in\pi^1} \left\{ \big((x,y), I \times J\big) : (y,J) \in \rho_x^2 \right\} \\ \tilde{\rho} &= \bigcup_{(x,I)\in\pi^1} \left\{ \big((x,y), I \times J\big) : (y,J) \in \tilde{\rho}_x^2 \right\}. \end{aligned} \tag{3.60}$$

Then $\rho$ and $\tilde{\rho}$ are contained in $\alpha$ so that by (3.45)

$$\left|U(\rho) - U(\tilde{\rho})\right| \le \frac{1}{4}\varepsilon. \tag{3.61}$$

Furthermore, we have

$$\begin{aligned} &\left|U(\rho) - W(\pi^1)\right| \le \\ &\quad \left| \sum_{\substack{((x,y),I\times J)\in\rho \\ x\notin T}} U\big((x,y), I \times J\big) - \sum_{\substack{(x,I)\in\pi^1 \\ x\notin T}} W(x,I) \right| + \\ &\qquad \left| \sum_{\substack{((x,y),I\times J)\in\rho \\ x\in T}} U\big((x,y), I \times J\big) - \sum_{\substack{(x,I)\in\pi^1 \\ x\in T}} W(x,I) \right|. \end{aligned} \tag{3.62}$$

Since $\pi^1 \subset \alpha^1$, from the way $\alpha^1$ was chosen, we get

$$\begin{aligned} &\left| \sum_{\substack{((x,y),I\times J)\in\rho \\ x\notin T}} U\big((x,y), I \times J\big) - \sum_{\substack{(x,I)\in\pi^1 \\ x\notin T}} W(x,I) \right| = \\ &\left| \sum_{\substack{((x,y),I\times J)\in\rho \\ x\notin T}} U_1(x,I)U_2(x,y,J) - \sum_{\substack{(x,I)\in\pi^1 \\ x\notin T}} U_1(x,I)g(x) \right| = \\ &\left| \sum_{\substack{(x,I)\in\pi^1 \\ x\notin T}} \sum_{(y,J)\in\rho_x^2} U_1(x,I)U_2(x,y,J) - \sum_{\substack{(x,I)\in\pi^1 \\ x\notin T}} U_1(x,I)g(x) \right| = \\ &\left| \sum_{\substack{(x,I)\in\pi^1 \\ x\notin T}} U_1(x,I)\left(\left(\sum_{(y,J)\in\rho_x^2} U_2(x,y,J)\right) - g(x)\right) \right| \le \\ &\sum_{r\in\mathbb{N}} \sum_{\substack{(x,I)\in\pi^1 \\ x\in Q_r}} r\left|U_1(x,I)\right| \le \\ &\sum_{r\in\mathbb{N}} \frac{r\varepsilon}{r\,2^{r+2}} = \frac{1}{4}\varepsilon. \end{aligned} \tag{3.63}$$

Now let $x \in T$ for $(x, I) \in \pi^1$. Suppose

$$U_1(x,I)\Big(\Big(\sum_{(y,J)\in\pi^2_x} U_2(x,y,J)\Big) - g(x)\Big) > 0. \tag{3.64}$$

Then

$$\begin{aligned}
&0 < U_1(x,I)\Big(\Big(\sum_{(y,J)\in\rho^2_x} U_2(x,y,J)\Big) - g(x)\Big) \le \\
&2U_1(x,I)\Big(\Big(\sum_{(y,J)\in\rho^2_x} U_2(x,y,J)\Big) - g(x) + g(x) - \Big(\sum_{(y,J)\in\widetilde{\rho}^2_x} U_2(x,y,J)\Big)\Big) = \\
&2U_1(x,I)\Big(\Big(\sum_{(y,J)\in\rho^2_x} U_2(x,y,J)\Big) - \Big(\sum_{(y,J)\in\widetilde{\rho}^2_x} U_2(x,y,J)\Big)\Big).
\end{aligned} \tag{3.65}$$

Suppose, however, that (3.64) does not hold. Then

$$\begin{aligned}
&\Big|U_1(x,I)\Big(\Big(\sum_{(y,J)\in\rho^2_x} U_2(x,y,J)\Big) - g(x)\Big)\Big| \le \\
&U_1(x,I)\Big(\Big(\sum_{(y,J)\in\rho^2_x} U_2(x,y,J)\Big) - g(x) - \Big(\sum_{(y,J)\in\widetilde{\rho}^2_x} U_2(x,y,J)\Big) + g(x)\Big) = \\
&U_1(x,I)\Big(\Big(\sum_{(y,J)\in\rho^2_x} U_2(x,y,J)\Big) - \Big(\sum_{(y,J)\in\widetilde{\rho}^2_x} U_2(x,y,J)\Big)\Big).
\end{aligned} \tag{3.66}$$

Combining (3.65) and (3.66), we get for $x \in T$, $(x, I) \in \pi^1$

$$\begin{aligned}
&\Big|U_1(x,I)\Big(\Big(\sum_{(y,J)\in\rho^2_x} U_2(x,y,J)\Big) - g(x)\Big)\Big| \le \\
&2U_1(x,I)\Big(\Big(\sum_{(y,J)\in\rho^2_x} U_2(x,y,J)\Big) - \Big(\sum_{(y,J)\in\widetilde{\rho}^2_x} U_2(x,y,J)\Big)\Big).
\end{aligned} \tag{3.67}$$

Now let us note the following

$$\begin{aligned}
&U(\rho) - U(\widetilde{\rho}) = \\
&\Big(\sum_{\substack{((x,y),I\times J)\in\rho \\ x\in T}} U\big((x,y), I\times J\big)\Big) - \Big(\sum_{\substack{((x,y),I\times J)\in\widetilde{\rho} \\ x\in T}} U\big((x,y), I\times J\big)\Big) = \\
&\sum_{\substack{(x,I)\in\pi^1 \\ x\in T}} U_1(x,I)\Big(\Big(\sum_{(y,J)\in\rho^2_x} U_2(x,y,J)\Big) - \Big(\sum_{(y,J)\in\widetilde{\rho}^2_x} U_2(x,y,J)\Big)\Big).
\end{aligned} \tag{3.68}$$

Therefore, by (3.61), (3.67), and (3.68)

$$
\begin{aligned}
&\left|\Big(\sum_{\substack{((x,y),I\times J)\in\rho\\ x\in T}} U\big((x,y),I\times J\big)\Big)-\Big(\sum_{\substack{(x,I)\in\pi^1\\ x\in T}} W(x,I)\Big)\right| = \\
&\left|\sum_{\substack{(x,I)\in\pi^1\\ x\in T}}\bigg(\Big(\sum_{(y,J)\in\rho^2_x} U_2(x,y,J)\Big)-\Big(\sum_{(y,J)\in\widetilde{\rho}^2_x} U_2(x,y,J)\Big)\bigg)\right| \le \\
&2\Big(\sum_{\substack{(x,I)\in\pi^1\\ x\in T}} U_1(x,I)\Big)\bigg(\Big(\sum_{(y,J)\in\rho^2_x} U_2(x,y,J)\Big)-\Big(\sum_{(y,J)\in\widetilde{\rho}^2_x} U_2(x,y,J)\Big)\bigg) \le \\
&2\left(U(\rho)-U(\widetilde{\rho})\right) \le \frac{1}{2}\varepsilon.
\end{aligned}
\tag{3.69}
$$

Combining (3.63) and (3.69) we get

$$|U(\rho)-W(\pi^1)| \le \frac{3}{4}\varepsilon \tag{3.70}$$

and this together with (3.44) gives the desired inequality (3.43). The proof is ended.

### 3.3. Corollaries to the Fubini Theorem.

**3.3.1. Corollary.** *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be $\Delta_1$-integrable (or $\widetilde{\Delta}_1$-integrable). Write*

$$T=\{x\in[a,b]: f(x,\cdot) \text{ is Denjoy-Perron integrable on } [c,d]\}. \tag{3.71}$$

*Then*

*(i) $[a,b]\setminus T$ is of measure zero;*

*(ii) If we define*

$$g(x)=\int_c^d f(x,y)\,dy \qquad \text{(Denjoy-Perron integral)} \tag{3.72}$$

*for $x\in T$, and choose $g(x)$ arbitrarily otherwise, then*

$$(\Delta_1)\int_{[a,b]\times[c,d]} f\,d\lambda = \int_a^b\int_c^d f(x,y)\,dy\,dx \tag{3.73}$$

*— the integrals on the right-hand side being Denjoy-Perron integrals.*

**Proof.** This follows from the theorem 3.2.1 and (3.11) (or (3.12)).

Note: A theorem equivalent to 3.3.1 appears in [31].

**3.3.2. Corollary.** *Let $f : [a,b] \times [c,d] \to \mathbf{R}$ be $\Delta_1^*$-integrable (i.e., Lebesgue integrable). Write*

$$T = \{x \in [a,b] : f(x,\cdot) \text{ is Lebesgue integrable on } [c,d]\}. \tag{3.74}$$

*Then*

*(i) $[a,b] \setminus T$ is of measure zero;*

*(ii) For $g(x)$ defined as*

$$g(x) = \int_c^d f(x,y)\,dy \qquad \text{(Lebesgue integral)} \tag{3.75}$$

*for $x \in T$, and arbitrarily otherwise, we have*

$$(\Delta_1^*)\int_{[a,b]\times[c,d]} f\,d\lambda = \int_a^b\int_c^d f(x,y)\,dydx \tag{3.76}$$

*— the integrals on the right-hand side being Lebesgue integrals.*

**Proof.** The corollary follows easily from the theorem 3.2.1 and (3.13).

Note: 3.3.2 is the classical Fubini Theorem for the Lebesgue integral.

### 3.4. Tolstov's counterexample.

**3.4.1. Example.** In [55] Tolstov constructed a real-valued function $H$, defined on the class of all subintervals $I \subset [0,1] \times [0,1]$, which is additive, continuous in the sense of Saks, and such that (for $\rho = \frac{1}{2}$)

$$\mathrm{D}_{\Delta_2} H_\lambda(x,y) = f(x,y) \text{ a.e.} \tag{3.77}$$

for a certain measurable $f : [0,1] \times [0,1] \to \mathbf{R}$, and

$$-\infty < \underline{\mathrm{D}}_{\Delta_2} H_\lambda(x,y) \le \overline{\mathrm{D}}_{\Delta_2} H_\lambda(x,y) < +\infty \tag{3.78}$$

for all $(x,y) \in [0,1] \times [0,1]$, and yet

$$\begin{aligned} &\frac{\partial_{ap}}{\partial x}\,(\Delta_2)\int_{[0,x]\times[0,y]} f\,d\lambda \\ &\frac{\partial_{ap}}{\partial y}\,(\Delta_2)\int_{[0,x]\times[0,y]} f\,d\lambda \end{aligned} \tag{3.79}$$

do not exist anywhere in $(0,1) \times (0,1)$.

In [27] it is shown that the conditions (3.77) and (3.78) force $f$ to be integrable in the sense of Kempisty (this notion is introduced and discussed in chapter 4). In theorem 4.5.2 we will show that any Kempisty-integrable function is also $\Delta_2$-integrable. Thus $f$ is $\Delta_2$-integrable.

However, if any of the equalities

$$\begin{aligned} (\Delta_2) \int_{[a,b]\times[c,d]} f\,d\lambda &= \int_c^d \int_a^b f(x,y)\,dx dy, \\ (\Delta_2) \int_{[a,b]\times[c,d]} f\,d\lambda &= \int_a^b \int_c^d f(x,y)\,dy dx \end{aligned} \tag{3.80}$$

holds for every $[a,b] \times [c,d] \subset [0,1] \times [0,1]$ (with the integrals on the right-hand side of (3.80) being the general Denjoy integrals, as considered in [55], or even more general approximately continuous Perron integrals of [5]), then the corresponding partial (3.79) exists a.e. (see [5], property III). Thus for the function constructed by Tolstov we do not have one of the equalities (3.80) for some $[a,b] \times [c,d] \subset [0,1] \times [0,1]$.

Combining this with the corollary 3.3.1 we see that $f$ is an example of a function which is $\Delta_2$-integrable, but not $\Delta_1$-integrable.

*Chapter 4*

# THE INTEGRAL OF KEMPISTY

We consider here the Denjoy-type integral in $\mathbb{R}^2$ introduced by Kempisty. It is less general than the $\Delta_2$-integral. Under a special assumption on minorants and majorants (which is satisfied for every finite regular derivative) $\Delta_2$-integrals which are continuous in the sense of Saks are also Kempisty integrals.

Any Kempisty-integrable function is Lebesgue-integrable on some nondegenerate closed interval. This shows that a certain class of $\Delta_2$ and $\Delta_1$-integrable functions also has this property. There exist, however, $\Delta_2$-integrable functions without it.

We consider also nonabsolute integrals of Mawhin and Pfeffer, for which the classical divergence theorem holds. They are less general than the $\Delta_2$-integral, but more general than the $\Delta_1$-integral.

## 4.1. Functions absolutely continuous in the sense of Kempisty.

**4.1.1.** In this chapter we will concentrate on the $\Delta_2$ base, with a fixed $\varrho \in (0,1)$ given. If $H : \Phi_\varrho \to \mathbb{R}$ we will write simply $\mathrm{D}\, H(x)$, $\underline{\mathrm{D}}\, H(x)$, $\overline{\mathrm{D}}\, H(x)$ for $\mathrm{D}_{\Delta_2} H_\lambda(x)$, $\underline{\mathrm{D}}_{\Delta_2} H_\lambda(x)$, $\overline{\mathrm{D}}_{\Delta_2} H_\lambda(x)$ (respectively).

If $E \subset \mathbb{R}^2$, we will write $R_E$ for the smallest interval containing $E$.

**4.1.2.** In [27] a function $H : \Phi_\varrho \to \mathbb{R}$ is termed $ACr$ on a bounded set $E \subset \mathbb{R}^2$ if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that if $I_1, I_2, \ldots, I_n$ are nonoverlapping elements of $\Phi_\varrho$, meeting $E$, and contained in $R_E$, then

$$\sum_{i=1}^{n} |H(I_i)| < \varepsilon \qquad \text{whenever} \qquad \sum_{i=i}^{n} \lambda(I_i) < \eta. \tag{4.1}$$

But — there is a difficulty here. If $E$ is contained in a vertical or horizontal segment, $R_E$ is degenerate and the definition becomes void.

The vertical or horizontal linear sets are precisely those which cause problems.

Consider the following example. Let

$$\begin{aligned} a_n &= \frac{(-1)^n}{n} \qquad \text{for } n \in \mathbf{N}, \\ I_0 &= [0,1] \times [0,1], \\ u_n &= 1 + \frac{1}{2^n},\ v_n = \frac{1}{2^n} \qquad \text{for } n \in \mathbf{N}, \\ F(I) &= \sum_{\substack{u_i \in [a,b) \\ v_i \in [c,d)}} a_i \qquad \text{for } I = [a,b] \times [c,d] \in \Phi_\varrho. \end{aligned} \tag{4.2}$$

$F$ is well-defined, finite, additive, and $F(I) = 0$ for any $I \subset I_0$. Thus $F$ is $ACr$ (in the above defined sense) on $I_0$.

On the other hand, if $I_n$, for $n \in \mathbf{N}$, is a $\frac{1}{3}$-regular interval with a vertex at $(u_{2n-1}, v_{2n-1})$, a side contained in $\{1\} \times [0,1]$ and not containing $(u_{2n}, v_{2n})$, then

$$\sum_{n=1}^{\infty} |F(I_n)| = +\infty, \tag{4.3}$$

and $I_n$'s are nonoverlapping. Since all $I_n$'s meet $\{1\} \times [0,1[$, $F$ is not $ACr$ on any interval $[1, 1+\delta] \times [0,1]$ for $\delta > 0$.

But $\{1\} \times [0,1[$ is a subset of a set on which $F$ is $ACr$.

In view of these difficulties we decided to alter Kempisty's definition slightly. The change as the one made here appears also in [7]. It does not effect the theory of integration of Kempisty, just straightens things up.

**4.1.3. Definition.** $H : \Phi_\varrho \to \mathbf{R}$ is $ACr$ on a bounded set $E \subset \mathbf{R}^2$ if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that if $I_1, \ldots, I_n$ are nonoverlapping elements of $\Phi_\varrho$, meeting $E$, then (4.1) is satisfied.

**4.1.4. Proposition.** *Let $H$ be additive and continuous in the sense of Saks. If $H$ is $ACr$ on a bounded set $E$ then it is also $ACr$ on $E^-$.*

**Proof.** Let $\varepsilon > 0$ be arbitrary. Choose $\eta > 0$ such that for every finite collection $I_1, \ldots, I_m$ of elements of $\Phi_\varrho$, containing points of $E$, and nonoverlapping, we have

$$\sum_{i=1}^{m} |H(I_i)| < \frac{1}{2}\varepsilon \tag{4.4}$$

whenever

$$\sum_{i=1}^{m} \lambda(I_i) < \eta. \tag{4.5}$$

Now let $I_1, \ldots, I_n \in \Phi_\varrho$ be nonoverlapping, containing points of $E^-$ and such that $\sum_{i=1}^{n} I_i < \eta$.

Since $H$ is continuous in the sense of Saks, there exists a $\delta$ such that for any nondegenerate closed intervals $J_1$ and $J_2$

$$|H(J_1) - H(J_2)| < \frac{\varepsilon}{2n} \tag{4.6}$$

whenever

$$\lambda(J_1 \triangle J_2) < \delta. \tag{4.7}$$

The intervals $I_1, \ldots, I_n$ might fail to meet $E$. We will construct intervals $I_1', \ldots, I_1'$ which will meet $E$, be nonoverlapping, and such that

$$\sum_{i=1}^{n} \lambda(I_i') < \eta \tag{4.8}$$

and

$$\lambda(I_i \triangle I_i') < \delta \qquad \text{for every } i. \tag{4.9}$$

The construction will be performed in $n$ steps, in each of them we will be either shifting or shrinking the intervals $I_i$, this way their total area will not be increased.

Define $I_i^0 = I_i$, for $i = 1, 2, \ldots, n$.

The $j$-th step $(1 \le j \le n)$ of the construction will be performed on the intervals $I_i^{j-1}$ for $i = 1, 2, \ldots, n$. After $j - 1$ steps we have intervals $I_1^{j-1}, \ldots, I_n^{j-1}$ which are $\varrho$-regular, nonoverlapping and such that

$$I_i^{j-1} \cap E \neq \emptyset \quad \text{for } i \le j-1, \tag{4.10}$$

$$I_i^{j-1} \cap E^- \neq \emptyset \quad \text{for } i \ge j, \tag{4.11}$$

$$\sum_{i=1}^{n} \lambda(I_i^{j-1}) < \eta, \tag{4.12}$$

and

$$\lambda(I_i \triangle I_i^{j-1}) < \frac{(j-1)\delta}{n}. \tag{4.13}$$

Now take the interval $I_j^{j-1}$. If it contains a point of E, let

$$I_i^j = I_i^{j-1} \qquad \text{for } i = 1, 2, \ldots, n. \tag{4.14}$$

Otherwise, it has a point of $E^-$ on its boundary. It is a cluster point of $E$. Therefore we can make $I_i^{j-1}$ meet $E$ by shifting it slightly toward that point. Note that any such move, even very small, will assure that the shifted interval meets $E$.

Denote the shifted interval by $I_j^j$. Make the move small enough for the area of the symmetric difference of $I_j^{j-1}$ and $I_j^j$ to be less than $\frac{\delta}{n}$.

If $I_j^j$ does not overlap with any of $I_i^{j-1}$ for $i \neq j$, put $I_i^j = I_i^{j-1}$ for $i \neq j$.

But if it does, more work is needed.

Let $I$ be an interval among $I_i^{j-1}$, $i \neq j$, such that $I$ and $I_j^j$ overlap. Shrink it to a proportional interval $J \subset I$ (being proportional to $I$, it will still be $\varrho$-regular) such that $J$ and $I_j^j$ do not overlap. If it so happens that $J$ does not contain points of $E$ (for $I = I_i^{j-1}$, $i \leq j-1$) or $E^-$ (for $I = I_i^{j-1}$, $i \geq j+1$) any more, then go back to the way $I_j^j$ was made out of $I_j^{j-1}$, and make the shift smaller, so that $J$ will contain points of $E$ or $E^-$ (respectively). This is possible unless $I_j^{j-1}$ and $I$ have a common boundary point belonging to $E^- \setminus E$ which is not a cluster point of $E$ — that obviously can't happen, or there is an isolated point of $E$ being a common boundary point of $I_j^{j-1}$ and $I$ — but then $I_j^{j-1}$ does not need to be shifted.

Finally, make the $I$ intervals shrink by not more than $\frac{\delta}{n}$ (in terms of area), and if that is not possible, go back to $I_j^{j-1}$ and curb its shift so that it is possible.

Do this for every interval $I$ among $I_i^{j-1}$, $i \neq j$, such that $I$ and $I_j^j$ overlap.

If $I_i^{j-1}$ is one of such intervals, let the corresponding $J$ (the shrunk interval) be $I_i^j$. Otherwise, leave $I_i^j = I_i^{j-1}$.

We obtain intervals $I_1^j, \ldots, I_n^j$ which are $\varrho$-regular, nonoverlapping, and such that

$$I_i^j \cap E \neq \emptyset \quad \text{for } i \leq j, \tag{4.15}$$

$$I_i^j \cap E^- \neq \emptyset \quad \text{for } i \geq j, \tag{4.16}$$

$$\sum_{i=1}^{n} \lambda(I_i^j) < \eta, \tag{4.17}$$

and

$$\lambda(I_i \Delta I_i^j) \leq \lambda(I_i \Delta I_i^{j-1}) + \lambda(I_i^{j-1} \Delta I_i^j) < \frac{(j-1)\delta}{n} + \frac{\delta}{n} = \frac{j\delta}{n}. \tag{4.18}$$

Let $I_i' = I_i^n$ for $i = 1, 2, \ldots, n$. We have from (4.17) and (4.7)

$$\sum_{i=1}^{n} |H(I_i')| < \frac{1}{2}\varepsilon. \tag{4.19}$$

On the other hand, for every $i$,

$$\lambda(I_i \Delta I_i') < \frac{n\delta}{n} = \delta \tag{4.20}$$

so that by (4.6) and (4.7)

$$|H(I_i) - H(I_i')| < \frac{\varepsilon}{2n}. \tag{4.21}$$

We get from (4.19) and (4.21)

$$\sum_{i=1}^{n} |H(I_i)| \leq \sum_{i=1}^{n} |H(I_i')| + \sum_{i=1}^{n} |H(I_i) - H(I_i')| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \tag{4.22}$$

and this ends the proof.

**4.1.5. Definition.** A function $H : \Phi_\varrho \to \mathbb{R}$ will be termed *generalized absolutely continuous in the sense of Kempisty* (or $ACGr$) on an interval $I_0$ if $I_0$ is expressible as a union of a sequence of sets on each of which $H$ is $ACr$.

Kempisty in [27] takes the sets in the sequence to be closed. Because of proposition 4.1.4 we will not do so, as we will deal mostly with additive continuous functions.

## 4.2. Burkill integral.

**4.2.1. Definition.** Let $I_0 \in \Phi_\varrho$, and $H : \Phi_\varrho \to \mathbf{R}$. The *Burkill integral* of $H$ over $I_0$ is the limit

$$\lim_{n(D)\to 0} \sum_{i=1}^{n} H(I_i) \tag{4.23}$$

where $D = \{I_1, \ldots, I_n\} \subset \Phi_\varrho$ is a division of $I_0$.

We will use the notation

$$(B)\int_{I_0} H \tag{4.24}$$

for the limit (4.23).

The *upper* and *lower Burkill integrals* of $H$ are defined in the natural manner, as the corresponding lower and upper limit.

**4.2.2. Definition.** If $H : \Phi_\varrho \to \mathbf{R}$ and $E \subset \mathbf{R}^2$ then we define

$$H_E(I) = \begin{cases} H(I) & \text{if } I \cap E \neq \emptyset; \\ 0 & \text{otherwise,} \end{cases} \tag{4.25}$$

and

$$H^E(I) = H(I) - H_E(I). \tag{4.26}$$

**4.2.3. Definition.** The *Burkill integral* of $H$ *over a* bounded *set* $E$ is defined as

$$(B)\int_E H = (B)\int_{R_E} H_E. \tag{4.27}$$

The *upper* and *lower Burkill integrals over a set* are defined similarly.

**4.2.4. Definition.** If $E \subset I_0$ is a closed set then its *portion* is a nonempty set of the form $I \cap E$, where $I \subset I_0$ is a nondegenerate closed interval.

We will say that $H : \Phi_\varrho \to \mathbf{R}$ is $Ir$ in $I_0$ if every closed set $F \subset I_0$ contains a portion on which $H$ is integrable in the sense of Burkill.

**4.2.5. Definition.** Let $I_0 \in \Phi$. We will say that $f : I_0 \to \mathbf{R}$ is *integrable in the sense of Kempisty* on $I_0$ if there exists an $H : \Phi \to \mathbf{R}$ which is

(i) additive,

(ii) continuous in the sense of Saks,

(iii) $Ir$ on $I_0$,

(iv) $ACGr$ on $I_0$,

(v) and such that

$$\mathrm{D}H(x) = f(x) \qquad \text{a.e. on } I_0. \tag{4.28}$$

The definition comes from [27]. The author does not specify there what he means by continuity of $H$, but a note after theorem 1.3 in [27] clearly implies that he means (ii).

## 4.3. Properties of derivatives and the Burkill integral.

**4.3.1. Lemma.** *Let $H : \Phi_\varrho \to \mathbf{R}$. Then the point functions*

$$\begin{aligned} x &\longmapsto \underline{\mathrm{D}}\, H(x), \\ x &\longmapsto \overline{\mathrm{D}}\, H(x) \end{aligned} \tag{4.29}$$

*are measurable.*

**Proof.** Let $r \in \mathbf{R}$ and

$$E = \{x : \overline{\mathrm{D}}\, H(x) > r\}. \tag{4.30}$$

A point $x$ belongs to $E$ if and only if there exists an $s \in \mathbf{R}$, $s > 0$, and a sequence of $\varrho$-regular intervals $\{I_n^x\}$ tending to $x$ (in the sense of the $\Delta_2$ base) such that

$$\frac{H(I_n^x)}{\lambda(I_n^x)} \geq r + s \tag{4.31}$$

for all $n \in \mathbf{N}$. Thus, if

$$E_{k,l} = \bigcup \Big\{ I \in \Phi_\varrho : d(I) \leq \frac{1}{k} \quad \text{and} \quad \frac{H(I)}{\lambda(I)} \geq r + \frac{1}{l} \Big\}, \tag{4.32}$$

then

$$E = \bigcup_{l \in \mathbf{N}} \bigcap_{k \in \mathbf{N}} E_{k,l}. \tag{4.33}$$

Every set $E_{k,l}$, as a union of a family of intervals in $\mathbf{R}^2$, is measurable (see [52], lemma 4.4.1, p. 112). Thus $E$ is measurable, and this ends the proof.

**4.3.2. Lemma.** *If $H$ is $ACr$ on $I_0$ then so is $|H|$.*

**Proof.** This is obvious.

**4.3.3. Corollary.** *Let $H$ be $ACr$ on $I_0$. Define*

$$\begin{aligned} P_H(I) &= \frac{1}{2}\Big(|H(I)| + H(I)\Big), \\ N_H(I) &= \frac{1}{2}\Big(|H(I)| - H(I)\Big). \end{aligned} \tag{4.34}$$

*Then both $P_H$ and $N_H$ are $ACr$ on $I_0$.*

**4.3.4. Lemma.** *If $H$ is $ACr$ on $I_0$, then all three Burkill integrals*

$$(B)\overline{\int}_{I_0} H, \tag{4.35}$$

$$(B)\underline{\int}_{I_0} H, \tag{4.36}$$

$$(B)\overline{\int}_{I_0} |H| \tag{4.37}$$

*are finite.*

**Proof.** Since we have

$$(B)\overline{\int}_{I_0} H \le (B)\overline{\int}_{I_0} P_H, \tag{4.38}$$

$$(B)\underline{\int}_{I_0} H \ge -(B)\overline{\int}_{I_0} N_H, \tag{4.39}$$

and

$$(B)\overline{\int}_{I_0} |H| \le (B)\overline{\int}_{I_0} P_H + (B)\overline{\int}_{I_0} N_H, \tag{4.40}$$

it suffices to show that

$$(B)\overline{\int}_{I_0} P_H \quad \text{and} \quad (B)\overline{\int}_{I_0} N_H \tag{4.41}$$

are finite.

Suppose not, and

$$(B)\overline{\int}_{I_0} N_H = +\infty. \tag{4.42}$$

By corollary 4.3.3, $N_H$ is $ACr$.

Let $\varepsilon > 0$. There exists an $\eta > 0$ such that if $I_1, \ldots, I_n \in \Phi_\varrho$ are nonoverlapping and

$$\sum_{i=1}^{n} \lambda(I_i) < \eta \tag{4.43}$$

then

$$\sum_{i=1}^{n} N_H(I_i) < \varepsilon. \tag{4.44}$$

Let $r$ be a real number such that

$$r > \frac{\varepsilon}{\eta}\lambda(I_0) + 1. \tag{4.45}$$

Because of (4.42) there exists a $\varrho$-regular division $\mathcal{D}$ of $I_0$ such that

$$n(\mathcal{D}) < \sqrt{\frac{\eta}{r}} \tag{4.46}$$

and

$$\sum_{I \in \mathcal{D}} N_H(I) > r + \varepsilon. \tag{4.47}$$

Then, by (4.46), for any $I \in \mathcal{D}$ we have

$$\lambda(I) \le \left(n(I)\right)^2 < \frac{\eta}{r}. \tag{4.48}$$

Write $\mathcal{D} = \{I_1, \ldots, I_n\}$. Set

$$\mathcal{D}_1 = \{I_1, \ldots, I_{k_1}\}, \tag{4.49}$$

where $k_1$ is the smallest integer $k$ such that

$$\frac{r-1}{r}\eta \le \sum_{i=1}^{k} \lambda(I_i) < \eta \tag{4.50}$$

(if there is no such $k$, let $\mathcal{D}_1 = \mathcal{D}$).

Let $\mathcal{D}_2$ be obtained from $\mathcal{D} \setminus \mathcal{D}_1$ the way $\mathcal{D}_1$ was obtained from $\mathcal{D}$.

Continuing that process, we will stop at some point, and get a family of classes of $\varrho$-regular intervals

$$\mathcal{D}_1, \ldots, \mathcal{D}_l \tag{4.51}$$

such that

$$\frac{r-1}{r}\eta \leq \sum_{I \in \mathcal{D}_i} \lambda(I) < \eta \qquad \text{for } i = 1, 2, \ldots, l-1 \tag{4.52}$$

and

$$\sum_{I \in \mathcal{D}_l} \lambda(I) < \eta. \tag{4.53}$$

Note that, since $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \ldots \cup \mathcal{D}_l = \mathcal{D}$, we have

$$l < \frac{\lambda(I_0)}{\frac{r-1}{r}\eta} + 1 = \frac{r\lambda(I_0)}{(r-1)\eta} + 1. \tag{4.54}$$

Now, since $N_H$ is $ACr$ and

$$\sum_{I \in \mathcal{D}_i} \lambda(I) < \eta, \tag{4.55}$$

we have

$$\sum_{I \in \mathcal{D}_i} N_H(I) < \varepsilon, \qquad \text{for } i = 1, 2, \ldots, l. \tag{4.56}$$

From (4.45), (4.54), and (4.56) we get

$$\begin{aligned} \sum_{I \in \mathcal{D}} N_H(I) = \sum_{i=1}^{l} \sum_{I \in \mathcal{D}_i} N_H(I) < \\ \frac{r\lambda(I_0)\varepsilon}{(r-1)\eta} + \varepsilon < r + \varepsilon, \end{aligned} \tag{4.57}$$

which contradicts (4.47). Thus (4.42) is impossible. The other half of the lemma is proved similarly.

**4.3.5. Proposition.** *If $H$ is nonnegative and*

$$(\mathcal{B})\overline{\int}_{I_0} H < +\infty \tag{4.58}$$

*then $\overline{\mathrm{D}}\, H(x)$ and $\underline{\mathrm{D}}\, H(x)$ are finite a.e. on $I_0$.*

**Proof.** Note that $\underline{\mathrm{D}}\, H(x) \geq 0$ so that we need only to show that $\overline{\mathrm{D}}\, H(x) < +\infty$ a.e. on $I_0$.

Let

$$(B)\overline{\int}_{I_0} H = r. \tag{4.59}$$

There exists an $\eta$ such that if $D$ is a $\varrho$-regular division of $I_0$ of norm less than $\eta$ then

$$\sum_{I \in D} H(I) < r + 1. \tag{4.60}$$

We will show that the set

$$E = \left\{x \in I_0 : \overline{\mathrm{D}}\, H(x) = +\infty\right\} \tag{4.61}$$

is of measure zero. Suppose not, i.e., $\lambda(E) > 0$. We can assume that the distance of $E$ from the boundary of $I_0$ is positive, since if it is zero, $E$ still contains a measurable subset of positive measure whose distance from $\partial I_0$ is positive.

Let

$$E_n = \left\{x \in I_0 : \overline{\mathrm{D}}\, H(x) > n\right\} \tag{4.62}$$

for $n \in \mathbb{N}$. Every $E_n$ is measurable, by lemma 4.3.1. There exists a $\kappa > 0$ such that

$$\lambda(E_n) > \kappa \tag{4.63}$$

for every $n \in \mathbb{N}$. Fix an $n \in \mathbb{N}$ such that

$$n > \frac{r+1}{\kappa}. \tag{4.64}$$

Choose an $\alpha \in \Delta_2$ such that for any $x \in E_n$, $(x, I) \in \alpha$

$$H(I) > n\lambda(I) \tag{4.65}$$

and

$$n(I) < \eta. \tag{4.66}$$

Let

$$\mathcal{E} = \left\{ I \in \Phi_\varrho : (x, I) \in \alpha \quad \text{for some } x \right\}. \tag{4.67}$$

Then $\mathcal{E}$ is a Vitali cover of $E_n$. Therefore by the Vitali Covering Theorem (see [52], theorem 4.3.1, p 109), there exists a finite class $\mathcal{M} \subset \mathcal{E}$ such that the elements of $\mathcal{M}$ are disjoint and

$$\sum_{I \in \mathcal{M}} \lambda(I) > \kappa. \tag{4.68}$$

From (4.67) and (4.68) we get

$$\sum_{I \in \mathcal{M}} H(I) > \sum_{I \in \mathcal{M}} n\lambda(I) > n\kappa > r + 1. \tag{4.69}$$

By extending $\mathcal{M}$ to a division $\mathcal{D}$ of $I_0$, of norm less than $\eta$, we get

$$\sum_{I \in \mathcal{D}} H(I) \geq \sum_{I \in \mathcal{M}} H(I) > r + 1. \tag{4.70}$$

This contradicts (4.60). Therefore $\lambda(E) = 0$.

**4.3.6. Corollary.** *If $H$ is $ACr$ in $I_0$ then $\overline{\mathrm{D}}\, H(x)$ and $\underline{\mathrm{D}}\, H(x)$ are finite a.e. in $I_0$.*

**Proof.** Using the notation of (4.34), write

$$H(I) = P_H(I) - N_H(I). \tag{4.71}$$

By lemmas 4.3.4 and 4.3.5, $\overline{\mathrm{D}}\, P_H(x)$, $\underline{\mathrm{D}}\, P_H(x)$, $\overline{\mathrm{D}}\, N_H(x)$, and $\underline{\mathrm{D}}\, N_H(x)$ are finite a.e., and that implies 4.3.6.

**4.3.7. Lemma.** *Let $H$ be $ACr$ in $I_0$. Write*

$$A_k = \left\{ x \in I_0 : \overline{\mathrm{D}}\, H(x) > k \right\}, \qquad k \in \mathbf{N}, \tag{4.72}$$

$$A'_k = \left\{ x \in I_0 : \underline{\mathrm{D}}\, H(x) > k \right\}, \qquad k \in \mathbf{N}, \tag{4.73}$$

$$B_n = \left\{ x \in I_0 : \overline{\mathrm{D}}\, H(x) < -n \right\}, \qquad n \in \mathbf{N}, \tag{4.74}$$

$$B'_n = \left\{ x \in I_0 : \underline{\mathrm{D}}\, H(x) < -n \right\}, \qquad n \in \mathbf{N}. \tag{4.75}$$

*Then*

$$\lim_{k\to\infty} k\lambda(A_k) = \lim_{k\to\infty} k\lambda(A'_k) = \lim_{n\to\infty} n\lambda(B_n) = \lim_{n\to\infty} n\lambda(B'_n) = 0. \tag{4.76}$$

**Proof.** We will show that

$$\lim_{k\to\infty} k\lambda(A_k) = 0. \tag{4.77}$$

Note that $A'_k \subset A_k$ and $B'_n \supset B_n$, so that the only other part requiring a proof is

$$\lim_{n\to\infty} n\lambda(B'_n) = 0. \tag{4.78}$$

It can be proved similarly as (4.77) will be.

Suppose that (4.77) does not hold. There exists an $\eta > 0$ such that

$$k\lambda(A_k) > \eta \tag{4.79}$$

for infinitely many $k$. Let $\widetilde{A}_k$ be a subset of $A_k$ which has a positive distance from the boundary of $I_0$ and is such that

$$\lambda(\widetilde{A}_k) > \frac{3}{4}\lambda(A_k). \tag{4.80}$$

Choose an $\alpha \in \Delta_2$ such that for any $x \in \widetilde{A}_k$, $(x, I) \in \alpha$

$$H(I) > k\lambda(I). \tag{4.81}$$

Let

$$\mathcal{E} = \{I \in \Phi_\varrho : (x, I) \in \alpha \quad \text{for some } x \in \widetilde{A}_k\}. \tag{4.82}$$

Then $\mathcal{E}$ is a Vitali cover of $\widetilde{A}_k$. By the Vitali Covering Theorem (see [52], theorem 4.3.1, p 109) there exists a finite class $\mathcal{M} \subset \mathcal{E}$ consisting of disjoint $\varrho$-regular intervals and such that

$$2\lambda(\widetilde{A}_k) > \sum_{I\in\mathcal{M}} \lambda(I) > \frac{3}{4}\lambda(\widetilde{A}_k). \tag{4.83}$$

But then

$$2\lambda(A_k) > \sum_{I\in\mathcal{M}} \lambda(I) > \frac{9}{16}\lambda(A_k) > \frac{1}{2}\lambda(A_k). \tag{4.84}$$

Thus we get

$$\sum_{I\in\mathcal{M}} H(I) > \sum_{I\in\mathcal{M}} k\lambda(I) > \frac{1}{2}k\lambda(A_k) > \frac{1}{2}\eta. \tag{4.85}$$

However, $H$ is $ACr$, and by corollary 4.3.6

$$\lim_{k\to\infty} \lambda(A_k) = 0. \tag{4.86}$$

Therefore, by choosing $k$ large enough, we can make

$$\sum_{I\in\mathcal{M}} \lambda(I) \tag{4.87}$$

small sufficiently to have

$$\sum_{I\in\mathcal{M}} H(I) < \frac{1}{2}\eta. \tag{4.88}$$

This contradicts (4.85). The proof is ended.

**4.3.8. Lemma.** *If $H$ is $ACr$ in $I_0$ then*

$$(\mathcal{B})\underline{\int}_{I_0} H \le \int_{I_0} \underline{\mathrm{D}}\, H(x)dx \le \int_{I_0} \overline{\mathrm{D}}\, H(x)dx \le (\mathcal{B})\overline{\int}_{I_0} H \tag{4.89}$$

*(the integrals in the middle are Lebesgue integrals).*

**Proof.** We will show that

$$\int_{I_0} \overline{\mathrm{D}}\, H(x)dx \le (\mathcal{B})\overline{\int}_{I_0} H, \tag{4.90}$$

the other half can be proved in a similar manner.

Define, for $n, k \in \mathbb{N}$

$$f_{n,k} = \min\big(k, \max(\overline{\mathrm{D}}\, H(x), -n)\big). \tag{4.91}$$

Then $f_{n,k}$ are bounded and measurable, therefore Lebesgue integrable. We will show that for $n, k$ sufficiently large

$$\int_{I_0} f_{n,k}(x)dx \le (\mathcal{B})\overline{\int}_{I_0} H. \tag{4.91}$$

This, obviously, will imply (4.90), since by lemma 4.3.4 the integral

$$(\mathcal{B})\overline{\int}_{I_0} H \tag{4.93}$$

is finite.

Let $\varepsilon > 0$ be arbitrary. Choose an $\eta > 0$ such that

(a) if $\mathcal{D}$ is a division of $I_0$ such that $n(\mathcal{D}) < \eta$ then

$$\sum_{I\in\mathcal{D}} H(I) \leq (\mathcal{B})\overline{\int}_{I_0} H + \varepsilon \tag{4.94}$$

and

(b) if $\mathcal{E}$ is a finite class of nonoverlapping $\varrho$-regular intervals such that

$$\sum_{I\in\mathcal{E}} \lambda(I) < \eta \tag{4.95}$$

then

$$\sum_{I\in\mathcal{E}} |H(I)| < \varepsilon. \tag{4.96}$$

Let $n, k \in \mathbb{N}$ be such that

$$\lambda(A_k) < \frac{1}{3}\eta, \tag{4.97}$$

$$\lambda(B_n) < \frac{1}{3}\eta, \tag{4.98}$$

and

$$k\lambda(A_k) < \varepsilon. \tag{4.99}$$

Let

$$-n = l_0 < l_1 < l_2 < \ldots < l_m = k \tag{4.100}$$

be such that for

$$E_i = \left\{x \in I_0 : l_{i-1} < \overline{\mathrm{D}}\, H(x) \leq l_i\right\}, \qquad i = 1, 2, \ldots, m \tag{4.101}$$

we have

$$\sum_{i=1}^{m} l_{i-1}\lambda(E_i) > \int_{I_0\setminus(A_k\cup B_n)} \overline{\mathrm{D}}\, H(x)dx - \varepsilon. \tag{4.102}$$

Note that

$$I_0 \setminus (A_k \cup B_n) = \bigcup_{i=1}^{m} E_i. \tag{4.103}$$

But

$$\int\limits_{I_0\setminus(A_k\cup B_n)} \overline{\mathrm{D}}H(x)dx = \int\limits_{I_0} f_{n,k}(x)dx - k\lambda(A_k) + n\lambda(B_n) > \int\limits_{I_0} f_{n,k}(x)dx - \varepsilon. \tag{4.104}$$

Thus

$$\sum_{i=1}^{m} l_{i-1}\lambda(E_i) > \int\limits_{I_0} f_{n,k}(x)dx - 2\varepsilon. \tag{4.105}$$

Let

$$\delta = \min\left(\frac{\varepsilon}{2m^2k}, \frac{\eta}{6m^2}\right). \tag{4.106}$$

Since the sets $E_i$ are measurable, for every $i$ there exists a finite collection of nonoverlapping intervals

$$J_1^i, \ldots, J_{k_i}^i \tag{4.107}$$

such that

$$E_i = \left(\bigcup_{l=1}^{k_i} J_l^i \cup G_i\right) \setminus L_i, \tag{4.108}$$

where $G_i$, $L_i$ are some measurable sets of measure less than $\delta$. Then

$$\lambda\left(\bigcup_{l=1}^{k_i} J_l^i \cap E_i\right) > \lambda(E_i) - \delta. \tag{4.109}$$

Some of $J_l^i$'s (for different $i$'s) might overlap. To avoid that, remove any nonvoid intersections of $J_l^i$'s and divide the remaining parts of intervals into finite number of $\varrho$-regular intervals.

For each $i$, let

$$P_1^i, \ldots, P_{s_i}^i \tag{4.110}$$

be so obtained elements of $\Phi_\varrho$. In each set

$$\bigcup_{l=1}^{k_i} J_l^i \cap E_i \tag{4.111}$$

we removed, in terms of measure, not more than the sum of all $\lambda(G_j)$, $\lambda(L_j)$ for $j \neq i$, i.e., $2\delta(m-1)$. We get then

$$\lambda\Big(E_i \cap \bigcup_{l=1}^{s_i} P_l^i\Big) > \lambda\Big(E_i \cap \bigcup_{l=1}^{k_i} J_l^i\Big) - 2\delta(m-1). \tag{4.112}$$

From (4.109) and (4.112) we have

$$\lambda\Big(E_i \cap \bigcup_{l=1}^{s_i} P_l^i\Big) > \lambda(E_i) - (2m-1)\delta. \tag{4.113}$$

Let $C_i$ be a subset of

$$E_i \cap \bigcup_{l=1}^{s_i} P_l^i \tag{4.114}$$

which has a positive distance from the boundary of $\bigcup_{l=1}^{s_i} P_l^i$ and is such that

$$\lambda(C_i) > \lambda\Big(E_i \cap \bigcup_{l=1}^{s_i} P_l^i\Big) - \frac{1}{2}\delta. \tag{4.115}$$

For every $x \in C_i$ we have

$$l_{i-1} < \overline{\mathrm{D}}\, H(x), \tag{4.116}$$

so that for every $\alpha \in \Delta_2[\{x\}]$ there exist $(x, I) \in \alpha$ such that

$$l_{i-1}\lambda(I) < H(I). \tag{4.117}$$

We can assume that for every such $(x, I)$, $n(I) < \eta$ and $I \subset \bigcup_{l=1}^{s_i} P_l^i$.

Let

$$\mathcal{M} = \Big\{I \in \Phi_\varrho : \text{for some } x \in C_i \text{ and } \alpha \in \Delta_2[\{x\}], \\ (x, I) \in \alpha \text{ and (4.117) holds}\Big\}. \tag{4.118}$$

Then $\mathcal{M}$ covers $C_i$ in the sense of Vitali. By the Vitali Covering Theorem (see [52], theorem 4.3.1, p. 109), there exists a finite class

$$\{R_1^i, \ldots, R_{t_i}^i\} \subset \mathcal{M} \tag{4.119}$$

consisting of disjoint intervals such that

$$\lambda\Big(\bigcup_{l=1}^{t_i} R_l^i\Big) > \lambda(C_i) - \frac{1}{2}\delta. \tag{4.120}$$

Then, by (4.115) and (4.120), we have

$$\lambda\Big(\bigcup_{l=1}^{t_i} R_l^i\Big) > \lambda\Big(E_i \cap \bigcup_{l=i}^{s_i} P_l^i\Big) - \delta. \tag{4.121}$$

Combining (4.113) and (4.121) we get

$$\lambda\Big(\bigcup_{l=1}^{t_i} R_l^i\Big) > \lambda(E_i) - 2m\delta. \tag{4.122}$$

Let

$$\mathcal{R} = \{R_l^i : l = 1, 2, \dots,\ i = 1, 2, \dots, m\}. \tag{4.123}$$

The elements of $\mathcal{R}$ are nonoverlapping. By (4.97), (4.98), (4.103) and (4.123) we have

$$\begin{aligned}\lambda\Big(\bigcup \mathcal{R}\Big) >& \lambda(I_0 \setminus (A_k \cup B_n)) - 2m^2\delta \geq \\ & \lambda(I_0) - \lambda(A_k) - \lambda(B_n) - \frac{1}{3}\eta > \lambda(I_0) - \eta.\end{aligned} \tag{4.124}$$

If $\mathcal{D}$ is a division of $I_0 \setminus \bigcup \mathcal{R}$ then by (b) and (4.124)

$$\Big|\sum_{I\in\mathcal{D}} H(I)\Big| < \varepsilon. \tag{4.125}$$

Since $n(I) < \eta$ for every $I \in \mathcal{R}$, (a) and (4.125) give

$$\sum_{I\in\mathcal{R}} H(I) \leq (\mathcal{B}) \overline{\int}_{I_0} H + 2\varepsilon. \tag{4.126}$$

On the other hand, by (4.106), (4.117), and (4.122) we have

$$\begin{aligned}\sum_{I\in\mathcal{R}} H(I) =& \sum_{i=1}^{m}\Big(\sum_{l=1}^{t_i} H(R_l^i)\Big) \geq \sum_{i=1}^{m} l_{i-1}\Big(\sum_{l=1}^{t_i} \lambda(R_l^i)\Big) \geq \\ & \sum_{i=1}^{m} l_{i-1}(\lambda(E_i) - 2m\delta) \geq \sum_{i=1}^{m} l_{i-1}\lambda(E_i) - 2m^2k\delta \geq \\ & \sum_{i=1}^{m} l_{i-1}\lambda(E_i) - \varepsilon.\end{aligned} \tag{4.127}$$

Thus (4.105), (4.126), and (4.127) give

$$\int_{I_0} f_{n,k}(x)dx < (\mathcal{B}) \overline{\int}_{I_0} H + 5\varepsilon. \tag{4.128}$$

Since $\varepsilon$ was arbitrary, this implies (4.92), and then (4.90). The proof is ended.

**4.3.9. Lemma.** *If $H$ is $ACr$ on $I_0$ then*

$$(\mathcal{B})\overline{\int}_{I_0} H = \int_{I_0} \overline{\mathrm{D}}\, H(x)dx \tag{4.129}$$

*and*

$$(\mathcal{B})\underline{\int}_{I_0} H = \int_{I_0} \underline{\mathrm{D}}\, H(x)dx. \tag{4.130}$$

**Proof.** By the lemma 4.3.8 it suffices to show that

$$(\mathcal{B})\overline{\int}_{I_0} H \leq \int_{I_0} \overline{\mathrm{D}}\, H(x)dx. \tag{4.131}$$

Suppose (4.131) is not true. Set

$$\kappa = \frac{1}{3\lambda(I_0)}\Big((\mathcal{B})\overline{\int}_{I_0} H - \int_{I_0} \overline{\mathrm{D}}\, H(x)dx\Big). \tag{4.132}$$

For every $n \in \mathbb{N}$ there exists a $\varrho$-regular division $\mathcal{D}_n$ of $I_0$ such that $n(\mathcal{D}_n) < \frac{1}{n}$ and

$$\sum_{I\in\mathcal{D}_n} H(I) - \int_{I_0} \overline{\mathrm{D}}\, H(x)dx > 2\kappa\lambda(I_0). \tag{4.133}$$

Let

$$\mathcal{E}_n = \Big\{I \in \mathcal{D}_n : H(I) - \int_{I} \overline{\mathrm{D}}\, H(x)dx > \kappa\lambda(I)\Big\}. \tag{4.134}$$

Let $\eta$ be a positive number such that if $I_1, \ldots, I_k$ are nonoverlapping $\varrho$-regular intervals such that

$$\sum_{i=1}^{k} \lambda(I_i) < \eta \tag{4.135}$$

then

$$\Big|\sum_{i=1}^{k} H(I_i) - \sum_{i=1}^{k}\int_{I_i} \overline{\mathrm{D}}\, H(x)dx\Big| < \kappa\lambda(I_0). \tag{4.136}$$

Such an $\eta$ exists since both $H$ and $\int \overline{\mathrm{D}}\, H(x)$ are $ACr$ on $I_0$.

We claim that

$$\lambda\Big(\bigcup_{I\in\mathcal{E}_n} I\Big) \geq \eta. \tag{4.137}$$

Suppose not. Then

$$\sum_{I\in\mathcal{D}_n} H(I) - \int_{I_0} \overline{\mathrm{D}}\, H(x)dx = \sum_{I\in\mathcal{E}_n} H(I) - \int_{\bigcup\mathcal{E}_n} \overline{\mathrm{D}}\, H(x)dx + \sum_{I\in\mathcal{D}_n\setminus\mathcal{E}_n} H(I) - \sum_{I\in\mathcal{D}_n\setminus\mathcal{E}_n} \int_I \overline{\mathrm{D}}\, H(x)dx < \kappa\lambda(I_0) + \sum_{I\in\mathcal{D}_n\setminus\mathcal{E}_n} \kappa\lambda(I) \leq 2\kappa\lambda(I_0), \tag{4.138}$$

a contradiction to (4.133).

Let

$$K_n = \bigcup_{I\in\mathcal{E}_n} I \tag{4.139}$$

for $n \in \mathbb{N}$. We have

$$\lambda(K_n) \geq \eta \tag{4.140}$$

for every $n \in \mathbb{N}$. Set

$$K = \limsup_{k\to\infty} K_n. \tag{4.141}$$

Then

$$\lambda(K) \geq \eta. \tag{4.142}$$

Let $x \in K$ be a point where

$$\mathrm{D}\int_I \overline{\mathrm{D}}\, H(x)dx = \overline{\mathrm{D}}\, H(x) \tag{4.143}$$

— almost every point of $K$ has this property.

Since $x \in K$, by the definition of $K$ and (4.134), there exists a sequence $\{I_m\}$ of $\varrho$-regular intervals, each of them containing $x$, $n(I_m) < \frac{1}{m}$, such that

$$H(I_m) - \int_{I_m} \overline{\mathrm{D}}\, H(x)dx > \kappa\lambda(I_m). \tag{4.144}$$

Then $\{I_m\}$ converges to $x$ and by (4.144)

$$\overline{\mathrm{D}}\, H(x) \geq \limsup_{m\to\infty} \frac{\int_{I_m} \overline{\mathrm{D}}\, H(x)dx}{\lambda(I_m)} + \kappa. \tag{4.145}$$

This contradicts (4.143).

**4.3.10. Lemma.** *If $E$ is a closed subset of $I_0$, and $H : \Phi_\varrho \to \mathbf{R}$, then*

$$\underline{\mathrm{D}}\, H_E(x) = \begin{cases} \underline{\mathrm{D}}\, H(x) & \text{for } x \in E, \\ 0 & \text{otherwise,} \end{cases} \tag{4.146}$$

$$\overline{\mathrm{D}}\, H_E(x) = \begin{cases} \overline{\mathrm{D}}\, H(x) & \text{for } x \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{4.147}$$

**Proof.** This is obvious.

**4.3.11. Corollary.** *If $H$ is $ACr$ on a closed set $E \subset I_0$ then*

$$(\mathcal{B}) \overline{\int}_E H = \int_E \overline{\mathrm{D}}\, H(x) dx \tag{4.148}$$

*and*

$$(\mathcal{B}) \underline{\int}_E H = \int_E \underline{\mathrm{D}}\, H(x) dx. \tag{4.149}$$

**Proof.** This is a consequence of lemmas 4.3.9 and 4.3.10.

### 4.4. Semi-absolutely-continuous functions.

**4.4.1. Definition.** We will say that $H : \Phi_\varrho \to \mathbf{R}$ is $SACr$ on $I_0$ if for every $\epsilon > 0$ there exists an $\eta > 0$ such that if $I_1, \ldots, I_n$ are nonoverlapping $\varrho$-regular intervals contained in $I_0$ and

$$\sum_{i=1}^{n} \lambda(I_i) < \eta \tag{4.150}$$

then

$$\sum_{i=1}^{n} H(I_i) < \epsilon. \tag{4.151}$$

If we replace (4.151) by

$$\sum_{i=1}^{n} H(I_i) > -\epsilon, \tag{4.152}$$

then we obtain a definition of an $IACr$ function on $I_0$.

**4.4.2. Definition.** $H : \Phi_\varrho \to \mathbf{R}$ will be termed $SACr$ on a set $E \subset I_0$ if $H_E$ is $SACr$ on $I_0$. $H$ is $IACr$ on $E$ if $H_E$ is $IACr$ on $I_0$.

**4.4.3. Definition.** $H$ is $SACGr$ ($IACGr$) on $I_0$ if $I_0$ is expressible as a union of a sequence of sets on each of which $H$ is $SACr$ ($IACr$).

**4.4.4. Lemma.** *If $H : \Phi_\varrho \to \mathbf{R}$ and for every $x \in I_0$*

$$\overline{\mathrm{D}}\, H(x) < +\infty \tag{4.153}$$

*then $H$ is $SACGr$ on $I_0$.*

*If*

$$\underline{\mathrm{D}}\, H(x) > -\infty \tag{4.154}$$

*for every $x \in I_0$ then $H$ is $IACGr$ on $I_0$.*

**Proof.** We will show the first part only; the other one may be proved in the same manner.

Write for $n \in \mathbf{N}$

$$E_n = \Big\{ x \in I_0 : H(I) \le n\lambda(I) \quad \text{for every } (x, I) \in \alpha^{\varrho}_{\frac{1}{n}} \in \Delta_2 \Big\}. \tag{4.155}$$

If $x \in I_0$ then (4.153) is satisfied so there exists an $m \in \mathbf{N}$ such that

$$\overline{\mathrm{D}}\, H(x) < m. \tag{4.156}$$

There exists an $\alpha = \alpha^{\varrho}_{p} \in \Delta_2$ such that for every $(x, I) \in \alpha$

$$H(I) < m\lambda(I). \tag{4.157}$$

Let $n \in \mathbf{N}$ be such that

$$n > \max\Big(m, \frac{1}{p(x)}\Big). \tag{4.158}$$

Then for every $(x, I) \in \alpha^{\varrho}_{\frac{1}{n}}$

$$H(I) \le m\lambda(I) \le n\lambda(I), \tag{4.159}$$

so that $x \in E_n$. Consequently

$$I_0 = \bigcup_{n \in \mathbf{N}} E_n. \tag{4.160}$$

We will show that $H$ is $SACr$ on every set $E_n$. Fix an $n \in \mathbb{N}$. Let $\varepsilon > 0$ and

$$\eta = \min\left(\frac{\varepsilon}{n}, \frac{\varrho}{2n^2}\right). \tag{4.161}$$

Let $J_1, \ldots, J_k$ be nonoverlapping, $\varrho$-regular intervals meeting $E_n$ such that

$$\sum_{j=1}^{k} \lambda(J_j) < \eta. \tag{4.162}$$

Because of (4.162) we have

$$\lambda(J_j) < \eta \tag{4.163}$$

for every $j$, so that

$$\begin{gathered} d(J_j) \leq \sqrt{2}\, n(J_j) \leq \sqrt{2}\sqrt{\frac{\lambda(J_j)}{\varrho}} < \\ \sqrt{2}\sqrt{\frac{\eta}{\varrho}} \leq \sqrt{2}\sqrt{\frac{\varrho}{2n^2\varrho}} = \frac{1}{n} \end{gathered} \tag{4.164}$$

for every $j = 1, 2, \ldots, k$. If $x \in E_n \cap J_j$ (by assumption on $J_j$'s, this intersection is nonempty) then by (4.164)

$$(x, J_j) \in \alpha^{\varrho}_{\frac{1}{n}}, \tag{4.165}$$

and by (4.155)

$$H(J_j) \leq n\lambda(J_j). \tag{4.166}$$

Consequently

$$\sum_{j=1}^{k} H(J_j) \leq n \sum_{j=1}^{k} \lambda(J_j) < n\eta \leq \varepsilon. \tag{4.167}$$

This shows that $H$ is $SACr$ on $E_n$, thus being $SACGr$ on $I_0$.

**4.5. The relationship between the Kempisty integral and the $\Delta_2$-integral.**

**4.5.1. Lemma.** *Suppose $H : \Phi \to \mathbf{R}$ is additive, $f : I_0 \to \mathbf{R}$ for an $I_0 \in \Phi$, and*

$$\mathrm{D}\, H(x) = f(x) \qquad \text{a.e. on } I_0. \tag{4.168}$$

*Moreover, let $H$ be $ACGr$ on $I_0$, and continuous in the sense of Saks. Then $H$ is $Ir$ on $I_0$.*

Note that in view of this lemma, Kempisty's definition of integral (see [27] and definition 4.2.5) contains an unnecessary condition (iii).

**Proof.** Let $\{E_n\}$ be a sequence of sets such that

$$I_0 = \bigcup_{n \in \mathbf{N}} E_n \tag{4.169}$$

and $H$ is $ACr$ on each $E_n$. Since $H$ is continuous in the sense of Saks, by proposition 4.1.4 we can assume that all $E_n$'s are closed. Let $E$ be an arbitrary closed subset of $I_0$. Then

$$E = \bigcup_{n \in \mathbf{N}} (E_n \cap E). \tag{4.170}$$

By the Baire Category Theorem one of the sets $E_n \cap E$ contains a portion of $E$. Let $P$ be such a portion. Then $H$ is $ACr$ on $P$, $P$ is closed, and

$$\mathrm{D}\, H(x) = f(x) \qquad \text{a.e. on } P. \tag{4.171}$$

By 4.3.4, 4.3.8, 4.3.10, and 4.3.11, $H$ is Burkill-integrable on $P$ with

$$(B) \int_P H = \int_P f(x)dx. \tag{4.172}$$

This ends the proof.

**4.5.2. Theorem.** *If $f : I_0 \to \mathbf{R}$ is integrable in the sense of Kempisty, and $H$ is its Kempisty integral, then $f$ is $\Delta_2$-integrable on $I_0$ and for $I \subset I_0$*

$$H(I) = (\Delta_2) \int_I f d\lambda. \tag{4.173}$$

**Proof.** Let

$$E = \{x \in I_0 : \mathrm{D}\, H(x) = f(x)\} \tag{4.174}$$

and

$$E_0 = I_0 \setminus E. \tag{4.175}$$

By proposition 1.6.4 it suffices to show that

$$V\Big(H - f\lambda, \Delta_2(I_0)[E_0]\Big) = 0. \tag{4.176}$$

From propositions 2.3.11 and 2.7.1 we have

$$V\Big(\lambda, \Delta_2(I_0)[E_0]\Big) = 0. \tag{4.177}$$

Let

$$E_n = \Big\{x \in E_0 : |f(x)| \le n\Big\} \tag{4.178}$$

for $n \in \mathbb{N}$. Then

$$E_0 = \bigcup_{n \in \mathbb{N}} E_n. \tag{4.179}$$

On the other hand, from (4.177) and (4.178) we get

$$V\Big(f\lambda, \Delta_2(I_0)[E_n]\Big) = 0 \tag{4.180}$$

for every $n \in \mathbb{N}$. Consequently

$$V\Big(f\lambda, \Delta_2(I_0)[E_0]\Big) = 0. \tag{4.181}$$

Therefore, in order to prove (4.176), it suffices to show that

$$V\Big(H, \Delta_2(I_0)[E_0]\Big) = 0. \tag{4.182}$$

Since $I_0$ is expressible as a union of a sequence of sets on each of which $H$ is $ACr$, and $\lambda_{\Delta_2}(E_0) = 0$, it is enough to show that if $A$ is a set on which $H$ is $ACr$ and such that $\lambda_{\Delta_2}(A) = 0$ then $H_{\Delta_2}(A) = 0$.

Let $\varepsilon > 0$. Choose an $\eta$ such that if $I_1, \ldots, I_n$ are nonoverlapping $\varrho$-regular intervals meeting $A$ and

$$\sum_{i=1}^{n} \lambda(I_i) < \eta \tag{4.183}$$

then

$$\sum_{i=1}^{n} |H(I_i)| < \varepsilon. \tag{4.184}$$

Since $\lambda_{\Delta_2}(A) = 0$ — and this, by proposition 2.3.11, means simply that $\lambda(A) = 0$ — there exists an open set $G \supset A$ such that

$$\lambda(G) < \eta. \tag{4.185}$$

As stated in observation 2.3.10, $\Delta_2$ is compatible with the Euclidean topology. Therefore we can find an $\alpha_2 \in \Delta_2$ such that

$$\alpha_2[G] \subset \alpha(G). \tag{4.186}$$

Let $\pi_2 \subset \alpha_2[A]$ be a partition. For an $(x, I) \in \pi_2$ we have $I \subset G$ so that

$$\sum_{(x,I)\in\pi_2} \lambda(I) \leq \lambda(G) < \eta. \tag{4.187}$$

Consequently

$$\sum_{(x,I)\in\pi_2} |H(I)| < \varepsilon. \tag{4.188}$$

This shows that

$$V(H, \alpha_2[A]) \leq \varepsilon, \tag{4.189}$$

so that

$$H_{\Delta_2}(A) = 0. \tag{4.190}$$

The proof is ended.

**4.5.3. Remark.** In [27], theorem 3, p. 36, Kempisty states that a $\Delta_2$-Perron-integrable function is integrable in his sense. The proof is based on his claim that if we have $A \in \overline{\mathcal{A}}(I_0)$, $B \in \underline{\mathcal{A}}(I_0)$, $H \in \mathcal{A}(I_0)$ such that

$$A(I_0) - B(I_0) \leq \varepsilon, \tag{4.191}$$

$$A(I) \geq H(I) \geq B(I), \tag{4.192}$$

and $A$ is $IACGr$, $B$ is $SACGr$ on $I_0$, then $H$ is $ACGr$ on $I_0$. This, however, is not shown by Kempisty, and does not appear obvious.

A weaker result may be proved, namely:

**4.5.4. Theorem.** *Let $f : I_0 \to \mathbb{R}$ be $\Delta_2$-integrable. Write*

$$H(I) = (\Delta_2)\int_I f\,d\lambda \tag{4.193}$$

*for $I \subset I_0$. Assume additionally that $H$ is continuous in the sense of Saks in $I_0$, and that for every $\varepsilon > 0$ the functions $A \in \overline{\mathcal{A}}(I_0)$, $B \in \underline{\mathcal{A}}(I_0)$ (defined for $\varrho$-regular intervals), existing by theorem 2.1.1, such that*

$$A(I_0) - B(I_0) \leq \varepsilon, \tag{4.194}$$

$$\underline{\mathrm{D}}\,A(x) \geq f(x) \geq \overline{\mathrm{D}}\,B(x), \tag{4.195}$$

*and*

$$H(I) = \inf_A A(I) = \sup_B B(I) \tag{4.196}$$

*can be chosen so that $A$ is $SACGr$ on $I_0$, $B$ is $IACGr$ on $I_0$.*

*Then $f$ is integrable in the sense of Kempisty, and $H$ is its Kempisty integral.*

Note: By lemma 4.4.4, the assumption on majorants and minorants is satisfied, in particular, if we can choose $A$, $B$ so that

$$\begin{aligned} \overline{\mathrm{D}}\,A(x) &< +\infty \\ \underline{\mathrm{D}}\,B(x) &> -\infty \end{aligned} \tag{4.197}$$

for $x \in I_0$.

**Proof.** We need only show that $H$ is $ACGr$ on $I_0$ — the rest will be given by proposition 2.7.1 and lemma 4.5.1.

Let $\{E_n\}$ be a sequence of sets such that

$$\bigcup_n E_n = I_0 \tag{4.198}$$

and $A$ is $SACr$, $B$ is $IACr$ on every $E_n$. We will show that $H$ is $ACr$ on each set $E_n$.

Let $\eta > 0$ be arbitrary. Find a $\delta > 0$ such that if $\mathcal{D}$ is a $\varrho$-regular division contained in $I_0$ such that each $I \in \mathcal{D}$ meets $E_n$ then

$$\sum_{I \in \mathcal{D}} \lambda(I) < \delta \tag{4.199}$$

implies

$$\sum_{I\in D} A(I) < \frac{1}{2}\eta \tag{4.200}$$

and

$$\sum_{I\in D} B(I) > -\frac{1}{2}\eta. \tag{4.201}$$

Then we have

$$-\frac{1}{2}\eta < \sum_{I\in D} B(I) \le \sum_{I\in D} H(I) \le \sum_{I\in D} A(I) < \frac{1}{2}\eta, \tag{4.202}$$

so that

$$\left|\sum_{I\in D} H(I)\right| < \frac{1}{2}\eta. \tag{4.203}$$

Let

$$D' = \{I \in D : H(I) \ge 0\} \tag{4.204}$$

and

$$D'' = D \setminus D'. \tag{4.205}$$

By what we have proved, since

$$\sum_{I\in D'} \lambda(I) < \delta \qquad \text{and} \qquad \sum_{I\in D''} \lambda(I) < \delta, \tag{4.206}$$

we get

$$\left|\sum_{I\in D'} H(I)\right| = \sum_{I\in D'} |H(I)| < \frac{1}{2}\eta \tag{4.207}$$

and

$$\left|\sum_{I\in D''} H(I)\right| = \sum_{I\in D''} |H(I)| < \frac{1}{2}\eta. \tag{4.208}$$

This gives

$$\sum_{I\in D} |H(I)| < \eta, \tag{4.209}$$

and thus $H$ is $ACr$ on $E_n$.

**4.5.5. Corollary.** *If $H$ is additive, continuous in the sense of Saks, and $DH(x) = f(x)$ for every $x \in I_0$ then $f$ is $\Delta_2$-integrable and Kempisty-integrable on $I_0$, with the values of both integrals equal.*

## 4.6. Lebesgue integrability on a nontrivial subinterval.

**4.6.1. Proposition.** *Let $f$ be Kempisty-integrable on $I_0$ and $H$ be its integral. Then for every closed set $E \subset I_0$ there exists its portion $P = I_1 \cap E$ for some $I_1 \in \Phi$ such that*

*(i) $f$ is Lebesgue-integrable on $P$,*

*(ii) $H^E$ is Burkill-integrable on every $I \subset I_1$, and*

*(iii) for every $I \subset I_1$*

$$H(I) = \int_{I\cap E} f(x)dx + (B)\int_I H^E. \tag{4.210}$$

**Proof.** Let $E \subset I_0$ be closed. There exists a sequence of closed sets $\{E_n\}$ such that

$$\bigcup_n E_n = I_0 \tag{4.211}$$

and $H$ is $ACr$ on every $E_n$. Then

$$\bigcup_n (E_n \cap E) = E. \tag{4.212}$$

By the Baire Category Theorem one of the sets $E_n \cap E$ contains a portion $P_1 = I' \cap E$ of $E$. $H$ is $ACr$ on $P_1$.

$P_1$ is closed, so it contains a portion $P = I_1 \cap P_1$, where $I_1 \subset I'$, on which $H$ is Burkill-integrable. It is also a portion of $E$, and $H$ is $ACr$ on it.

Let $I \subset I_1$, and let $D$ be its arbitrary division with elements of $\Phi_\varrho$. We have

$$H(I) = H_E(D) + H^E(D), \tag{4.213}$$

because $H$ is additive. As $n(D) \to 0$

$$H_E(D) \to (B)\int_{E\cap I} H \tag{4.214}$$

and $H(I)$ remains constant. Therefore

$$\lim_{n(D)\to 0} H^E(D) \tag{4.215}$$

exists, and it is, by definition, equal to

$$(B)\int_I H^E. \tag{4.216}$$

(4.213) gives then

$$H(I) = (B)\int_{E\cap I} H + (B)\int_I H^E. \tag{4.217}$$

Since $H$ is $ACr$ on $E\cap I$, and Burkill integrable there, corollary 4.3.11 implies

$$(B)\int_{E\cap I} H = \int\limits_{E\cap I} f(x)dx, \tag{4.218}$$

so that (4.210) holds.

**4.6.2. Corollary.** *If $f$ is Kempisty-integrable on $I_0$ then there exists a nondegenerate subinterval $I_1$ of $I_0$ on which $f$ is Lebesgue-integrable.*

**Proof.** It suffices to take $E = I_0$ in 4.6.1.

**4.6.3. Corollary.** *If $f : I_0 \to \mathbf{R}$ is $\Delta_2$-integrable and its integral*

$$H(I) = (\Delta_2)\int_I f d\lambda \tag{4.219}$$

*is continuous in the sense of Saks, and if the minorants and majorants in the Perron integral can be chosen to be $SACGr$, $IACGr$, respectively, then there exists a nondegenerate interval $I_1 \subset I_0$ such that $f$ is Lebesgue-integrable on $I_1$.*

**Proof.** This follows from 4.5.4 and 4.6.2.

**4.6.4. Remark.** If $f$ is $\Delta_1$-integrable, then it is also $\Delta_2$-integrable. Therefore if the assumption about majorants and minorants holds for a $\Delta_1$-integrable function whose integral is continuous in the sense of Saks, then the conclusion of corollary 4.6.3 is true for it.

The question posed by Karták in [23] — whether for a Perron-integrable function one can find a nondegenerate interval on which it is Lebesgue-integrable, remains unanswered.

**4.6.5. Corollary.** *If $\mathrm{D}H(x) = f(x)$ for every $x \in I_0$ then there exists a nondegenerate interval $I_1 \subset I_0$ such that $f$ is Lebesgue-integrable on $I_1$.*

**4.7. The integrals of Mawhin and Pfeffer.**

**4.7.1. Definition.** Let $f : I_0 \to \mathbb{R}$. We will say that $f$ is *integrable in the sense of Mawhin* (or *Mawhin-integrable*) on $I_0$, with the value of the integral written as $(\mathcal{M}) \int_{I_0} f d\lambda$, if for every $\varepsilon > 0$ and every $\varrho \in (0,1)$ there exists a $p \in \mathcal{P}$ such that if $\pi$ is a $\varrho$-regular partition of $I_0$ satisfying

$$x \in I \subset D(x, p(x)) \qquad \text{for every} \quad (x, I) \in \pi, \tag{4.220}$$

then

$$\left| \sum_{(x,I)\in\pi} f(x)\lambda(I) - (\mathcal{M}) \int_{I_0} f d\lambda \right| < \varepsilon. \tag{4.221}$$

The notion introduced here originates from [35].

**4.7.2. Observation.** *If $f$ is Mawhin-integrable then it is $\Delta_2$-integrable with the values of both integrals equal.*

*If $f$ is $\Delta_1$-integrable then it is Mawhin-integrable, and the values of both integrals are equal.*

**4.7.3. Theorem.** *Let $I_0 \in \Phi$ and let $V$ be a vector field which is differentiable in an open domain containing $I_0$. Then $\nabla \cdot V$ (the divergence of $V$) is Mawhin-integrable over $I_0$ and*

$$(\mathcal{M}) \int_{I_0} \nabla \cdot V = \int_{\partial I_0} V \cdot n, \tag{4.222}$$

*where $n$ is the exterior normal, and the integral on the right-hand side is a Lebesgue integral (continuity of $V$ implies that we may even take it to be the classical Riemann integral).*

**Proof.** This is proved in [35], theorem 5.1, p. 625.

**4.7.4. Definition.** Let $I \in \Phi$ and let $\mathcal{H}$ be a family of straight lines. The *regularity* of $I$ relative to $\mathcal{H}$ is the number $r(I, \mathcal{H})$ defined as follows:

(i) if $\mathcal{H} = \emptyset$ then $r(I, \mathcal{H}) = r(I)$;

(ii) if $\mathcal{H}$ consists of a single line $H$

$$r(I, \mathcal{H}) = \frac{\lambda_1(I \cap H)}{n(I)} \tag{4.223}$$

whenever $I \cap H \neq \emptyset$, and $r(I, \mathcal{H}) = r(I)$ otherwise;

(iii) if $\mathcal{H}$ is arbitrary then

$$r(I, \mathcal{H}) = \sup_{H \in \mathcal{H}} r(I, H). \tag{4.224}$$

Obviously, $r(I) \leq r(I, \mathcal{H})$ for any $\mathcal{H}$.

**4.7.5. Definition.** Let $\varepsilon$ and $p$ be real-valued positive funtions on $I_0 \in \Phi$. Let $\mathcal{H}$ be a family of straight lines. A partition $\pi$ of $I_0$ is

(i) *p-fine* if for every $(x, I) \in \pi$ we have $x \in I \subset D(x, p(x))$;

(ii) an *$(\varepsilon, \mathcal{H})$-partition* if for every $(x, I) \in \pi$, $r(I, \mathcal{H}) \geq \varepsilon(x)$.

**4.7.6. Proposition.** *Let $P(\varepsilon, \mathcal{H}, p)$ be the set of all p-fine $(\varepsilon, \mathcal{H})$-partitions of $I_0$. Then $P(\varepsilon, \mathcal{H}, p) \neq \emptyset$ whenever $\varepsilon < 1$ and $p > 0$.*

**Proof.** See [47], corollary 2.5.

**4.7.7. Definition.** A pair $(\varepsilon, \mathcal{H})$ where $\varepsilon$ is a constant function, $0 < \varepsilon < \frac{1}{2}$, and $\mathcal{H}$ is a finite family of straight lines, is called a *regulator*.

**4.7.8. Definition.** A function $f : I_0 \to \mathbb{R}$ is *integrable in the sense of Pfeffer* (or *Pfeffer-integrable*), with the value of the integral written as

$$(P) \int_{I_0} f d\lambda, \tag{4.225}$$

if for every regulator $(\varepsilon, \mathcal{H})$ there exists a positive function $p : I_0 \to \mathbb{R}$ such that for every $p$-fine $(\varepsilon, \mathcal{H})$-partition $\pi$ of $I_0$ we have

$$\left| (P) \int_{I_0} f d\lambda - \sum_{(x,I) \in \pi} f(x) \lambda(I) \right| < \varepsilon. \tag{4.226}$$

This definition comes from [47].

**4.7.9. Proposition.** *If $f$ is Pfeffer-integrable then it is $\Delta_2$-integrable and the values of both integrals are equal.*

*If $f$ is $\Delta_1$-integrable then it is Pfeffer-integrable and the values of both integrals are equal.*

**Proof.** Let $\frac{1}{2} > \epsilon > 0$. Put $\mathcal{N} = \emptyset$. Find $p \in \mathcal{P}$ which corresponds to $(\epsilon, \mathcal{N})$ in the definition of the Pfeffer integral. Let $\alpha = \alpha_p^\epsilon \in \Delta_2$. Then for every $\pi \subset \alpha$, a partition of $I_0$, we have (4.226). This ends the proof of the first part. The second part is obvious.

**4.7.10. Remark.** As shown in [47], remark 7.3, if we take

$$I_0 = [0,1] \times [0,1], \tag{4.227}$$

$$I_1 = [-1,0] \times [0,1], \tag{4.228}$$

$$A_+^n = \left[\frac{3}{2^{n+1}}, \frac{1}{2^{n-1}}\right] \times \left[0, \frac{1}{2^{2n}}\right], \tag{4.229}$$

$$A_-^n = \left[0, \frac{1}{2^n}\right] \times \left[\frac{3}{2^{n+1}}, \frac{1}{2^{n-1}}\right]. \tag{4.230}$$

and let $f : I_0 \cup I_1 \to \mathbf{R}$ be defined as follows:

$$f(x) = \begin{cases} \pm n\, 2^{3n+1} & \text{if } x \in A_\pm^n \\ 0 & \text{otherwise,} \end{cases} \tag{4.231}$$

then $f$ is Mawhin-integrable on $I_0$ but not on $I_0 \cup I_1$ or $[0, \frac{1}{2}] \times [0,1]$. On the other hand, as Pfeffer shows in [47], if a function $g$ is Pfeffer-integrable on $I'$ and $I''$, then it is Pfeffer-integrable on $I' \cup I''$.

The function defined in (4.231) is not Pfeffer-integrable.

**4.7.11. Theorem.** *Let $I_0 \in \Psi$ and let $V$ be a continuous vector field which is differentiable on $I_0^\circ$. Then $\nabla \cdot V$ is Pfeffer-integrable and*

$$(\mathcal{P}) \int_{I_0} \nabla \cdot V = \int_{\partial I_0} V \cdot n, \tag{4.232}$$

*where the integral on the right-hand side is a Lebesgue integral.*

**Proof.** This is shown in [47], corollary 5.5.

**4.7.12. Example.** Let

$$I_n = \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \qquad n = 0, 1, \ldots \tag{4.233}$$

Using a technique of [39], 1.3, we can construct continuously differentiable functions $\phi_n$ on $\mathbf{R}$ such that

(i) $0 \le \phi_n \le 1$, $\phi_n(t) = 0$ for $t \le \frac{4}{3}2^{-n-1}$, and $\phi_n(t) = 1$ for $t \ge \frac{5}{3}2^{-n-1}$, and

(ii) there exists a $\kappa > 0$ such that

$$\int_{I_n} \phi_n \ge \frac{\kappa}{2^n} \qquad \text{for } n = 0, 1, \ldots \tag{4.234}$$

Given $(u, v) \in \mathbf{R}^2$, put

$$f(u,v) = \begin{cases} 0 & \text{if } v \le 0, \\ v^2 \sin u & \text{if } v \ge 1, \\ \phi_n(v)\, v^2 \sin(8^n u) + & \\ \qquad (1 - \phi_n(v))\, v^2 \sin(8^{n+1} u) & \text{if } v \in I_n,\ n = 0, 1, \ldots \end{cases} \tag{4.235}$$

As shown is [47], example 5.7, since the field $V = (f, 0)$ is differentiable, $\frac{\partial^2 f}{\partial u \partial v}(u, v)$ is Pfeffer-integrable on $[0, 2\pi] \times [0, 1]$. On the other hand

$$\int_0^1 \frac{\partial^2 f}{\partial u \partial v}(u, v) dv \tag{4.236}$$

does not exist for almost all $u \in [0, 2\pi]$. By the corollary 3.3.1 $\frac{\partial^2 f}{\partial u \partial v}(u, v)$ is not $\Delta_1$-integrable on $[0, 2\pi] \times [0, 1]$.

**4.7.13. Example.** Tolstov in [56] constructed a continuous function $g : [0, 1] \times [0, 1] \to \mathbf{R}$ such that the second mixed partial derivatives of $g$ exist everywhere on $(0, 1) \times (0, 1)$, are finite, and

$$\frac{\partial^2 g}{\partial u \partial v} = \frac{\partial^2 g}{\partial v \partial u}, \tag{4.237}$$

and yet for

$$f(u, v) = \frac{\partial^2 g}{\partial u \partial v}(u, v), \tag{4.238}$$

$f$ is not Lebesgue-integrable on any nondegenerate subinterval of $[0, 1] \times [0, 1]$.

By corollary 4.6.2, $f$ is not Kempisty-integrable.

It might be interesting to note that if we put

$$V = \left(0, \frac{\partial g}{\partial u}\right) \tag{4.239}$$

then

$$f = \nabla \cdot V. \tag{4.240}$$

We do not know, however, whether $V$ is differentiable, and we do not know if $f$ is Mawhin-integrable, or Pfeffer-integrable.

*Chapter 5*

# APPROXIMATE DERIVATION BASES

We define three density topologies on the plane.

Then we introduce the notion of a filtered base, and define approximate bases $\Delta_5$,$\Delta_6$, and $\Delta_7$ as filtered bases generated by the density topologies on the plane.

We also define a product approximate base $\Delta_4$, finer than $\Delta_1$, for which the Fubini Theorem proved in chapter 3 holds.

We show that $\Delta_7 \preceq \Delta_6 \preceq \Delta_5 \preceq \Delta_4$ which in turn gives us the relationships between integrals generated by these bases.

The nonabsolute integral of Chelidze and Dzhvarsheĭshvili is then defined. We show that a certain subclass of functions integrable in the sense of Chelidze and Dzhvarsheĭshvili consists of functions which are $\Delta_7$-integrable.

## 5.1. Density topologies on the plane.

**5.1.1. Definition.** We will say that $x \in \mathbf{R}^2$ is an ***ordinary density point*** of $E \subset \mathbf{R}^2$ if

$$\lim_{Q \to x} \frac{\lambda(E \cap Q)}{\lambda(Q)} = 1, \tag{5.1}$$

where $Q$ is a square centered at $x$, and $Q \to x$ is interpreted with respect to the $\Delta_1$ base.

A point $x$ is a ***strong density point*** of $E \subset \mathbf{R}^2$ if

$$\lim_{I \to x} \frac{\lambda(E \cap I)}{\lambda(I)} = 1, \tag{5.2}$$

where $I$ is and interval centered at $x$, and $I \to x$ is interpreted as above.

A point $x_0 = (u_0, v_0)$ is a *$\tilde{d}$-density point* of $E \subset \mathbf{R}^2$ if it is its strong density point, and for

$$L_1 = \{(u_0, v) : v \in \mathbf{R}\},\qquad L_2 = \{(u, v_0) : u \in \mathbf{R}\} \tag{5.3}$$

we have

$$d_e(E \cap L_1, (u_0, v_0)) = d_e(E \cap L_2, (u_0, v_0)) = 1, \tag{5.4}$$

where $d_e$ denotes the outer linear density (see [2], p. 18).

**5.1.2. Definition.** Let us write, for $E \subset \mathbf{R}^2$, $d_o(E)$ for the set of its ordinary density points, $d_s(E)$ for the set of its strong density points, and $\tilde{d}(E)$ for the set of its $\tilde{d}$-density points. Let

$$\begin{aligned} \mathcal{T}_o &= \{E \subset \mathbf{R}^2 : E \text{ is measurable and } E \subset d_o(E)\},\\ \mathcal{T}_s &= \{E \subset \mathbf{R}^2 : E \text{ is measurable and } E \subset d_s(E)\},\\ \tilde{\mathcal{T}} &= \{E \subset \mathbf{R}^2 : E \text{ is measurable and } E \subset \tilde{d}(E)\}. \end{aligned} \tag{5.5}$$

Then $\mathcal{T}_o$, $\mathcal{T}_s$, and $\tilde{\mathcal{T}}$ are topologies on $\mathbf{R}^2$ (see [10]).

**5.1.3.** Every $\tilde{d}$-density point is a strong density point, and every strong density point is an ordinary density point, so that we have

$$\tilde{\mathcal{T}} \subset \mathcal{T}_s \subset \mathcal{T}_o. \tag{5.6}$$

Furthermore, the set

$$E_1 = \{(u, v) \in \mathbf{R}^2 : u \neq 0 \text{ whenever } v \neq 0\} \tag{5.7}$$

belongs to $\mathcal{T}_s$ but not to $\tilde{\mathcal{T}}$, and

$$E_2 = \{(u, v) \in \mathbf{R}^2 : |v| > u^2\} \cup \{(0,0)\} \tag{5.8}$$

belongs to $\mathcal{T}_o$ but not to $\mathcal{T}_s$, so that the containments in (5.6) are proper. All three topologies are finer than the natural topology on $\mathbf{R}^2$. As shown in [10], $\mathcal{T}_o$ is completely regular, but not normal, $\mathcal{T}_s$ is not even regular. It is not known whether $\tilde{\mathcal{T}}$ is completely regular.

## 5.2. Filtered bases.

**5.2.1.** We will assume that $X = \mathbf{R}^2$ (although the definition 5.2.2 may be generalized). We will say that an interval $I$ is generated by $x_1$ and $x_2$, elements of $X$, if $x_1$ and $x_2$ are opposite vertices of $I$.

**5.2.2. Definition.** Let a system

$$\mathcal{N} = \{N(x) : x \in X\} \tag{5.9}$$

of nontrivial filters $N(x)$ of subsets of $X$, converging to $x$, be given. A *filtered base* $\Delta$ generated by it is defined as

$$\Delta = \left\{\alpha_\eta : \eta \in \prod_{x \in X} N(x)\right\} \tag{5.10}$$

where

$$\alpha_\eta = \{(x, I) : I \text{ is generated by } x \text{ and some } x' \in \eta(x)\}. \tag{5.11}$$

An element $\eta$ of the Cartesian product in (5.10) will be called a *choice*.

**5.2.3.** Let $\mathcal{T}$ be a Hausdorff topology on the plane. Then

$$N(x) = \{G \in \mathcal{T} : x \in G\} \tag{5.12}$$

is a filter satisfying the assumptions of 5.2.2. Therefore any Hausdorff topology naturally generates a filtered base. We will write $\Delta_{\mathcal{T}}$ for that base.

**5.2.4. Observation.** *Suppose $\mathcal{T}'$ and $\mathcal{T}''$ are Hausdorff topologies on the plane and $\mathcal{T}' \subset \mathcal{T}''$. Then*

$$\Delta_{\mathcal{T}'} \succeq \Delta_{\mathcal{T}''}. \tag{5.13}$$

**5.2.5. Definition.** Let us write

$$\begin{aligned} \Delta_5 &= \Delta_{\tilde{\mathcal{T}}}, \\ \Delta_6 &= \Delta_{\mathcal{T}_*}, \\ \Delta_7 &= \Delta_{\mathcal{T}_\circ}. \end{aligned} \tag{5.14}$$

**5.2.6. Observation.** *Let $\mathcal{T}_{nat}$ be the natural topology on the plane. Then*

$$\tilde{\Delta}_1 = \Delta_{\mathcal{T}_{nat}}. \tag{5.15}$$

**5.2.7. Corollary.**

$$\Delta_7 \preceq \Delta_6 \preceq \Delta_5 \preceq \tilde{\Delta}_1. \tag{5.16}$$

**Proof.** This follows from 5.2.4 and 5.2.6.

**5.2.8. Observation.** *Any filtered base has a local character and is filtering down.*

**5.2.9. Corollary.** *Each of the bases $\Delta_5$, $\Delta_6$, and $\Delta_7$ has a local character and is filtering down.*

### 5.3. Approximate bases.

**5.3.1. Definition.** If $\mathcal{D}$ is the density topology on the line (see [10]), then we can define, just as it was done in 5.2.3, the filtered base on $\mathbb{R}$ generated by it. This base (see [53], p. 85) will be denoted by $\mathcal{A}$ and called the *approximate base* on the real line.

**5.3.2. Proposition.** *$\mathcal{A}$ is filtering down, has local character and the partitioning property.*

**Proof.** The partitioning property of $\mathcal{A}$ is shown in [19], and discussed in [53], p. 85.

**5.3.3. Proposition.** *The integral generated by $\mathcal{A}$ is the approximately continuous Perron integral of* [5].

**Proof.** See [19].

**5.3.4. Definition.**

$$\Delta_4 = \mathcal{A} \times \mathcal{A}. \tag{5.17}$$

**5.3.5. Proposition.** *$\Delta_4$ has the partitioning property.*

**Proof.** This follows from the proposition 3.1.5.

**5.3.6. Observation.** *If $u_0 \in \mathbf{R}$ is a density point of $E_1 \subset \mathbf{R}$, and $v_0 \in \mathbf{R}$ is a density point of $E_2 \subset \mathbf{R}$, then $(u_0, v_0)$ is a $\tilde{d}$-density point of $E_1 \times E_2$.*

**5.3.7. Corollary.**

$$\Delta_7 \preceq \Delta_6 \preceq \Delta_5 \preceq \Delta_4 \preceq \tilde{\Delta}_1. \tag{5.18}$$

**5.3.8. Definition.** The bases $\Delta_i$, for $i = 4, 5, 6, 7$, will be termed *approximate derivation bases* on the plane.

**5.3.9. Observation.** *$\lambda_A$ is the Lebesgue outer measure on $\mathbf{R}$.*

**Proof.** This is proved just as it was done in 2.3.11.

**5.3.10. Corollary.** *Let $f : [a,b] \times [c,d] \to \mathbf{R}$ be $\Delta_4$-integrable. Write*

$$T = \{u \in [a,b] : \text{ the approximately continuous Perron integral of } f \text{ exists on } [a,b]\}. \tag{5.19}$$

*Then $[a,b] \setminus T$ is of Lebesgue measure zero. If we define*

$$g(u) = \int_c^d f(u,v)dv \qquad \text{(approximately continuous Perron integral)} \tag{5.20}$$

*for $u \in T$, and choose $g(u)$ arbitrarily otherwise, then*

$$(\Delta_4) \int_{[a,b]\times[c,d]} f\, d\lambda = \int_a^b g(u)du, \tag{5.21}$$

*the integral on the right-hand side being the approximately continuous Perron integral.*

**Proof.** This follows from the theorem 3.2.1 and the proposition 5.3.3.

**5.3.10. Example.** Looking back at 3.4.1 we observe that the function constructed by Tolstov is $\Delta_2$-integrable, but not $\Delta_4$-integrable.

**5.3.11. Remark.** It is not known whether the bases $\Delta_5$, $\Delta_6$, and $\Delta_7$ have the partitioning property. However, by taking any of the forms (iii), (iv), or (v) of the equivalent definitions discussed in the theorem 1.6.1, we can talk about the Henstock (or — variational) integral generated by them, since the equivalences (iii)$\Leftrightarrow$(iv) $\Leftrightarrow$(v) do not require the partitioning property.

### 5.4. The Integral of Chelidze and Dzhvarsheĭshvili

**5.4.1. Definition.** We will say that a real-valued function of interval $H$ is *absolutely continuous in the sense of Chelidze-Dzhvarsheĭshvili* (or $AC-CD$) on a bounded set $E \subset \mathbf{R}^2$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $I_1, \ldots, I_n$ is a finite collection of nonoverlapping intervals, each of which has some two opposite vertices in $E$, such that

$$\sum_{i=1}^{n} \lambda(I_i) < \delta, \tag{5.22}$$

then

$$\sum_{i=1}^{n} |H(I_i)| < \varepsilon. \tag{5.23}$$

**5.4.2. Definition.** We will say that $H$ is *generalized absolutely continuous in the sense of Chelidze-Dzhvarsheĭshvili* (or $ACG-CD$) on $I_0$ if $I_0$ is expressible as a union of a sequence of sets on each of which $H$ is $AC-CD$.

**5.4.4. Definition.** If $H$ is a real-valued function defined on $\Phi$, the class of all closed, nondegenerate intervals in $\mathbf{R}^2$, then $D_{\Delta_7} H_\lambda(x)$ will be called the *approximate derivative* of $H$ at $x \in \mathbf{R}^2$, and denoted by $D_{ap}H(x)$.

**5.4.5. Definition.** A function $f : I_0 \to \mathbf{R}$ will be called *integrable in the sense of Chelidze and Dzhvarsheĭshvili* (or *CD-integrable*) if there exists a $ACG-CD$ function $H$, continuous in the sense of Saks and such that

$$D_{ap}H(x) = f(x) \qquad \text{a.e. on } I_0. \tag{5.24}$$

The definitions 5.4.1, 5.4.2, and 5.4.5 come from [6] and [7].

### 5.5. The relationship of the CD-integral to the other integrals.

**5.5.1. Example.** [7] (p. 163) contains an example of a function which is CD-integrable, but not integrable in the sense of Kempisty.

Let $g$ be a real-valued function on $[0,1]$ which is integrable in general Denjoy sense and such that for

$$G(x) = \int_0^x g(t)dt \tag{5.25}$$

$G$ does not have a finite derivative on a set $E \subset [0,1]$ of positive measure. The existence of such functions is well-known and shown in [52]. By the theorem 33, p. 161 of [7], $f(x,y) = g(x)g(y)$ is $CD$-integrable. It is not Kempisty-integrable.

**5.5.2. Example.** As shown in [7], p. 155, there exists a function of interval $H$ such that $\mathrm{D}_{ap}H(x)$ exists everywhere, but $H$ is not $ACG-CD$. Obviously, by 1.6.3, $f(x) = \mathrm{D}_{ap}H(x)$ is then $\Delta_7$-integrable to $H$. However, not being $ACG-CD$, $H$ cannot be the CD-integral of $f$.

The situation is more complicated, though. It is not clear whether there is no other function of interval $G$ which is the CD-integral of $f$. The equality $\mathrm{D}_{ap}G(x) = \mathrm{D}_{ap}H(x)$ a.e. does not guarantee that $G = H$ (compare it with the theorem 2.13.21, p. 153, in [7]). It is, in fact, an interesting question — is the function constructed in [7], p. 155, CD-integrable?

**5.5.3. Example.** As shown in [7], p. 194, the function constructed by Tolstov, and discussed in 3.3.1, is CD-integrable. The reason is that finiteness of lower and upper regular derivatives on an intervals function forces it to be $ACG-CD$ (see [7], theorem 2.14.22, p. 133). We already know that function is neither $\Delta_1$-integrable, nor $\Delta_4$-integrable.

**5.5.4. Theorem.** *Let $H : \Phi \to \mathbb{R}$ be such that for an $I_0 \in \Phi$, and $f : I_0 \to \mathbb{R}$,*

*(i)* $D_{ap}H(x) = f(x)$ a.e.,

*(ii) There exists a sequence of $\mathcal{T}_o$-open sets $\{E_n\}$ such that*

$$\bigcup_{n \in \mathbb{N}} E_n = I_0 \tag{5.26}$$

*and $H$ is AC-CD on each of the $E_n$'s.*

*Then $f$ is $\Delta_7$-integrable to $H$ on $I_0$.*

**Proof.** Exactly as it was done in the proof of the theorem 4.5.2, we observe that it is enough to show

$$V\big(H, \Delta_7(I_0)[E]\big) = 0 \tag{5.27}$$

whenever $E$ is a set of measure zero contained in a $\mathcal{T}_o$-open set $M$ on which $M$ is $AC - CD$.

Note that the problem mentioned in 5.3.11 does not cause difficulties, as in 4.5.2, 1.6.4 and hence 1.6.3 were used, which, for sake of consistency with 1.6.1, hypothesize but do not use the partitioning property.

Let $\varepsilon > 0$. There exists a $\delta$ such that whenever $\{I_1, \ldots, I_n\}$ is a finite system of nonoverlapping intervals with two opposite vertices in $M$ and

$$\sum_{i=1}^{n} \lambda(I_i) < \delta \tag{5.28}$$

then

$$\sum_{i=1}^{n} \left|H(I_i)\right| < \varepsilon. \tag{5.29}$$

Since $E$ is of measure zero, there exists a set $U$, open in the Euclidean topology, $U \supset E$, such that $\lambda(U) < \delta$.

For every $x \in E$ there exists an open ball $B_x$ centered at $x$, of radius $r(x)$, such that $B_x \subset U$.

Let, for $x \in E$,

$$\eta(x) = B_x \cap M \tag{5.30}$$

— note that $B_x \cap M$ is a $\mathcal{T}_o$-neighborhood of $x$.

For $x \in I_0 \setminus E$, let $\eta(x)$ be defined arbitrarily.

Let $\alpha_\eta$ be an element of $\Delta_7$ generated by $\eta$. Then, if $\pi$ is a partition contained in $\alpha_\eta[E]$, we have

$$\bigcup_{(x,I)\in\pi} I \subset \bigcup_{x\in E} B_x \subset U, \tag{5.31}$$

so that

$$\sum_{(x,I)\in\pi} \lambda(I) < \varepsilon. \tag{5.32}$$

On the other hand, if $(x, I) \in \pi \subset \alpha_\rho$, then $x$ and the opposite vertex of $I$ belong to $M$, so that

$$\sum_{(x,I)\in\pi} |H(I)| < \epsilon. \tag{5.33}$$

Consequently, $V\left(H, \Delta_7(I_0)[E]\right) = 0$. This completes the proof.

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