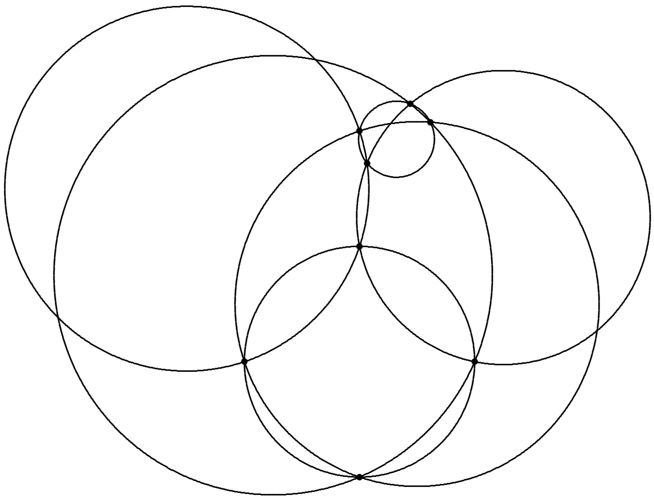
MA2219 An Introduction to Geometry



Wong Yan Loi

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The online software GeoGebra can be used to draw most of the figures.
Visit https://www.geogebra.org/geometry?lang=en

Chapter 1

A Brief History of Greek Mathematics

1.1 Early Greek mathematics

At the dawn of civilization, man discovered two mathematical concepts: "*multiplicity*" and "*space*". The first notion involved counting (of animals, days, etc.) and the second involved areas and volumes (of land, water, crop yield, etc). These evolved into two major branches of mathematics: arithmetic and geometry. (The word "*geometry*" is derived from the Greek roots "*geo*" meaning "*earth*" and "*metrein*" meaning "*measure*".) The mathematics of the Egyptians and the Babylonians was essentially empirical in nature. It has been traditional to state that demonstrative mathematics first appeared in the sixth century B.C. The Greek geometer, Thales of Miletus, is credited for giving some logical reasoning (rather than by intuition and experimentation) for several elementary results involving circles and angles of triangles. The next major Greek mathematician is Pythagoras (born ca. 572 B.C.) of Samos. He founded a scholarly society called the Pythagorean brotherhood (it was an academy for the study of philosophy, mathematics and natural science; it was also a society with secret rites). The Pythagoreans believed in the special role of "*whole number*" as the foundation of all natural phenomena. The Pythagoreans gave us 2 important results: Pythagorean theorem, and more importantly (albeit reluctantly), the irrational quantities (which struck a blow against the supremacy of the whole numbers). The Pythagorean, Hippasus, is credited with the discovery that the side of a square and its diagonal are incommensurable (i.e. a square of length 1 has a diagonal with irrational length). This showed that the whole numbers are inadequate to represent the ratios of all geometric lengths. This discovery established the supremacy of geometry over arithmetic in all subsequent Greek mathematics. For all the trouble that Hippasus caused, the Pythagoreans supposedly took him far out into the Mediterranean and tossed him overboard to his death - thereby indicating the dangers inherent in free thinking, even in the relatively austere discipline of mathematics! Hippocrates of Chios (born ca. 440 B.C.) is credited with two significant contributions to geometry. The first was his composition of the first Elements: the first exposition developing the theorems of geometry precisely and logically from a few given axioms and postulates. This treatise (which has been lost to history) was rendered obsolete by that of Euclid. His other contribution was

the quadrature of the lune.
Plato (427-347 B.C.) studied philosophy in Athens under Socrates. He then set out on his travels, studying mathematics under Theodorus of Cyrene in North Africa. On his return to Athens in 387 B.C. he founded the Academy. Plato's influence on mathematics was not due to any mathematical discoveries, but rather to his conviction that the study of mathematics provides the best training for the mind, and was hence essential for the cultivation of philosophers. The renowned motto over the door of his Academy states:

Let no one ignorant of geometry enter here.

Aristotle, a pupil of Plato, was primarily a philosopher. His contribution to mathematics is his analysis of the roles of definitions and hypotheses in mathematics. Plato's Academy did however produce some great mathematicians, one of whom is Eudoxus (ca. 408-355 B.C.). With the discovery of the incommensurables, certain proofs of the Pythagoreans in geometric theorems (such as those on similar triangles) were rendered false. Eudoxus developed the theory of proportions which circumvented these problems. This theory led directly to the work of Dedekind (Dedekind cuts) in the nineteenth century. His other great contribution, the method of exhaustion, has applications in the determination of areas and volumes of sophisticated geometric figures. This process was used by Archimedes to determine the area of the circle. The method of exhaustion can be considered the geometric forerunner of the modern notion of "*limit*" in integral calculus. Menaechmus (ca. 380 - 320 B.C.), a pupil of Eudoxus, discovered the conic sections. There is a legendary story told about Alexander the Great (356-323 B.C.) who is said to have asked his tutor, Menaechmus, to teach him geometry concisely, to which the latter replied,

> *O king, through the country there are royal roads and roads for common citizens, but in geometry there is one road for all.*

Alexander entered Egypt and established the city of Alexandria at the mouth of the Nile in 332 B.C. This city grew rapidly and reached a population of half a million within three decades. Alexander's empire fell apart after his death. In 306 B.C. one of his generals, Ptolemy, son of Lagos, declared himself King Ptolemy I (thereby establishing the Ptolemaic dynasty). The Museum and Library of Alexandria were built under Ptolemy I. Alexandria soon supplanted the Academy as the foremost center of scholarship in the world. At one point, the Library had over 600,000 papyrus rolls. Alexandria remained the intellectual metropolis of the Greek race until its destruction in A.D. 641 at the hands of the Arabs. Among the scholars attracted to Alexandria around 300 B.C. was Euclid, who set up a school of mathematics. He wrote the "*Elements*". This had a profound influence on western thought as it was studied and analyzed for centuries. It was divided into thirteen books and contained 465 propositions from plane and solid geometry to number theory. His genius was not so much in creating new mathematics but rather in the presentation of old mathematics in a clear, logical and organized manner. He provided us with an axiomatic development of the subject. The Elements begins with 23 definitions, 5 postulates and 5 common notions or general axioms. From these he proved his first proposition. All subsequent results were obtained from a blend of his definitions, postulates, axioms and previously proven propositions. He thus avoided circular arguments. There is a story about Euclid (reminiscent of the one about Menaechmus and Alexander): Ptolemy I once asked Euclid if there was in geometry any shorter way than that of the Elements, to which Euclid replied, *There is no royal road to geometry.*

The greatest mathematician of antiquity is Archimedes (287-212 B.C.) of Syracuse, Sicily. He made great contributions in applied mechanics (especially during the second Punic War against the Romans - missile weapons, crane-like beaks and iron claws for seizing ships, spinning them around, sinking or shattering them against cliffs,...), astronomy and hydrostatics. He devised methods for computing areas of curvilinear plane figures, volumes bounded by curved surfaces and methods of approximating π. Using the method of exhaustion (of Eudoxus), he anticipated the integral calculus of Newton and Leibniz by more than 2000 years; in one of his problems he also anticipated their invention of differential calculus. The above-mentioned problem involved constructing a tangent at any point of his spiral. There are numerous stories told about Archimedes. According to Plutarch, Archimedes would

> *... forget his food and neglect his person, to that degree that when he was occasionally carried by absolute violence to bathe or have his body anointed, he used to trace geometrical figures in the ashes of the fire, and diagrams in the oil of his body, being in a state of entire preoccupation, and, in the truest sense, divine possession with his love and delight in science.*

The death of Archimedes as told by Plutarch:

> *...as fate would have it, intent upon working out some problem by a diagram, and having fixed his mind alike and his eyes upon the subject of his speculation, he never noticed the incursion of the Romans, nor that the city was taken. In this transport of study and contemplation, a soldier, unexpectedly coming up to him, commanded him to follow to Marcellus; which he declined to do before he had worked out his problem to a demonstration, the soldier, enraged, drew his sword and ran him through.*

The next major mathematician of the third century B.C. was Apollonius (262-190 B.C.) of Perga, Asia Minor. His claim to fame rests on his work Conic Sections in eight books. It contains 400 propositions and supersedes the work in that subject of Menaechmus, Aristaeus and Euclid. The names "*ellipse,*" "*parabola*" and "*hyperbola*" were supplied by Apollonius. His methods are similar to modern methods and he is said to have anticipated the analytic geometry of Descartes by 1800 years.

The end of the third century B.C. saw the end of the Golden Age of Greek Mathematics. In the next three centuries only one mathematician made a significant contribution, Hipparchus of Nicaea (180-125 B.C.), who founded trigonometry.

SOME QUOTATIONS

Carl Friedrich Gauss: "*... that the sum of the angles cannot be less than* $180°$*: this is the critical point, the reef on which all the wrecks occur.*"

Janos Bolyai: "*Out of nothing I have created a strange new universe.*"

Bernhard Riemann: "*The unboundedness of space possesses a greater empirical certainty than any external experience. But its infinite extent by no means follows from this.*"

Bertrand Russell: "*... what matters in mathematics ... is not the intrinsic nature of our terms but the logical nature of their interrelations.*"

1.2 Euclid's Elements

EXTRACT FROM BOOK I OF EUCLID'S ELEMENTS

To illustrate the systematic approach that Euclid used in his elements, we include below an extract from Book 1 of the Elements.

DEFINITIONS

1. A point is that which has no part.
2. A line is breadthless length.
3. The extremities of a line are points.
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight lines on itself.
8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And where the lines containing the angles are straight, the angle is called rectilineal .
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other side is called a perpendicular to that on which it stands.
11. An obtuse angle is an angle greater than a right angle.
12. An acute angle is an angle less than a right angle.
13. A boundary is that which is an extremity of anything.
14. A figure is that which is contained by any boundary or boundaries.
15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;
16. And the point is called the centre of the circle.
17. A diameter of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.
18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.
19. Rectilineal figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.
20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.
22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.
23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

COMMON NOTIONS

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

POSTULATES or AXIOMS

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

There is much that can be said about Euclid's "*definitions*". However, we shall refrain from doing so. We shall instead examine more closely the controversial fifth postulate. Over a hundred years ago, a postulate was supposed to be a "*self-evident truth.*" Observe that the statements of the first four postulates are short and "*self-evident,*" and thus, readily acceptable by most people. However, the statement of the fifth postulate is rather long and sounds complicated. Nevertheless, it was deemed to be necessarily true, and hence one should be able to derive it from the other four postulates and the definitions. The problem then is to prove the fifth postulate as a theorem.

Let us first take another look at Postulate 2. It pertains to extensions of a finite straight line. Euclid implicitly assumed that Postulate 2 implies that straight lines must be "*unbounded in extent*", or are "*infinitely long.*" As it turns out, this is a hidden postulate (i.e. it should be considered a separate postulate). Euclid made tacit use of this assumption in proving Proposition 16 of Book I of *Elements*:

Proposition I.16. *In any triangle, if any one of the sides is produced, the exterior angle is greater than either of the interior and opposite angles.*

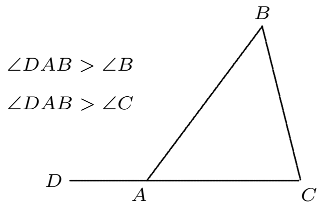


Figure 1.1: Proposition 16

This proposition was then used to prove:

Proposition I.17. *In any triangle, two angles taken together in any manner are less than two right angles.*

The following proposition also uses Proposition I.16 and hence requires that straight lines be infinitely long:

Proposition I.27. *If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.*

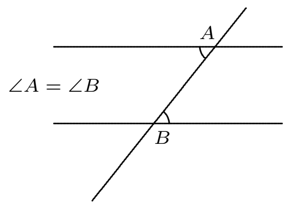


Figure 1.2: Proposition 27

1.3 The 5th postulate

Let us state Postulate 5 in modern language and notation.

Postulate 5. *If AB and CD are cut by EF so that $\alpha + \beta < 180°$, then AB and CD meet in the direction of B and D.*

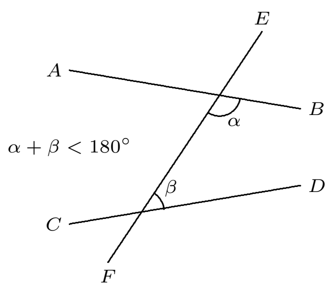


Figure 1.3: The 5th postulate

Euclid did not make use of Postulate 5 until Proposition 29 of Book I:
Proposition I.29. *A straight line falling on parallel straight lines makes the alternate angles equal to one another.*
He used this to prove the following (stated here in modern language and notation):
Proposition I.32. *The exterior angle of a triangle is equal to the sum of the two interior and opposite angles; the sum of the interior angles of a triangle is* $180°$.
Mathematicians, as early as Proclus (410-485 A.D.), have tried to prove Postulate 5 from Postulates 1-4 directly. However, they make implicit use of "*unstated axioms*" in their "*proofs.*" These unstated axioms were later found to be logically equivalent to the fifth postulate. This means that
(1) Postulates 1-4 + "*unstated axiom*" implies Postulate 5; and conversely,
(2) Postulates 1-5 implies "*unstated axiom.*"
Thus, the "*proofs*" involved circular reasoning.
Proclus' proof involved the use of the following unstated axiom:
Proclus' Axiom. *If a straight line cuts one of two parallels, it must cut the other one also.*
Other famous axioms logically equivalent to Postulate 5 include:
Playfair's Axiom. *If P is a point not on a line ℓ, then there is exactly one line through P that is parallel to ℓ.*
(Playfair's Axiom is actually a restatement of Euclid's Proposition I.31 in modern language and notation.)
Equidistance Axiom. *Parallel lines are everywhere equidistant.*
It turns out the second part of Proposition I.32 is also logically equivalent to Postulate 5:
Angle Sum of Triangle Axiom. *The sum of the interior angles of a triangle is* $180°$.

1.4 The work of Gerolamo Saccheri

Gerolamo Saccheri (1667-1733) was the first mathematician who attempted to prove the fifth postulate via an indirect method - reductio ad absurdum. This means that he assumed that Postulate 5 was not true and he attempted to derive a contradiction. In his book *Euclides ad omni naevo vindicatus* (Euclid vindicated of all flaws), he introduced what is now called the Saccheri quadrilateral. It is a quadrilateral $ABCD$ such that AB forms the base, AD and BC the sides such that $AD = BC$, and the angles at A and B are right angles. We shall refer to the $\angle C$ and $\angle D$ as summit angles.

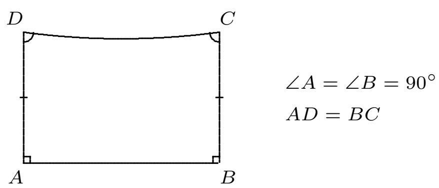


Figure 1.4: A Saccheri quadrilateral

Saccheri first proved that $\angle C = \angle D$. He then considered three mutually exclusive hypotheses regarding the summit angles:

HRA (Hypothesis of the Right Angle.) The summit angles are right angles. ($\angle C = \angle D = 90°$.)
HAA (Hypothesis of the Acute Angle.) The summit angles are acute angles. ($\angle C = \angle D < 90°$.)
HOA (Hypothesis of the Obtuse Angle.) The summit angles are obtuse angles. ($\angle C = \angle D > 90°$.)
HRA turns out to be logically equivalent to Postulate 5. To use the method of reductio ad absurdum Saccheri assumed that HRA is false. First, he would assume that HOA is true and show that this leads to a contradiction. Next, he would assume that HAA is true and show that this also leads to a contradiction.

When he adopted HOA, Saccheri indeed reached a contradiction. But in doing so, he made use of Proposition I.16, hence the hidden assumption that straight lines are infinitely long. It is precisely this hidden postulate that the HOA is contradicting. Saccheri thus proved (without knowing it) that Postulates 1-4, together with the additional postulate that straight lines are infinitely long, implies that the sum of the interior angles of a quadrilateral is equal to or less than $360°$.

When Saccheri attempted to eliminate the possibility of HAA, he experienced greater difficulties. Assuming HAA, he derived the fact that the sum of the interior angles of a triangle is less than $180°$, and other "*strange results*"; however, he never reached a contradiction. Nevertheless, he rejected HAA because "*is repugnant to the nature of the straight line*". He thus missed his chance of being a founder of the first non-euclidean geometry. Instead, the honors belong to three men - who discovered it independently.

Carl Friedrich Gauss (1777-1855) developed the geometry implied by HAA between the years 1810-1820, but he did not publish his results as he was too "*timid*". We know this only because of his correspondence and his private papers which became available after his death.

While Gauss and others were working in Germany, and had arrived independently at some of the results of non-euclidean geometry, there was a considerable interest in the subject in France and Britain inspired chiefly by A.M.Legendre (1752-1833). Assuming all Euclid's definitions, axioms and postulates except the 5th postulate, he proves an important result.

Theorem 1.1 (Legendre) *The sum of the three angles of a triangle is less than or equal to* $180°$.

Proof. Let $A_1A_2C_1$ be a triangle. Denote its angles and lengths of its sides as indicated in the figure. Along the line A_1A_2 (we assume it can be extended indefinitely), place the triangles $A_2A_3C_2, \ldots,$ $A_nA_{n+1}C_n$, each congruent to $A_1A_2C_1$ next to each other as shown in the figure. Then the triangles $A_2C_1C_2, \cdots, A_nC_{n-1}C_n$ are all congruent.

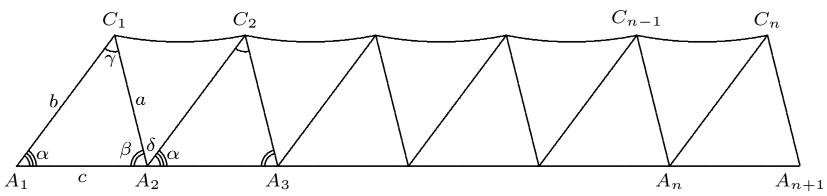


Figure 1.5: The angle sum of a triangle is less than or equal to $180°$

We prove the result by contradiction. Suppose $180° < \alpha + \beta + \gamma$. Then at the point A_2 on the straight line A_1A_2, $\alpha + \delta + \beta = 180° < \alpha + \beta + \gamma$ so that $\delta < \gamma$. As $C_1A_1 = A_2C_2$,

$C_1A_2 = A_2C_1$ and $\delta < \gamma$, we have $C_1C_2 < A_1A_2$. Let $C_iC_{i+1} = p$ for all $i = 1, \dots, n-1$. Then $d \equiv c - p > 0$. The total length $A_1C_1 + C_1C_2 + \cdots C_{n-1}C_n + C_nA_{n+1}$ is $b + (n-1)p + a$. By triangle inequality, it is greater than or equal to $A_1A_{n+1} = nc$. Thus $b + (n-1)p + a \geq nc$. That is $b - p + a \geq n(c-p) = nd$. By taking n sufficiently large, we have a contradiction. (Here we have used the Archimedean property of real numbers.)

1.5 Non-Euclidean geometry

Two contemporaries of Gauss, Janos Bolyai (1802-1860) and Nicolai Ivanovitch Lobachevsky (1793-1856), who worked independently of each other, are officially credited with the discovery of the first non-euclidean geometry. In 1829 Lobachevsky published his results (in Russian) on the new geometry in the journal "Kazan Messenger". Because this is a rather obscure journal, his results went unnoticed by the scientific community. In 1832 Janos Bolyai's results were published in an appendix to a work called "Tentamen", written by his father Wolfgang Bolyai, a lifelong friend of Gauss. Gauss received the manuscript and instantly recognized it as a work of a genius. However, Gauss was determined to avoid all controversy regarding non-euclidean geometry, and he not only suppressed his own work on the subject, but he also remained silent about the work of others on that subject.

In 1837 Lobachevsky published a paper on that subject in Crelle's journal and in 1840 he wrote a small book in German on the same subject. Again Gauss recognized genius and in 1842 he proposed Lobachevsky for membership in the Royal Society of Göttingen. But Gauss made no statement regarding Lobachevsky's work on non-euclidean geometry. He did not tell Lobachevsky about Janos Bolyai. Bolyai learned about Lobachevsky around 1848, but Lobachevsky died without knowing that he had a codiscoverer. Just before he died, when he was blind, Lobachevsky dictated an account of his revolutionary ideas about geometry. This was translated into French by Jules Houel in 1866, who also translated the works of Bolyai the following year. Thus began the widespread circulation of the ideas on the new non-euclidean geometry. Euclid was finally vindicated - Postulate 5 is indeed an independent postulate.

Consequence of HAA include:

- the sum of the angles of any triangle is less than $180°$,
- the existence of many parallel lines through a point not on a given line.

We now return to the hidden postulate on infinitely long lines. Bernhard Riemann (1826-1866) diluted Euclid's implicit assumption that lines are infinitely long and replaced it with endlessness or unboundedness. As a result, he discovered that

(1) Proposition I.16 no longer holds;

(2) Proposition I.17 no longer holds;

(3) Proposition I.27 no longer holds.

Now Proposition I.27 allows us to construct at least one parallel through a point not on a given line. A consequence of the non-validity of Proposition I.27 is that under HOA, there are no parallel lines! In this new geometry, the sum of the angles of a triangle is greater than $180°$.

By 1873 Felix Klein classified, unified and named the three geometries:

	Parabolic Geometry	**Hyperbolic Geometry**	**Elliptic Geometry**
Saccheri	HRA	HAA	HOA
Angle sum of triangle	$= 180°$	$< 180°$	$> 180°$
Playfair's	exactly 1 parallel	more than 1 parallel	no parallels
Curvature	zero	negative	positive
Founder	Euclid	Gauss, Bolyai, Lobachevski	Riemann

Exercise 1.1 Prove that Playfair's axiom implies Euclid's 5th Axiom.

Chapter 2

Basic Results in Book I of the Elements

2.1 The first 28 propositions

A plane geometry is "*neutral*" if it does not include a parallel postulate or its logical consequences. The first 28 propositions of Book I of Euclid's *Elements* are results in a neutral geometry that are proved based on the first 4 axioms and the common notions.

Proposition I.1. To construct an equilateral triangle.

Proposition I.2. To place a straight line equal to a given straight line with one end at a given point.

Proposition I.3. To cut off from the greater of two given unequal straight lines a straight line equal to the less.

Proposition I.4. (SAS) If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals to the triangle, and the remaining angles equal the remaining angles respectively.

Proposition I.5. In isosceles triangles, the angles at the base equal one another; and if the equal straight lines are produced further, then the angles under the base equal one another.

Proposition I.6. If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

Proposition I.7. Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.

Proposition I.8. (SSS) If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

Proposition I.9. To bisect a given rectilinear angle.

Proposition I.10. To bisect a given finite straight line.

Proposition I.11. To draw a straight line at right angles to a given straight line from a given point on it.

Proposition I.12. To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

Proposition I.13. If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

Proposition I.14. If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

Proposition I.15. If two straight lines cut one another, then they make the vertical angles equal to one another.

Proposition I.16. (Exterior Angle Theorem) In any triangle, if any one of the sides is produced, the exterior angle is greater than either of the interior and opposite angles.

Proposition I.17. In any triangle, two angles taken together in any manner are less than two right angles.

Proposition I.18. In any triangle, the angle opposite the greater side is greater.

Proposition I.19. In any triangle, the side opposite the greater angle is greater.

Proposition I.20. In any triangle, the sum of any two sides is greater than the remaining one.

Proposition I.21. If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangles, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

Proposition I.22. To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

Proposition I.23. To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

Proposition I.24. If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

Proposition I.25. If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have one of the angles contained by the equal straight lines greater than the other.

Proposition I.26. (ASA or AAS) If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angles equals the remaining angle.

Proposition I.27. If a straight line falling on two straight lines make the alternate angles equal to one another, then the straight lines are parallel to one another.

Proposition I.28. If a straight line falling on two straight lines make the exterior angles equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angle, then the straight lines are parallel to one another.

Proposition I.1, I.2, and I.3 are basically proved by construction using straightedge and compass. Proposition I.4 (SAS) is deduced by means of the uniqueness of straight line segment joining two points. Apparently Euclid places it early in his list so that he can make use it in proving later results. Before we proceed, let's state the definition of "*congruent triangle*."

Definition 2.1 *Two triangles are "congruent" if and only if there is some "way" to match vertices of one to the other such that corresponding sides are equal in length and corresponding angles are equal in size.*

If $\triangle ABC$ is congruent to $\triangle XYZ$, we shall use the notation $\triangle ABC \cong \triangle XYZ$. Thus $\triangle ABC \cong \triangle XYZ$ if and only if $AB = XY$, $AC = XZ$, $BC = YZ$ and $\angle BAC = \angle YXZ, \angle CBA = \angle ZYX, \angle ACB = \angle XZY$.

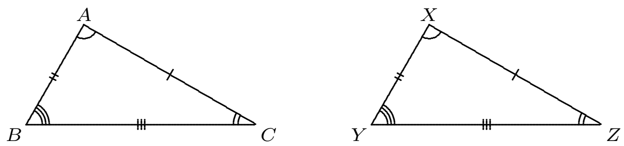


Figure 2.1: Congruent triangles

Let's state and prove proposition I.5 and I.6 in modern language

Proposition I.5. In $\triangle ABC$, if $AB = AC$, then $\angle ABC = \angle ACB$, same for the exterior angles at B and C.
Proof. Let the angle bisector of $\angle A$ meet BC at D. Then by (SAS), $\triangle BAD \cong \triangle CAD$. Thus $\angle ABC = \angle ACB$. (Alternatively, take D to be the midpoint of BC and use (SSS) to conclude that $\triangle BAD \cong \triangle CAD$.)

2.2 Pasch's axiom

There is a hidden assumption that the bisector actually intersects the third side of the triangle. This seems intuitively obvious to us, as we see that any triangle has an "*inside*" and an "*outside*." That is "*the triangle separates the plane into two regions*" which is a simple version of the Jordan curve theorem! In fact, Euclid assumes this separation property without proof and does not include it as one of his axioms. Pasch (1843-1930) was the first to notice this hidden assumption of Euclid. Later he formulates this property specifically; and it is now known as "*Pasch's axiom*".

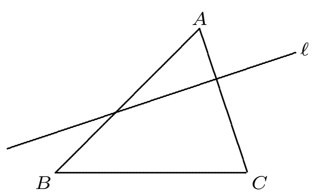


Figure 2.2: Pasch's axiom

Pasch's axiom *Let ℓ be a line passing through the side AB of a triangle ABC. Then ℓ must pass through a either a point on AC or on BC.*
Proposition I.6. In $\triangle ABC$, if $\angle ABC = \angle ACB$, then $AB = AC$.
Proof. Suppose $AB \neq AC$. Then one of them is greater. Let $AB > AC$. Mark off a point D on AB such that $DB = AC$. Also $CB = BC$ and $\angle ACB = \angle DBC$. Thus triangles ACB is congruent to triangle DBC, the less equal to the greater, which is absurd. Therefore $AB = AC$.

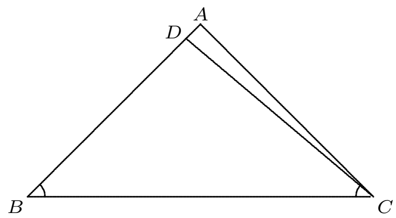


Figure 2.3: Proposition 6

Similarly, it is not true that $AB < AC$. Consequently, $AB = AC$.

Propositions I.7 and I.8 are the (SSS) congruent criterion. Proposition I.7 is self-evident by construction and proposition I.8 follows from I.7. Propositions I.9 to I.15 follow from definitions and construction. Propositions I.16 and I.17 are discussed in chapter 1. The proofs use crucially axioms 1 and 2.

Proposition I.18. In the triangle ABC, if $AB > AC$, then $\angle C > \angle B$.
Proof. Mark off a point D on AB such that $AD = AC$.

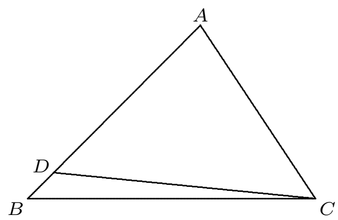


Figure 2.4: Proposition 18

By proposition I.5, $\angle ADC = \angle ACD$. Thus $\angle C > \angle ACD = \angle ADC > \angle B$ by the exterior angle theorem (proposition I.16).

Proposition I.19. In the triangle ABC, if $\angle B > \angle C$, then $AC > AB$.
Proof. If $AB = AC$, then by proposition I.5 we have $\angle B = \angle C$. If $AB > AC$, then by proposition 18 we have $\angle C > \angle B$. Thus both cases lead to a contradiction. Hence, we must have $AC > AB$.

Proposition I.20. (Triangle Inequality) For any triangle ABC, $AB + BC > AC$.
Proof. Exercise.

Proposition I.21. Let D be a point inside the triangle ABC. Then $AB + AC > DB + DC$ and $\angle BDC > \angle BAC$.

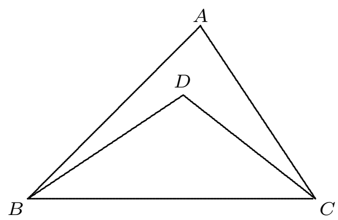


Figure 2.5: Proposition 21

Proof. This follows from the triangle inequality (proposition I.20) and the exterior angle theorem (proposition I.16).

Also Proposition I.22 follows from the triangle inequality (proposition I.20). Proposition 23 is on copying an angle by means of a straightedge and a compass. It can be justified using (SSS) condition.

Proposition I.24. For the triangles ABC and PQR with $AB = PQ$ and $AC = PR$, if $\angle A > \angle P$ then $BC > QR$.

Proof. Stack the triangle PQR onto ABC so that PQ matches with AB. Since $\angle A > \angle P$, the ray AR is within $\angle BAC$. Join BR and CR. Suppose R is outside the triangle ABC.

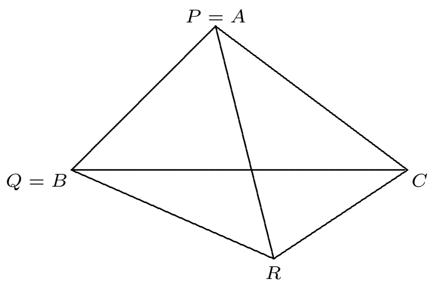


Figure 2.6: Proposition 24

As $AC = AR$ (or PR), $\angle ARC = \angle ACR$. Thus $\angle BRC > \angle ARC = \angle ACR > \angle BCR$. Therefore, $BC > QR$. We leave it as an exercise for the case where R is inside ABC.

Proposition I.25. For the triangles ABC and PQR with $AB = PQ$ and $AC = PR$, if $BC > QR$, then $\angle A > \angle P$.

Proof. If $\angle A = \angle P$, then by (SAS) the two triangles are congruent. But $BC \neq QR$, we have a contradiction. If $\angle A < \angle P$, then by proposition I.24, $BC < QR$, which also contradicts the given condition. Thus we must have $\angle A > \angle P$.

Proposition I.26. (ASA) For the triangles ABC and PQR, if $\angle B = \angle Q, \angle C = \angle R$ and $BC = QR$ then $\triangle ABC \cong \triangle PQR$.

Proof. Suppose $AB > PQ$. Mark off a point D on AB such that $BD = QP$. Then by (SAS), $DBC \cong PQR$ so that $\angle BCD = \angle QRP = \angle R$. But then $\angle BCD < \angle C = \angle R$, a contradiction.

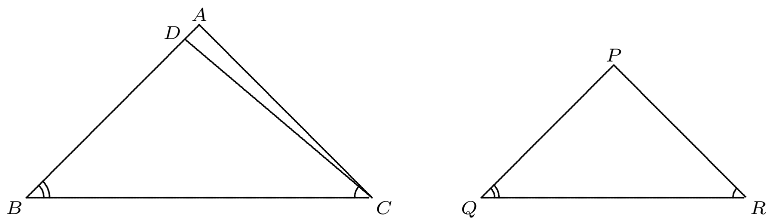


Figure 2.7: Proposition 26

Similarly we get a contradiction if $AB < PQ$. Thus $AB = PQ$. Then by (SAS), $\triangle ABC \cong \triangle PQR$. The (AAS) case is left as an exercise.

Finally proposition I.27 is proved in chapter 1 and proposition I.28 is a reformation of proposition I.27.

Chapter 3

Triangles

In this chapter, we prove some basic properties of triangles in Euclidean geometry.

3.1 Basic properties of triangles

Theorem 3.1 (Congruent Triangles) *Given two triangles ABC and $A'B'C'$,*

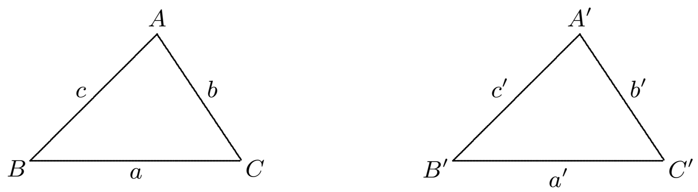


Figure 3.1: Congruent Triangles

the following statements are equivalent.
(a) $\triangle ABC$ is congruent to $\triangle A'B'C'$. ($\triangle ABC \cong \triangle A'B'C'$)
(b) $a = a', b = b', c = c'$. (SSS)
(c) $b = b'$, $\angle A = \angle A'$, $c = c'$. (SAS)
(d) $\angle A = \angle A'$, $b = b'$, $\angle C = \angle C'$. (ASA)
(e) $\angle A = \angle A'$, $\angle B = \angle B'$, $a = a'$. (AAS)

Theorem 3.2 *Given two triangles ABC and $A'B'C'$ where $\angle C = \angle C' = 90°$,*

Figure 3.2: Congruent right Triangles

the following statements are equivalent.

(a) $\triangle ABC \cong \triangle A'B'C'$.
(b) $\angle C = \angle C' = 90°$, $a = a', c = c'$. *(RHS)*
(b) $\angle C = \angle C' = 90°$, $b = b', c = c'$. *(RHS)*

Theorem 3.3 (Similar triangles) *Given two triangles* ABC *and* $A'B'C'$,

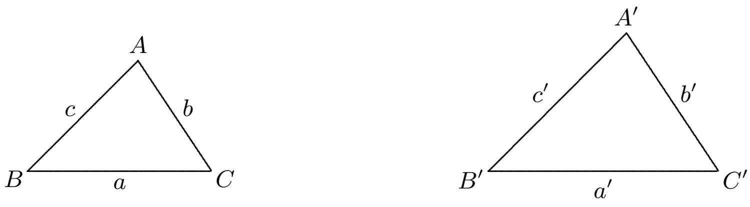


Figure 3.3: Similar Triangles

the following are equivalent.

(a) $\triangle ABC$ *is similar to* $\triangle A'B'C'$. $(\triangle ABC \sim \triangle A'B'C')$
(b) $\angle A = \angle A'$ *and* $\angle B = \angle B'$.
(c) $\angle A = \angle A'$ *and* $b : b' = c : c'$.
(d) $a : a' = b : b' = c : c'$.

Theorem 3.4 (The midpoint theorem) *Let* D *and* E *be points on the sides* AB *and* AC *of the triangle* ABC *respectively. Then* $AD = DB$ *and* $AE = EC$ *if and only if* DE *is parallel to* BC *and* $DE = \frac{1}{2}BC$.

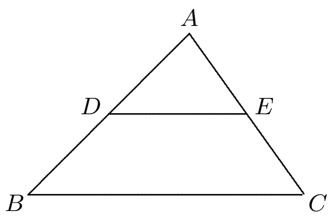


Figure 3.4: The midpoint theorem

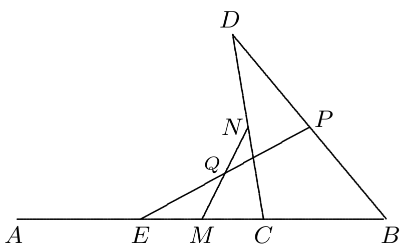


Figure 3.5: The midpoint of AC is E

Example 3.1 In figure 3.5, M, N, and P are respectively the mid-points of the line segments AB, CD and BD. Let Q be the mid-point of MN and let PQ be extended to meet AB at E. Show that $AE = EC$.

Solution. Join NP. Because N is the mid-point of CD and P is the midpoint of BD, we have NP is parallel to AB. Since $NQ = MQ$, we see that $\triangle NPQ$ is congruent to $\triangle MEQ$. Thus $EM = NP = \frac{1}{2}BC$. Therefore, $2EM = BC = MB - MC = AM - MC = AC - 2MC = AC - 2(EC - EM) = AC - 2EC + 2EM$. Thus $AC = 2EC$ and E is the mid-point of AC.

Definition 3.1 *For any polygonal figure $A_1A_2 \cdots A_n$, the area bounded by its sides is denoted by $(A_1A_2 \cdots A_n)$.*

For example if ABC is a triangle, then (ABC) denotes the area of $\triangle ABC$; and if $ABCD$ is a quadrilateral, then $(ABCD)$ denotes its area, etc.

Theorem 3.5 (Varignon) *The figure formed when the midpoints of the sides of a quadrilateral are joined is a parallelogram, and its area is half that of the quadrilateral.*

Proof. Let P, Q, R, S be the midpoints of the sides AB, BC, CD, DA of a quadrilateral respectively. The fact that $PQRS$ is a parallelogram follows from the midpoint theorem. Even $ABCD$ is a "*cross-quadrilateral*", the result still holds.

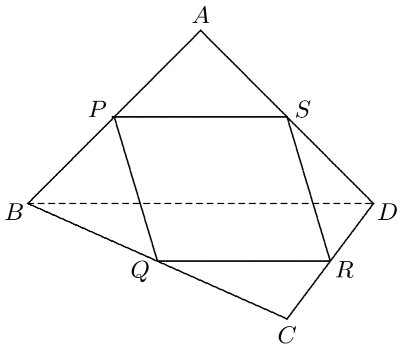


Figure 3.6: Varignon's theorem

As for the area, we have

$$
\begin{aligned}
(PQRS) &= (ABCD) - (PBQ) - (RDS) - (QCR) - (SAP) \\
&= (ABCD) - \tfrac{1}{4}(ABC) - \tfrac{1}{4}(CDA) - \tfrac{1}{4}(BCD) - \tfrac{1}{4}(DAB) \\
&= (ABCD) - \tfrac{1}{4}(ABCD) - \tfrac{1}{4}(ABCD) \\
&= \tfrac{1}{2}(ABCD).
\end{aligned}
$$

If "*sign area*" is used, the result still holds.

Theorem 3.6 (Steiner-Lehmus) *Let BD be the bisector of $\angle B$ and let CE be the bisector of $\angle C$. The following statements are equivalent:*
(a) $AB = AC$
(b) $\angle B = \angle C$
(c) $BD = CE$

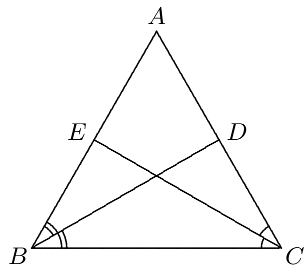


Figure 3.7: Steiner-Lehmus Theorem

The result on (c) implies (a) is called the Steiner-Lehmus Theorem. The proof relies on two lemmas.

Lemma 3.7 *If two chords of a circle subtend different acute angles at points on the circle, the smaller angle belongs to the shorter chord.*

Proof. Two equal chords subtend equal angles at the center and equal angles (half as big) at suitable points on the circumference. Of two unequal chords, the shorter, being farther from the center, subtends a smaller angle there and consequently a smaller acute angle at the circumference.

Lemma 3.8 *If a triangle has two different angles, the smaller angle has the longer internal angle bisector.*

Proof. Let ABC be the triangle with $\angle B > \angle C$. Let's take $\beta = \frac{1}{2}\angle B$ and $\gamma = \frac{1}{2}\angle C$. Thus $\beta > \gamma$. Let BE and CF be the internal angle bisectors at angles B and C respectively. Since $\angle EBF = \beta > \gamma$, we can mark off a point M on CF such that $\angle EBM = \gamma$. Then B, C, E, M lie on a circle.

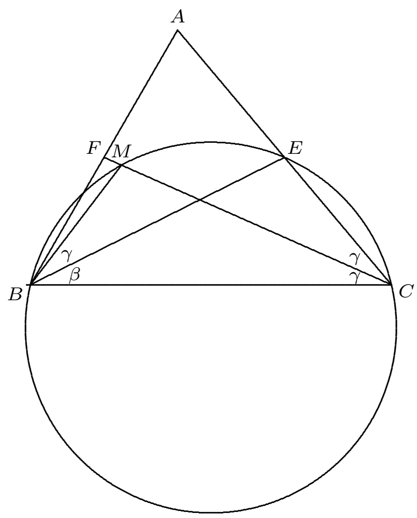


Figure 3.8: The smaller angle has the longer internal angle bisector

Note that $\beta+\gamma < \beta+\gamma+\frac{1}{2}\angle A = 90°$. Also $\angle C = 2\gamma < \beta+\gamma = \angle CBM$. Hence $CF > CM > BE$. To prove the theorem, we prove by contradiction. Suppose $AC > AB$. Then $\angle B > \angle C$. By lemma 2, $CF > BE$, a contradiction. Can you produce a constructive proof of this result?

Theorem 3.9 (The angle bisector theorem) *If AD is the (internal or external) angle bisector of $\angle A$ in a triangle ABC, then $AB : AC = BD : DC$.*

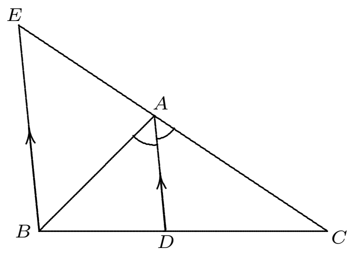


Figure 3.9: Angle bisectors

Proof. The theorem can be proved by applying sine law to $\triangle ABD$ and $\triangle ACD$. An alternate proof is as follow. Construct a line through B parallel to AD meeting the extension of CA at E. Then $\angle ABE = \angle BAD = \angle DAC = \angle AEB$. Thus $AE = AB$. Since $\triangle CAD$ is similar to $\triangle CEB$,we have $AB/AC = AE/AC = BD/DC$. The proof for the external angle bisector is similar.

Theorem 3.10 (Stewart) *If $\dfrac{BP}{PC} = \dfrac{m}{n}$, then $nAB^2 + mAC^2 = (m+n)AP^2 + \dfrac{mn}{m+n}BC^2$.*

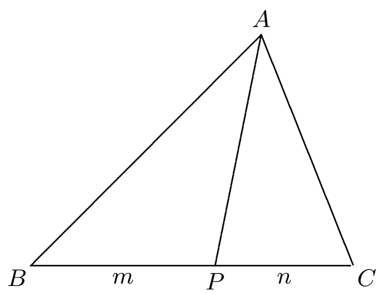


Figure 3.10: Stewart's theorem

Proof. Apply cosine law to the triangles ABP and APC for the two complementary angles at P.

Theorem 3.11 (Pappus' theorem) *Let P be the midpoint of the side BC of a triangle ABC. Then*

$$AB^2 + AC^2 = 2(AP^2 + BP^2).$$

3.2 Special points of a triangle

1. Perpendicular bisectors. The three perpendicular bisectors to the sides of a triangle ABC meet at a common point O, called the *circumcentre* of the triangle. The point O is equidistant to the three vertices of the triangle. Thus the circle centred at O with radius OA passes through the three vertices of the triangle. This circle is called the *circumcircle* of the triangle and the radius R is called the *circumradius* of the triangle.

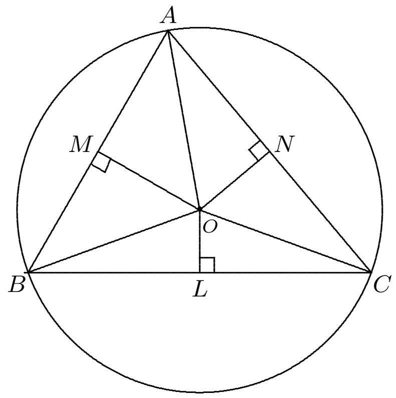


Figure 3.11: Perpendicular bisectors

For any triangle ABC with circumradius R, we have the *sine rule*: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R.$

2. Medians. The 3 medians AD, BE and CF of $\triangle ABC$ are concurrent. Their common point, denoted by G, is called the *centroid* of $\triangle ABC$.

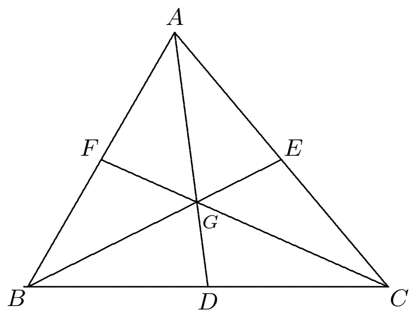


Figure 3.12: Medians

We have

(1) $(AGF) = (BGF) = (BGD) = (CGD) = (CGE) = (AGE)$.

(2) $AG : GD = BG : GE = CG : GF = 2 : 1$.

(3) (Apollonius'theorem)

$AD^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2,$

$BE^2 = \frac{1}{2}(c^2 + a^2) - \frac{1}{4}b^2,$

$CF^2 = \frac{1}{2}(a^2 + b^2) - \frac{1}{4}c^2.$

3. Angle bisectors. The internal bisectors of the 3 angles of $\triangle ABC$ are concurrent. Their common point, denoted by I, is called the *incentre* of $\triangle ABC$. It is equidistant to the sides of the triangle. Let r denote the distance from I to each side. The circle centred at I with radius r is called the *incircle* of $\triangle ABC$, and r is called the *inradius*.

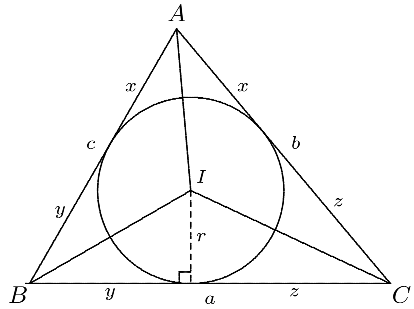


Figure 3.13: Angle bisectors

Let $s = \frac{1}{2}(a + b + c)$ be the *semi-perimeter*. We have

(1) $x = s - a$, $y = s - b$ and $z = s - c$.

(2) $(ABC) = sr$.

(3) $abc = 4srR$.

To prove (3), we have $4srR = 4(ABC)R = 2(ab \sin C)R = abc$.

Exercise 3.1 Prove that $\sin A = (2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4)^{\frac{1}{2}}/(2bc)$.

4. Altitudes. The 3 altitudes AD, BE and CF of $\triangle ABC$ are concurrent. The point of concurrence, denoted by H, is called the *orthocentre* of $\triangle ABC$. The triangle DEF is called the *orthic triangle* of $\triangle ABC$. We have the following result.

Theorem 3.12 *The orthocentre of an acute-angled triangle is the incentre of its orthic triangle.*

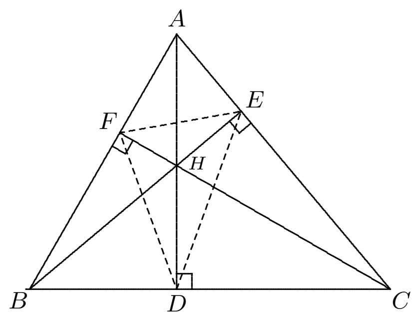


Figure 3.14: Altitudes

Example 3.2 Show that the three altitudes of a triangle are concurrent.

Solution. Draw lines PQ, QR, RP through C, A, B and parallel to AB, BC, CA respectively. Then PQR forms a triangle whose perpendicular bisectors are the altitudes of the triangle ABC.

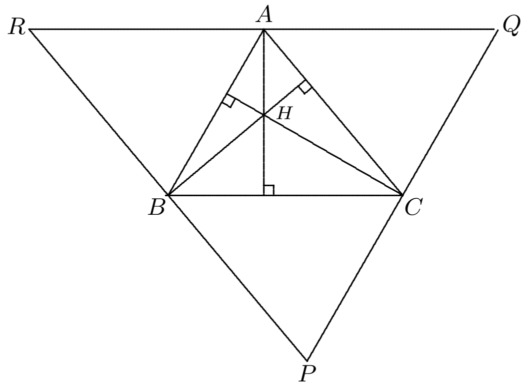


Figure 3.15: The three altitudes of a triangle are concurrent

Exercise 3.2 In an acute-angled $\triangle ABC$, $AB < AC$, BD and CE are the altitudes. Prove that
(i) $BD < CE$
(ii) $AD < AE$
(iii) $AB^2 + CE^2 < AC^2 + BD^2$
(iv) $AB + CE < AC + BD$.
(v) Is it true that $AB^n + CE^n < AC^n + BD^n$ for all positive integer n?

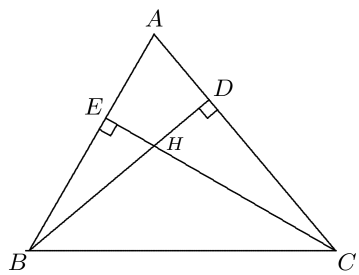


Figure 3.16: $AB^2 + CE^2 < AC^2 + BD^2$

Exercise 3.3 Prove Heron's formula that for a triangle ABC, we have

$$(ABC) = \sqrt{s(s-a)(s-b)(s-c)}.$$

Exercise 3.4 Prove that if I is the incentre of the triangle ABC, then $AI^2 = bc(s-a)/s$.

Exercise 3.5 Prove that for any triangle ABC,

$$\cos^2 \frac{A}{2} = \frac{s(s-a)}{bc} \quad \text{and} \quad \sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{bc}.$$

5. External bisectors. The external bisectors of any two angles of $\triangle ABC$ are concurrent with the internal bisector of the third angle.

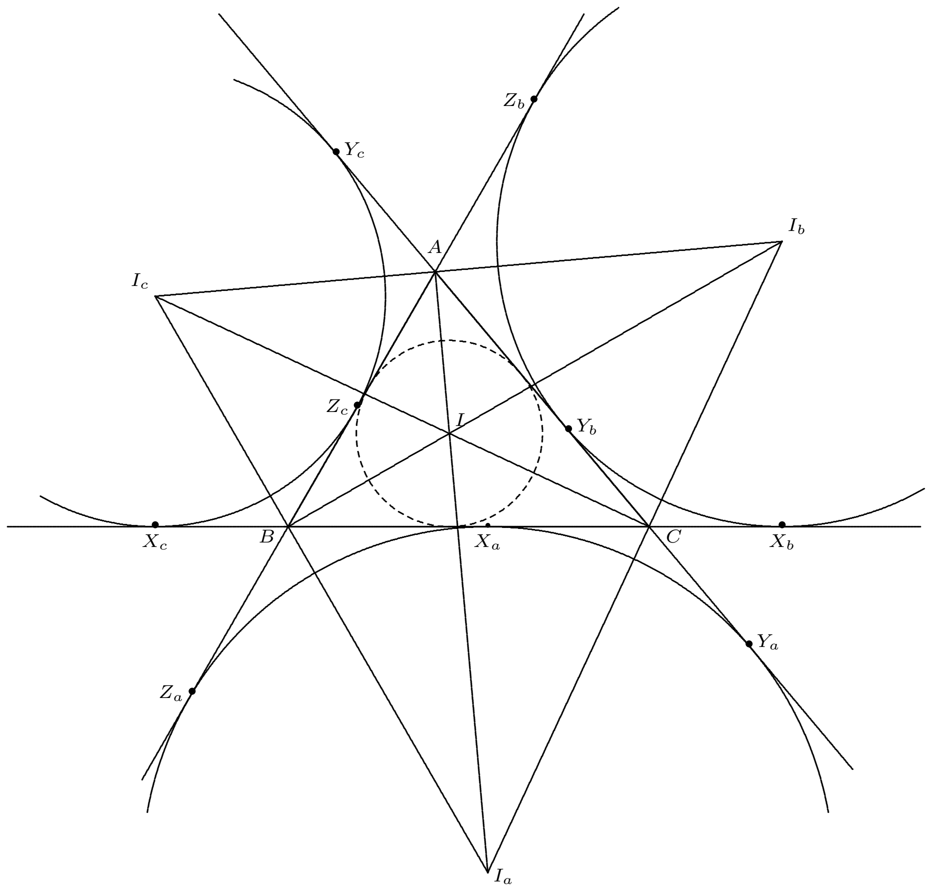


Figure 3.17: External angle bisectors

We call the circles centred at I_a, I_b, I_c with radii r_a, r_b, r_c respectively the *excircles* of the $\triangle ABC$, their centres I_a, I_b, I_c, the *excentres* and their radii r_a, r_b, r_c the *exradii*. Note that

(1) $AY_a = AZ_a = BZ_b = BX_b = CX_c = CY_c = s$.

$[2AY_a = AY_a + AZ_a = AB + BZ_a + AC + CY_a = AB + BX_a + X_aC + AC = AB + BC + AC = 2s.]$

(2) $BX_c = BZ_c = CX_b = CY_b = s - a$. $[BX_c = CX_c - BC = s - a.]$

$CY_a = CX_a = AY_c = AZ_c = s - b$.

$AZ_b = AY_b = BZ_a = BX_a = s - c$.

(3) $(ABC) = (s - a)r_a = (s - b)r_b = (s - c)r_c$.

$[(ABC) = \frac{1}{2}I_aZ_a \cdot AB + \frac{1}{2}I_aY_a \cdot AC - \frac{1}{2}I_aX_a \cdot BC = \frac{1}{2}r_a(c + b - a) = r_a(s - a).]$

(4) $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$.

(5) $\triangle ABC$ is the orthic triangle of $\triangle I_aI_bI_c$.

Exercise 3.6 Prove that $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$.

Exercise 3.7 Prove the identity

$$abc = s(s-b)(s-c)+s(s-c)(s-a)+s(s-a)(s-b)-(s-a)(s-b)(s-c),$$

where $2s = a+b+c$.

Exercise 3.8 Prove that $4R = r_a + r_b + r_c - r$

3.3 The nine-point circle

Theorem 3.13 *Let L be the foot of the perpendicular from O to BC. Then $AH = 2OL$.*

Proof. As $\triangle AEB$ is similar to $\triangle OLB$ with $AB : OB = c : R = 2\sin C$, we have $AE : OL = 2\sin C$. On the other hand, $\angle AHE = \angle C$ so that $AE : AH = AD : AC = \sin C$. Consequently, $AH = 2OL$. Alternatively, extend CO meeting the circumcircle of $\triangle ABC$ at the point P. Then $APBH$ is a parallelogram. Thus $AH = PB = 2OL$.

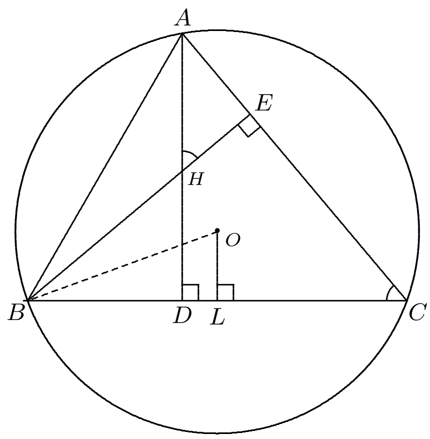


Figure 3.18: $AH = 2OL$

Theorem 3.14 *The circumcentre O, centroid G and orthocentre H of $\triangle ABC$ are collinear. The centroid G divides the segment OH into the ratio $1 : 2$.*

The line on which O, G, H lie is called the *Euler line* of $\triangle ABC$.

Proof. Since AH and OL are parallel, $\angle HAG = \angle OLG$. Also $AH = 2LO$ and $AG = 2LG$. Thus $\triangle HAG$ is similar to $\triangle OLG$ so that $\angle AGH = \angle LGO$. Therefore O, G, H are collinear.

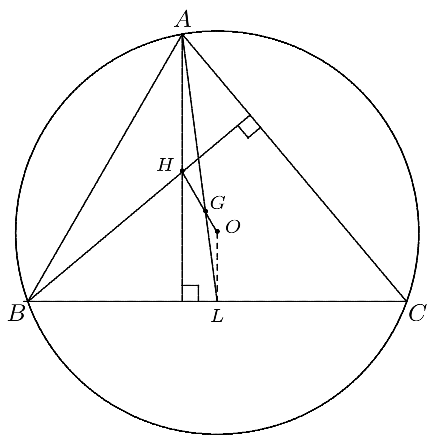


Figure 3.19: G divides OH in the ratio $1 : 2$

Let N be the midpoint of OH, where O is the circumcentre and H is the orthocentre of $\triangle ABC$. Using the fact that $OG : GH = 1 : 2$, we have $NG : GO = 1 : 2$. Since $GL : GA = 1 : 2$ and $\angle NGL = \angle OGA$, we see that $\triangle NGL$ is similar to $\triangle OGA$. Thus NL is parallel to OA and

$NL : OA = 1 : 2$. If we take H_1 to be the midpoint of AH, then L, N, H_1 are collinear, NH_1 is parallel to OA and $NH_1 = \frac{1}{2}OA$. Since N is the midpoint of OH, we also have $ND = NL$. Consequently, $ND = NL = NH_1 = \frac{1}{2}OA = \frac{1}{2}R$.
Alternatively, if we take H_1=midpoint of AH, then $\triangle NHH_1$ is congruent to $\triangle NOL$ because $HH_1 = \frac{1}{2}AH = OL, NH = NO, \angle H_1HN = \angle LON$. Then L, N, H_1 are collinear. Thus $NH_1 = NL = ND = \frac{1}{2}OA$. [Here G is not involved in the proof.]

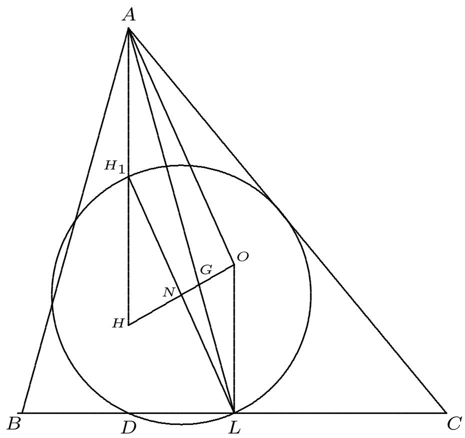


Figure 3.20: The Nine-point Circle

Theorem 3.15 (The Nine-point Circle) *The feet of the three altitudes of any triangle, the midpoints of the three sides, and the midpoints of the segments from the three vertices to the orthocentre, all lie on the same circle of radius $\frac{1}{2}R$ with centre at the midpoint of the Euler line. This circle is known as the nine-point circle or the Euler circle of the triangle.*

Exercise 3.9 Suppose the Euler line passes through a vertex of the triangle. Show that the triangle is either right-angled or isosceles or both.

Exercise 3.10 In a triangle ABC, $\angle C = 90°$, D is a point on AB such that the inradius of $\triangle ACD$ equals to the inradius of $\triangle BCD$. Prove that D is the midpoint of AB if and only if $AC = BC$.

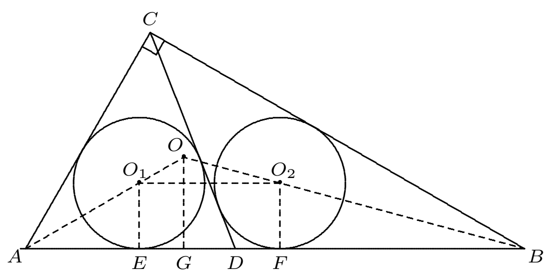


Figure 3.21: The triangles ACD and BCD have equal inradii.

Chapter 4

Quadrilaterals

Quadrilaterals are 4-sided polygons. Among them those whose vertices lie on a circle are called cyclic quadrilaterals. Cyclic quadrilaterals are the simplest objects like triangles in plane geometry and they possess remarkable properties. In this chapter, we shall explore some basic properties of quadrilaterals in Euclidean geometry.

4.1 Basic properties

1. For a quadrilateral $ABCD$, the following statements are equivalent:
(i) $ABCD$ is a parallelogram.
(ii) $AB \parallel DC$ and $AD \parallel BC$.
(iii) $AB = DC$ and $AD = BC$.
(iv) $AB \parallel DC$ and $AB = DC$.
(v) AC and BD bisect each other.

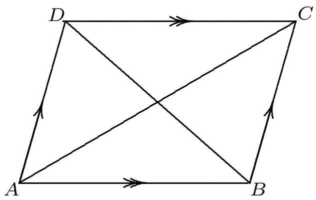


Figure 4.1: A parallelogram

2. For a parallelogram $ABCD$, the following statements are equivalent:
(i) $ABCD$ is a rectangle.
(ii) $\angle A = 90°$.
(iii) $AC = BD$.

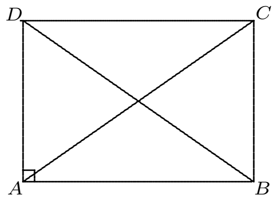


Figure 4.2: A rectangle

3. For a parallelogram $ABCD$, the following statements are equivalent:
(i) $ABCD$ is a rhombus
(ii) $AB = BC$.
(iii) $AC \perp BD$.
(iv) AC bisects $\angle A$.

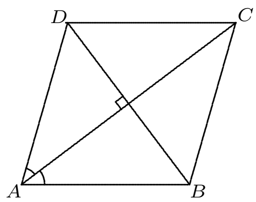


Figure 4.3: A rhombus

Example 4.1 In the figure, E, F are the midpoints of AB and BC respectively. Suppose DE and DF intersect AC at M and N respectively such that $AM = MN = NC$. Prove that $ABCD$ is a parallelogram.

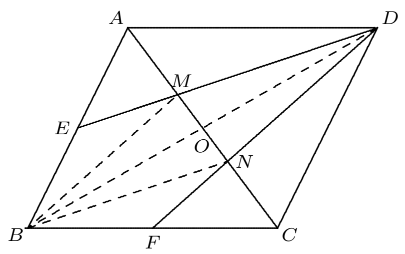


Figure 4.4

Solution. Join BM and BN. Let BD intersect AC at O. As $AE = EB, AM = MN$, we have EM is parallel to BN. Similarly, BM is parallel to FN. Therefore, $BMDN$ is a parallelogram. From this, we have $OB = OD$ and $OM = ON$. Since $AM = NC$, we also have $OA = OC$. Now the diagonals of $ABCD$ bisect each other. This means that $ABCD$ is a parallelogram.

Theorem 4.1 *The segments joining the midpoints of pairs of opposite sides of a quadrilateral and the segment joining the midpoints of the diagonals are concurrent and bisect one another.*

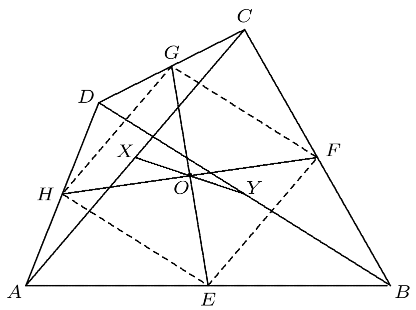


Figure 4.5: XY passes through O

Proof. Consider a quadrilateral $ABCD$ with midpoints E, F, G, H of its sides as shown in the figure. By Varignon's theorem, $EFGH$ is a parallelogram. Thus the diagonals EG and FH of this parallelogram bisect each other. Now consider the quadrilateral (a crossed-quadrilateral in the figure) $ABDC$. By Varignon's theorem, the midpoints E, Y, G, X of its sides form a parallelogram. Thus EG and XY bisect each other. Consequently, EG, FH and XY are concurrent at their common midpoint O.

Definition 4.1 *A quadrilateral $ABCD$ is called a cyclic quadrilateral if its 4 vertices lie on a common circle. In this case the 4 points A, B, C, D are said to be concyclic.*

Regarding cyclic quadrilaterals, we have the following characterizations.

Theorem 4.2 *Let $ABCD$ be a convex quadrilateral. The following statements are equivalent.*
(a) $ABCD$ is a cyclic quadrilateral.
(b) $\angle BAC = \angle BDC$.
(c) $\angle A + \angle C = 180°$.
(d) $\angle ABE = \angle D$.

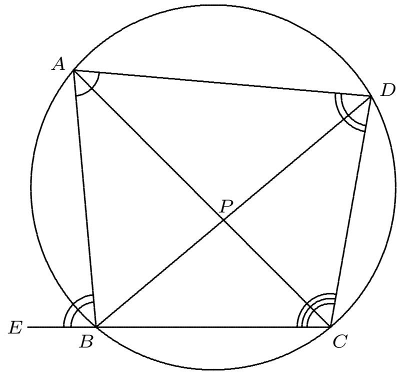


Figure 4.6: A cyclic quadrilateral

Proof. That (a) implies (b) follows from the property of circles, namely the angle subtended by a chord at any point on the circumference and on one side of the chord is a constant. To prove (b) implies (c), observe that $\triangle APB$ is similar to $\triangle DPC$. This in turn implies that $\triangle APD$ is similar to $\triangle BPC$. Thus $\angle BAC = \angle BDC$, $\angle ABD = \angle ACD$, $\angle CAD = \angle CBD$ and $\angle ADB = \angle ACB$. Therefore, $\angle A + \angle C = \angle BAC + \angle CAD + \angle ACB + \angle ACD = \frac{1}{2}(\angle A + \angle B + \angle C + \angle D) = 180°$. That (c) is equivalent to (d) is obvious. The part that (d) implies (a) is left as an exercise.

Exercise 4.1 Suppose the diagonals of a cyclic quadrilateral $ABCD$ intersect at a point P. Prove that $AP \cdot PC = BP \cdot PD$.

Theorem 4.3 *If a cyclic quadrilateral has perpendicular diagonals intersecting at* P*, then the line through* P *perpendicular to any side bisects the opposite side.*

Proof. Let XH be the line through P perpendicular to BC. We wish to prove X is the midpoint of AD.

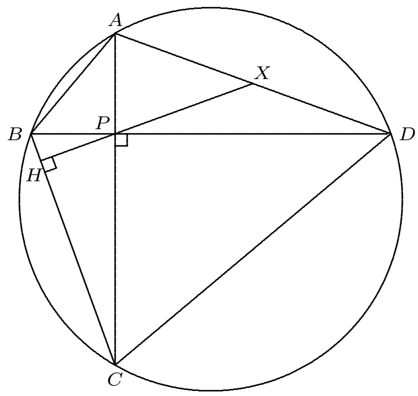


Figure 4.7: A cyclic quadrilateral with perpendicular diagonals

We have $\angle DPX = \angle BPH = \angle PCH = \angle ACB = \angle ADB = \angle XDP$. Thus the triangle XPD is isosceles. Similarly, the triangle XAP is isosceles. Consequently $XA = XP = XD$.

4.2 Ptolemy's theorem

Theorem 4.4 (The Simson line) *The feet of the perpendiculars from any point P on the circumcircle of a triangle ABC to the sides of the triangle are collinear.*

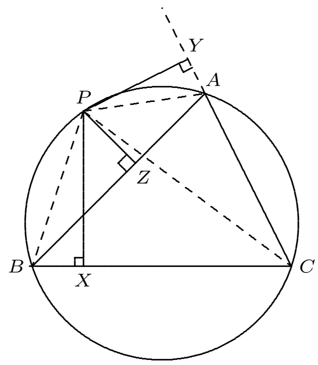


Figure 4.8: The Simson line

Proof. Referring to figure 4.8, we see that $PZAY, PXCY$ and $PACB$ are cyclic quadrilaterals. Therefore, $\angle PYZ = \angle PAZ = \angle PCX = \angle PYX$. This shows that Y, Z, X are collinear.

(Note that the converse of the statement in this theorem is also true. That is, if the feet of the perpendiculars from a point P to the sides of the triangle ABC are collinear, then P lies on the circumcircle of $\triangle ABC$.) The line containing the feet is known as the *Simson line*.

Theorem 4.5 (Ptolemy) *For any cyclic quadrilateral, the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.*

Proof. Let $PBCA$ be a cyclic quadrilateral and let X, Y, Z be the feet of the perpendiculars from P onto the sides BC, AC, AB respectively. By previous theorem, X, Y, Z lie on the Simson line.

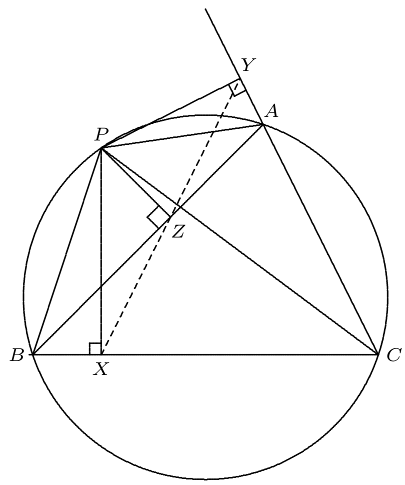


Figure 4.9: Ptolemy's theorem

The quadrilateral $AYPZ$ is cyclic. Since $\angle PYA = 90°$, the circle passing through A, Y, P, Z has diameter PA. Thus

$$\frac{YZ}{PA} = \sin\angle YAZ = \sin\angle BAC = \frac{a}{2R}.$$

That is $YZ = aPA/(2R)$. Similarly, by considering the cyclic quadrilaterals $PZXB$ and $PXCY$, we have $XZ = bPB/(2R)$ and $XY = cPC/(2R)$. As X, Y, Z lie on the Simson line, we have $XZ + ZY = XY$ so that $bPB/(2R) + aPA/(2R) = cPC/(2R)$. Canceling the common factor $2R$, we get $bPB + aPA = cPC$. That is

$$AC \cdot PB + BC \cdot PA = AB \cdot PC.$$

Ptolemy's Theorem can be strengthened by observing that if P is any point not on the circumcircle of $\triangle ABC$, then the equality $XZ + ZY = XY$ has to be replaced by the inequality $XZ + ZY > XY$ so that $AC \cdot PB + BC \cdot PA > AB \cdot PC$.

Theorem 4.6 *If P is a point not on the arc CA of the circumcircle of the triangle ABC, then*

$$AC \cdot PB + BC \cdot PA > AB \cdot PC.$$

Example 4.2 Let P be a point of the minor arc CD of the circumcircle of a square $ABCD$. Prove that

$$PA(PA + PC) = PB(PB + PD).$$

Solution. Refer to figure 4.10. Let $AB = a$. Applying Ptolemy's theorem to the cyclic quadrilaterals $PDAB$ and $PABC$, we have $PD \cdot BA + PB \cdot DA = PA \cdot DB$, and $PA \cdot BC + PC \cdot AB = PB \cdot AC$. That is $a(PD + PB) = \sqrt{2}a \cdot PA$ and $a(PA + PC) = \sqrt{2}a \cdot PB$. Canceling a common factor of a for both equations, we get $PD + PB = \sqrt{2}PA$ and $PA + PC = \sqrt{2}PB$. Thus $PA(PA + PC) = \sqrt{2}PA \cdot PB = PB(PB + PD)$.

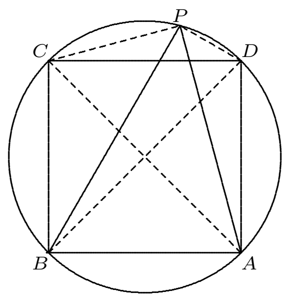


Figure 4.10

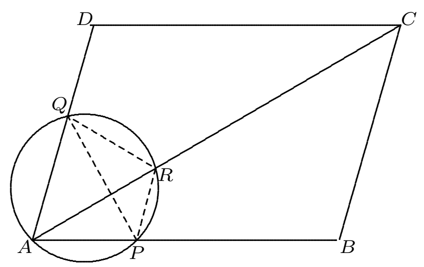


Figure 4.11

Exercise 4.2 In a parallelogram $ABCD$, a circle passing through A meets AB, AD and AC at P, Q and R respectively. Prove that $AP \cdot AB + AQ \cdot AD = AR \cdot AC$. See figure 4.11.

4.3 Area of a quadrilateral

Theorem 4.7 (Brahmagupta's Formula) *If a cyclic quadrilateral has sides a, b, c, d and semi-perimeter s, then its area K is given by*

$$K^2 = (s-a)(s-b)(s-c)(s-d).$$

Proof. Let $ABCD$ be a cyclic quadrilateral. Let the length of BD be n. First note that $\angle A + \angle C = 180°$ so that $\cos A = -\cos C$ and $\sin A = \sin C$. Thus by Cosine law,

$$a^2 + b^2 - 2ab\cos A = n^2 = c^2 + d^2 - 2cd\cos C,$$

giving

$$2(ab+cd)\cos A = a^2 + b^2 - c^2 - d^2. \tag{4.1}$$

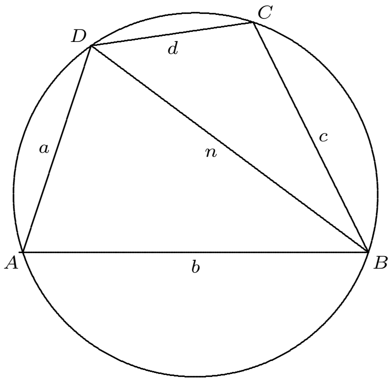


Figure 4.12: Brahmagupta's Formula

Since

$$K = \frac{1}{2}ab\sin A + \frac{1}{2}cd\sin C = \frac{1}{2}(ab+cd)\sin A,$$

we also have

$$2(ab+cd)\sin A = 4K. \tag{4.2}$$

Adding the squares of (4.1) and (4.2), we obtain

$$4(ab+cd)^2 = (a^2+b^2-c^2-d^2)^2 + 16K^2,$$

giving

$$16K^2 = (2ab+2cd)^2 - (a^2+b^2-c^2-d^2)^2.$$

Thus

$$\begin{aligned} 16K^2 &= (2ab+2cd)^2 - (a^2+b^2-c^2-d^2)^2 \\ &= (2ab+2cd+a^2+b^2-c^2-d^2)(2ab+2cd-a^2-b^2+c^2+d^2) \end{aligned}$$

$$= ((a+b)^2 - (c-d)^2)((c+d)^2 - (a-b)^2)$$
$$= (a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b)$$
$$= (2s-2d)(2s-2c)(2s-2b)(2s-2a).$$

Therefore, $K^2 = (s-a)(s-b)(s-c)(s-d)$.
Setting $d = 0$, we obtain Heron's formula for the area of a triangle:

$$(ABC)^2 = s(s-a)(s-b)(s-c).$$

Exercise 4.3 In a trapezium $ABCD$, AB is parallel to DC and E is the midpoint of BC. Prove that $2(AED) = (ABCD)$.

Exercise 4.4 Suppose the quadrilateral $ABCD$ has an inscribed circle. Show that $AB + CD = BC + DA$.

Exercise 4.5 Suppose the cyclic quadrilateral $ABCD$ has an inscribed circle. Show that $(ABCD) = \sqrt{abcd}$.

Exercise 4.6 Let $ABCD$ be a convex quadrilateral. Prove that its area K is given by

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{A+C}{2}\right).$$

Exercise 4.7 Let $ABCDE$ be the pentagon whose vertices are intersections of the extensions of non-neighboring sides of a pentagon $HIJKL$. Prove that the neighboring pairs of the circumcircles of the triangles ALH, BHI, CIJ, DJK, EKL intersect at 5 concyclic points P, Q, R, S, T.

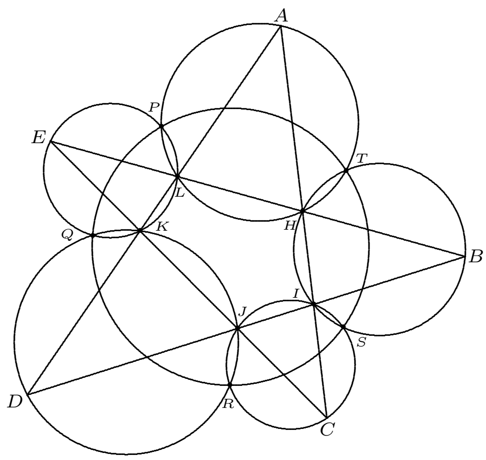


Figure 4.13: Miquel's 5-circle theorem

[Hint: Note that J, S, B, E are concyclic since $\angle EBS = \angle HBS = \angle CIS = \angle CJS$. Similarly, J, Q, E, B are concyclic. Thus J, S, B, E, Q are concyclic. Now try to show P, T, S, Q are concyclic by showing that $\angle QPT + \angle QST = 180°$.]

Remark 4.1 This is Miquel's 5-circle theorem first proved by Miquel in 1838. This problem was proposed by president Jiang Zemin of PRC to the students of Háo Jiāng Secondary School in Macau during his visit to the school in 20 December 2000.

4.4 Pedal triangles

Definition 4.2 *For any point P on the plane of a triangle ABC, the foot of the perpendiculars from P onto the sides of the triangle ABC form a triangle $A_1B_1C_1$ called the pedal triangle of the point P with respect to the triangle ABC.*

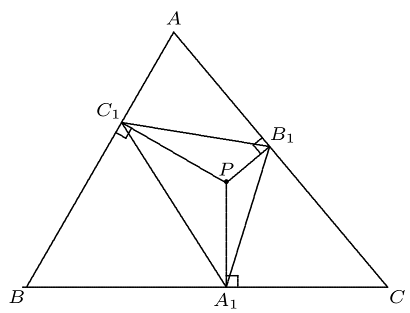


Figure 4.14: Pedal triangle

Theorem 4.8 *Let $A_1B_1C_1$ be the pedal triangle of the point P with respect to the triangle ABC. Then*

$$(A_1B_1C_1) = \frac{R^2 - OP^2}{4R^2}(ABC),$$

where O is the circumcentre and R is the circumradius of the triangle ABC.

Proof. Extend BP meeting the circumcircle of $\triangle ABC$ at B_2. Join B_2C. As in the figure, $\angle A_1 = \alpha + \beta = \angle B_2CP$.

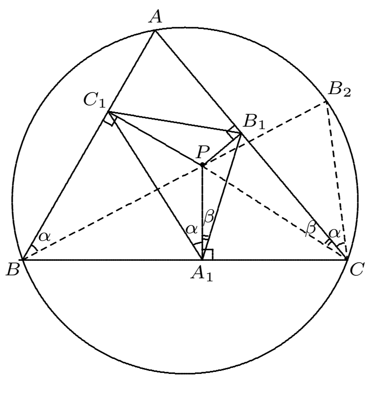


Figure 4.15: Area of the pedal triangle

Thus

$$(A_1B_1C_1) = \frac{1}{2}A_1B_1 \cdot A_1C_1 \cdot \sin A_1 = \frac{1}{2}(PC\sin C)(PB\sin B)\sin\angle B_2CP.$$

Also,

$$\frac{\sin\angle B_2CP}{\sin A} = \frac{\sin\angle B_2CP}{\sin\angle BB_2C} = \frac{PB_2}{PC}.$$

Thus,

$$\begin{aligned}(A_1B_1C_1) &= \frac{1}{2}PB_2 \cdot PB\sin A\sin B\sin C\\ &= \frac{1}{2}(R^2 - OP^2)\sin A\sin B\sin C\\ &= \frac{R^2 - OP^2}{4R^2}(ABC).\end{aligned}$$

The above result is a generalization of Simson's theorem.

Corollary 4.9 *The point P lies on the circumcircle of $\triangle ABC$ if and only if the area of the pedal triangle is zero if and only if A_1, B_1, C_1 are collinear.*

Exercise 4.8 Show that the third pedal triangle is similar to the original triangle.

Exercise 4.9 Let P be a point on the circumcircle of the triangle ABC. Prove that its Simson line with respect to the triangle ABC bisects PH, where H is the orthocentre of the triangle ABC.

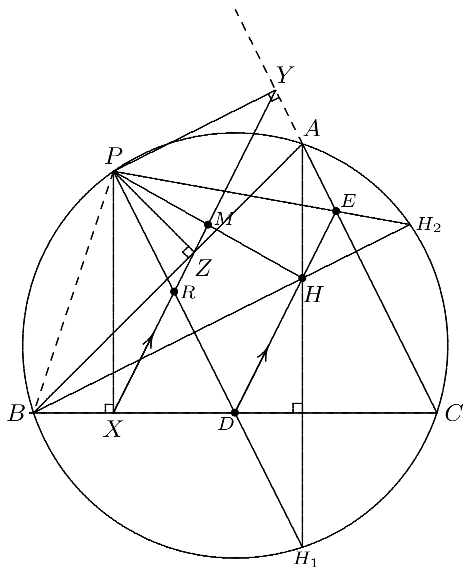


Figure 4.16: The Simson line bisects PH.

[Hint: Let X, Y and Z be the feet of perpendiculars from P onto the sides BC, CA and AB respectively. It is well-known that X, Y and Z are collinear. The line on which they lie is called the Simson line. Extend AH, BH and CH meeting the circumcircle of the triangle ABC at H_1, H_2 and H_3 respectively. Let PH_1 intersect BC at D, PH_2 intersect CA at E and PH_3 intersect AB at F. Join PB.]

Exercise 4.10 Let P and P' be diametrically opposite points on the circumcircle of the triangle ABC. Prove that the Simson lines of P and P' meet at right angle on the nine-point circle of the triangle.

Exercise 4.11 Prove Brahmagupta-Mahavira formula: Let $ABCD$ be a cyclic quadrilateral with $AB = b, BC = c, CD = d, DA = a$ and $AC = m, BD = n$. Then

$$\frac{m}{n} = \frac{ab+cd}{ad+bc}.$$

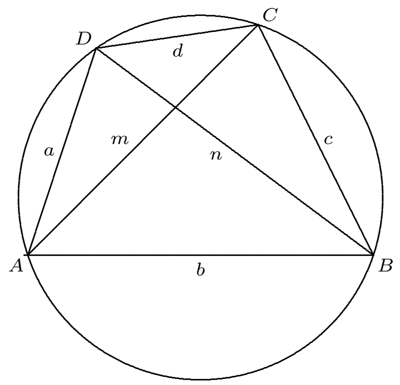


Figure 4.17: Brahmagupta-Mahavira formula

[Hint: Interchange the sides with lengths a and b, also a and d. Apply Ptolemy's theorem.]

Exercise 4.12 A cyclic quadrilateral $ABCD$ is called a *harmonic quadrilateral* if $AB \cdot CD = BC \cdot AD$. Show that $ABCD$ is a harmonic quadrilateral if and only the tangent at B, the tangent at D and the line AC are concurrent or parallel.

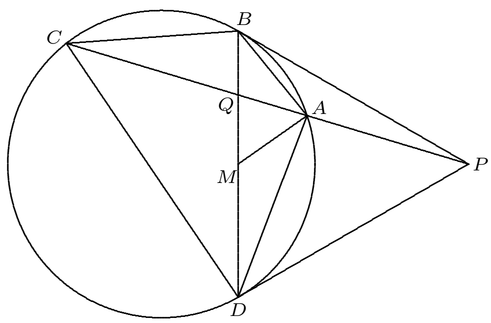


Figure 4.18: Harmonic quadrilateral

Exercise 4.13 Let $ABCD$ be a harmonic quadrilateral and let AC intersect BD at Q. Suppose the tangent at B intersects the extension of CA at P.
(a) Prove that $\frac{2}{PQ} = \left(\frac{1}{PA} + \frac{1}{PC}\right)$.
(b) Let M be the midpoint of BD. Then $\angle CAB = \angle DAM$.

Chapter 5

Concurrence

When several lines meet at a common point, they are said to be *concurrent*. The concurrence of lines occurs very often in many geometric configurations. The point of concurrence usually plays a significant and special role in the geometry of the figure. In this chapter, we will introduce several of these points and the classical Ceva's theorem which gives a necessary and sufficient condition for three cevians of a triangle to be concurrent. We will illustrate with many applications that stem out from Ceva's theorem.

5.1 Ceva's theorem

Definition 5.1 *The line segment joining a vertex of $\triangle ABC$ to any given point on the opposite side (or extended) is called a cevian.*

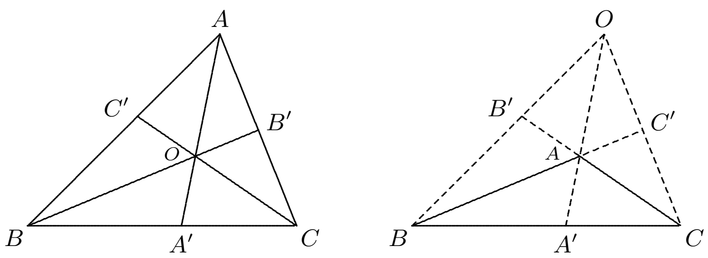


Figure 5.1: Three cevians meet a point

Theorem 5.1 (Ceva) *Three cevians AA', BB', CC' of $\triangle ABC$ are concurrent if and only if*

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1.$$

[*Here directed segments are used.*]

Proof. First suppose the 3 cevians AA', BB', CC' are concurrent. Draw a line through A parallel to BC meeting the extension of BB' and CC' at D and E respectively. See Figure 5.2. Then

$$\frac{CB'}{B'A} = \frac{BC}{AD}, \quad \frac{AC'}{C'B} = \frac{EA}{BC}.$$

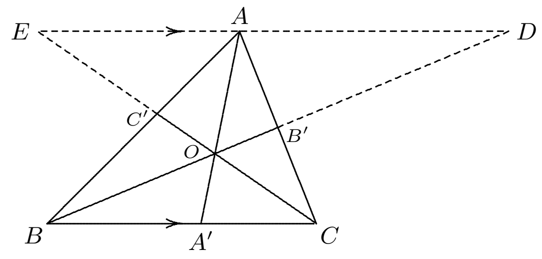


Figure 5.2: Ceva's Theorem

Since $\frac{BA'}{AD} = \frac{A'O}{OA} = \frac{A'C}{EA}$, we have $\frac{BA'}{A'C} = \frac{AD}{EA}$. Thus

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = \frac{AD}{EA} \cdot \frac{BC}{AD} \cdot \frac{EA}{BC} = 1.$$

To prove the converse, suppose

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1. \tag{5.1}$$

Let's consider the case where A', B', C' lie in the interior of BC, CA, AB, respectively. The case that two of them are outside is similar. Let BB' and CC' meet at a point O. Then connect AO meeting BC at a point A''. It suffices to prove $A' = A''$. By the forward implication of Ceva's theorem ,we have

$$\frac{BA''}{A''C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1. \tag{5.2}$$

Comparing equations (5.1) and (5.2), we have $\frac{BA'}{A'C} = \frac{BA''}{A''C}$. Thus $A' = A''$.
There is an alternate proof using area. As

$$\frac{BA'}{A'C} = \frac{(ABA')}{(AA'C)} = \frac{(OBA')}{(OA'C)} = \frac{(ABO)}{(ACO)}, \ \frac{CB'}{B'A} = \frac{(BCO)}{(BAO)}, \ \frac{AC'}{C'B} = \frac{(CAO)}{(CBO)}$$

we have

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1.$$

There is a trigonometric version of Ceva's theorem in terms of the sines of the angles that the cevians make with the sides of the triangles at the vertices. Refer to Figure 5.3. Let $\angle CAA' = \alpha_1$, $\angle A'AB = \alpha_2$, $\angle ABB' = \beta_1$, $\angle B'BC = \beta_2$, $\angle BCC' = \gamma_1$ and $\angle C'CA = \gamma_2$.
Then $\sin\alpha_1 = A'C \cdot \frac{\sin C}{AA'}$, $\sin\alpha_2 = BA' \cdot \frac{\sin B}{AA'}$, so that $\frac{\sin\alpha_2}{\sin\alpha_1} = \frac{BA'}{A'C}\frac{\sin B}{\sin C}$. Similarly,

$$\frac{\sin\beta_2}{\sin\beta_1} = \frac{CB'}{B'A}\frac{\sin C}{\sin A} \quad \text{and} \quad \frac{\sin\gamma_2}{\sin\gamma_1} = \frac{AC'}{C'B}\frac{\sin A}{\sin B}.$$

Therefore, by Ceva's theorem, $AA', BB'CC'$ are concurrent if and only if

$$\frac{\sin\alpha_2}{\sin\alpha_1} \cdot \frac{\sin\beta_2}{\sin\beta_1} \cdot \frac{\sin\gamma_2}{\sin\gamma_1} = 1.$$

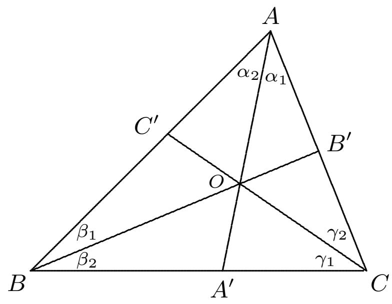


Figure 5.3: Trigonometric version of Ceva's Theorem

Example 5.1 We can use the trigonometric version of Ceva's theorem to deduce that the three altitudes of a triangle are concurrent.

5.2 Common points of concurrence

The common points of concurrence that arise from a triangle consist of the following.

1. The 3 medians of $\triangle ABC$ are concurrent. Their common point, denoted by G, is called the **centroid** of $\triangle ABC$.

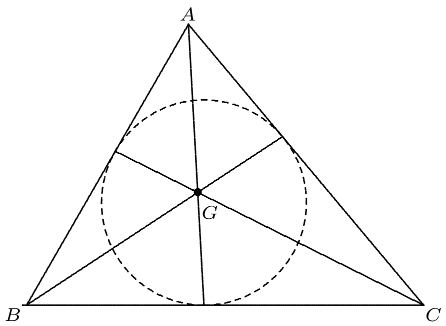
2. The 3 altitudes of $\triangle ABC$ are concurrent. Their common point, denoted by H, is called the **orthocentre** of $\triangle ABC$.

3. The internal bisectors of the 3 angles of $\triangle ABC$ are concurrent. Their common point, denoted by I, is called the **incentre** of $\triangle ABC$.

4. The internal bisector of $\angle A$ and the external bisectors of the other two angles of $\triangle ABC$ are concurrent. Their common point, denoted by I_a, is called the **excentre** of $\triangle ABC$. Similarly, there are excentres I_b and I_c.

5. The three perpendicular bisectors of a triangle $\triangle ABC$ are concurrent. Their common point, denoted by O is called the **circumcentre** of $\triangle ABC$.

6. The cevians where the feet are the tangency points of the incircle (or excircle) of a triangle are concurrent. This common point is called the **Gergonne** point. Thus there are 4 Gergonne points for a triangle.



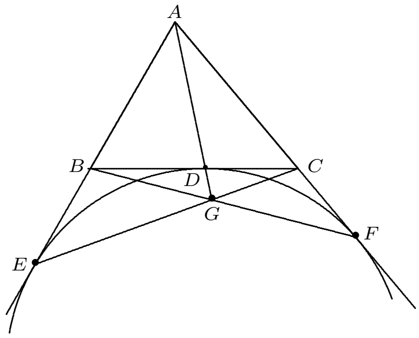


Figure 5.4: Gergonne point

Example 5.2 In $\triangle ABC$, D, E and F are the feet of the altitudes from A, B and C onto the sides BC, CA and AB respectively. Prove that the perpendiculars from A onto EF, from B onto DF and from C onto EF are concurrent.

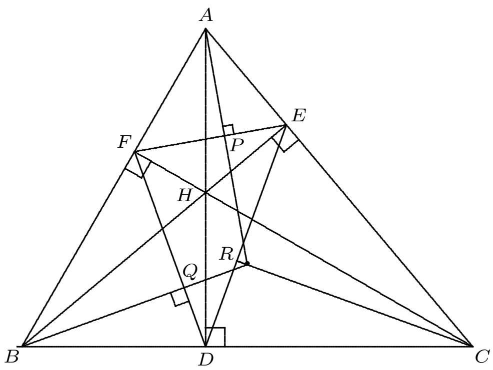


Figure 5.5: A point of concurrence

Solution. We shall use the trigonometric version of Ceva's theorem. First $\sin\angle FAP = \cos\angle AFP = \cos C$. Similarly, $\sin\angle PAE = \cos B$, $\sin\angle ECR = \cos B$, $\sin\angle RCD = \cos A$, $\sin\angle DBQ = \cos A$ and $\sin\angle QBF = \cos C$. Thus

$$\frac{\sin\angle FAP}{\sin\angle PAE}\cdot\frac{\sin\angle ECR}{\sin\angle RCD}\cdot\frac{\sin\angle DBQ}{\sin\angle QBF} = 1,$$

and by Ceva's theorem, AP, BQ and CR are concurrent. In fact the point of concurrence is the circumcentre of the triangle ABC.

Example 5.3 In an acute-angled triangle ABC, N is a point on the altitude AM. The line CN, BN meet AB and AC respectively at F and E. Prove that $\angle EMN = \angle FMN$.

Solution. Construct a line through A parallel to BC meeting the extensions of MF and ME at P and Q respectively. Thus $\angle MAP = 90°$. As $\triangle PAF$ is similar to $\triangle MBF$ and $\triangle QAE$ is similar to $\triangle MCE$, we have

$$PA = \frac{AF}{FB}\cdot BM,\quad AQ = \frac{EA}{EC}\cdot MC.$$

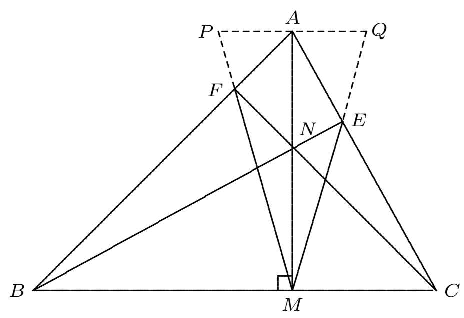


Figure 5.6: $\angle EMN = \angle FMN$

Thus $\dfrac{PA}{AQ} = \dfrac{AF}{FB} \cdot \dfrac{BM}{MC} \cdot \dfrac{CE}{EA} = 1$, by Ceva's Theorem. Therefore, $PA = AQ$. It follows that $\angle EMN = \angle FMN$.

Example 5.4 On the plane, there are 3 mutually and externally disjoint circles Γ_1, Γ_2 and Γ_3 centred at X_1, X_2 and X_3 respectively. The two internal common tangents of Γ_2 and Γ_3, (Γ_3 and Γ_1, Γ_1 and Γ_2) meet at P, (Q, R respectively). Prove that X_1P, X_2Q and X_3R are concurrent.

Solution. Let the radii of Γ_1, Γ_2 and Γ_3 be r_1, r_2 and r_3 respectively.

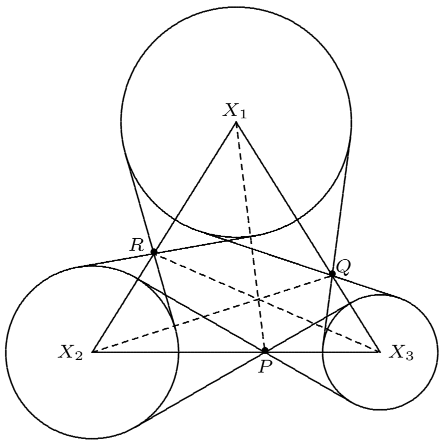


Figure 5.7: X_1P, X_2Q and X_3R are concurrent

Then $X_1R : RX_2 := r_1 : r_2$, $X_2P : PX_3 := r_2 : r_3$ and $X_3Q : QX_1 := r_3 : r_1$. Thus

$$\frac{X_1R}{RX_2} \cdot \frac{X_2P}{PX_3} \cdot \frac{X_3Q}{QX_1} = 1.$$

By Ceva's Theorem, X_1P, X_2Q and X_3Z are concurrent.

Example 5.5 Prove that the 3 cevians of a triangle ABC such that each of them bisects the perimeter of the triangle ABC are concurrent.

Solution. Let $BC = a, AC = b, AB = c$ and $s = \frac{1}{2}(a + b + c)$. Let A', B', C' be the points on BC, AC, AB such that AA', BB', CC' each bisects the perimeter of $\triangle ABC$. Then $BA'+A'C = a$ and $c + BA' = b + A'C$. Thus $BA' = s - c$ and $A'C = s - b$. Similarly, $CB' = s - a, B'A = s - c, AC' = s - b$ and $C'B = s - a$. Thus

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1,$$

so that by Ceva's Theorem, AA', BB', CC' are concurrent. The point of concurrence is called the **Nagel point** of $\triangle ABC$. It is also the point of concurrence of the cevians that join the vertices of the triangle to the points of tangency of the excircles on the opposite sides.

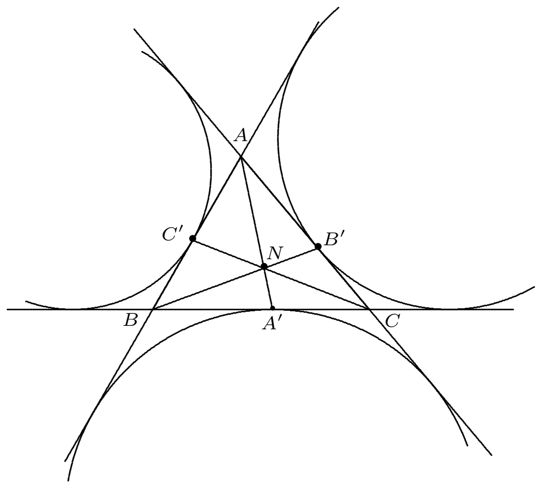


Figure 5.8: Nagel point

Remark 5.1 If D, E, F are the points of tangency of the incircle to the sides BC, CA and AB, and DX, EY, FZ are the diameters of the incircle respectively, then AX, BY, CZ concurs at the Nagel point. In fact we can prove that the extension of AX, BY and CZ meet BC, CA and AB at A', B' and C', respectively. To see this, we show that the point A' on BC which is the point of tangency of the excircle with the side BC together with the points X and A are collinear. This is because a *homothety* mapping the incircle to this excircle must map the highest point X of the incircle to the highest point A' of the excircle.

Exercise 5.1 Let $ABCD$ be a trapezium with AB parallel to CD. Let M and N be the midpoints of AB and CD respectively. Prove that MN, AC and BD are concurrent.

Exercise 5.2 Suppose a circle cuts the sides of a triangle $A_1A_2A_3$ at the points $X_1, Y_1, X_2, Y_2, X_3, Y_3$. Show that if A_1X_1, A_2X_2, A_3X_3 are concurrent, then A_1Y_1, A_2Y_2, A_3Y_3 are concurrent.

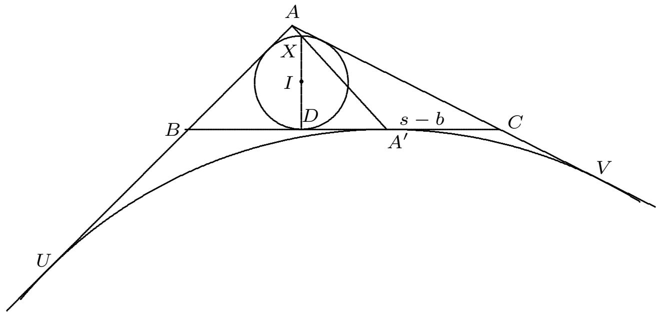


Figure 5.9: The incircle and excircle

[Hint: Observe that $X_1A_2 \cdot Y_1A_2 = X_3A_2 \cdot Y_3A_2$.]

Exercise 5.3 Let P be a point inside the triangle ABC. The bisector of $\angle BPC$, $\angle CPA$, and $\angle APB$ meet BC, CA and AB at X, Y and Z, respectively. Prove that AX, BY, CZ are concurrent.

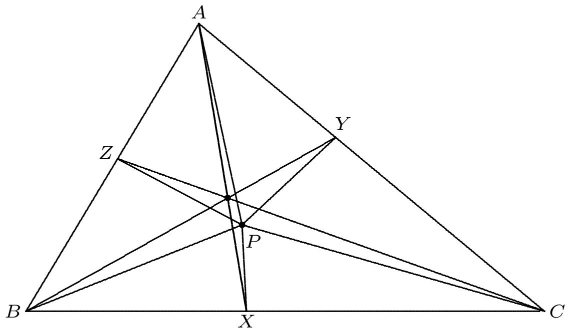


Figure 5.10: AX, BY, CZ are concurrent

Exercise 5.4 Let Γ be a circle with center I, the incentre of triangle ABC. Let D, E, F be points of intersection of Γ with the lines from I that are perpendicular to the sides BC, CA, AB respectively. Prove that AD, BE, CF are concurrent.

[Hint: Let the intersection of AD, BE, CF with BC, CA, AB be D', E', F' respectively. It is easy to establish that $\angle FAF' = \angle EAE'$, $\angle FBF' = \angle DBD'$, $\angle DCD' = \angle ECE'$. Also $AE = AF$, $BF = BD$, $CD = CE$. The ratio $AF'/F'B$ equals to the ratio of the altitudes from A and B on CF of the triangles AFC and BFC and hence equals to the ratio of their areas. Now apply Ceva's theorem.]

Exercise 5.5 Let A_1, B_1 and C_1 be points in the interiors of the sides BC, CA and AB of a triangle ABC respectively. Prove that the perpendiculars at the points A_1, B_1, C_1 are concurrent if and only if $BA_1^2 - A_1C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = 0$. This is known as Carnot's lemma.

Solution. Suppose the three perpendiculars concur at a point O. Note that O is inside the triangle ABC. As $BA_1^2 - A_1C^2 = (OB^2 - OA_1^2) - (OC^2 - OA_1^2) = OB^2 - OC^2$, $CB_1^2 - B_1A^2 =$

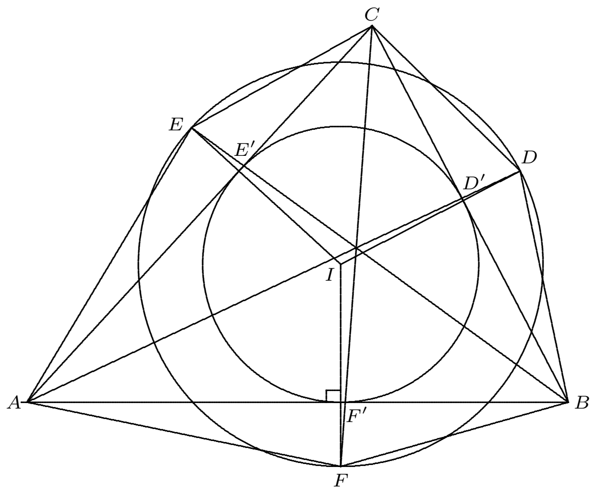


Figure 5.11: A generalization of the Gergonne point

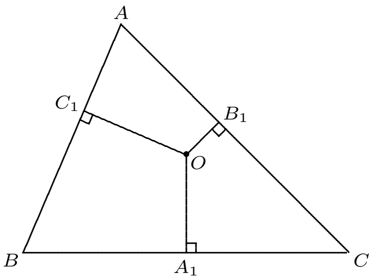


Figure 5.12: Carnot's lemma

$(OC^2 - OB_1^2) - (OA^2 - OB_1^2) = OC^2 - OA^2$, and $AC_1^2 - C_1B^2 = (OA^2 - OC_1^2) - (OB^2 - OC_1^2) = OA^2 - OB^2$, we thus have $BA_1^2 - A_1C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = 0$.
Conversely, suppose $BA_1^2 - A_1C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = 0$. Let the perpendiculars at B_1 and C_1 meet at a point O. Note that O is inside the triangle ABC. Drop the perpendicular OA' from O onto BC. We want to prove $A' = A_1$. By the proven forward implication, we know that $BA'^2 - A'C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = 0$. Together with the given relation, we obtain $BA'^2 - A'C^2 = BA_1^2 - A_1C^2$. That is $(BA' + A'C)(BA' - A'C) = (BA_1 + A_1C)(BA_1 - A_1C)$. As $BA' + A'C = BC = BA_1 + A_1C$, we have $BA' - A'C = BA_1 - A_1C$. From these equations, we deduce that $BA' = BA_1$ and $A'C = A_1C$. Thus $A' = A_1$ and the three perpendiculars are concurrent.

Chapter 6

Collinearity

Problems on collinearity of points and concurrence of lines are very common in elementary plane geometry. To prove that 3 points A, B, C are collinear, the most straightforward technique is to verify that one of the angles $\angle ABC, \angle ACB$ or $\angle BAC$ is 180°. We could also try to verify that the given points all lie on a specific line which is known to us. These methods have been applied in earlier chapters to prove that the Simson line and the Euler line are lines of collinearity of certain special points of a triangle. In this chapter, we shall explore more results such as Desargues' theorem, Menelaus' theorem and Pappus' theorem which give conditions on when three points are collinear.

The concept of collinearity and concurrence are dual to each other. For instance, suppose we wish to prove that 3 lines PQ, MN, XY are concurrent. Let PQ intersect MN at Z. Now it reduces to prove that X, Y, Z are collinear. Conversely, to prove that X, Y, Z are collinear, it suffices to show that the 3 lines PQ, MN, XY are concurrent.

6.1 Menelaus' theorem

Theorem 6.1 (Menelaus) *The three points P, Q, R on the sides AC, AB and BC respectively of a triangle ABC are collinear if and only if*

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = -1,$$

where directed segments are used. That is either 1 or 3 points among P, Q, R are outside the triangle.

Proof. Suppose that P, Q, R are collinear. Construct a line through C parallel to AB intersecting the line containing P, Q, R at a point D. See figure 6.1. Since $\triangle DCR \sim \triangle QBR$ and $\triangle PDC \sim \triangle PQA$, we have

$$\frac{QB \cdot RC}{BR} = DC = \frac{AQ \cdot CP}{PA}.$$

From this, the result follows.

Conversely, suppose

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = -1.$$

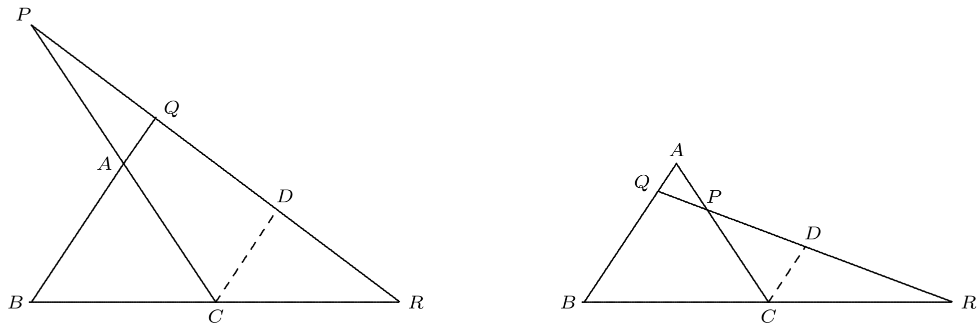


Figure 6.1: Menelaus' theorem

Let the line containing R and Q intersect AC at P'. Now P', Q, R are collinear. Hence,

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP'}{P'A} = -1.$$

Therefore, $CP'/P'A = CP/PA$. This implies that P and P' must coincide.

Definition 6.1 *The line PQR that cuts the sides of a triangle is called a transversal of the triangle.*

Example 6.1 The side AB of a square $ABCD$ is extended to P so that $BP = 2AB$. Let M be the midpoint of CD and Q the point of intersection between AC and BM. Find the position of the point R on BC such that P, R, Q are collinear.

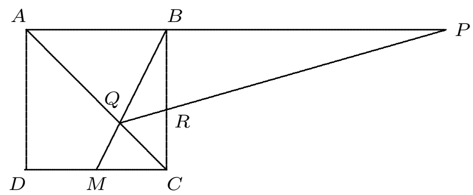


Figure 6.2

Solution. First we know that $AP : PB = 3 : -2$. Next we have $\triangle ABQ \sim \triangle CMQ$. Hence, $CQ : QA = CM : AB = \frac{1}{2}$. By Menelaus' theorem applied to triangle ABC, the points P, R, Q are collinear if and only if

$$\frac{AP}{PB} \cdot \frac{BR}{RC} \cdot \frac{CQ}{QA} = -1.$$

That is $BR : RC = 4 : 3$.

Example 6.2 In the figure, a line intersects each of the three sides of a triangle ABC at D, E, F. Let X, Y, Z be the midpoints of the segments AD, BE, CF respectively. Prove that X, Y, Z are collinear.

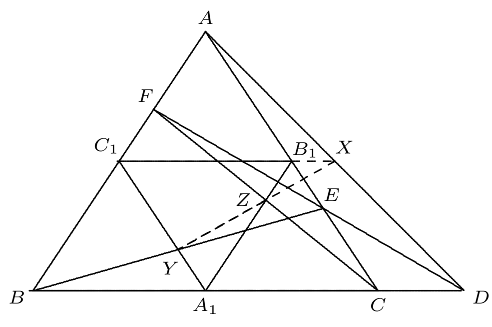


Figure 6.3

Solution. Let A_1, B_1 and C_1 be the midpoints of BC, AC and AB respectively. Then B_1C_1 is parallel to BC and B_1, C_1 and X are collinear. Hence, $BD/DC = C_1X/XB_1$. Similarly, $CE/EA = A_1Y/YC_1$ and $AF/FB = B_1Z/ZA_1$. Now apply Menelaus' theorem to $\triangle ABC$ and the straight line DEF. We have

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1,$$

That is

$$\frac{C_1X}{XB_1} \cdot \frac{B_1Z}{ZA_1} \cdot \frac{A_1Y}{YC_1} = -1.$$

Then, by Menelaus' theorem applied to $\triangle A_1B_1C_1$ and the points X, Y, Z, the points X, Y, Z are collinear.
(*The line XYZ is called the Gauss line.*)

Example 6.3 A line through the centroid G of $\triangle ABC$ cuts the sides AB at M and AC at N. Prove that

$$AM \cdot NC + AN \cdot MB = AM \cdot AN.$$

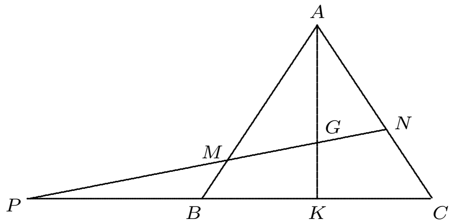


Figure 6.4

Solution. The above relation is equivalent to $NC/AN + MB/AM = 1$. If MN is parallel to BC, then $NC/AN = MB/AM = GK/AK = \frac{1}{2}$. Therefore the result is true.
Next consider the case where MN meets BC at a point P. Apply Menelaus' theorem to $\triangle AKB$ and the line PMG. We have $(BP/PK) \cdot (KG/GA) \cdot (AM/MB) = 1$ in absolute value. As $KG/GA = \frac{1}{2}$, we have $BP = (2MB \cdot PK)/AM$. Similarly, by applying Menelaus' theorem to $\triangle ACK$ and the line PGN, we have $PC = (2CN \cdot KP)/NA$.
Note that $PC - PK = KC = BK = PK - PB$. Substituting the above relations into this equation, we obtain the desired expression.

Theorem 6.2 *In the convex quadrilateral $ACGE$, AG intersects CE at H, the extension of AE intersects the extension of CG at I, the extension of EG intersects the extension of AC at D, and the line IH meets EG at F and AD at B. Then*
(i) $AB/BC = -AD/DC$,
(ii) $EF/FG = -ED/DG$.
Here directed line segments are used.

Proof. (i) Refer to Figure 6.5. Applying Ceva's Theorem to $\triangle ACI$, we have

$$\frac{IE}{EA}\frac{AB}{BC}\frac{CG}{GI} = 1.$$

Next by Menelaus' Theorem applied to $\triangle ACI$ with transversal EGD, we have

$$\frac{AD}{DC}\frac{CG}{GI}\frac{IE}{EA} = -1.$$

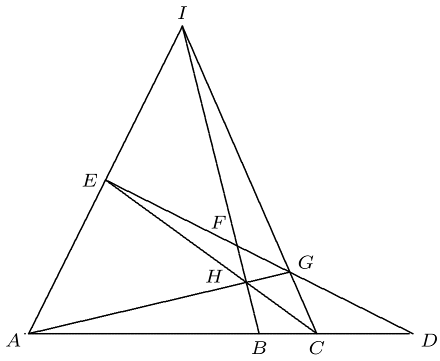


Figure 6.5: A complete quadrilateral

Thus, $AB/BC = -AD/DC$.
(ii) To prove the second assertion, apply Ceva's Theorem to $\triangle IEG$ with cevians IF, EC and GA. They concur at H. Thus, we have

$$\frac{IA}{AE}\frac{EF}{FG}\frac{GC}{CI} = 1.$$

By Menelaus' Theorem applied to $\triangle IEG$ with transversal ACD,

$$\frac{ED}{DG}\frac{GC}{CI}\frac{IA}{AE} = -1.$$

Thus $EF/FG = -ED/DG$.

6.2 Desargues' theorem

Theorem 6.3 (Desargues) *Let ABC and $A_1B_1C_1$ be two triangles such that AA_1, BB_1, CC_1 meet at a point O. (The two triangles are said to be perspective from the point O.) Let L be the intersection of BC and B_1C_1, M the intersection of CA and C_1A_1 and N the intersection of AB and A_1B_1. Then L, M and N are collinear.*

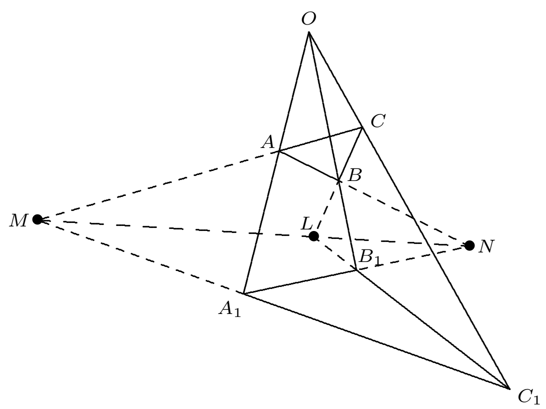


Figure 6.6: Two triangles in perspective from a point

Proof. The line LB_1C_1 cuts $\triangle OBC$ at L, B_1 and C_1. By Menelaus' theorem,

$$\frac{BL}{LC} \cdot \frac{CC_1}{C_1O} \cdot \frac{OB_1}{B_1B} = -1.$$

Similarly, the lines MA_1C_1 and NB_1A_1 cut $\triangle OCA$ and $\triangle OAB$ respectively. By Menelaus' theorem, we have

$$\frac{CM}{MA} \cdot \frac{AA_1}{A_1O} \cdot \frac{OC_1}{C_1C} = -1 \text{ and } \frac{AN}{NB} \cdot \frac{BB_1}{B_1O} \cdot \frac{OA_1}{A_1A} = -1.$$

Multiplying these together, we obtain

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = -1.$$

By Menelaus' theorem applied to $\triangle ABC$, the points L, M and N are collinear.

Exercise 6.1 Prove the converse of Desargues' theorem: Let ABC and $A_1B_1C_1$ be two triangles such that BC intersects B_1C_1 at L, CA intersects C_1A_1 at M and AB intersects A_1B_1 at N. Suppose L, M, N are collinear. Then AA_1, BB_1 and CC_1 are concurrent.

[Hint: Refer to figure 6.6. Let AA_1 intersect BB_1 at O. It suffices to prove O, C, C_1 are collinear. To do so, apply Desargues' theorem to the triangles MAA_1 and LBB_1 which are perspective from the point N.]

6.3 Pappus' theorem

Theorem 6.4 (Pappus) *If A, C, E are three points on one line, B, D, F on another, and if the three lines AB, CD, EF meet DE, FA, BC respectively at points L, M, N, then L, M, N are collinear.*

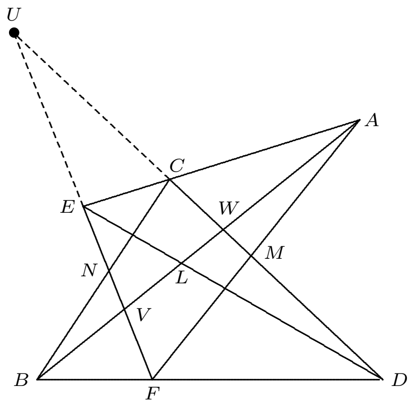


Figure 6.7

Proof. Extend FE and DC meeting at a point U as in the figure. If FE and DC are parallel, then the point U is at infinity. The proof is still valid if the problem is suitably translated in terms of projective geometry. Let's not worry about this situation as this would take us too far in the direction of projective geometry. We may as well consider the intersection point between BC and FA if they are not parallel. The case where $FE \parallel DC$ and $BC \parallel FA$ can be proved directly. The reader is invited to try by himself or herself.

Apply Menelaus' theorem to the five triads of points L, D, E; A, M, F; B, C, N; A, C, E; B, D, F on the sides of the triangle UVW. We obtain

$$\frac{VL}{LW} \cdot \frac{WD}{DU} \cdot \frac{UE}{EV} = -1, \frac{VA}{AW} \cdot \frac{WM}{MU} \cdot \frac{UF}{FV} = -1, \frac{VB}{BW} \cdot \frac{WC}{CU} \cdot \frac{UN}{NV} = -1,$$

$$\frac{VA}{AW} \cdot \frac{WC}{CU} \cdot \frac{UE}{EV} = -1, \frac{VB}{BW} \cdot \frac{WD}{DU} \cdot \frac{UF}{FV} = -1.$$

Dividing the product of the first three expressions by the product of the last two, we have

$$\frac{VL}{LW} \cdot \frac{WM}{MU} \cdot \frac{UN}{NV} = -1.$$

By Menelaus' theorem, N, L, M are collinear.

Exercise 6.2 Prove that the interior angle bisectors of two angles of a non-isosceles triangle and the exterior angle bisector of the third angle meet the opposite sides in three collinear points.

Exercise 6.3 (Monge's Theorem) Prove that the three pairs of common external tangents to three circles, taken two at a time, meet in three collinear points.

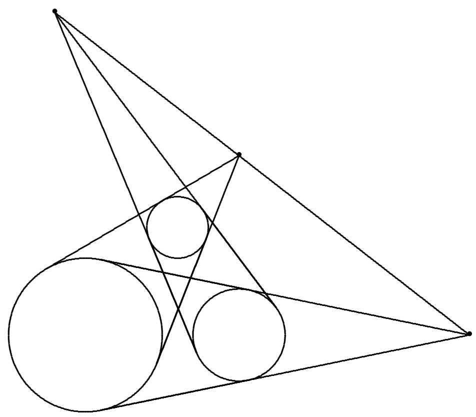


Figure 6.8: Monge's theorem

Exercise 6.4 Let I be the centre of the inscribed circle of the non-isosceles triangle ABC, and let the circle touch the sides BC, CA, AB at the points A_1, B_1, C_1 respectively. Prove that the centres of the circumcircles of $\triangle AIA_1, \triangle BIB_1$ and $\triangle CIC_1$ are collinear.

[Hint: Let the line perpendicular to CI and passing through C meet AB at C_2. By analogy, we have the points A_2 and B_2. It is obvious that the centres of the circumcircles of $\triangle AIA_1$, $\triangle BIB_1$ and $\triangle CIC_1$ are the midpoints of A_2I, B_2I and C_2I, respectively. So it is sufficient to prove that A_2, B_2 and C_2 are collinear.]

Chapter 7

Circles

A circle consists of points on the plane which are of fixed distance r from a given point O. Here O is the centre and r is the radius of the circle. It has long been known to the Pythagoreans such as Antiphon and Eudoxus that the area of the circle is proportional to the square of its radius. Inevitably the value of the proportionality π is of great importance to science and mathematics. Many ancient mathematicians spent tremendous effort in computing its value. Archimedes was the first to calculate the value of π to 4 decimal places by estimating the perimeter of a 96-gon inscribed in the circle. He obtained $223/71 < \pi < 22/7$. Around 265AD, Liu Hui in China came up with a simple and rigorous iterative algorithm to calculate π to any degree of accuracy. He himself carried out the calculation to 3072-gon and obtained $\pi = 3.1416$. The Chinese mathematician Zu Chongzhi (429-500) gave the incredible close rational approximation $\frac{355}{113}$ to π, which is often referred to as "Milu".

7.1 Basic properties

Circles are the most symmetric plane figures and they possess remarkable geometric properties. In this chapter, we shall explore some of these results as well as coaxal families of circles. In addition, figures inscribed in a circle or circumscribing a circle also enjoy interesting properties. We begin with some basic results about circles which we will leave them for the readers to supply the proofs.

1. Let AB and CD be two chords in a circle. The followings are equivalent.

(i) $\overset{\frown}{AB}=\overset{\frown}{CD}$, where $\overset{\frown}{AB}$ is the length arc of AB.
(ii) $AB = CD$.
(iii) $\angle AOB = \angle COD$.
(iv) $OE = OF$.

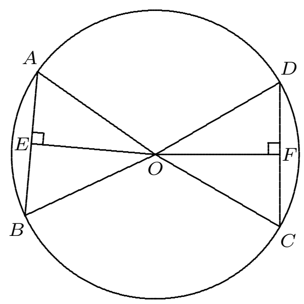


Figure 7.1

2. Let AB and CD be two chords in a circle. The followings are equivalent.

(i) $\overset{\frown}{AB} > \overset{\frown}{CD}$
(ii) $AB > CD$.
(iii) $\angle AOB > \angle COD$.
(iv) $OE < OF$.

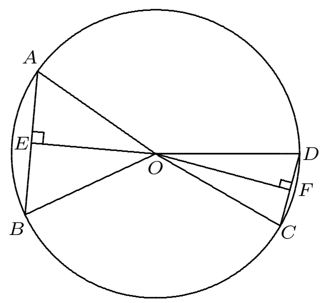


Figure 7.2

3. Let D be a point on the arc AB. The followings are equivalent.

(i) $\overset{\frown}{AD} = \overset{\frown}{DB}$.
(ii) $AC = CB$.
(iii) $\angle AOD = \angle BOD$.
(iv) $OD \perp AB$.

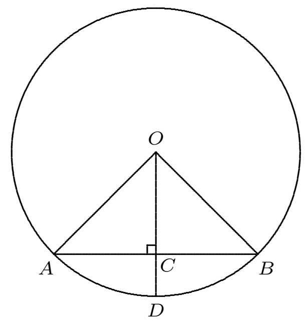


Figure 7.3

4. The angle subtended by an arc BC at a point A on a circle is half the angle subtended by the arc BC at the centre of the circle.

That is $\angle BOC = 2\angle BAC$.

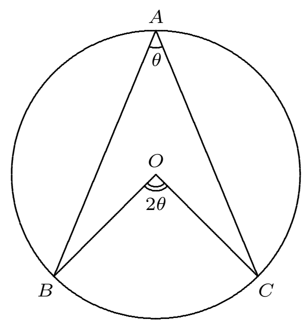


Figure 7.4

5. The angle subtended by the same segment at any point on the circle is constant.

That is $\angle BAC = \angle BDC$.

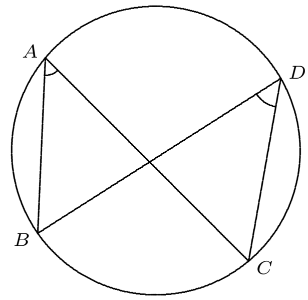


Figure 7.5

6. A chord BC is a diameter if and only if the angle subtended by it at point on the circle is a right angle.

That is $\angle BAC = 90°$ for any point $A \neq B$ or C on the circle.

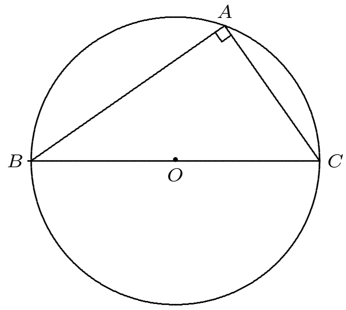


Figure 7.6

7. Let $ABCD$ be a convex quadrilateral. The followings are equivalent.

(i) $ABCD$ is a cyclic quadrilateral
(ii) $\angle BAC = \angle BDC$.
(iii) $\angle A + \angle C = 180°$.
(iv) $\angle ABE = \angle D$.

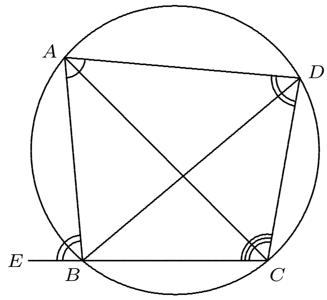


Figure 7.7

8. Alternate Segment Theorem. Let A, B, C be three points on a circle. Let TA be a line through A with T and B lying on the same side of the line AC. Then the followings the equivalent.

(i) AT is tangent to the circle at A.
(ii) $OA \perp AT$.
(iii) $\angle BAT = \angle BCA$.

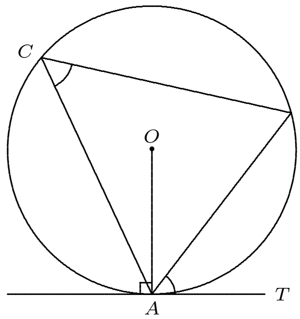


Figure 7.8

9. Let PS and PT be tangents to the circle. Then

(i) $PS = PT$,
(ii) OP bisects $\angle SPT$
(iii) OP bisects $\angle SOT$
(iv) OP is the perpendicular bisector of the segment ST.

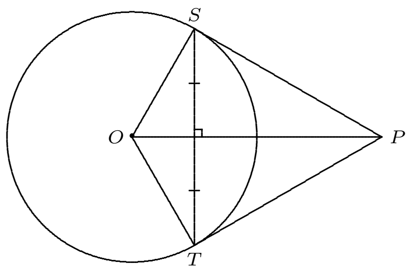


Figure 7.9

Definition 7.1 *Four points are concyclic if they lie on a circle.*

Theorem 7.1 (Euclid's theorem) *Let A, B, C, D be 4 points on the plane such AB and CD or their extensions intersect at the point P. Then A, B, C, D are concyclic if and only if*

$$PA \cdot PB = PC \cdot PD.$$

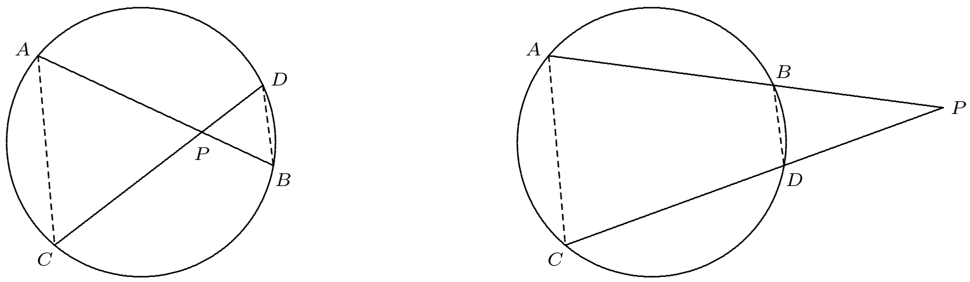


Figure 7.10: Euclid's theorem

Proof. The result follows from the fact the triangles APC and DBC are similar.

Definition 7.2 *The power of a point P with respect to the circle centred at O with radius R is defined as $OP^2 - R^2$.*

(i) If P is outside the circle, then
the Power of P
= $OP^2 - R^2$
= $PT^2 = PA \cdot PB$,
which is positive.

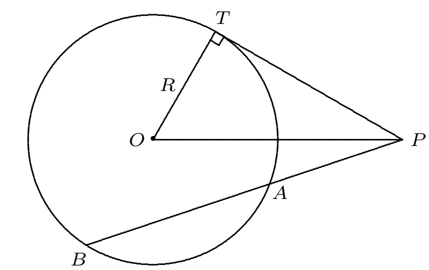


Figure 7.11

(ii) If P lies on the circumference, then
the power of $P = OP^2 - R^2 = 0$.

(iii) If P is inside the circle, then
the power of P
= $OP^2 - R^2 = -PZ^2$
= $-PX \cdot PY$
= $-PA \cdot PB$,
which is negative.

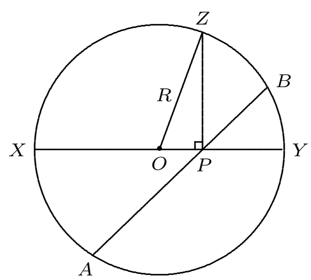


Figure 7.12

Exercise 7.1 Let D, E and F be three points on the sides BC, CA and AB of a triangle ABC respectively. Show that the circumcircles of the triangles AEF, BDF and CDE meet a common point. This point is called the Miquel point.

Theorem 7.2 (Euler's formula for OI)
Let O and I be the circumcentre and the incentre, respectively, of $\triangle ABC$ with circumradius R and inradius r. Then

$$OI^2 = R^2 - 2rR.$$

Proof. As $\angle CBQ = \frac{1}{2}\angle A$, it follows that $\angle QBI = \angle QIB$ and $QB = QI$. The absolute value of the power of I with respect to the circumcircle of ABC is $R^2 - OI^2$, which is also equal to $IA \cdot QI = IA \cdot QB = \frac{r}{\sin\frac{A}{2}} \cdot 2R\sin\frac{A}{2} = 2Rr$.

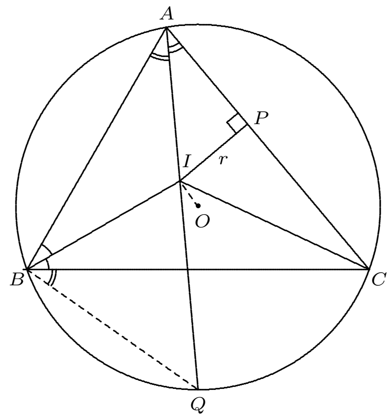


Figure 7.13

Corollary 7.3 $R \geq 2r$. *Equality holds if and only if ABC is equilateral.*

Exercise 7.2 Prove the *isoperimetric inequality* $s^2 \geq 3\sqrt{3}A$, where A is the area and s is the semiperimeter of the triangle. Show that equality holds if and only if the triangle is equilateral.

7.2 Coaxal circles

Let C be a circle and P a point. Suppose AA' and BB' are two chords of C intersecting at P. Then $PA \cdot PA' = PB \cdot PB'$. Let R be the radius of C and d the distance from P to the centre of C. We have $PA \cdot PA' = d^2 - R^2$ or $R^2 - d^2$, depending on whether P is outside or inside C. Recall that the quantity $d^2 - R^2$ is called the *power* of P with respect to the circle C. Note that the power of P with respect to C is positive if and only if P is outside C.

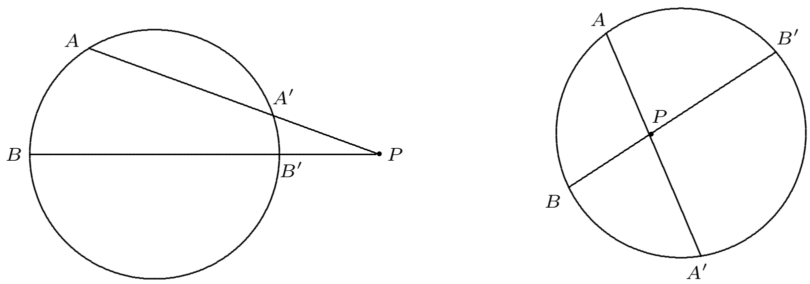


Figure 7.14: The power of a point with respect to a circle

If P is outside C and PT is a tangent to C at T, then the power of P with respect to C is PT^2. The power of P with respect to C can also be expressed in terms of the equation of C. (The coefficients of x^2 and y^2 are both 1.)
The standard equation of a circle centred at $(-f, -g)$ is of the form

$$C(x, y) = x^2 + y^2 + 2fx + 2gy + h = 0.$$

Theorem 7.4 *The power of a point $P(a,b)$ with respect to a circle $C = 0$ is also given by $C(a,b)$.*

Definition 7.3 *The locus of the points having equal power with respect to two non-concentric circles C_1 and C_2 is called the radical axis of C_1 and C_2.*

Theorem 7.5 *For any two non-concentric circles $C_1 = 0$ and $C_2 = 0$, the radical axis is given by*

$$C_1 - C_2 = 0.$$

Proof. If $P(a,b)$ is on the radical axis, then $C_1(a,b) = C_2(a,b)$, i.e, P is on the line $C_1 - C_2 = 0$. Conversely, any point $P(a,b)$ on the line has equal power with respect to the two circles.

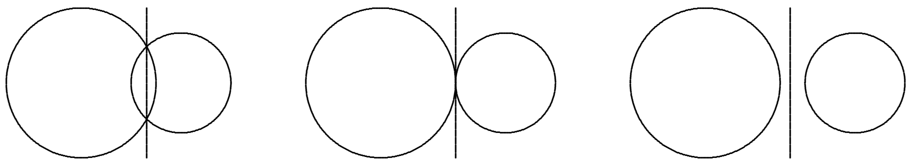


Figure 7.15: Radical axis

Exercise 7.3 Show that the radical axis of 2 circles is perpendicular to the line joining the centres of the 2 circles.

Theorem 7.6 *Let $C_3 = \lambda C_1 + \mu C_2 = 0$, where $\lambda + \mu = 1$.*
(i) Any point $P(a,b)$ on the line $C_1(x,y) - C_2(x,y) = 0$ has equal power with respect to the three circles C_1, C_2, C_3.
(ii) For any point $Q(c,d)$ on C_3, the ratio of the powers of Q w.r.t C_1 and C_2 is $-\mu/\lambda$, which is a constant.

Proof. (i) The power of P with respect to C_1 and C_2 are equal to $k = C_1(a,b) = C_2(a,b)$. Its power with respect to C_3 is

$$\lambda C_1(a,b) + \mu C_2(a,b) = (\lambda + \mu)k = k.$$

(ii) Since Q is on C_3, we have $\lambda C_1(c,d) + \mu C_2(c,d) = 0$ or $C_1(c,d)/C_2(c,d) = -\mu/\lambda$.

Definition 7.4 *The collection of all circles of the form $C_3 = \lambda C_1 + \mu C_2$, where $\lambda + \mu = 1$, forms a so-called pencil of circles. Any two such circles have the same radical axis, and they are called coaxal circles.*

Theorem 7.7 *Suppose C_1, C_2, C_3 are three circles such that for any point $P(a,b)$ on C_3, the ratio of the powers of P w.r.t to C_1, C_2 is a constant $k(\neq 1)$, then $C_3 = \lambda C_1 + \mu C_2$, where $\mu = k/(k-1)$ and $\lambda = -1/(k-1)$.*

Proof. We have $C_1(a,b)/C_2(a,b) = k$. So $C_1(a,b) - kC_2(a,b) = 0$. Thus $C_3 = \lambda C_1 + \mu C_2$. Note that for the above statement to be true we need the condition to hold for 3 points on C_3 because 3 points determine a unique circle, i.e. if $C_1(a_i,b_i)/C_2(a_i,b_i) = k$ for 3 distinct points (a_i,b_i), $i = 1,2,3$, then C_3 above is the circumcircle of the triangle whose vertices are (a_i,b_i).

Example 7.1 Let C_1 and C_2 be two circles tangent at a point M. If A is any point on C_1, with AP as the tangent to C_2, then AP/AM is a constant as A varies on C_1.

Solution. Regard M as a circle of 0 radius. Then the 3 circles C_1, M, C_2 are coaxal with the tangent at M as the radical axis. Thus, AP/AM is the ratio of the powers of A with respect to C_2 and M which is constant.

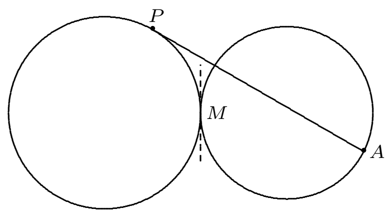


Figure 7.16: AP/AM is a constant as A varies on C_1

Theorem 7.8 *The three radical axes of three non-concentric circles C_1, C_2, C_3, taken in pairs, are either parallel or concurrent.*

Proof. The three radical axes are $C_1 - C_2 = 0, C_2 - C_3 = 0, C_3 - C_1 = 0$. Any point that satisfies two of the equations must satisfy the third. Thus if two of the lines intersect, then the third must also pass through the point of intersection, i.e., they are concurrent. Otherwise, they are pairwise parallel.

Definition 7.5 *The point of concurrence of the 3 radical axes of 3 circles is called the radical centre of the 3 circles.*

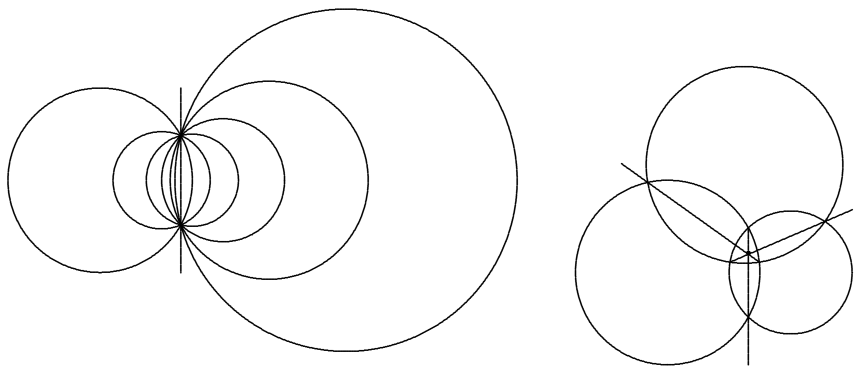


Figure 7.17: Coaxal circles and the radical centre of three non-coaxal circles

Exercise 7.4 Consider the pencil of circles $x^2 + y^2 - 2ax + c = 0$, where c is fixed and a is the parameter. (If $c > 0$, a varies in the range $\mathbb{R} \setminus (-\sqrt{c}, \sqrt{c})$.) Any two of its members have the same line of centres and the same radical axis. Hence it is a pencil of coaxal circles. Prove the following.

(a) If $c < 0$, each circle in the pencil meets the y-axis at the same two points $(0, \pm\sqrt{-c})$, and the pencil consists of circles through these two points.

(b) If $c = 0$, the pencil consists of circles touching the y-axis at the origin.

(c) If $c > 0$, the pencil consists of non-intersecting circles. Also when $a = \pm\sqrt{c}$ ($c > 0$), the circle degenerates into a point at $(\pm\sqrt{c}, 0)$.

7.3 Orthogonal pair of pencils of circles

Two non-intersecting circles give rise to a pencil of non-intersecting coaxal circles together with two degenerate circles, called the *limit points* of the pencil. For any point on the radical axis of this pencil of circles, the tangents to these circles are all of the same length. Therefore, the circle centred at that point with radius equal to the length of the tangent is orthogonal to all the circles in this pencil. All such circles form another pencil and any two of them uniquely determine the original pencil. Moreover, each circle in one pencil is orthogonal to each circle of the other pencil.

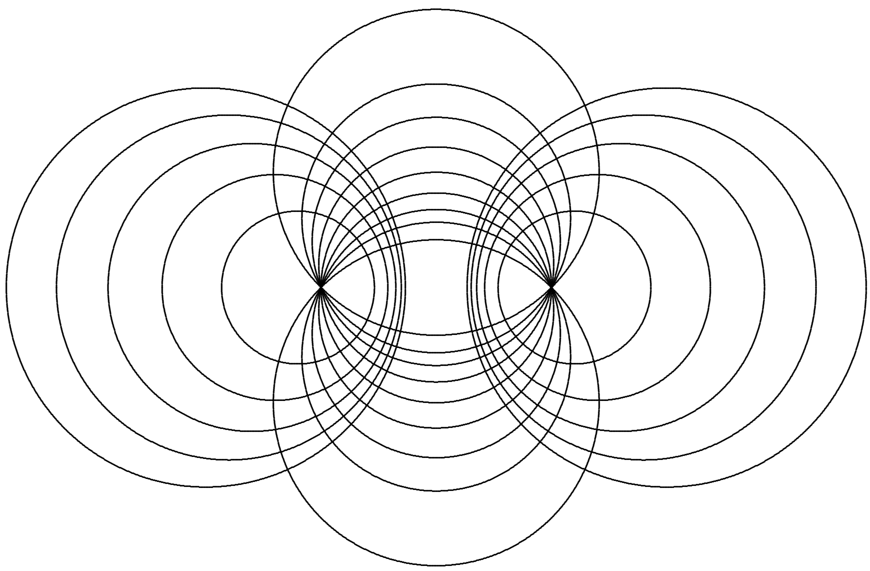


Figure 7.18: Two orthogonal pencils of coaxal circles

Exercise 7.5 Consider the two pencils of circles $\mathcal{P}_1 : x^2 + y^2 - 2ax + c = 0$ and $\mathcal{P}_2 : x^2 + y^2 - 2by - c = 0$ where $c > 0$ is fixed, a and b are the parameters.

(a) Show that $\mathcal{P}_1$ consists of non-intersecting circles, and $\mathcal{P}_2$ consists of intersecting circles all passing through the points $(\pm\sqrt{c}, 0)$.

(b) Show that each circle in $\mathcal{P}_1$ is orthogonal to each circle in $\mathcal{P}_2$.

7.4 The orthocentre

Theorem 7.9 *Let AD, BE and CF be the altitudes of the triangle ABC. The circle with diameter AB passes through D and E. Hence $HA \cdot HD = HB \cdot HE$. Similarly, $HB \cdot HE = HC \cdot HF$.*

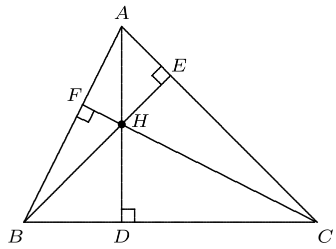


Figure 7.19

Theorem 7.10 *If X, Y, Z are any points on the respective sides BC, CA, AB of a triangle ABC, then the circles constructed on the cevians AX, BY, CZ as diameters will pass through the feet of the altitudes D, E, F respectively.*

Theorem 7.11 *If circles are constructed on 2 cevians of a triangle as diameters, then their radical axis passes through the orthocentre of the triangle.*

Theorem 7.12 *For any 3 non-coaxal circles having cevians of a triangle ABC as diameters, their radical centre is the orthocentre of $\triangle ABC$.*

Theorem 7.13 *If circles are constructed having the medians, (or altitudes or angle bisectors) of $\triangle ABC$ as diameters, then their radical centre is the orthocentre of $\triangle ABC$.*

7.5 Pascal's theorem and Brianchon's theorem

Theorem 7.14 (Pascal) *If all 6 vertices of a hexagon lie on a circle and the 3 pairs of opposite sides intersect, then the three points of intersection are collinear.*

Theorem 7.15 (Brianchon) *If all 6 sides of a hexagon touch a circle, then the three diagonals are concurrent (or possibly parallel).*

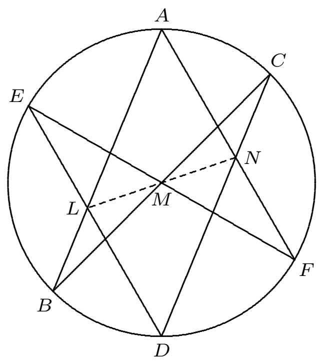


Figure 7.20: Pascal's Theorem

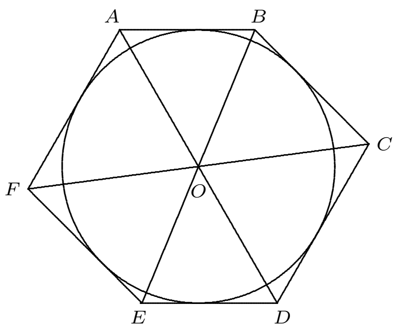


Figure 7.21: Brianchon's theorem

Proof of Pascal's theorem. We assume the lines AB, CD, EF form a triangle. Let AB intersect CD at W. The intersection points between various lines are shown in the figure.

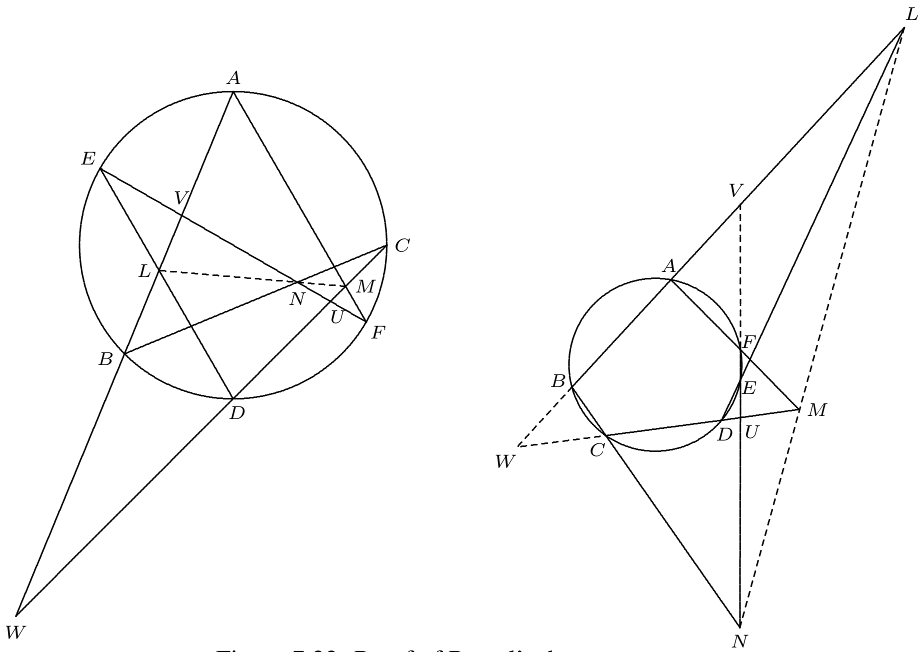


Figure 7.22: Proof of Pascal's theorem

Apply Menelaus' theorem to the transversals ELD, AMF, BNC with respect to $\triangle UVW$. We have

$$\frac{VL}{LW}\frac{WD}{DU}\frac{UE}{EV} = -1, \frac{VA}{AW}\frac{WM}{MU}\frac{UF}{FV} = -1, \frac{VB}{BW}\frac{WC}{CU}\frac{UN}{NV} = -1.$$

Therefore

$$\frac{VL}{LW}\frac{WM}{MU}\frac{UN}{NV} = \frac{DU}{WD}\frac{EV}{UE}\cdot\frac{AW}{VA}\frac{FV}{UF}\cdot\frac{BW}{VB}\frac{CU}{WC} = -1,$$

since $DU \cdot CU = UE \cdot UF$, $EV \cdot FV = VA \cdot VB$ and $AW \cdot BW = WC \cdot WD$. By Menelaus' theorem, L, N, M are collinear.

Note that the 3 equations obtained by applying Menelaus' theorem to the transversals ELD, AMF, BNC with respect to $\triangle UVW$ are the same as those in the proof of Pappus' theorem. In Pappus' theorem, there are two more such equations arising from the 2 original lines which are also transversals to $\triangle UVW$. In Pascal's theorem, these are replaced by the 3 equations arising from the condition on equality of powers of the three vertices of $\triangle UVW$ with respect to the circle.

Proof of Brianchon's theorem. Let R, Q, T, S, P, U be the points of contact of the six tangents AB, BC, CD, DE, EF, FA, as in the figure. For simplicity, we assume the hexagon is convex so that all three diagonals AD, BE, CF are chords of the inscribed circles and they are not parallel. On the lines, FE, BC, BA, DE, DC, FA extended, take points P', Q', R', S', T', U' so that

$$PP' = QQ' = RR' = SS' = TT' = UU',$$

with any convenient length, and construct circles I touching PP' and QQ' at P' and Q', II touching RR' and SS' at R' and S', and III touching TT' and UU' at T' and U'. This is possible because $ABCDEF$ has an incircle.

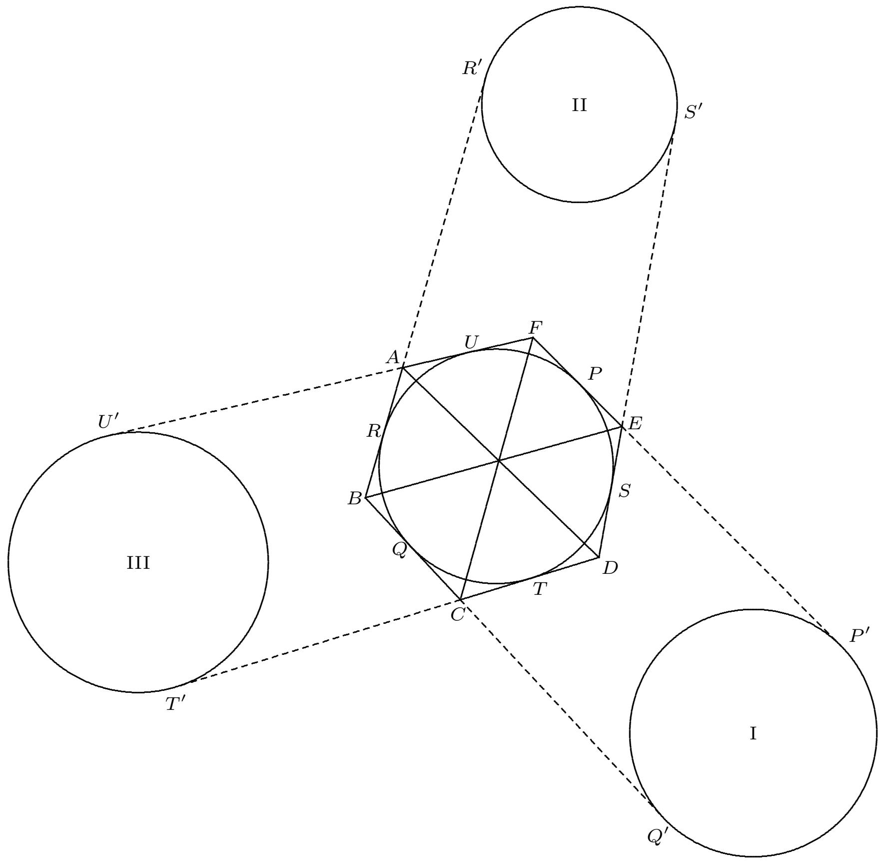


Figure 2.23: Proof of Brianchon's theorem

Now $AU' = UU' - AU = RR' - AR = AR'$ and $DT' = DT + TT' = DS + SS' = DS$ so that A and D are of equal power with respect to the circles II and III. Thus AD is the radical axes of II and III. Similarly, BE is the radical axis of I and II, and CF is the radical axis of I and III. Consequently, AD, BE and CF are concurrent.

Example 7.2 Tangents to the circumcircle of $\triangle ABC$ at points A, B, C meet sides BC, AC, and AB at points P, Q and R respectively. Prove that points P, Q and R are collinear.

Solution. As $\triangle RCA$ is similar to $\triangle RBC$, we have $RB/RC = RC/RA = BC/AC$. Hence, $RB/RA = (RC/RA)^2 = (BC/AC)^2$. Similarly, we have $QA/QC = (BA/BC)^2$ and PC/PB=$(AC/BA)^2$. Consequently, $(BR/RA) \cdot (AQ/QC) \cdot (CP/PB) = 1$. Therefore, by Menelaus' theorem, P, Q, R are collinear.

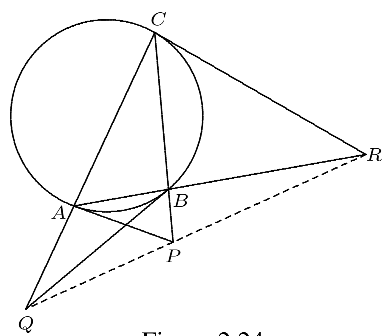


Figure 2.24

2nd solution. Alternatively, the result can be proved by applying Pascal's theorem to the degenerate 'hexagon' $AABBCC$.

Example 7.3 Let ABC be any triangle and P any point in its interior. Let P_1, P_2 be the feet of the perpendiculars from P to the sides AC and BC. Draw AP and BP and from C drop perpendiculars to AP and BP. Let Q_1 and Q_2 be the feet of these perpendiculars. Prove that the lines Q_1P_2, Q_2P_1 and AB are concurrent.

Solution. Since $\angle CP_1P$, $\angle CP_2P$, $\angle CQ_2P$, $\angle CQ_1P$ are all right angles, one sees that the points C, Q_1, P_1, P, P_2, Q_2 lie on a circle with CP as diameter. CP_1 and Q_1P intersect at A and Q_2P and CP_2 intersect at B. If we apply Pascal's Theorem to the crossed hexagon $CP_1Q_2PQ_1P_2$, we see that P_2Q_1 and P_1Q_2 intersect at a point X on the line AB.

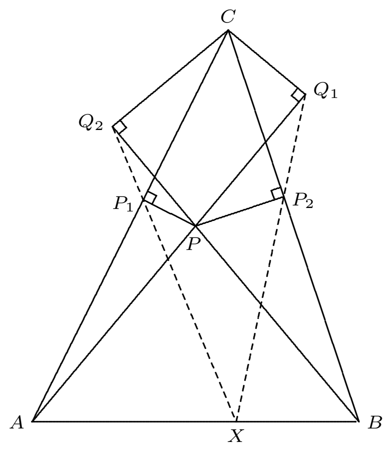


Figure 2.25

Example 7.4 A, E, B, D are points on a circle in a clockwise sense. The tangents at E and B meet at a point N, lines AE and DB meet at M and the diagonals AB and DE meet at L. Prove that L, N, M are collinear.

Solution. Apply Pascal's theorem to the degenerate hexagon $ABCDEF$ with $B = C$ and $E = F$. The sides BC and EF degenerate into the tangents at B and E respectively.

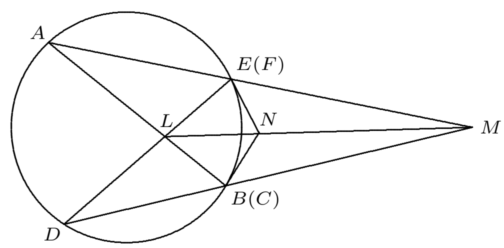


Figure 2.26

Example 7.5 Prove that the lines joining the tangency point of the incircle of a triangle to its opposite vertices concur at a common point.

Solution. The result is obvious by Ceva's theorem. Alternatively, the result follows by applying Brianchon's theorem to the hexagon $AC'BA'CB'$, where A', B', C' are the tangency points of the incircle of $\triangle ABC$ to its sides. This point is called the Gergonne point of $\triangle ABC$. See also example 5.1 in chapter 5.

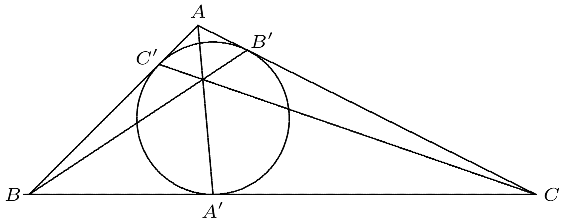


Figure 2.27

Example 7.6 Suppose $ABCD$ has an inscribed circle. Show that the lines joining the points of tangency of the inscribed circle on opposite sides are concurrent with the two diagonals.

Solution. The proof is by a degenerate case of Brianchon's theorem. For example, by taking the hexagon $ABYCDW$, we see that AC, BD, YW are concurrent; and by taking the hexagon $AXBCZD$, AC, BD, XZ are concurrent. Consequently, AC, BD, YW and XZ are concurrent. Moreover, WZ, AC, XY (same for WX, ZY, DB) are concurrent by suitably applying Brianchon's theorem.

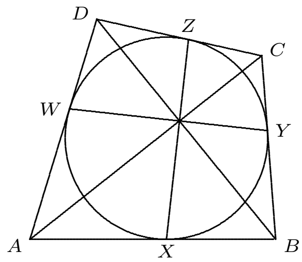


Figure 2.28

Exercise 7.6 Let ABC be a triangle, and draw isosceles triangles BCD,CAE,ABF externally to ABC, with BC,CA,AB as their respective bases. Prove that the lines through A,B,C perpendicular to lines EF, FD, DE, respectively, are concurrent.

[Hint: Draw three circles with centres D, E, F and radii DB, EC and AF respectively.]

Exercise 7.7 A convex quadrilateral $ABCD$ is inscribed in a circle centred at O. The diagonals AC and BD meet at P. Points E and F, distinct from A, B, C, D, are chosen on this circle. The circle determined by A, P, F and the circle determined by B, P, E meet at a point Q distinct from P. Suppose AF and BE intersect at R, and DF and CE intersect at X. Prove that the four points P, Q, X, R are collinear.

[Hint: Apply Pascal's theorem to the crossed hexagon $AFDBEC$.]

7.6 Homothety

Definition 7.6 *A bijective map $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is called a homothety if there exist a point C and a number $k \neq 0$, such that $h(C) = C$, and for any point P distinct from C, the point $P' = h(P)$ lies on the line CP with $\vec{CP'} = k\vec{CP}$. The point C is called the homothetic centre and k is the similitude ratio. A homothety is also called a similitude.*

Exercise 7.8 Show that if $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a homothety with homothetic centre C and similitude ratio k, then h^{-1} is a homothety with homothetic centre C and similitude ratio $1/k$.

Two figures are said to be *homothetic* if there exists a homothety mapping one to the other. If two figures are homothetic, then the lines joining the corresponding points all meet at the homothetic centre C. Note the corresponding sides of two homothetic figures are parallel. If $k = 1$, h is just the identity map. If $|k| > 1$, h enlarges any figure. If $|k| < 1$, h contracts any figure. If $k > 0$, h preserves orientation. If $k < 0$, h reverses orientation.
A common homothety arises when two circles touch at a point C. Then the ray from C meeting the two circles at two corresponding points is a homothety that maps the circles to each other.

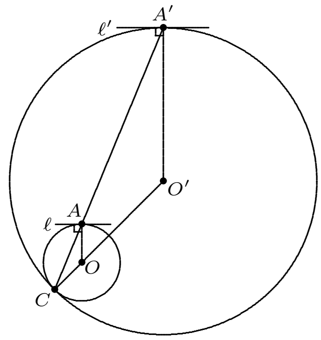


Figure 7.29: A homothety between two circles

In the above figure, one can show using similar triangles that $CA'/CA = CO'/CO = O'A'/OA = r'/r = k$, where r and r' are radii of the circles. As OA is parallel to $O'A'$, we have ℓ is parallel to ℓ'. Thus the homothety also maps ℓ to ℓ'.

Example 7.7 Three circles $\Gamma_1, \Gamma_2, \Gamma_3$ of equal radius passing through a common point O lie inside a triangle ABC such that Γ_1 touches the sides AC and AB, Γ_2 touches the sides BA and BC, and Γ_3 touches the sides CA and CB. Prove that the point O, the incentre and the circumcentre of $\triangle ABC$ are collinear.

Solution.

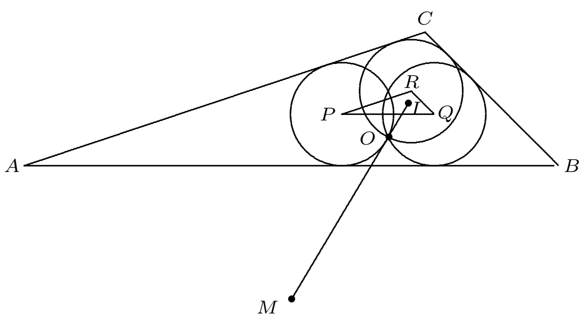


Figure 7.30: A homothety

Let the centres of the circles $\Gamma_1, \Gamma_2, \Gamma_3$ be P, Q and R respectively. Since the circles are of equal radius, we have PR is parallel to AC, RQ is parallel to CB and PQ is parallel to AB. Thus,

$\triangle PQR$ is similar to $\triangle ABC$. Note that AP, BQ and CR are the angle bisectors of $\triangle ABC$. They meet at the incentre I of $\triangle ABC$. So I is the centre of the homothety that maps $\triangle PQR$ to $\triangle ABC$. Note that O being equidistant to P, Q and R is the circumcentre of $\triangle PQR$. Therefore, under the homothety, O is mapped to the circumcentre M of $\triangle ABC$. In other words, I, O and M are collinear.

Exercise 7.9 Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three circles inside and tangent to a circle Γ such that Γ_2 is tangent to Γ_1 and Γ_3 externally; and Γ_1, Γ_3 lie outside each other. Suppose the common tangent between Γ_1, Γ_2 and the common tangent between Γ_2, Γ_3 meet at a point P on Γ. Prove that $\Gamma_1, \Gamma_2, \Gamma_3$ touch a line.

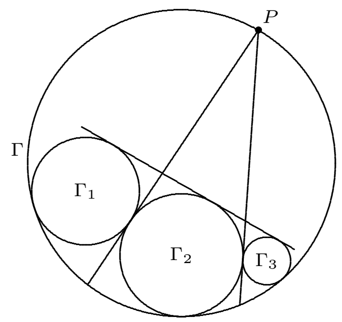


Figure 7.31: A common tangent to the three circles

7.7 The Apollonius circle of two points

Theorem 7.16 *Let A and A' be two distinct points on the plane. The locus of the points P such that $PA : PA' = \lambda$, where λ is a positive constant, is a circle with centre O on the line AA' and radius $r = (OA \cdot OA')^{\frac{1}{2}}$.*

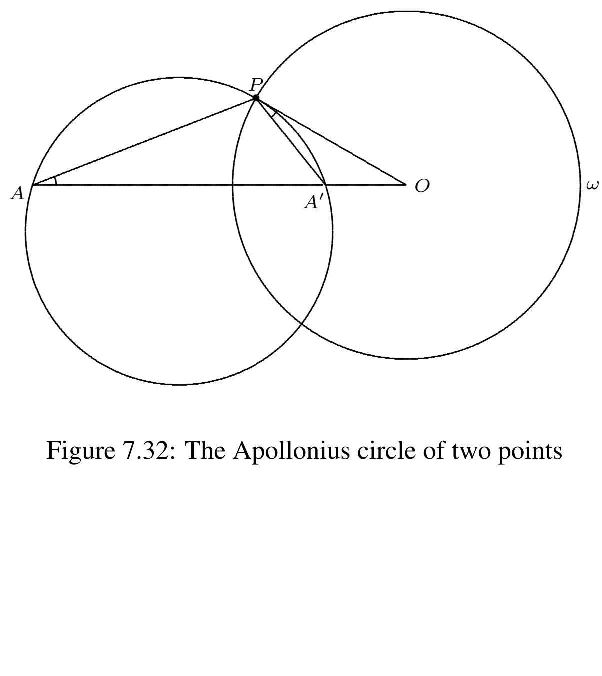


Figure 7.32: The Apollonius circle of two points

Proof. Let P be a point such that $PA : PA' = \lambda$. Consider the circle passing through P, A, A'. Let the tangent at P to the circle intersect the line AA' at a point O. Then $\angle PAO = \angle A'PO$. Thus the triangles PAO and $A'PO$ are similar so that $\lambda = AP : PA' = OP : OA' = AO : PO$. Therefore,

$$\frac{AO}{OA'} = \frac{AO}{PO} \cdot \frac{OP}{OA'} = \lambda^2.$$

Thus the point O, which lies outside the segment AA', is a fixed point on the line AA'. As $OP^2 = OA \cdot OA'$, the length OP is a constant. This means P moves along a circle ω with centre on the line AA' and radius $r = (OA \cdot OA')^{\frac{1}{2}}$. Conversely, one can chase back the above argument to prove that any point on this circle ω satisfies the property that $PA : PA' = \lambda$.

Definition 7.7 *The circle in theorem 7.16 is called the Apollonius circle of the two points.*

Remark 7.1 Since $OA \cdot OA' = r^2$, the points A and A' are inverses of each other under the inversion in ω.

Remark 7.2 If $\lambda = 1$, the Apollonius circle is the perpendicular bisector of the segment AA'.

Remark 7.3 Let the Apollonius circle intersect the line AA' at two points E and E', where E and E' are respectively inside and outside the segment AA'. Then by the angle bisector theorem, PE and PE' are respectively the internal and external angle bisectors of $\angle APA'$. Also EE' is a diameter of the Apollonius circle of the two points A and A' since $\angle EPE' = 90°$.

Exercise 7.10 Let $a > 0$. Let $A(a, 0)$ and $A'(-a, 0)$ be two distinct points on the xy-plane.
(a) Let $\lambda > 0$ and $\lambda \neq 1$. Show that the equation of the Apollonius circle ω_λ of A and A' (with ratio $PA : PA' = \lambda$) is

$$x^2 + y^2 - 2ax\left(\frac{\lambda^2+1}{\lambda^2-1}\right) + a^2 = 0.$$

(b) Show that for any $\lambda > 0, \lambda \neq 1$, the pencil of circles ω_λ are coaxal with radical axis $x = 0$.
(c) Show that the pencil of circles ω_μ

$$x^2 + y^2 - 2\mu y - a^2 = 0$$

is orthogonal to the Apollonius circle ω_λ of A and A' in (a).

7.8 Soddy's theorem

Frederick Soddy (1877-1956) being a famous chemist and an economist discovered the following beautiful formula relating the radii of 4 mutually tangent circles. He was so fascinated by this formula that he stated the result in the form of a poet in an article in *Nature* **137** (1936), p1021. See [3, page 158].

Theorem 7.17 (Soddy) *Let the radii of four mutually and externally tangent circles be r_1, r_2, r_3 and r_4. Then*

$$2\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2}\right) = \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right)^2.$$

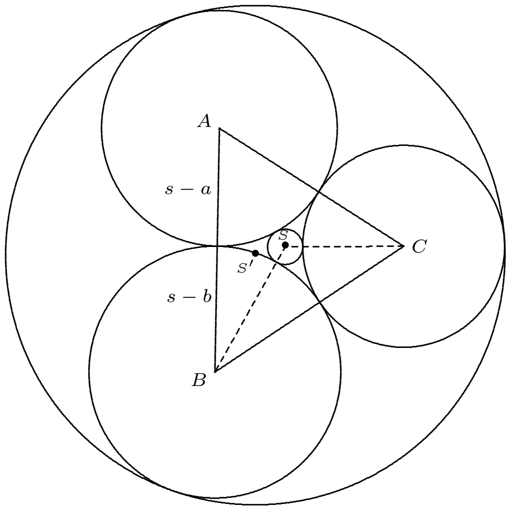


Figure 7.33: Soddy's theorem

Proof. Let the centres of the four circles be A, B, C and S. We may suppose the circle centred at S is the one which is inscribed inside the region bounded by the other three circles. There is another circle with centre S' containing and tangent to these three circles. In the triangle ABC, let's take $r_1 = s - a$, $r_2 = s - b$ and $r_3 = s - c$. Also let the radii of the circles centred at S and S' be r_4 and r_4' respectively. Then

$$SA = r_4 + s - a,\ SB = r_4 + s - b,\ SC = r_4 + s - c.$$

If we consider the circle centred at S' as the fourth circle, then $S'A = r_4 - (s - a)$ etc., and the calculations below give the same result with r_4 negative.
Let α, β, γ denote the angles at S in the three triangles SBC, SCA, SAB respectively. Applying to these triangles the formulas

$$\cos^2 \frac{A}{2} = \frac{s(s-a)}{bc} \quad \text{and} \quad \sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{bc}$$

for the angle A of any triangle ABC, we obtain

$$\cos^2 \frac{\alpha}{2} = \frac{(r_4 + a)r_4}{(r_4 + s - b)(r_4 + s - c)} \quad \text{and} \quad \sin^2 \frac{\alpha}{2} = \frac{(s-b)(s-c)}{(r_4 + s - b)(r_4 + s - c)},$$

and similar expressions for β and γ.
Since $\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 180°$, they form the angles of a triangle. By cosine law applied to this triangle, we have

$$\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\beta}{2} - \sin^2 \frac{\gamma}{2} + 2 \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \cos \frac{\alpha}{2} = 0.$$

Thus

$$\frac{(s-b)(s-c)}{(r_4+s-b)(r_4+s-c)} - \frac{(s-c)(s-a)}{(r_4+s-c)(r_4+s-a)} - \frac{(s-a)(s-b)}{(r_4+s-a)(r_4+s-b)}$$
$$+\ 2\left[\frac{(s-c)(s-a)}{(r_4+s-c)(r_4+s-a)} \cdot \frac{(s-a)(s-b)}{(r_4+s-a)(r_4+s-b)} \cdot \frac{(r_4+a)r_4}{(r_4+s-b)(r_4+s-c)}\right]^{\frac{1}{2}} = 0.$$

Multiplying throughout by $\frac{(r_4+s-a)(r_4+s-b)(r_4+s-c)}{(s-a)(s-b)(s-c)}$ and writing $r_4+a = r_4+s-b+s-c$, we get

$$\frac{r_4+s-a}{s-a} - \frac{r_4+s-b}{s-b} - \frac{r_4+s-c}{s-c} + 2\left[\frac{r_4(r_4+s-b+s-c)}{(s-b)(s-c)}\right]^{\frac{1}{2}} = 0.$$

Dividing by r_4, we obtain

$$\frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4} + 2(\frac{1}{r_2r_3} + \frac{1}{r_3r_4} + \frac{1}{r_4r_3})^{\frac{1}{2}} = 0.$$

Thus

$$(\frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4})^2 = 4(\frac{1}{r_2r_3} + \frac{1}{r_3r_4} + \frac{1}{r_4r_3}).$$

Using this, we obtain

$$\begin{aligned}(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4})^2 &= 4(\frac{1}{r_1r_2} + \frac{1}{r_1r_3} + \frac{1}{r_1r_4} + \frac{1}{r_2r_3} + \frac{1}{r_3r_4} + \frac{1}{r_4r_2}) \\ &= 2(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4})^2 - 2(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2}).\end{aligned}$$

Therefore,

$$2(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2}) = (\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4})^2.$$

Exercise 7.11 Given 3 circles of radii r_1, r_2, r_3 touching each other, show that the radii of the circles touching all three of them are given by

$$\left[r_1^{-1} + r_2^{-1} + r_3^{-1} \pm 2\sqrt{(r_1r_2)^{-1} + (r_2r_3)^{-1} + (r_3r_1)^{-1}}\right]^{-1}.$$

7.9 A generalized Ptolemy theorem

One can prove a generalized Ptolemy theorem by replacing the 4 points on a circle Γ with four circles tangent to it. The 4 circles can be inside or outside Γ. The distance between two points will then by replaced by the length of the common external or internal tangent between the two circles. In this section, we only treat the case[1] where all the 4 circles are inside and tangent to Γ.

Let's denote the length of one of the external common tangents of two circles with centres O_1 and O_2 which do not contain each other properly by $\widehat{O_1O_2}$. Whenever we refer to $\widehat{O_1O_2}$, we assume the two circles do not contain each other properly. If the two circles are tangent *internally* at a point, then $\widehat{O_1O_2} = 0$.

Lemma 7.18 *Let Γ_1, Γ_2 and Γ be three circles centred at O_1, O_2 and O with radii r_1, r_2 and r respectively. Suppose Γ_1 and Γ_2 are both inside Γ and touch Γ at points P_1 and P_2 respectively. Then $\widehat{O_1O_2} = \frac{P_1P_2}{r}\sqrt{(r-r_1)(r-r_2)}$.*

[1] The discussion for other cases can be found in the reference: Shay Gueron, Two Applications of the generalized Ptolemy Theorem, *Amer. Math. Monthly*, Volume 109, **4** (2002) 362-370.

Proof. Let T_1 on Γ_1 and T_2 on Γ_2 be the points of tangency of an external common tangent of Γ_1 and Γ_2. Then $\widehat{O_1O_2}^2 = T_1T_2^2 = O_1O_2^2 - (r_1 - r_2)^2$. By Cosine Law applied to $\triangle OO_1O_2$, we have $O_1O_2^2 = OO_1^2 + OO_2^2 - 2OO_1 \cdot OO_2 \cos\angle O_1OO_2 = (r - r_1)^2 + (r - r_2)^2 - 2(r - r_1)(r - r_2)\cos\angle O_1OO_2$. Also using Cosine Law on $\triangle P_1OP_2$, we have $\cos\angle O_1OO_2 = (2r^2 - P_1P_2^2)/(2r^2)$. Substituting these expressions for $O_1O_2^2$ and $\cos\angle O_1OO_2$ into the expression for $\widehat{O_1O_2}$ and simplifying, we obtain the require result.

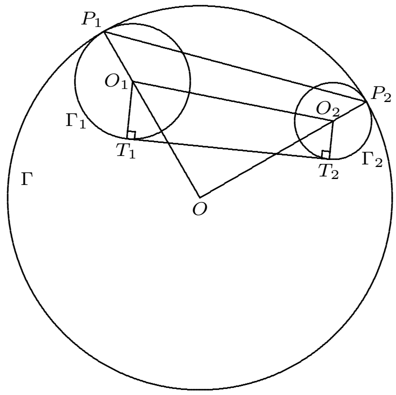


Figure 7.34: External common tangent

Theorem 7.19 (Generalized Ptolemy theorem - John Casey) *Let Γ be a circle, and let P_1, P_2, P_3, P_4 be four distinct points in this order on Γ. Let four circles centred at O_1, O_2, O_3, O_4 be inside and tangent to Γ at P_1, P_2, P_3, P_4 respectively. Then*

$$\widehat{O_1O_2} \cdot \widehat{O_3O_4} + \widehat{O_1O_4} \cdot \widehat{O_2O_3} = \widehat{O_1O_3} \cdot \widehat{O_2O_4}.$$

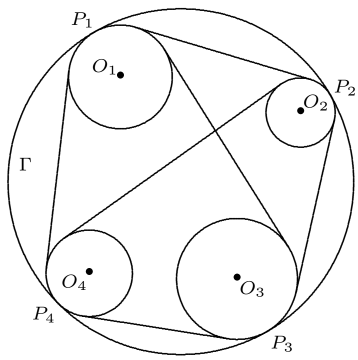


Figure 7.35: A generalized Ptolemy theorem

Proof. Let the radii of the 4 circles be r_1, r_2, r_3, r_4 and the radius of Γ be r. By Ptolemy's theorem, we have

$$P_1P_2 \cdot P_3P_4 + P_1P_4 \cdot P_2P_3 = P_1P_3 \cdot P_2P_4.$$

Multiplying throughout by $\frac{1}{r^2}\sqrt{(r-r_1)(r-r_2)(r-r_3)(r-r_4)}$ and using lemma 7.9, we get

$$\widehat{O_1O_2}\cdot\widehat{O_3O_4}+\widehat{O_1O_4}\cdot\widehat{O_2O_3}=\widehat{O_1O_3}\cdot\widehat{O_2O_4}.$$

Theorem 7.20 (The parallel tangent theorem) *Let Γ_1, Γ_2 be 2 circles lying outside each other, both inside and tangent to a circle Γ. Let ℓ be one of the external common tangent to Γ_1 and Γ_2. Let the two internal common tangents meet Γ at points X and Y where X and Y are both on the same side of ℓ. Then XY is parallel to ℓ.*

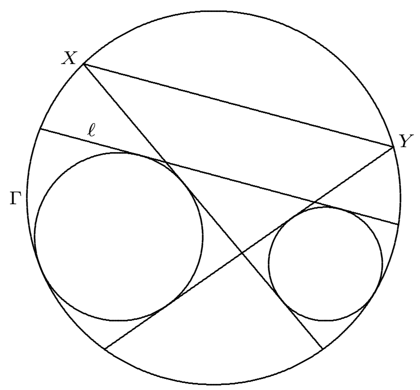


Figure 7.36: The parallel tangent theorem

Proof. Let Γ_1 and Γ_2 be tangent to Γ at S and T respectively. Let ℓ touch Γ_1 at G and Γ_2 at H. Consider the homothety centred at S mapping Γ_1 to Γ. It maps G to a point M on Γ and also ℓ to the tangent α to Γ at M. Similarly the homothety centred at T mapping Γ_2 to Γ maps H to a point M' on Γ and ℓ to the tangent β to Γ at M'. Since α and β are both parallel to ℓ and on the same side of ℓ, they must coincide. Thus $M = M'$ and SG, TH meet at the point M on Γ. Note that $\angle MST = \angle EMT = \angle MHG$ so that the triangles MHG and MST are similar. This means $MG \cdot MS = MH \cdot MT$ so that M is of equal power with respect to Γ_1 and Γ_2. If we can show M is the midpoint of the arc XY, then XY is parallel to the tangent to Γ at M, and thus parallel to ℓ. This can be achieved by using the generalized Ptolemy theorem.

Let the internal common tangent of the two circles through X meet Γ_1 at A and Γ_2 at B, and the one through Y meet Γ_1 at C and Γ_2 at D respectively. Denote the length of the tangents from M to Γ_1 and to Γ_2 by t. (M is of equal power with respect to Γ_1 and Γ_2.) Regarding M, X, Y as circles with zero radii and applying the generalized Ptolemy theorem to M, X, Γ_1, Y and M, X, Γ_2, Y, we have

$$MX \cdot YD + MY \cdot XA = tXY,$$

$$MX \cdot YC + MY \cdot XB = tXY.$$

Subtracting, we get

$$MX \cdot (YD - YC) = MY \cdot (XB - XA).$$

That is $MX \cdot CD = MY \cdot AB$. Since $AB = CD$, we have $MX = MY$ proving the result.

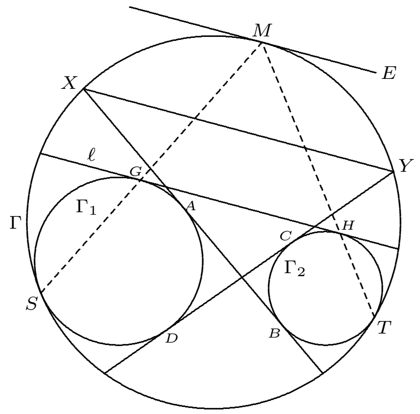


Figure 7.37: The tangent at M is parallel to XY.

Exercise 7.12 Circles Γ_1 and Γ_2 intersect at two points A and B. A line through B intersects Γ_1 and Γ_2 at points C and D, respectively. Another line through B intersects Γ_1 and Γ_2 at points E and F, respectively. Line CF intersects Γ_1 and Γ_2 at points P and Q, respectively. Let M and N be the middle points of the minor arcs $\overset{\frown}{PB}$ and $\overset{\frown}{QB}$ respectively. Prove that if $CD = EF$, then C, F, M, N are concyclic.

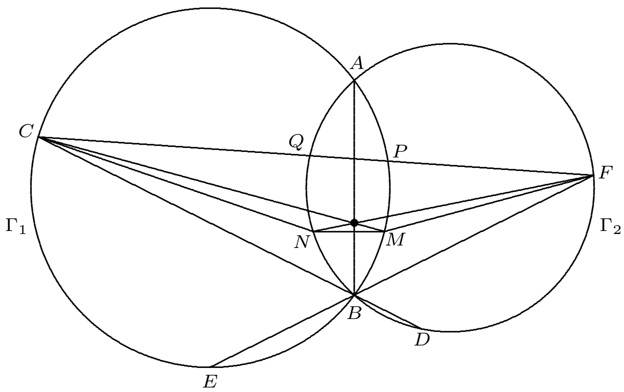


Figure 7.38: The points C, F, M, N are concyclic.

[Hint: This problem appears in China Mathematical Olympiad 2010. Let O_1 and O_2 be the centres of Γ_1 and Γ_2 respectively. Let AB intersect O_1O_2 at X. Note that O_1O_2 is perpendicular to AB. Join O_1 to the midpoints K_1 of the chord BC and L_1 of the chord BE, and join O_2 to the midpoints K_2 of the chord BD and L_2 of the chord BF, respectively. Let O_1S be the perpendicular to O_2K_2, and O_2R the perpendicular to O_1L_1. Thus $O_1K_1K_2S$ and $O_2RL_1L_2$ are rectangles. Since $CD = EF$, we have $L_1L_2 = K_1K_2$. Thus $RO_2 = SO_1$. This implies that the two right-angled triangles RO_1O_2 and SO_2O_1 are congruent. Hence $\angle RO_1O_2 = \angle SO_2O_1$. Also $\angle K_1O_1R = \angle CBE = \angle FBD = \angle L_2O_2S$. Consequently $\angle K_1O_1X = \angle K_2O_2X$, and this implies that $\angle ABC = \angle ABF$. Therefore, BA is the bisector of $\angle CBF$. Now complete the proof by showing that the incenter I of the triangle BCF is of equal power with respect to Γ_1 and Γ_2.]

Exercise 7.13 Let ABC be a triangle, H its orthocentre, O its circumcentre and R its circumradius. Let D, E, F be the reflections of A, B, C across BC, CA, AB respectively. Prove that D, E, F are collinear if and only if $OH = 2R$.

[Hint: Let G be the centroid of the triangle ABC, and A', B' and C' be the midpoints of BC, CA and AB respectively. Let $A''B''C''$ be the triangle for which A, B and C are the midpoints of $B''C''$, $C''A''$ and $A''B''$, respectively. Then G is the centroid and H is the circumcentre of triangle $A''B''C''$. Let D', E' and F' denote the projections of O on the lines $B''C''$, $C''A''$ and $A''B''$, respectively. Consider the homothety h with centre G and ratio $-1/2$. It maps A, B, C, A'', B'' and C'' into A', B', C', A, B and C respectively. Note that $A'D'$ is perpendicular to BC which implies $AD : A'D' = 2 : 1 = GA : GA'$ and $\angle DAG = \angle D'A'G'$. We conclude that $h(D) = D'$ and similarly, $h(E) = E'$, $h(F) = F'$. Thus D', E' and F' are the projections of O on the sides $B''C''$, $C''A''$ and $A''B''$, respectively. Now apply Simon's theorem.]

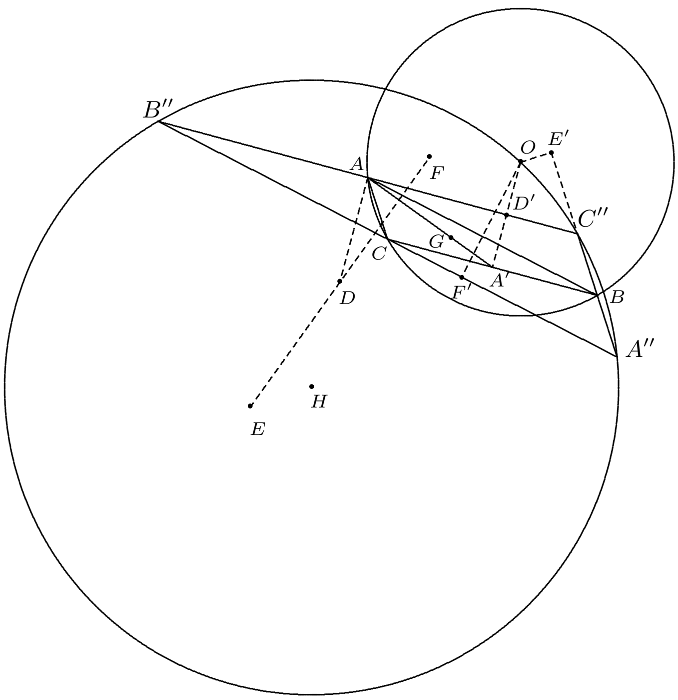


Figure 7.39: D, E, F are collinear if and only if $OH = 2R$.

Exercise 7.14 A circle ω is tangent to the circumcircle of a triangle ABC at P internally, and the sides AB and AC at U and V respectively. Prove that the line UV passes through the incentre of the triangle ABC. ω is called the *mixtilinear incircle* oppoiste to A.

Exercise 7.15 For $i = 1, 2, 3$, a circle ω_i is drawn internally tangent to the circumcircle of a triangle ABC, and two sides of the triangle ABC at U_i and V_i respectively. Prove that the lines

U_1V_1, U_2V_2, U_3V_3 are concurrent.

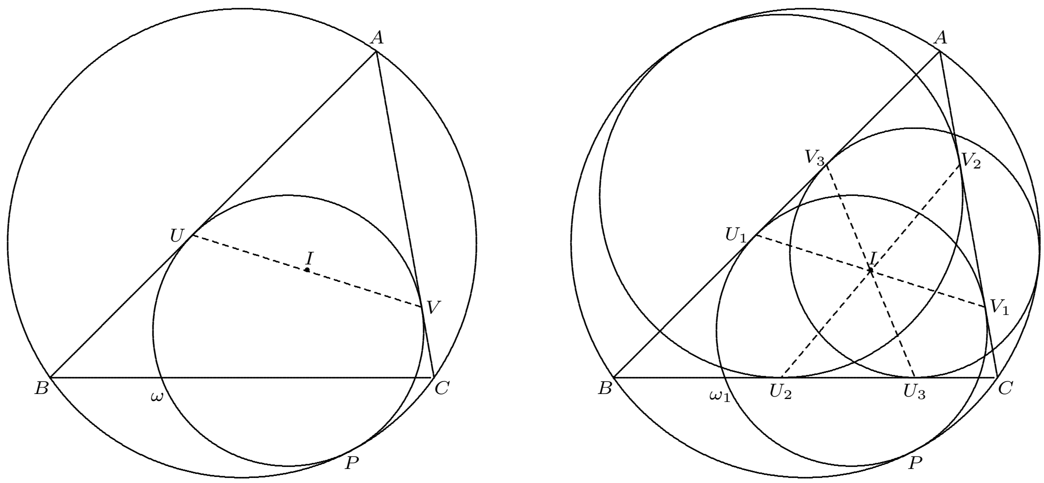


Figure 7.40: U, I, V are collinear.

Exercise 7.16 Show that the power of the centroid with respect to the circumcircle of a triangle is $\frac{1}{9}(a^2 + b^2 + c^2)$.

Exercise 7.17 Two circles δ_1 and δ_2 intersect at two points X and Y. A circle ω_1 inside δ_1 is tangent to δ_1 internally and tangent to δ_2 externally. Another circle ω_2 inside δ_2 is tangent to δ_2 internally and tangent to δ_1 externally. The two internal common tangents of ω_1 and ω_2 intersect at P. Prove that P lies on the line XY.

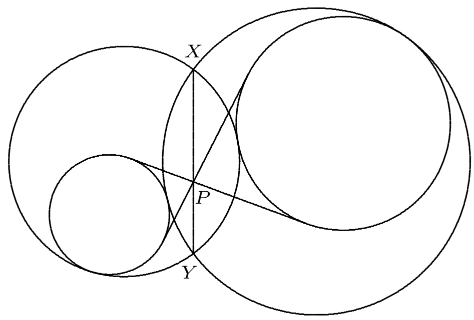


Figure 7.41: P lies on the line XY.

Exercise 7.18 Let θ be one of the angles between the circumcircle and the nine-point circle of an obtuse angled triangle ABC. Prove that

$$\cos\theta = \frac{a^2 + b^2 + c^2 - 4R^2}{4R^2},$$

where a, b, c are the lengths of the sides of the triangle and R is its circumradius.

[Hint: For an obtuse angled triangle, the circumcircle and the nine-point circle always intersect at two points. Let P be one of the intersection points. Apply cosine rule to the triangle PON and use the fact that $ON = 3OG/2$.]

Chapter 8

Using Coordinates

Coordinate geometry is invented and developed by Ren Descartes (1596-1650). First a coordinate system in which two mutually perpendicular axes intersecting at the origin is set up. In such a system, points are denoted by ordered pairs of real numbers while lines are represented by linear equations. Other objects such as circles can be represented by algebraic equations. Finding intersections between lines and curves reduces to solving equations. It has the advantage of translating geometry into purely algebra. For instance, concurrence of lines and collinearity of points can also be expressed in terms of algebraic conditions.

8.1 Basic coordinate geometry

In this section, we shall review some basic formulas in coordinate geometry.

1. **Ratio formula**. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. If P is the point that divides the line segment AB in the ratio $r : s$, (i.e. $AP : PB = r : s$), then the coordinates of P is given by
$$(\frac{sa_1 + rb_1}{r+s}, \frac{sa_2 + rb_2}{r+s}).$$

2. **Incentre**. Let the coordinates of the vertices of a triangle ABC be $(x_A, y_A), (x_B, y_B), (x_C, y_C)$ respectively. The coordinates of the incentre I of $\triangle ABC$ are
$$x_I = \frac{ax_A + bx_B + cx_C}{a+b+c} \quad \text{and} \quad y_I = \frac{ay_A + by_B + cy_C}{a+b+c}.$$

   Proof. Let the sides BC, AC, AB of $\triangle ABC$ be a, b, c respectively. Let BI meet AC at B'. Then using the Angle Bisector Theorem, $AB' : B'C = c : a$, and $BI : IB' = (a + c) : b$. (For the second ratio, extend AB to AB_1 so that $BB_1 = a$ and extend AI to meet B_1C at I'. Then B_1C is parallel to BB'. Hence $BI : IB' = B_1I' : I'C = (a + c) : b$. From this, we obtain the coordinates of I.

3. **Family of lines**. If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are two lines intersecting at a point P (i.e. $a_1b_2 \neq a_2b_1$), then the family of lines passing though P can be expressed as
$$\lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) = 0.$$

4. **Area**. The algebraic area of a triangle with vertices $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$ is given by $\frac{1}{2}(y_A + y_B)(x_A - x_B) + \frac{1}{2}(y_B + y_C)(x_B - x_C) + \frac{1}{2}(y_C + y_A)(x_C - x_A) = \frac{1}{2}(x_B y_C - x_C y_B + x_C y_A - x_A y_C + x_A y_B - x_B y_A)$ which can be expressed as a determinant

$$\frac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}.$$

This is only the algebraic area. If the ordering of the vertices of the triangle ABC is changed to ACB, then the value of this area changes by a sign. Thus (ABC) is the absolute value of this determinant.

The determinant can also be expressed as

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x_B - x_A & y_B - y_A & 0 \\ x_C - x_A & y_C - y_A & 0 \end{vmatrix},$$

which is just $\frac{1}{2}(\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{k}$.

5. **Tangent to a circle**. Let C be the circle with equation $x^2 + y^2 + 2fx + 2gy + h = 0$ and $P = (x_0, y_0)$ be a point on C. The equation of the tangent line to the circle C at P is given by

$$x_0 x + y_0 y + f(x + x_0) + g(y + y_0) + h = 0.$$

Proof. The center of the circle is $(-f, -g)$. Thus if (x, y) is a point on the tangent line, then $\langle (x - x_0, y - y_0), (x_0 + f, y_0 + g) \rangle = 0$. Using $x_0^2 + y_0^2 + 2fx_0 + 2gy_0 + h = 0$, the result follows.

6. **Coaxal circles**. The standard equation of a circle is of the form

$$C(x, y) = x^2 + y^2 + 2fx + 2gy + h = 0.$$

The power of a point $P(a, b)$ with respect to a circle $C = 0$ is also given by $C(a, b)$.

The locus of the points having equal power with respect to C_1 and C_2 is called the *radical axis* of C_1 and C_2. For any 2 circles $C_1 = 0$ and $C_2 = 0$, the radical axis is given by

$$C_1 - C_2 = 0$$

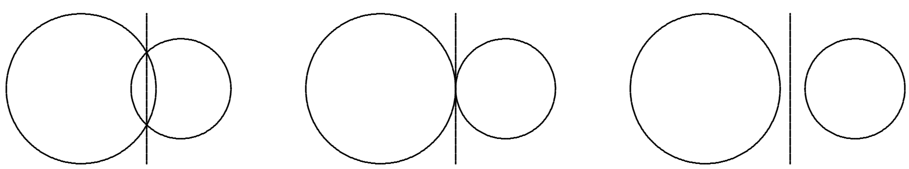


Figure 8.1: Coaxal circles

The collection of all circles of the form $C_3 = \lambda C_1 + \mu C_2$, where $\lambda + \mu = 1$, forms a so-called *pencil of circles*. Any two such circles have the same radical axes, and they are called *coaxal circles*.

Example 8.1 Let $C_1 : x^2 + y^2 = 10$ and $C_2 : x^2 + y^2 - 2x + y = 10$. Find the equation of the circle passing through the points of intersection of C_1 and C_2 and the point $(5, 5)$.

Solution. The radical axis of C_1 and C_2 has the equation given by $(x^2 + y^2 - 10) - (x^2 + y^2 - 2x + y - 10) = 0$. That is $y - 2x = 0$. Thus the equation of the required circle is of the form $(x^2 + y^2 - 10) + \lambda(y - 2x) = 0$. Since it passes through the point $(5, 5)$, we find that $\lambda = 8$. Consequently, the equation is $x^2 + y^2 - 10 + 8y - 16x = 0$.

Example 8.2 Let ℓ be a line outside a circle $\mathcal{C}$. Take any point T on ℓ. Let TA and TB be the two tangents from T to $\mathcal{C}$. Prove that the chord AB passes through a fixed point.

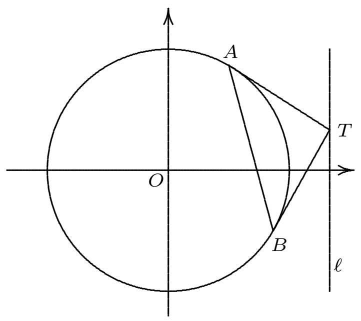


Figure 8.2: The chord AB passes through a fixed point

Solution. Let the centre O of the circle be the origin. Choose coordinate axes so that ℓ is parallel to the y-axis. Let r be the radius of the circle and (c, t) the coordinates of T. Here $r < c$. The equation of the circle is $x^2 + y^2 = r^2$.

Next, we wish to find the equation of the chord AB. To do this, it is not necessary to find the coordinates of A and B. Let the coordinates of A be (x_A, y_A). The equation of the tangent line TA is $x_A x + y_A y = r^2$. (It is a straight line passing through A and perpendicular to OA.) As it passes through T, we have $x_A c + y_A t = r^2$. Therefore, A lies on the straight line

$$cx + ty = r^2. \tag{8.1.1}$$

Similarly, B lies on (8.1.1). So (8.1.1) is the equation of AB! Clearly, the line defined by (8.1.1) passes through the point $(r^2/c, 0)$ which is independent of t.

There is also another easy way to find the equation of the line AB. Observe that O, A, T, B lie on a circle $\mathcal{C}'$ with diameter OT. The equation of this circle is $(x - c)x + (y - t)y = 0$. (Take a point X on $\mathcal{C}'$. Then OX is perpendicular to TX. The scalar product gives the equation satisfied by X.) Now $\mathcal{C}$ and $\mathcal{C}'$ both pass through A and B. Hence the difference of their equations is the equation of AB. Note that the line AB is the radical axis of $\mathcal{C}$ and $\mathcal{C}'$.

Exercise 8.1 Let $P = (a, b)$ be a point outside the unit circle $x^2 + y^2 = r^2$ and let PT_1 and PT_2 be the tangents to it. Show that the coordinates of T_1 and T_2 are given by

$$\left(\frac{ar^2-br\sqrt{a^2+b^2-r^2}}{a^2+b^2}, \frac{br^2+ar\sqrt{a^2+b^2-r^2}}{a^2+b^2}\right), \left(\frac{ar^2+br\sqrt{a^2+b^2-r^2}}{a^2+b^2}, \frac{br^2-ar\sqrt{a^2+b^2-r^2}}{a^2+b^2}\right).$$

Exercise 8.2 Let C be the circle with equation $x^2 + y^2 + 2ax + 2by + f = 0$ and $P = (x_0, y_0)$ a point outside C. The tangents from P touch C at the points X and Y. Show that the equation of the line XY is given by

$$x_0x + y_0y + a(x + x_0) + b(y + y_0) + f = 0.$$

Exercise 8.3 Show that the equation of the circle passing through the points $(p_1, p_2), (q_1, q_2), (r_1, r_2)$ is given by

$$\begin{vmatrix} x - p_1 & y - p_2 & p_1^2 + p_2^2 - x^2 - y^2 \\ p_1 - q_1 & p_2 - q_2 & q_1^2 + q_2^2 - p_1^2 - p_2^2 \\ q_1 - r_1 & q_2 - r_2 & r_1^2 + r_2^2 - q_1^2 - q_2^2 \end{vmatrix} = 0.$$

[Hint: The form of this determinant shows that it is an equation of a circle. The substitution of the coordinates of each of the three points clearly makes the determinant zero. Consider the 4 points: $(x, y), (p_1, p_2), (q_1, q_2), (r_1, r_2)$ on the circle, the perpendicular bisectors of any three of the chords among these 4 points must concur at the centre of the circle. Thus 4 points are concyclic if and only if the above determinant is zero.]

8.2 Barycentric and homogeneous coordinates

Let $A_1A_2A_3$ be a triangle on the plane. For any point M, the ratio of the (signed) areas

$$[MA_2A_3] : [MA_3A_1] : [MA_1A_2]$$

is called the *barycentric coordinates* or *areal coordinates* of M.
Here $[MA_2A_3]$ is the signed area of the triangle MA_2A_3. It is positive, negative or zero according to both M and A_1 lie on the same side, opposite side, or on the line A_2A_3. Generally, we use $(\mu_1 : \mu_2 : \mu_3)$ to denote the barycentric coordinates of a point M.

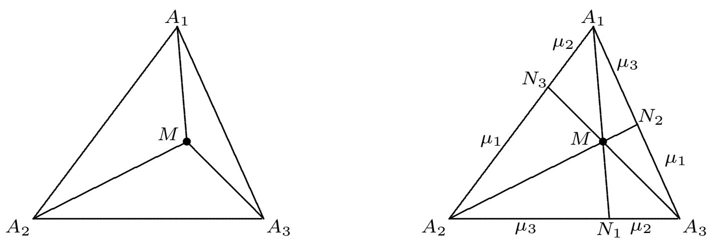


Figure 8.3: Barycentric coordinates

Theorem 8.1 *Let* $[MA_2A_3] = \mu_1$, $[MA_3A_1] = \mu_2$, $[MA_1A_2] = \mu_3$ *and* $[A_1A_2A_3] = 1$ *so that* $\mu_1 + \mu_2 + \mu_3 = 1$. *Then*
1. $A_3N_2 : N_2A_1 = \mu_1 : \mu_3$, *etc.*
2. $A_1M : MN_1 = (\mu_2 + \mu_3) : \mu_1$.
3. $\mathbf{A_2M} = \mu_3\mathbf{A_2A_3} + \mu_1\mathbf{A_2A_1}$.

Proof. Let prove 2. Let $[MN_1A_3] = \alpha, [MA_2N_1] = \beta$. Then $\frac{A_1M}{MN_1} = \frac{\mu_2}{\alpha}$ and $\frac{A_1M}{MN_1} = \frac{\mu_3}{\beta}$. Thus

$$\frac{A_1M}{MN_1} = \frac{\mu_2 + \mu_3}{\alpha + \beta} = \frac{\mu_2 + \mu_3}{\mu_1}.$$

Properties

1. The barycentric coordinates of a point are homogeneous. That is $(\mu_1 : \mu_2 : \mu_3) = (k\mu_1 : k\mu_2 : k\mu_3)$ for any nonzero real number k. As such, it can also be identified with the homogeneous coordinates of the point.

2. For the points A_1, A_2 and A_3, we have $A_1 = (1 : 0 : 0), A_2 = (0 : 1 : 0)$ and $A_3 = (0 : 0 : 1)$ respectively.

3. Let the Cartesian coordinates of A, B, C be (x_A, y_A), $(x_B, y_B), (x_C, y_C)$ respectively. If the barycentric coordinates of M is $(\mu_1 : \mu_2 : \mu_3)$, then the Cartesian coordinates of M is $\left(\frac{\mu_1 x_A+\mu_2 x_B+\mu_3 x_C}{\mu_1+\mu_2+\mu_3}, \frac{\mu_1 y_A+\mu_2 y_B+\mu_3 y_C}{\mu_1+\mu_2+\mu_3}\right)$.

4. The centroid of $\triangle A_1A_2A_3$ is the point $G = (1 : 1 : 1)$.

5. The circumcentre of $\triangle A_1A_2A_3$ is the point $O = (\sin 2A_1 : \sin 2A_2 : \sin 2A_3)$.

6. Suppose $A_1M_3/M_3A_2 = m_1$ and $A_2M_1/M_1A_3 = m_2$. Then $M = (1 : m_1 : m_1m_2)$.

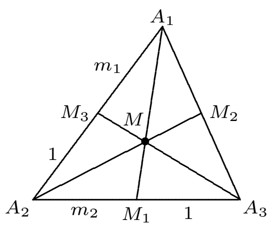


Figure 8.4: The barycentric coordinates of M

Proof. As $[MA_2A_3] : [MA_3A_1] = M_3A_2 : A_1M_3 = 1 : m_1$, and $[MA_3A_1] : [MA_1A_2] = 1 : m_2 = m_1 : m_1m_2$, we have $[MA_2A_3] : [MA_3A_1] : [MA_1A_2] = 1 : m_1 : m_1m_2$.

7. The incentre of $\triangle A_1A_2A_3$ is the point $(a_1 : a_2 : a_3)$, where a_1, a_2, a_3 are lengths of the sides $\triangle A_1A_2A_3$. This follows from the angle bisector theorem.

8. For the excentres of $\triangle A_1A_2A_3$, we have

$$I_1 = (-a_1 : a_2 : a_3), \quad I_2 = (a_1 : -a_2 : a_3), \quad I_3 = (a_1 : a_2 : -a_3).$$

9. The orthocentre of $\triangle A_1A_2A_3$ is the point

$$H = (\tan A_1 : \tan A_2 : \tan A_3) = \left(\frac{1}{-a_1^2 + a_2^2 + a_3^2} : \frac{1}{a_1^2 - a_2^2 + a_3^2} : \frac{1}{a_1^2 + a_2^2 - a_3^2}\right).$$

10. The Gergonne point of $\triangle A_1A_2A_3$ is the point

$$\left(\frac{1}{s - a_1} : \frac{1}{s - a_2} : \frac{1}{s - a_3}\right).$$

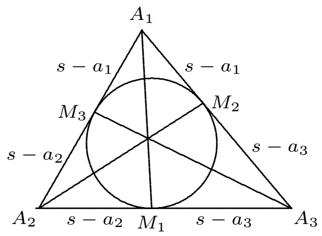


Figure 8.5: Gergonne point

11. The Nagel point of $\triangle A_1A_2A_3$ is the point $N = (s-a_1 : s-a_2 : s-a_3)$.

12. The symmedian point of $\triangle A_1A_2A_3$ is the point $(a^2 : b^2 : c^2)$.

13. The equation of the line passing through the points $(a_1 : a_2 : a_3)$ and $(b_1 : b_2 : b_3)$ is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} x_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} x_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} x_3 = 0.$$

This is a linear relation of the homogeneous coordinates of a point. In general the equation of a straight line in homogeneous coordinates is of the form

$$\ell : \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 = 0.$$

Usually, the coefficients are used to denote such a line. In notation, we write

$$\ell = [\mu_1 : \mu_2 : \mu_3].$$

Thus the line passing through $(a_1 : a_2 : a_3)$ and $(b_1 : b_2 : b_3)$ is given by

$$\ell = [a_2b_3 - a_3b_2 : -a_1b_3 + a_3b_2 : a_1b_2 - a_2b_1].$$

14. Three points $A = (a_1 : a_2 : a_3), B = (b_1 : b_2 : b_3), C = (c_1 : c_2 : c_3)$ are collinear if and only if

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

15. The intersection of the lines $\ell_1 = [a_1 : a_2 : a_3]$ and $\ell_2 = [b_1 : b_2 : b_3]$ is given by

$$P = (a_2b_3 - a_3b_2 : -a_1b_3 + a_3b_2 : a_1b_2 - a_2b_1).$$

16. Three lines $\ell = [a_1 : a_2 : a_3], m = [b_1 : b_2 : b_3], n = [c_1 : c_2 : c_3]$ are concurrent if and only if

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Proof. The intersection of ℓ and m is $(a_2b_3 - a_3b_2 : -a_1b_3 + a_3b_2 : a_1b_2 - a_2b_1)$. It lies on n if and only if $\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 = 0$.

17. Let $P = (p_1 : p_2 : p_3)$ and $Q = (q_1 : q_2 : q_3)$ with $p_1+p_2+p_3 = 1$ and $q_1+q_2+q_3 = 1$. If M divides PQ in the ratio $PM : MQ = \beta : \alpha$, then the point M has homogeneous coordinates $(\alpha p_1 + \beta q_1 : \alpha p_2 + \beta q_2 : \alpha p_3 + \beta q_3)$.

Example 8.3 In any triangle $A_1A_2A_3$, the centroid G, the incentre I and the Nagel point N are collinear.

Solution. This is because $\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ s - a_1 & s - a_2 & s - a_3 \end{vmatrix} = 0$. In fact G divides the segment IN in the ratio 1:2.

Theorem 8.2 (Menelaus) *In the triangle $A_1A_2A_3$, points B_1, B_2, and B_3 are on the sides A_2A_3, A_3A_1 and A_1A_2 respectively such that $A_2B_1 : B_1A_3 = \alpha_1 : \beta_1$, $A_3B_2 : B_2A_1 = \alpha_2 : \beta_2$ and $A_1B_3 : B_3A_2 = \alpha_3 : \beta_3$. Then B_1, B_2 and B_3 are collinear if and only if $\alpha_1\alpha_2\alpha_3 = -\beta_1\beta_2\beta_3$.*

Proof. Take $A_1 = (1 : 0 : 0), A_2 = (0 : 1 : 0), A_3 = (0 : 0 : 1)$. Then $B_1 = (0 : \beta_1 : \alpha_1)$, $B_2 = (\alpha_2 : 0 : \beta_2)$ and $B_3 = (\beta_3 : \alpha_3 : 0)$. Thus

$$\begin{vmatrix} 0 & \beta_1 & \alpha_1 \\ \alpha_2 & 0 & \beta_2 \\ \beta_3 & \alpha_3 & 0 \end{vmatrix} = \alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3.$$

Therefore, B_1, B_2 and B_3 are collinear if and only if $\alpha_1\alpha_2\alpha_3 = -\beta_1\beta_2\beta_3$.

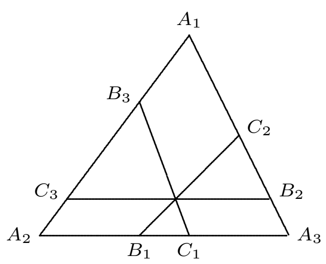


Figure 8.6: A generalization of Ceva's theorem

Theorem 8.3 *Let B_1 and C_1, B_2 and C_2, B_3 and C_3 be respective pairs of points on the sides A_2A_3, A_3A_1, A_1A_2 or their extensions of $\triangle A_1A_2A_3$ such that*

$$\frac{A_2B_1}{B_1A_3} = \lambda_1, \; \frac{A_3B_2}{B_2A_1} = \lambda_2, \; \frac{A_1B_3}{B_3A_2} = \lambda_3,$$

$$\frac{A_3C_1}{C_1A_2} = \mu_1, \; \frac{A_1C_2}{C_2A_3} = \mu_2, \; \frac{A_2C_3}{C_3A_1} = \mu_3.$$

Then B_1C_2, B_2C_3, B_3C_1 are concurrent if and only if

$$\lambda_1\lambda_2\lambda_3 + \mu_1\mu_2\mu_3 + \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 - 1 = 0.$$

Remark 8.1 Suppose $C_1 = A_3, C_2 = A_1, C_3 = A_2$ so that $\mu_1 = \mu_2 = \mu_3 = 0$. The conclusion is that B_1A_2, B_2A_3, B_3A_1 are concurrent if and only if $\lambda_1\lambda_2\lambda_3 = 1$, which is Ceva's Theorem.

Proof. Take $A_1 = (1 : 0 : 0), A_2 = (0 : 1 : 0), A_3 = (0 : 0 : 1)$. Then

$$B_1 = (0 : 1 : \lambda_1), B_2 = (\lambda_2 : 0 : 1), B_3 = (1 : \lambda_3 : 0)$$

and

$$C_1 = (0 : \mu_1 : 1), C_2 = (1 : 0 : \mu_2), C_3 = (\mu_3 : 1 : 0).$$

The line B_1C_2 is given by $\begin{vmatrix} x_1 & x_2 & x_3 \\ 0 & 1 & \lambda_1 \\ 1 & 0 & \mu_2 \end{vmatrix} = 0$. That is $B_1C_2 = [\mu_2 : \lambda_1 : -1]$. Similarly, $B_2C_3 = [-1 : \mu_3 : \lambda_2]$ and $B_3C_1 = [\lambda_3 : -1 : \mu_1]$. They are concurrent if and only if

$$\begin{vmatrix} \mu_2 & \lambda_1 & -1 \\ -1 & \mu_3 & \lambda_2 \\ \lambda_3 & -1 & \mu_1 \end{vmatrix} = 0,$$

which is the required expression.

Exercise 8.4 Prove that in any triangle the 3 lines each of which joins the midpoint of a side to the midpoint of the altitude to that side are concurrent.

[Hint. Take $A_1 = (1 : 0 : 0), A_2 = (0 : 1 : 0), A_3 = (0 : 0 : 1)$. Let F_1, F_2 and F_3 be the midpoints of the altitudes A_1N_1, A_2N_2 and A_3N_3 respectively. If M_1, M_2 and M_3 are the midpoints of the sides A_2A_3, A_3A_1 and A_1A_2 respectively, show that $M_1F_1 = [\tan A_3 - \tan A_2 : \tan A_2 + \tan A_3 : -\tan A_2 - \tan A_3]$, $M_2F_2 = [-\tan A_1 - \tan A_3 : \tan A_1 - \tan A_3 : \tan A_1 + \tan A_3]$, and $M_3F_3 = [\tan A_1 + \tan A_2 : -\tan A_1 - \tan A_2 : \tan A_2 - \tan A_1]$.]

Exercise 8.5 (Euler line) Prove that the circumcentre, the centroid and the orthocentre of a triangle are collinear.

Exercise 8.6 (Newton line) Let $ABCD$ be a quadrilateral. Let H, I, G, J, E, F be the midpoints of AB, BC, CD, DA, BD, CA respectively. Let IJ intersect HG at M, AB intersect CD at U, BC intersect AD at V. Let N be the midpoint of UV. Prove that E, F, M, N are collinear.

Exercise 8.7 In triangle ABC, let D, E, F be the midpoints of BC, CA, AB, respectively. Let M, N, P be points on the segments FD, FB, DC, respectively, such that $FM : FD = FN : FB = DP : DC$. Prove that AM, EN, FP are concurrent.

8.3 Projective plane

The real projective plane usually denoted by $\mathbb{P}^2$ consists of all lines in $\mathbb{R}^3$ passing through the origin. That is

$$\mathbb{P}^2 = \{L \ : \ L \text{ is a line through } O \text{ in } \mathbb{R}^3\}.$$

We can represent each line L through O by any non-zero vector $\mathbf{OA}$ along L. This suggests we can represent L by homogeneous coordinates consisting of a triple of three numbers $(\alpha : \beta : \gamma)$. (That is $(\alpha : \beta : \gamma) = (k\alpha : k\beta : k\gamma)$ for any non-zero k.) Thus

$$\mathbb{P}^2 = \{(\alpha : \beta : \gamma) \ : \ \alpha, \beta, \gamma \in \mathbb{R} \text{ and not all } \alpha, \beta, \gamma = 0\}.$$

For any two distinct lines L_1 and L_2 through O, it determines a plane $ax + by + xz = 0$ through O. We can represent this plane by the three coefficients a, b, c. As any non-zero multiple of $ax + by + xz = 0$ represents the same plane, this plane can be represented by the homogeneous coordinates $[a : b : c]$. Furthermore, the vector $\langle a, b, c\rangle$ is a normal vector to this plane. Thus if $L_1 = (\alpha_1 : \beta_1 : \gamma_1)$ and $L_2 = (\alpha_2 : \beta_2 : \gamma_2)$, the plane ℓ determined by L_1 and L_2 has a normal vector given by the *cross product* of $\langle \alpha_1, \beta_1, \gamma_1\rangle$ and $\langle \alpha_2, \beta_2, \gamma_2\rangle$. That is the homogeneous coordinates of the plane ℓ is

$$\left[\begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix} : - \begin{vmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{vmatrix} : \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \right].$$

If we denote the collection of all planes through the origin by $\mathbb{P}^{2^*}$, then

$$\mathbb{P}^{2^*} = \{[a : b : c] \ : \ a, b, c \in \mathbb{R} \text{ and not all } a, b, c = 0\}.$$

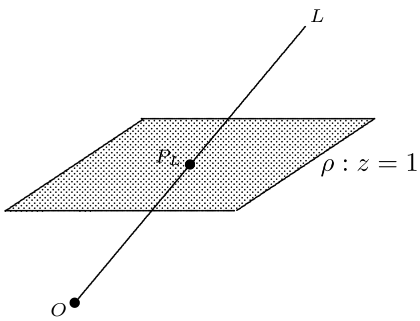


Figure 8.7: The projective plane

There is a one-to-one correspondence between $\mathbb{P}^2$ and $\mathbb{P}^{2^*}$ given by associating a line L the plane perpendicular to L.

Consider the plane $\rho : z = 1$, or any plane not containing the origin. Any element L of $\mathbb{P}^2$ not contained in the xy plane intersects ρ in a unique point P_L. See figure 8.7. In this way we can think of $\mathbb{R}^2 \equiv \rho$ lying inside $\mathbb{P}^2$. Any plane containing two distinct lines L_1 and L_2 in $\mathbb{P}^2$ (both L_1 and L_2 are not contained in the xy-plane) intersects ρ in a line ℓ joining P_{L_1} and P_{L_2}. Thus we can represents a point in $\mathbb{R}^2 \equiv \rho$ by the homogeneous coordinates $(\alpha : \beta : \gamma)$, and a line in $\mathbb{R}^2 \equiv \rho$ by $[a : b : c]$.

In fact we can think of $\mathbb{P}^2$ as $\mathbb{R}^2$ with a "line" ω added at infinity. This "line" ω corresponds to the xy-plane. With this correspondence, every "line" in $\mathbb{P}^2$ meets ω in a unique point, and any two "lines" in $\mathbb{P}^2$ meet.

If S^2 denotes the unit sphere in $\mathbb{R}^3$, then every line in $\mathbb{R}^3$ intersects S^2 in a pair of antipodal (diametrically opposite) points. In this way, we can regard $\mathbb{P}^2$ as the *space* obtained by identifying antipodal points of the unit sphere.
Though the geometry of $\mathbb{P}^2$ is different from $\mathbb{R}^2$, the properties of concurrence and collinearity are equivalent in both $\mathbb{P}^2$ and $\mathbb{R}^2$. Thus many of the results involving concurrence and collinearity in $\mathbb{R}^2$ can be stated and proved in $\mathbb{P}^2$.

8.4 Quadratic curves

A quadratic curve (or a conic) is a curve with equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$. Thus the general equation of a quadratic curve is determined by 6 coefficients. So it only requires 5 points to determine a quadratic curve. Quadratic curves are classified into the following types: parabola, circle, ellipse, hyperbola, and 2-straight line. They are the possible cross-sections obtained by slicing a double cone with a plane, thus they are also called conics. If $F_1(x, y) = 0$ and $F_2(x, y) = 0$ are two such curves, their intersection points are given by the roots of the system of these two equations. Since F_1 and F_2 are quadratic, there are generally 4 solutions for this system. Thus two quadratic curves generally intersect in 4 points (or less). Suppose $F_1(x, y) = 0$ and $F_2(x, y) = 0$ intersect in P_1, P_2, P_3, P_4. Then for any real numbers λ_1 and λ_2 not both equal to 0, $\lambda_1 F_1 + \lambda_2 F_2 = 0$ is also a quadratic curve, and it passes through P_1, P_2, P_3, P_4. Conversely, any quadratic curve passing through P_1, P_2, P_3, P_4 is of the form $\lambda_1 F_1 + \lambda_2 F_2 = 0$ for some suitable λ_1 and λ_2.

Theorem 8.4 (Butterfly theorem) *Through the midpoint O of a chord GH of a circle, two other chords AB and CD are drawn; chords AC and BD meet GH at E and F respectively. Then O is the midpoint of EF.*

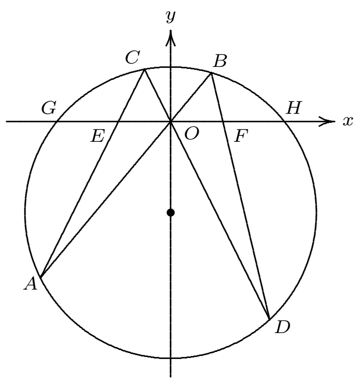


Figure 8.8: Butterfly theorem

Proof. Let the equation of the circle be $x^2+y^2-2by+f = 0$. Let the equations of the lines AB and CD be $y = k_1 x$ and $y = k_2 x$ respectively. Therefore the pair of lines $(y-k_1x)(y-k_2x) = 0$ passes through the 4 points A, B, C, D. Each quadratic curve passing through the 4 points A, B, C, D is represented by

$$x^2 + y^2 - 2by + f + \lambda(y - k_1x)(y - k_2x) = 0.$$

In particular the pair of lines AC and BD is of this form for some suitable λ. Setting $y = 0$ for the equation of this pair of lines, we get $(1 + \lambda k_1 k_2)x^2 + f = 0$. From this we see that the roots of this equation give the intercepts E and F of this pair of lines with the x-axis, and they are of equal magnitude but opposite sign. Thus $OE = OF$.

Remark 8.2 We can also take the lines AD and BC meeting the x-axis at E' and F' respectively. Then $OE' = OF'$.

Example 8.4 Suppose AB and CD are non-intersecting chords in a circle and that P is a point on the arc AB remote from C and D. Let PC and PD intersect AB at Q and R respectively. Prove that $AQ \cdot RB/QR$ is a constant independent of the position of P.

Solution. Let $AQ = x$, $QR = y$ and $RB = z$. Suppose we draw the circle through P, Q and D to cross AB extended at E. In this circle, the chord QD will subtend equal angles θ at P and E. Now, as P varies, $\angle CPD = \theta$ remains the same in the given circle, implying that, for all positions of P, this second circle through P, Q and D always goes through the same point E on AB extended. Consequently, the segment BE always has the same length k.

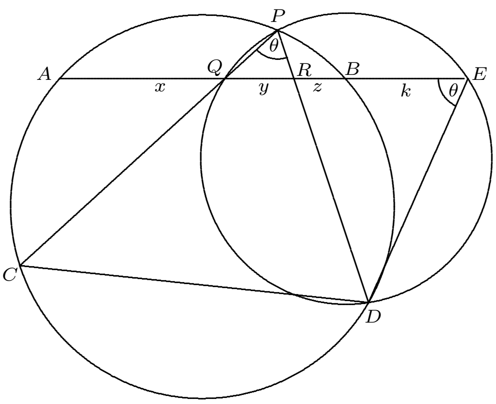


Figure 8.9: Haruki's lemma

Therefore,$(x + y)z = PR \cdot RD = y(z + k)$ giving $xz = yk$, thus $xz/y = k$ is a constant. [This is called Haruki's lemma and can be used to prove the Butterfly Theorem and the double Butterfly Theorem. See Mathematics Magazine vol 63, No 4, October 1990, pp256.]

Exercise 8.8 Using the result of Example 8.4, deduce the Butterfly theorem 8.4.

Exercise 8.9 In Figure 8.10, the point O is the midpoint of BC. Prove that $OX = OY$.

Exercise 8.10 Let A, B, C, D, E, F be 6 points on the plane such that AB intersects DE at L, BC intersects EF at N and CD intersects FA at M. Prove that if L, N, M are collinear, then there is a conic passing through A, B, C, D, E, F.

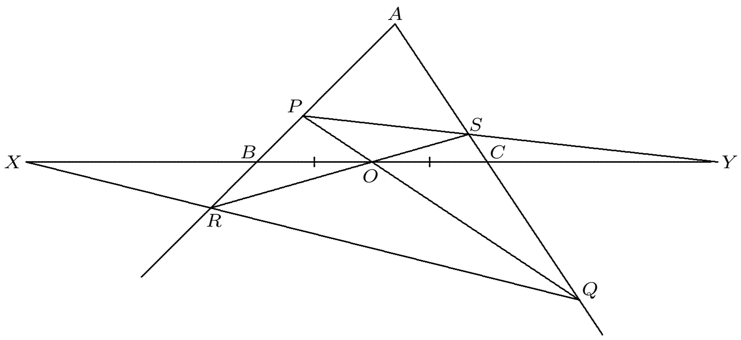


Figure 8.10: Butterfly theorem for 2-straight lines

[Hint: Use the fact that for any 5 points in general position, there is a conic passing through them. Let α be a conic passing through A, B, C, D, E. Let EN meet α at F' and let the intersection of AF' and CD be M'. By Pascal's theorem which also holds for the six points A, B, C, D, E, F' on the conic α, L, N, M' are collinear. Show that $M' = M$ and hence $F' = F$. This is the converse of Pascal's theorem.]

Exercise 8.11 Show that the Butterfly theorem holds for any quadratic curve $ax^2 + bxy + cy^2 + dx + ey + f = 0$.

[Hint: Position the chord PQ of the quadratic curve so that P and Q lie on the x-axis with the origin as their midpoint. Show that in this coordinate system, the coefficient $d = 0$. Then follow the proof of theorem 8.4.]

Remark 8.3 A direct proof of the Butterfly theorem is as follow.

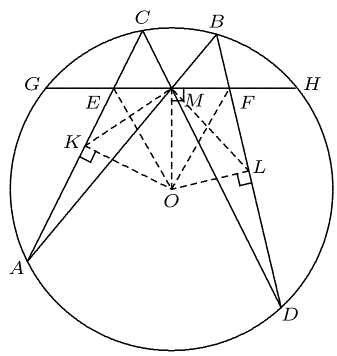


Figure 8.11: A direct proof of the Butterfly Theorem

In figure 8.11, M is the midpoint of GH. Let K and L be the midpoints of the chords AC and BD respectively. Join K, E, M, F, L to the centre O of the circle. Join KM and LM. Since the triangles AMC and DMB are similar, we have $2CK/CM = CA/CM = BD/BM = 2BL/MB$ so that the triangles KCM and LBM are similar. Thus $\angle CKM = \angle BLM$.
As O, K, E, M and O, L, F, M are concyclic, we have $\angle EOM = \angle CKM = \angle BLM = \angle FOM$. Since OM is perpendicular to GH, we conclude that M is the midpoint of EF.

Exercise 8.12 (**A generalized Butterfly theorem**) Let AB be a chord of a circle with midpoint P, and let the chords XW and ZY intersect AB at M and N respectively. Let AB intersects XY at C and ZW at D. Prove that if $MP = PN$, then $CP = PD$.

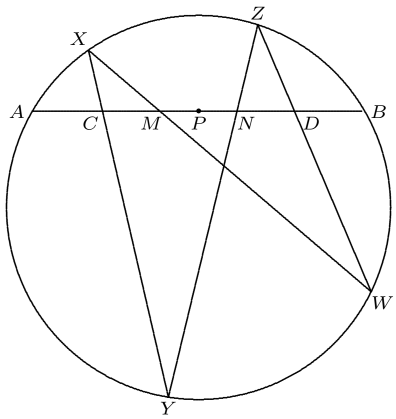


Figure 8.12: A generalized Butterfly theorem

[Hint: Use Haruki's lemma.]

Exercise 8.13 (**A Double Butterfly theorem**) If a chord PP' of a circle cuts two "*butterflies*" at points A, B, C, D and A', B', C', D' such that $PA = P'A', PB = P'B', PC = P'C'$, then $PD = P'D'$.

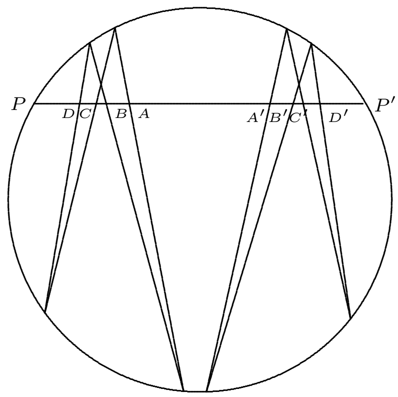


Figure 8.13: A double Butterfly theorem

Chapter 9

Inversive Geometry

Given a circle on the plane, it is possible to turn it inside out by a mapping. It has the effect of mapping circles into circles and preserving angles. This map called "*inversion*" was first introduced by J. Steiner in 1830. In many situations, solutions to the geometric problems become clear after doing the inversion.

9.1 Cross ratio

Definition 9.1 *Two pairs of points AC and BD are said to "separate each other" if they all lie on a circle (or on a straight line) in such an order that either of the arcs AC (or the line segment AC) contains one but not both of the remaining points B and D.*

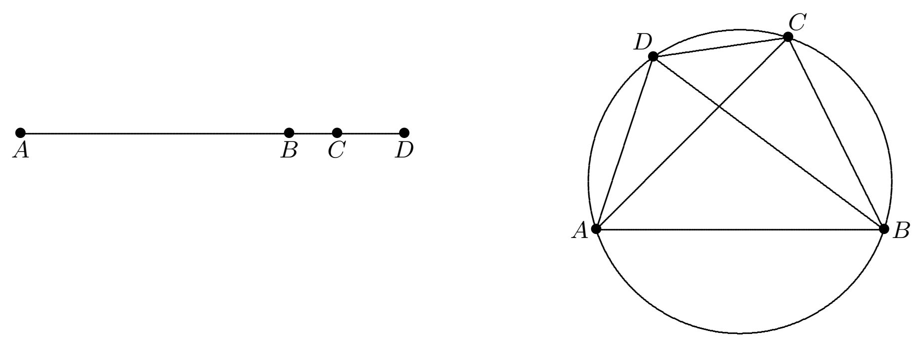


Figure 9.1: Separation of points

Definition 9.2 *Any 4 distinct points A, B, C, D determine a number $\{AB, CD\}$ called the "cross ratio" of the points in this order; it is defined by*

$$\{AB, CD\} = \frac{AC \times BD}{AD \times BC}.$$

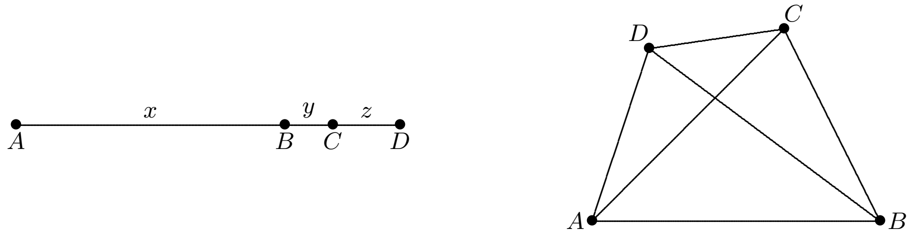


Figure 9.2: The cross ratio of 4 points

Theorem 9.1 *Let A, B, C, D be 4 distinct points on a line. Then AC and BD separate each other if and only if $\{AD, BC\} + \{AB, DC\} = 1$.*

Proof. Consider A, B, C, D in this order. Let $AB = x, BC = y$ and $CD = z$. Then $\{AD, BC\} + \{AB, DC\} = \frac{AB \times DC}{AC \times DB} + \frac{AD \times BC}{AC \times BD} = \frac{AB \times DC + AD \times BC}{AC \times BD} = \frac{xz+(x+y+z)y}{(x+y)(y+z)} = 1$. The other cases can be checked similarly.

Theorem 9.2 *Let A, B, C, D be 4 distinct points on a circle. Then AC and BD separate each other if and only if $\{AD, BC\} + \{AB, DC\} = 1$.*

Proof. By Ptolemy's theorem and its converse, $AB \times CD + BC \times AD \geq AC \times BD$ with equality if and only if D lies on the arc AC not containing B. Dividing by $AC \times BD$, we have $\frac{AB \times CD}{AC \times BD} + \frac{BC \times AD}{AC \times BD} = 1$ if and only if D lies on the arc AC not containing B.

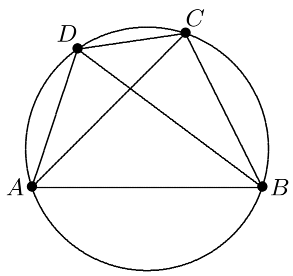


Figure 9.3: Ptolemy's theorem

That is $\{AD, BC\} + \{AB, DC\} = 1$ if and only if AB separates CD.

Corollary 9.3 *The cross ratio of 4 distinct points A, B, C, D satisfies*

$$\{AD, BC\} + \{AB, DC\} = 1$$

if and only if AC separates BD.

Exercise 9.1 Prove that $\{AB, CD\} = \{BA, DC\} = \{CD, AB\} = \{DC, BA\}$

Remark 9.1 Any 3 distinct points A, B, C determine a unique circle (or a line), which may be described as consisting of the 3 points themselves along with all the points X such that BC separates AX, or CA separates BX, or AB separates CX.

Exercise 9.2 Let P, A, B, C, D be five distinct points on a circle. Prove that

$$\{AB, CD\} = \frac{\sin\angle APC \times \sin\angle BPD}{\sin\angle APD \times \sin\angle BPC}.$$

Exercise 9.3 Let ω and ω' be two circles tangent at a point P. Let A, B, C, D be 4 distinct points on ω such that the lines PA, PB, PC, PD intersect ω' at A', B', C', D' respectively. Prove that $\{A'B', C'D'\} = \{AB, CD\}$.

Theorem 9.4 *Let P, A, B, C, D be five distinct points on a circle. Suppose a chord of the circle intersects the segments PA, PB, PC, PD at A', B', C', D' respectively. Then $\{A'B', C'D'\} = \{AB, CD\}$.*

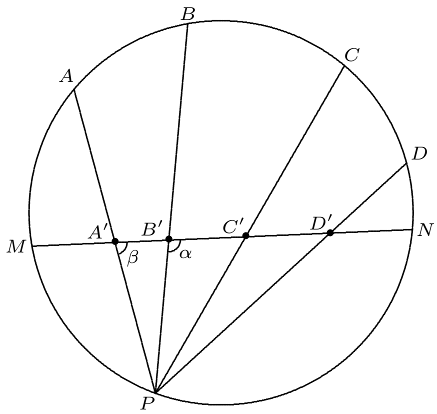


Figure 9.4: $\{A'B', C'D'\} = \{AB, CD\}$

Proof. Using exercise 9.2, we have

$$\{A'B', C'D'\} = \frac{A'C' \times B'D'}{A'D' \times B'C'} = \frac{\left(\frac{PC'}{\sin\alpha}\sin\angle A'PC'\right)\cdot\left(\frac{PD'}{\sin\beta}\sin\angle B'PD'\right)}{\left(\frac{PD'}{\sin\alpha}\sin\angle A'PD'\right)\cdot\left(\frac{PC'}{\sin\beta}\sin\angle B'PC'\right)}$$

$$= \frac{\sin\angle A'PC'\cdot\sin\angle B'PD'}{\sin\angle A'PD'\cdot\sin\angle B'PC'} = \frac{\sin\angle APC\cdot\sin\angle BPD}{\sin\angle APD\cdot\sin\angle BPC} = \{AB, CD\}.$$

Remark 9.2 The cross ratio of 4 rays PA, PB, PC, PD is defined as $\{AB, CD\}_P = \frac{\sin\angle APC\cdot\sin\angle BPD}{\sin\angle APD\cdot\sin\angle BPC}$.

Example 9.1 The Haruki's lemma in example 8.4 is the consequence of the fact that $\{AR, QB\}$ is a constant.

Exercise 9.4 Use 9.4 to prove the Butterfly theorem 8.4.

Exercise 9.5 Using cross ratio, prove the result in exercise 8.9.

9.2 Inversion

Definition 9.3 *Given a circle ω with centre O and radius k and a point P different from O, we define the "inverse" of P to be the point P' on the ray OP such that $OP \cdot OP' = k^2$.*

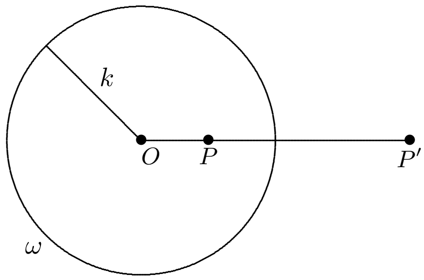


Figure 9.5: Inversion

Basic Properties.

1. Inversion is a map from $\mathbb{R}^2 \setminus \{O\}$ onto itself.
2. $(P')' = P$. Thus inversion is a bijection of order 2.
3. If P is inside (outside) ω, then P' is outside (inside) ω.
4. If P lies on ω, then $P' = P$. That is P is fixed under the inversion with respect to ω.
5. If α is a circle centred at O with radius r, then α' is also a circle centred at O with radius k^2/r.
6. Any line ℓ through O is its own inverse. That is $\ell' = \ell$.
7. Let $P \neq O$ be a point inside ω. To determine the position of P' on the ray OP, draw a chord through P perpendicular to OP meeting ω at T and S. Then the tangents to ω at T and S meet at the inverse point P'.

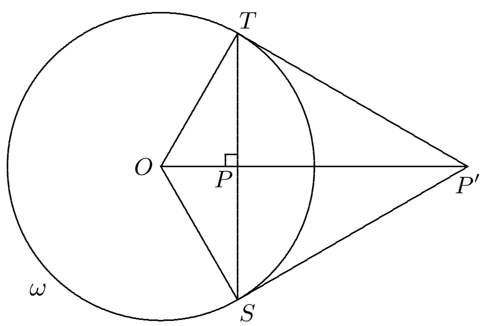


Figure 9.6: The inverse of a point

Proof. As $\triangle OPT$ is similar to $\triangle OTP'$, we have $OP/OT = OT/OP'$ so that $OP \cdot OP' = k^2$.

8. If P is a point outside ω, then the two tangents from P to ω determine the chord TS. The intersection between TS and the ray OP gives the inverse point P'.

   Alternatively, draw the circle with diameter OP intersecting ω at T and S. Then P' is the midpoint of TS.

9. The inverse of any line ℓ, not through O, is a circle through O (minus the point O itself), and the diameter through O of this circle is perpendicular to ℓ.

   Proof. Let A be the foot of the perpendicular from O onto ℓ, and let A' be the inverse of A. Then $OA{\cdot}OA' = k^2$. Consider the circle with diameter OA'. Let P be a point on ℓ and let OP intersect this circle at P'. As $\triangle OAP$ is similar to $\triangle OP'A'$, we have $OP/OA = OA'/OP'$ so that $OP \cdot OP' = OA \cdot OA' = k^2$. Therefore the inverse of P is the point P' on this circle.

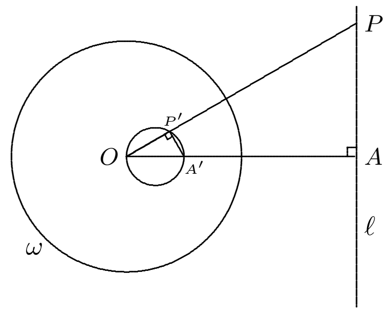


Figure 9.7: The inverse of a circle

10. The inverse of any circle through O (with O omitted) is a line perpendicular to the diameter through O. *That is a line parallel to the tangent at O to the circle.*

11. A pair of intersecting circles α and β, with common points O and P inverts into a pair of intersecting lines α' and β' through the inverse point P'.

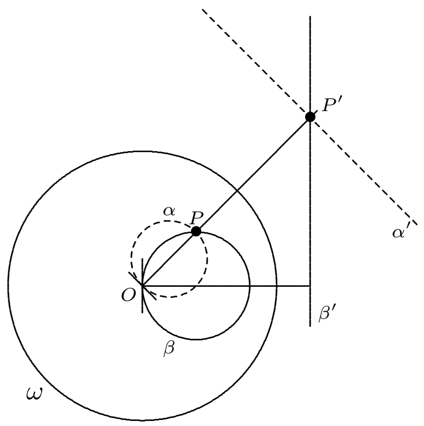


Figure 9.8: The inverses of a pair of circles intersecting at the centre of inversion

12. If α and β are tangent to each other at O, then α' and β' are parallel.

Theorem 9.5 *For a suitable circle of inversion, any 3 distinct points A, B, C can be inverted into the vertices of a triangle $A'B'C'$ congruent to a given triangle DEF.*

Proof. Construct isosceles triangles BCO_1 and ACO_2 on the outside of $\triangle ABC$ such that the base angles $\angle O_1BC = \angle O_1CB = \angle A + \angle D - 90°$ and $\angle O_2AC = \angle O_2CA = \angle B + \angle E - 90°$. If the base angle is negative, then the point O_1 or O_2 is on the other side of the base. Let the circle centred at O_1 with radius O_1C intersect the circle centred at O_2 with radius O_2C at the point O (and C). Now consider the inversion with respect to the circle centred at O with radius k, where

$$k^2 = \frac{OA \cdot OB \cdot DE}{AB}.$$

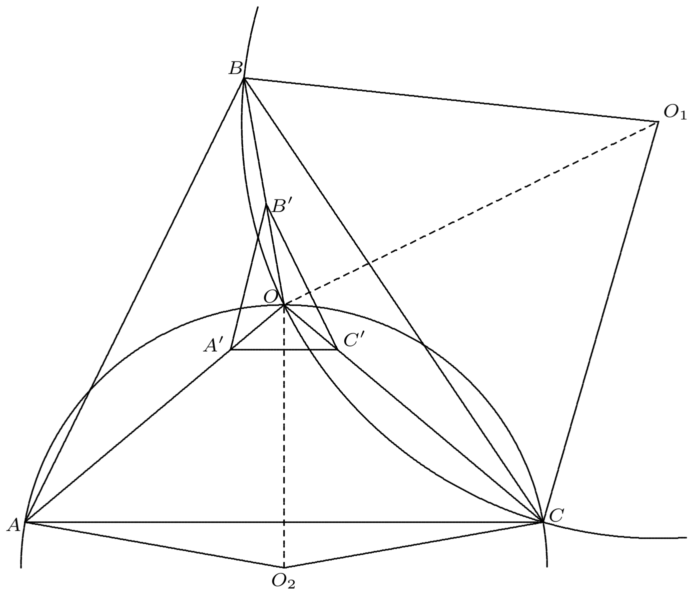


Figure 9.9: Inverting a triangle to another triangle

Let A', B', C' be the inverses of A, B, C respectively under this inversion. First observe that $\triangle OBA$ is similar to $\triangle OA'B'$ and $\triangle OBC$ is similar to $\triangle OC'B'$. Thus $\angle ABO = \angle B'A'O$ and $\angle CBO = \angle B'C'O$. Therefore, $\angle B + \angle B' = \angle B' + \angle B'A'O + \angle B'C'O = \angle AOC$. Similarly, $\angle A + \angle A' = \angle BOC$ and $\angle C + \angle C' = \angle AOB$.

By the construction of the isosceles triangles BCO_1 and ACO_2, we find that $\angle AOC = \angle B + \angle E$, and $\angle BOC = \angle A + \angle D$. Thus $\angle E = \angle B'$ and $\angle D = \angle A'$. From this $\angle F = \angle C'$. Thus $\triangle DEF$ is similar to $\triangle A'B'C'$.

Lastly, $A'B'/AB = OA'/OB$ so that

$$A'B' = (AB/OB) \cdot OA' = (AB/OB) \cdot (k^2/OA) = \frac{AB \cdot OA \cdot OB \cdot DE}{OB \cdot OA \cdot AB} = DE.$$

Therefore $\triangle DEF$ is congruent to $\triangle A'B'C'$.

9.3 The inversive plane

Theorem 9.6 *If two points A and B are inverted into the points A' and B' respectively, then*

$$A'B' = \frac{k^2 AB}{OA \cdot OB}.$$

Proof. Since $\triangle OAB$ is similar to $\triangle OB'A'$, we have

$$\frac{A'B'}{AB} = \frac{OA'}{OB} = \frac{OA \cdot OA'}{OA \cdot OB} = \frac{k^2}{OA \cdot OB}.$$

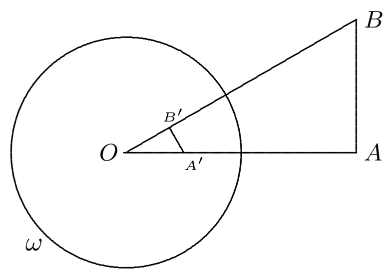


Figure 9.10: The inversive distance

Theorem 9.7 *If A, B, C, D invert into A', B', C', D' respectively, then*

$$\{A'B', C'D'\} = \{AB, CD\}.$$

Proof.

$$\{A'B', C'D'\} = \frac{A'C' \cdot B'D'}{A'D' \cdot B'C'} = \frac{\frac{k^2 AC}{OA \cdot OC} \cdot \frac{k^2 BD}{OB \cdot OD}}{\frac{k^2 AD}{OA \cdot OD} \cdot \frac{k^2 BC}{OB \cdot OC}} = \frac{AC \cdot BD}{AD \cdot BC} = \{AB, CD\}.$$

Theorem 9.8 *If A, B, C, D invert into A', B', C', D' respectively, and AC separates BD, then $A'C'$ separates $B'D'$.*

Proof. Since inversion preserves the cross ratio, we have

$$\{A'B', C'D'\} + \{A'B', D'C'\} = \{AB, CD\} + \{AB, DC\} = 1$$

so that $A'C'$ separates $B'D'$.

Theorem 9.9 *The inverse of a circle not passing through O is a circle not passing through O.*

Proof. Any given circle can be described, in terms of three of its points, as consisting of A, B, C and all points X satisfying BC separates AX or CA separates BX or AB separates CX. Hence the inverse of a given circle consists of A', B', C', and all points X' satisfying $B'C'$ separates $A'X'$ or $C'A'$ separates $B'X'$ or $A'B'$ separates $C'X'$. That is, the inverse is the circle (or line) $A'B'C'$. Also we know that the inverse is a line if and only if the given circle passes through O. Therefore we have proved the result.

Remark 9.3 If we regard a line as a circle of infinite radius, then the terminology "circle" includes "line" as a special case. At the same time, let's add a point P_∞ at infinity which corresponds to the inverse of the centre of any circle of inversion. The plane, so completed, is called the "*inversive plane*". Since a circle with centre O inverts any circle through O into a line, we regard a line as a circle through P_∞. Since two circles tangent to each other at O invert into parallel lines, we regard parallel lines as circles tangent to each other at P_∞. With this convention, we can state our result for the inversive plane as the following.

Theorem 9.10 *The inverse of a circle is a circle.*

Remark 9.4 Note that the centre of α' is usually not the centre of α under an inversion.

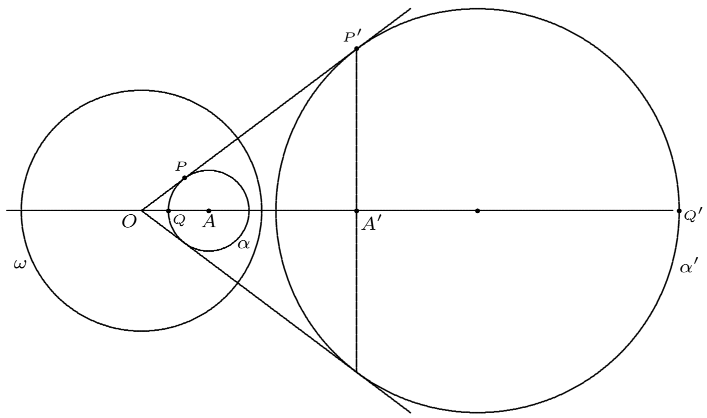


Figure 9.11: The inverse of the centre of a circle is not necessarily the centre of the inverse circle

Exercise 9.6 In the above figure, show that P, P', A', A are concyclic. Show also that P, P', Q', Q are concyclic.

Now let's investigate how the equation of a circle is transformed under inversion. Let ω be the circle with equation $x^2 + y^2 = k^2$. Then the inverse of the point $P(x, y)$ is the point $P'(x', y')$, where

$$x' = \frac{xk^2}{x^2 + y^2}, \quad y' = \frac{yk^2}{x^2 + y^2}.$$

Theorem 9.11 *If α is the circle with equation $x^2 + y^2 + 2fx + 2gy + h = 0$, then the inverse circle α' under the inversion by $\omega : x^2 + y^2 = k^2$ has the equation*

$$hx^2 + hy^2 + 2fk^2x + 2gk^2y + k^4 = 0.$$

Remark 9.5 If $h = 0$, then the original circle α passes through O so that α' is a straight line with equation: $2fx + 2gy + k^2 = 0$.

If $h \neq 0$, then α' is a circle centred at $\frac{k^2}{h}(-f, -g)$, with radius $r' = \frac{k^2 r}{|h|}$, where $r = \sqrt{f^2 + g^2 - h}$ is the radius of α.

Proof of the theorem. Let (x, y) be a point on α. Then $x' = \frac{xk^2}{x^2+y^2}$ and $y' = \frac{yk^2}{x^2+y^2}$. Since (x, y) satisfies the equation of α, we can replace $x^2 + y^2$ by $-(2fx + 2gy + h)$. Thus $x' = \frac{-xk^2}{2fx+2gy+h}$ and $y' = \frac{-yk^2}{2fx+2gy+h}$. Solving for x and y, we obtain

$$x = \frac{-hx'}{2fx' + 2gy' + k^2}, \quad \text{and} \quad y = \frac{-hy'}{2fx' + 2gy' + k^2}.$$

Substituting this into the equation of α and simplifying, we obtain the required equation satisfied by (x', y').

Exercise 9.7 Let three circles be mutually tangent to each other. Show that there are exactly two circles tangent to all the three circles.

9.4 Orthogonality

In this section, we shall study circles which intersect at right angles. As we shall see, such circles gives a generalization of the definition of inversion. The concept of orthogonal circles plays an important role in the theory of inversion.
The two supplementary angles between two circles are naturally defined as the angles between their tangents at a point of intersection. If two circles intersect at P and Q, then the angles at P and Q are easily seen to be equal by the reflection in the line of the centres.

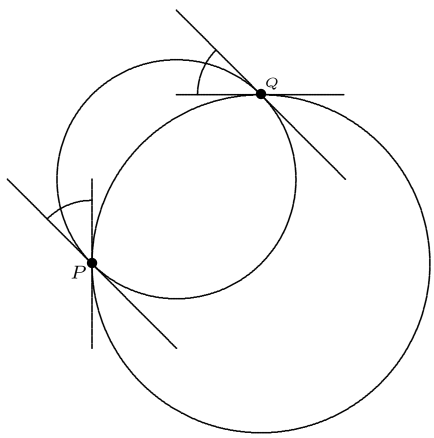


Figure 9.12: The angles between two intersecting circles.

To see how angles are affected by inversion in a circle ω with centre O, let θ be one of the angles between two lines a and b through the point P. See Figure 9.13. The line a is inverted to a circle α passing through O whose tangent at O is parallel to a. Similarly, the line b is inverted to a circle β passing through O whose tangent at O is parallel to b. Since θ is one of the angles between these tangents at O, it is one of the angles of intersection of α and β. As α and β also intersect P', the inverse of P, the same angle θ also appears at P'.

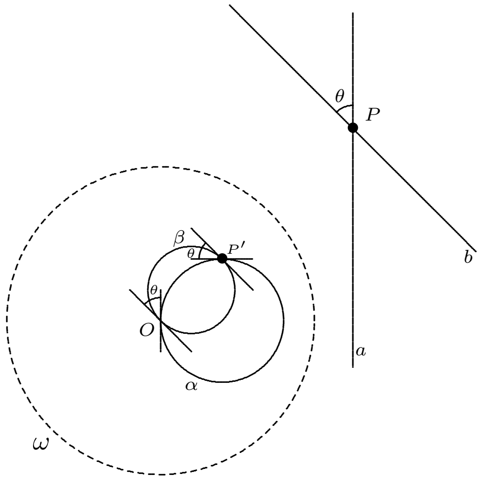


Figure 9.13: Inversion preserves angles

Exercise 9.8 Investigate the case when a or b happens to pass through O.

Now for any two circles through P, we can let a and b be their tangents at P. The inverse circles touch α and β respectively at P' so that they intersect at the same angle as a and b.

Theorem 9.12 *If two circles intersect at an angle θ, then their inverses intersect at the same angle.*

Definition 9.4 *Two circles are said to be orthogonal if they intersect (twice) at right angles, so that at either point of intersection, the tangent to each is a diameter of the other.*

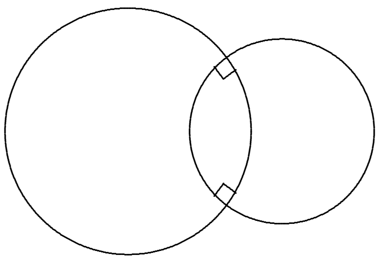


Figure 9.14: Orthogonal circles

Theorem 9.13 *Orthogonal circles invert into orthogonal circles.*

Exercise 9.9 Let ω be the circle $x^2 + y^2 = k^2$, and α the circle $x^2 + y^2 + 2fx + 2gy + h = 0$. Prove that α is orthogonal to ω if and only if $h = k^2$. Show that, in this case, the inverse circle α' under the inversion in ω has the same equation as α.

Suppose α is orthogonal to ω. Let T be one of the points of their intersection. See figure 9.15. For any ray emanating from the centre O of ω intersecting α at two points P and P', we have $OP \cdot OP' = OT^2 = k^2$ so that P and P' is a pair of inverse points.

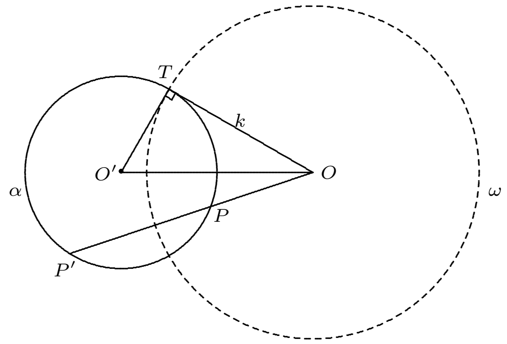


Figure 9.15: Orthogonal circles are invariant under inversion

Theorem 9.14 *Every circle orthogonal to ω is its own inverse.*

Theorem 9.15 *Any circle through two distinct points, inverses of each other in ω, is its own inverse, and is orthogonal to ω.*

Proof. Suppose P and P' is a pair of inverse points under the inversion in ω. Then $OP \cdot OP' = k^2$, where k is the radius of ω. Let α be any circle with centre O' passing through P and P'. Let OT be a tangent from the centre O of ω to α. Then $OP \cdot OP' = OT^2$. Thus $OT = k$. Therefore $\angle O'TO = 90°$ and α is orthogonal to ω. Furthermore, α is its own inverse.
We can redefine inversion in terms of orthogonality as follow.

Theorem 9.16 *Any point on ω is its own inverse; the inverse of any other point P is the second intersection of any two circles through P orthogonal to ω.*

Proof. Let P be a point not on ω, and let α and β be two circles through P orthogonal to ω.

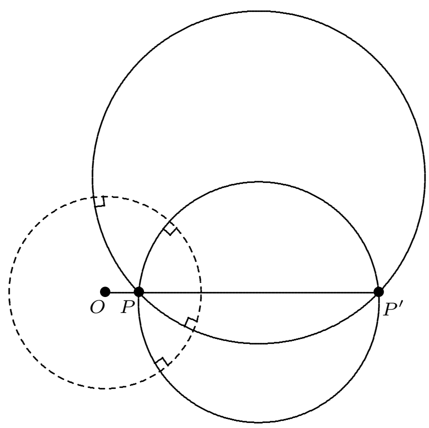


Figure 9.16: Inversion by means of orthogonal circles

As α inverts back to itself, P' lies on α. Similarly P' lies on β. Thus P' is the second point of intersection between α and β. (Thus if O is the centre of ω, then O, P, P' are collinear and $OP \cdot OP' = k^2$, where k =radius of ω.)

Exercise 9.10 Let P and Q be two points which are not inverse of each other with respect to the circle ω. Give a construction of the circle through P and Q orthogonal to ω.

Remark 9.6 Using this definition, one can regard reflection about a line as an inversion.

9.5 Concentric circles

Theorem 9.17 *Any two non-intersecting circles can be inverted into concentric circles.*

Proof. Let α and β be two disjoint circles. Take a point on the radical axis of α and β and draw the tangents from it to α and β. Thus the two tangents are of equal length. Let γ be the circle using this point as centre and the tangent as radius. Then γ is orthogonal to α and β. Construct a similar circle δ orthogonal to both α and β so that δ and γ meet at two points O and P.

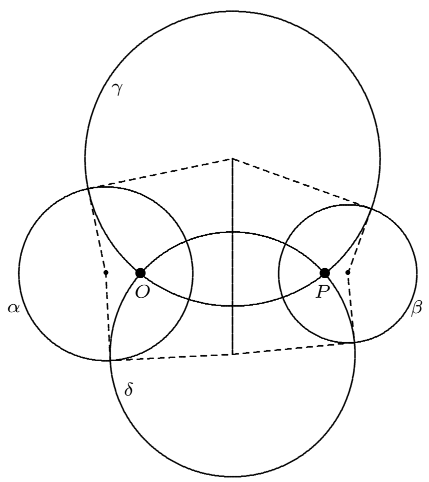


Figure 9.17: Inverting two circle into concentric circles

Consider the inversion in the circle centred at O with certain radius. The circles γ and δ are inverted into two lines γ' and δ' intersecting at a point P', which is the inverse of P. As α and β are orthogonal to both γ and δ, their inverses α' and β' are also orthogonal to both lines γ' and δ'. However, the circles orthogonal to two intersecting lines are those centred at the point of the intersection, and are all concentric. This proves the result.
If α and ω are two distinct circles, the inverse of α in ω belongs to the pencil (denoted by $\alpha\omega$) of coaxal circles determined by α and ω. If α inverts into α', we call ω is a "*mid-circle*" of α and α'. Since α' belongs to the pencil $\alpha\omega$, ω belongs to the pencil $\alpha\alpha'$.

Theorem 9.18 *Any two circles have at least one mid-circle. Two non-intersecting or tangent circles have just one mid-circle. Two intersecting circles have two mid-circles, orthogonal to each other.*

Proof. If α and β intersect, we can invert them into intersecting lines, which are transformed into each other by reflection in either of their angle bisectors. Inverting back again, the intersecting circles α and β have two mid-circles, orthogonal to each other and bisecting the angles between α and β.

If α and β are tangent, we can invert them into parallel lines. Thus they have a unique mid-circle.

If α and β are non-intersecting, we can invert them into concentric circles, of radii a and b. These concentric circles can be transformed into one another by an inversion in a concentric circle of radius $\sqrt{ab}$. Inverting back again, we see that the two non-intersecting circles α and β have a unique mid-circle. If α and β are congruent, their mid-circle coincides with their radical axis.

Exercise 9.11 *Prove that any two circles can be inverted into congruent circles.*

[Hint: Invert the figure in a circle centred at a point on a mid-circle of the two given circles.]

9.6 Steiner's porism

Given two non-concentric circles with one inside the other, one can draw circles within the ring-shaped region bounded by these two circles, touching one another successively and all touching the original two circles. It may happen that the sequence of tangent circles closes so as to form a ring of n circles with the last touching the first. *Steiner's porism* is the result that gives the condition on n that such a chain is possible.

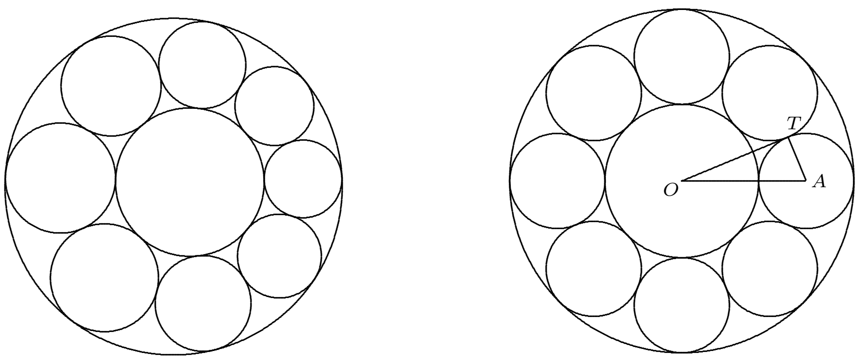


Figure 9.18: Steiner's Porism

The solution is very simple. Simply invert the two non-intersecting circles into concentric ones as in the above figure on the right. Let the radii of the outer and inner concentric circles in this figure be a and b respectively. Then OTA is a right-angled triangle with

$$OA = (a+b)/2, \quad \text{and} \quad AT = (a-b)/2.$$

Suppose it is possible to inscribe n small circles between these two concentric circles. By symmetry, all these small circles are congruent. Then $\angle TOA = \frac{\pi}{n}$. Thus

$$\sin\frac{\pi}{n} = \frac{AT}{OA} = \frac{a-b}{a+b} = \frac{a/b-1}{a/b+1},$$

or

$$\frac{a}{b} = \frac{1 + \sin\frac{\pi}{n}}{1 - \sin\frac{\pi}{n}}.$$

When $n = 8$, the result is shown in the above figure with $a/b = 2.24$. When $n = 4$, $a/b = (\sqrt{2} + 1)^2$. In this case, there are 6 circles, each touching four others.

Exercise 9.12 Given 4 circles touching one another to form a chain. Show that the 4 tangency points lie on a circle.

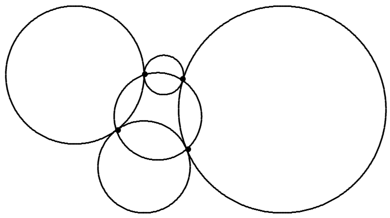


Figure 9.19: Four circles touching one another

[Hint: Invert the figure in a circle centred at one of these tangency points.]

Exercise 9.13 (The six-circles theorem) A pair of circles intersects another pair of circles in two sets of 4 points each. Prove that if the 4 intersection points in one set are concyclic, then the 4 intersection points in the other set are also concyclic.

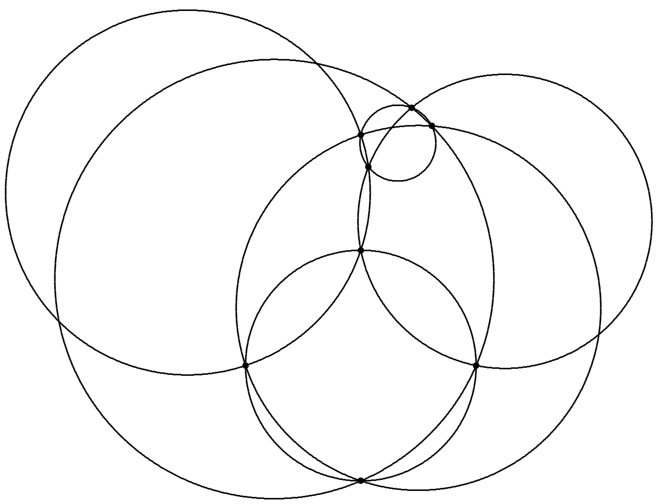


Figure 9.20: Two intersecting pairs of circles

[Hint: Invert in a circle centred at one of the intersection points of the first set. Then use the result in exercise 7.1.]

9.7 Stereographic projection

The map from the unit sphere with the north pole deleted to the plane given by

$$\zeta(x,y,z) = (\frac{x}{1-z}, \frac{y}{1-z})$$

is called the stereographic projection. That is $\zeta : S^2\backslash\{(0,0,1)\} \longrightarrow \mathbb{R}^2$, and if $(X,Y) = \zeta(x,y,z)$, we have

$$X = \frac{x}{1-z}, Y = \frac{y}{1-z}.$$

It is obtained by drawing a ray from the north pole to the point (x,y,z) on the sphere so that its extension meets the equatorial plane at the point $\zeta(x,y,z)$. The inverse of ζ is given by

$$\zeta^{-1}(X,Y) = (\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2}).$$

The fact that stereographic projection preserves angles was first proved by Edmond Halley (known for his comet) in 1695.

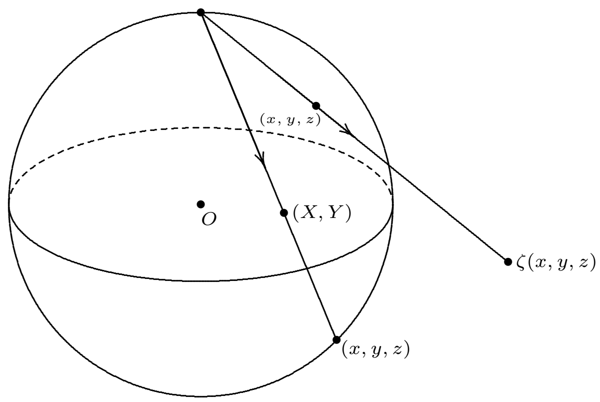


Figure 9.21: Stereographic Projection

Theorem 9.19 *The stereographic projection maps circles on the unit sphere to circles on the plane.*

Proof. A circle ω on the unit sphere S^2 of $\mathbb{R}^3$ can be defined as the intersection of a plane P : $ax+by+cz+d=0$ with $S^2 : x^2+y^2+z^2=1$. For $P\cap S^2 \neq \emptyset$, we shall require $a^2+b^2+c^2=1$ and $|d|<1$. Let (x,y,z) be a point on ω and let its image under ζ be (X,Y). Thus

$$x = \frac{2X}{1+X^2+Y^2},\ y = \frac{2Y}{1+X^2+Y^2},\ z = \frac{-1+X^2+Y^2}{1+X^2+Y^2}.$$

Since (x,y,z) lies on P, we have

$$a(\frac{2X}{1+X^2+Y^2}) + b(\frac{2Y}{1+X^2+Y^2}) + c(\frac{-1+X^2+Y^2}{1+X^2+Y^2}) + d = 0.$$

Simplifying, we get

$$(c+d)X^2 + (c+d)Y^2 + 2aX + 2bY + d - c = 0,$$

which is the equation of a circle Ω.

The centre of Ω is $(\frac{-a}{c+d}, \frac{-b}{c+d})$, and the radius is $\left[\frac{1-d^2}{(c+d)^2}\right]^{\frac{1}{2}}$. If $c+d=0$, then the plane P passes through the north pole $(0,0,1)$ so that Ω is a straight line $aX+bY=c$.

In particular, if $d=0$ so that the plane P passes through the origin and the circle $\omega = P \cap S^2$ is a great circle, then the circle Ω has centre $(-\frac{a}{c}, -\frac{b}{c})$ and radius $|c|^{-1}$.

It is easy to see that any tangent vector to the unit sphere at a point is the tangent vector to a great circle at that point. Suppose $\mathbf{u}$ and $\mathbf{v}$ are two unit tangent vectors to S^2 at a point p. Consider the great circles ω and ω' through the point p such that the tangent vector to ω at p is $\mathbf{u}$ and the tangent vector to ω' at p is $\mathbf{v}$. Let the planes containing ω and ω' have unit normal vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ respectively. Thus their equations are given by $a_1x + a_2y + a_3z = 0$ and $b_1x + b_2y + b_3z = 0$. We may assume $a_3 > 0$ and $b_3 > 0$. It is easy to observe that angle between $\mathbf{u}$ and $\mathbf{v}$ is the same as the angle between $\mathbf{a}$ and $\mathbf{b}$. Thus $\cos(\mathbf{u}, \mathbf{v}) = \cos(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$.

The image Ω of the circle ω has centre $(-\frac{a_1}{a_3}, -\frac{a_2}{a_3})$ and radius a_3^{-1}. Similarly, the image Ω' of the circle ω' has centre $(-\frac{b_1}{b_3}, -\frac{b_2}{b_3})$ and radius b_3^{-1}.

Let $p = (x, y, z)$. Note that p satisfies the equations $x^2 + y^2 + z^2 = 1$, $a_1x + a_2y + a_3z = 0$ and $b_1x + b_2y + b_3z = 0$. Recall that the point $\zeta(p)$ is given by $(\frac{x}{1-z}, \frac{y}{1-z})$. The radius vector from the centre of Ω to $\zeta(p)$ is given by $\langle \frac{x}{1-z} + \frac{a_1}{a_3}, \frac{y}{1-z} + \frac{a_2}{a_3}\rangle$. Similarly, the radius vector from the centre of Ω' to $\zeta(p)$ is given by $\langle \frac{x}{1-z} + \frac{b_1}{b_3}, \frac{y}{1-z} + \frac{b_2}{b_3}\rangle$. The lengths of these vectors are respectively the radii a_3^{-1} and b_3^{-1} of Ω and Ω'. Thus the inner product between these two vectors is equal to

$$
\begin{aligned}
&\langle \frac{x}{1-z} + \frac{a_1}{a_3}, \frac{y}{1-z} + \frac{a_2}{a_3}\rangle \cdot \langle \frac{x}{1-z} + \frac{b_1}{b_3}, \frac{y}{1-z} + \frac{b_2}{b_3}\rangle. \\
= &\frac{x^2}{(1-z)^2} + \left(\frac{a_1}{a_3} + \frac{b_1}{b_3}\right)\frac{x}{1-z} + \frac{a_1b_1}{a_3b_3} + \frac{y^2}{(1-z)^2} + \left(\frac{a_2}{a_3} + \frac{b_2}{b_3}\right)\frac{x}{1-z} + \frac{a_2b_2}{a_3b_3} \\
= &\frac{x^2+y^2}{(1-z)^2} + \frac{a_1b_3x + a_3b_1x + a_2b_3y + a_3b_2y}{a_3b_3(1-z)} + \frac{a_1b_1 + a_2b_2}{a_3b_3} \\
= &\frac{1-z^2}{(1-z)^2} + \frac{-(a_2b_3y + a_3b_3z) - (a_3b_2y + a_3b_3z) + a_2b_3y + a_3b_2y}{a_3b_3(1-z)} + \frac{a_1b_1 + a_2b_2}{a_3b_3} \\
= &\frac{1-z^2}{(1-z)^2} - \frac{2z}{1-z} + \frac{a_1b_1 + a_2b_2}{a_3b_3} \\
= &1 + \frac{a_1b_1 + a_2b_2}{a_3b_3} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{a_3b_3} = (\mathbf{a} \cdot \mathbf{b})/(a_3b_3).
\end{aligned}
$$

This shows that the angle between ω and ω' equals the angle between Ω and Ω'. The conclusion holds also if $a_3 = 0$ or $b_3 = 0$.

Corollary 9.20 *The stereographic projection preserves angles.*

Exercise 9.14 Prove that if a quadrilateral (not necessarily planar) touches the unit sphere at four points, then these four points are coplanar.

[Hint: First show that there exists a chain of 4 circles where the points of tangency are the 4 points of tangency of the quadrilateral with the unit sphere. Now apply the stereographic projection and use the result of exercise 9.12.]

9.8 Feuerbach's theorem

Theorem 9.21 (Feuerbach) *The nine-point circle is tangent to the incircle and the excircles of the triangle.*

Proof. Let M_a, M_b, M_c be the midpoints of the sides of the triangle ABC, and let G_a, G_b, G_c be the tangency points of the incircle with the sides of the triangle. Let the internal bisector of $\angle A$ meet the side BC at V. Let $B'C'$ be the reflection of the line segment BC about AV. Note that $B'C'$ is tangent to the incircle at the point G'_a. Join CC' and let the extension of AV meet CC' at P. Note that P is the midpoint of CC'. Thus M_bP is parallel to AC'. Also M_bP passes through M_a. Let M_bP intersect $B'C'$ at Q.

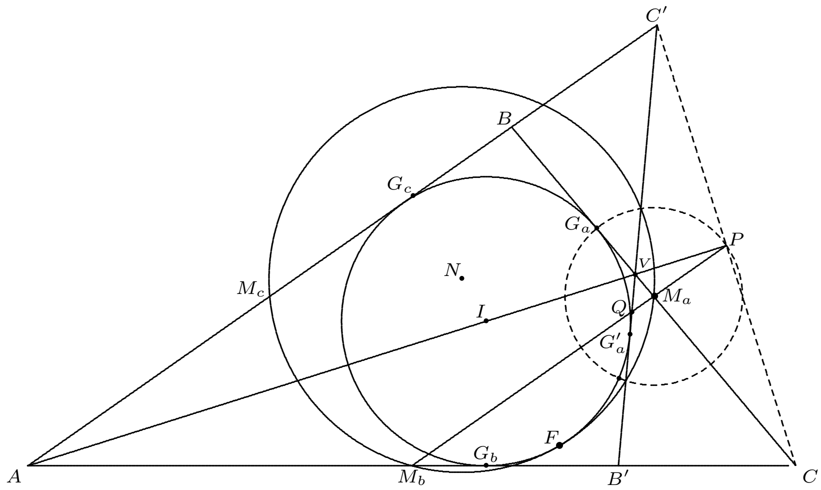


Figure 9.22: The nine-point circle is tangent to the incircle of the triangle

Then $M_aP = |M_aM_b - PM_b| = \frac{1}{2}|AB - AC| = \frac{1}{2}|BG_c - CG_b| = \frac{1}{2}|BG_a - CG_a| = M_aG_a$ which is also equal to $\frac{1}{2}|b-c|$. As $\triangle M_bM_aC$ is similar to $\triangle ABC$, $\triangle ABV$ is similar to $\triangle PM_aV$, and $\triangle BC'V$ is similar to $\triangle M_aQV$, we have

$$\frac{M_aP}{M_aM_b} = \frac{BC'}{BA} = \frac{M_aQ}{M_aP}.$$

Thus $M_aG_a^2 = M_aP^2 = M_aQ \cdot M_aM_b$. Let ω be the circle centred at M_a with radius M_aG_a. Therefore, the inverse of M_b under the inversion in ω is the point Q lying on the line $B'C'$. The same is true for the point M_c. Hence inverse of the nine-point circle is the line $B'C'$. Note that the incircle is orthogonal to ω so that it is inverted onto itself. Since the line $B'C'$ is tangent to the incircle, the nine-point circle is tangent to the incircle.
The proof for the excircle is similar.

Exercise 9.15 Show that the nine-point circle is tangent to the excircles of the triangle.

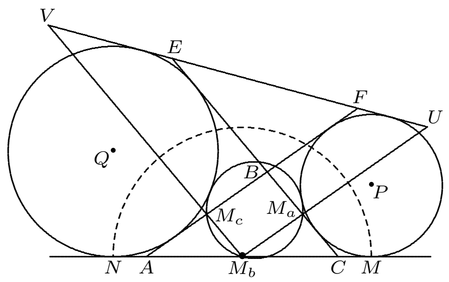


Figure 9.23: The nine-point circle is tangent to the excircles of the triangle

[Hint: Let M_a, M_b, M_c be the midpoints of the sides of the triangle ABC. Let UV be the other common external tangent to the two excircles whose centres lies on the external bisector of $\angle B$. Let ω be the circle centred at M_b with radius $\frac{1}{2}(a+c)$. These two excircles are orthogonal to ω. Also $M_bM_a \cdot M_bU = M_bM_c \cdot M_bV = \frac{1}{4}(a+c)^2$. Thus the nine-point circle is inverted under ω to the line UV which is tangent to the two excircles.]

Exercise 9.16 Prove Ptolemy's theorem 4.5 by using inversion.

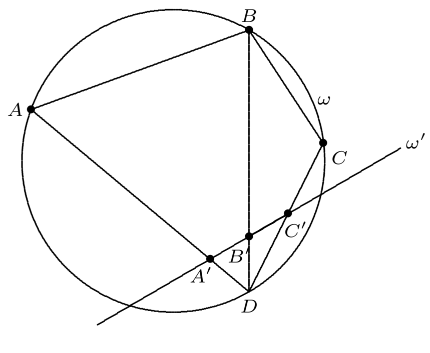


Figure 9.24: Proof of Ptolemy's theorem using inversion

[Hint: Let $ABCD$ be a quadrilateral inscribed in a circle ω. If we invert in a circle centred at D, the circle ω is inverted into a line ω', the points A, B, C are inverted into points A', B', C' on ω' respectively. Then $A'B' = (k^2/DA \cdot DB) \cdot AB$, $B'C' = (k^2/DB \cdot DC) \cdot BC$, and $A'C' = (k^2/DA \cdot DC) \cdot AC$. The result follows from $A'C' = A'B' + B'C'$.]

Chapter 10

Models of Hyperbolic Geometry

Non-Euclidean geometry, hyperbolic geometry in particular, was discovered independently by Janos Bolyai (1802-1860) and Nicolai Ivanovitch Lobachevsky (1793- 1856), in an attempt to prove Euclid's 5th Postulate by way of contradiction. In their work, results of a consistent but new geometry were discovered by assuming the negation of Euclid's 5th Postulate. In this chapter, we shall assume the following form of the negation of the 5th Postulate.
Given a line and a point not on the line, it is possible to construct more than one line through the given point parallel to the line.
This postulate has become known as the *Bolyai-Lobachevsky Postulate*, and is also known as the *hyperbolic postulate*. A geometry constructed from the first 4 Euclidean postulates, plus the hyperbolic postulate is known as the hyperbolic geometry.
Recall that the first 28 propositions in Euclid's Elements are valid in a neutral geometry in which the 5th Postulate is not assumed. In particular they are valid in the hyperbolic geometry. Here as in the Elements, we assume Pasch's axiom and that straight lines can be extended infinitely. These rule out the spherical geometry and non-orientability. In this chapter, we shall introduce some models of hyperbolic geometry.

10.1 The Poincaré model

In the Poincaré Model for 2-dimensional hyperbolic geometry, a point is taken to be any point in the interior of the unit disk

$$\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \;:\; x^2 + y^2 < 1\}.$$

The collection $\mathbb{D}$ of all such points will be called the *Poincaré disk*.

Definition 10.1 *A hyperbolic line is a Euclidean circular arc, or a Euclidean line segment, within the Poincaré disk that meets the boundary circle at right angles. Thus if it is a Euclidean line segment, then it must be a diameter of* $\mathbb{D}$. *See Figure 10.1.*

Let's examine the first 4 postulates of Euclid.

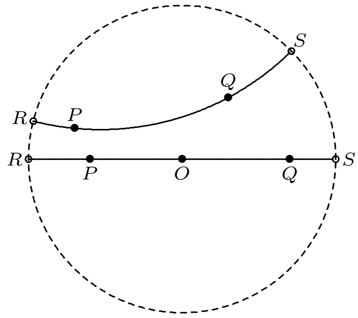


Figure 10.1: Hyperbolic lines in the Poincaré model

1. *To draw a straight line from any point to any point.*
2. *To produce a finite straight line continuously in a straight line.*
3. *To describe a circle with any centre and distance.*
4. *That all right angles are equal to one another.*

For the first axiom, let P, Q be two points in $\mathbb{D}$. If P and Q lie on a diameter. Then that diameter is the unique line passing through P and Q. Suppose P and Q do not lie on a diameter. Let P' be the inverse of P under the inversion in the boundary circle of $\mathbb{D}$. Then the circle passing through P, P' and Q is the unique circle orthogonal to the boundary circle of $\mathbb{D}$. Thus the circular arc in the interior of $\mathbb{D}$ is the unique hyperbolic line passing through P and Q. Since the boundary of $\mathbb{D}$ is excluded, lines can always be extended continuously. Thus the second axiom is satisfied. To define circles for the 3rd postulate, we need a notion of distance. As the boundary of the Poincaré disk is not reachable, we need a distance that approaches infinity as we approach the boundary.

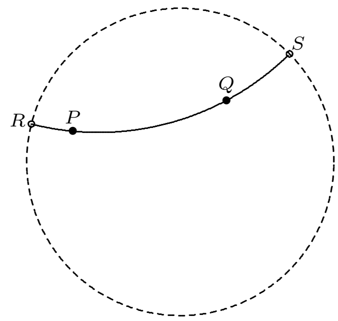


Figure 10.2: The hyperbolic distance between two points in the Poincaré model

Definition 10.2 *The hyperbolic distance from P to Q in the Poincaré Model is defined to be*

$$d_P(P, Q) = \left| \ln \left(\frac{PS \cdot QR}{PR \cdot QS} \right) \right|,$$

where R and S are the points where the hyperbolic line through P and Q meets the boundary circle. Here PS, QR etc are the usual Euclidean distances of the line segments. That is $d_P(P, Q) = |\ln\{PQ, SR\}| = |\ln\{PQ, RS\}|$.

It can be proved that d_P satisfies the usual properties of a metric (or distance). That is
I. $d_P(P,Q) \geq 0$, and $d_P(P,Q) = 0$ if and only if $P = Q$.
II. $d_P(P,Q) = d_P(Q,P)$.
III. (Triangle Inequality) $d_P(P,R) \leq d_P(P,Q) + d_P(Q,R)$.
Note that when P approaches R or Q approaches S, the distance approaches infinity.

Theorem 10.1 *Let ℓ be a hyperbolic line in the Poincaré disk, which intersects the boundary circle at R and S. Then*

$$f(P) = \ln(PR/PS),$$

for $P \in \ell$ defines a bijection $f : \ell \longrightarrow \mathbb{R}$, for which

$$d_P(P,Q) = |f(Q) - f(P)|$$

for all points $P, Q \in \ell$.

Proof. The function f is a strictly increasing map from ℓ to $\mathbb{R}$, such that $f(P)$ tends to ∞ when P approaches S, and $f(P)$ tends to $-\infty$ when P approaches R.
This shows that if X, Y, Z are 3 points on a hyperbolic line in this order so that $f(X) < f(Y) < f(Z)$, then $d_P(X,Z) = d_P(X,Y) + d_P(Y,Z)$.

Definition 10.3 *A hyperbolic circle α of radius r centred at a point O in the Poincaré disk is the set of points in the Poincaré disk whose hyperbolic distance to O is r.*

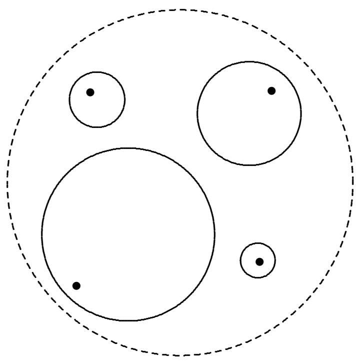


Figure 10.3: Hyperbolic circles in the Poincaré model

To construct the circle of radius r at O, we note that on any line passing through O, we can find points that are r units away (measured in the hyperbolic distance function). The locus of a hyperbolic circle in the Poincarè model is a Euclidean circle. Note that the hyperbolic centre of a hyperbolic circle is not necessary the Euclidean centre of the circle.

Theorem 10.2 *Let $r > 0$ and C a point in the Poincaré disk. Then the locus of points X such that $d_P(C,X) = r$ in the Poincaré disk is a Euclidean circle.*

Proof. Let $\alpha = \{X \in \mathbb{D} \, : \, d_P(C, X) = r\}$. That is, α is a hyperbolic circle with hyperbolic centre at C. Denote the center of $\mathbb{D}$ by O and the boundary of $\mathbb{D}$ by δ. Let $X \in \alpha$ and let ℓ be a hyperbolic line through X and C with endpoints R and S on δ. We may suppose R, C, X, S are in this order along ℓ. Thus $d_P(C, X) = |\ln\{CX, SR\}|$.

If $C = O$, then RS is a diameter of δ. Then $d_P(O, X) = |\ln\{CX, SR\}| = \ln\left(\frac{1+OX}{1-OX}\right)$. Thus $d_P(O, X) = r$ is equivalent to $\ln\left(\frac{1+OX}{1-OX}\right) = r$, or $OX = k$, where $k = (e^r - 1)/(e^r + 1)$. Therefore α is a Euclidean circle centred at O of radius k.

Let ω be a circle orthogonal to δ such that C is inverted into O in ω. (Give a construction of ω). Consider the inversion in ω. We have $C' = O$, $\delta' = \delta$ and ℓ' passes through $O = C'$ and is orthogonal to $\delta' = \delta$. Thus ℓ' is a diameter of δ.

Since inversion preserves cross ratio, the inverse point X' of X satisfies $d_P(O, X') = d_P(C', X') = d_P(C, X) = r$. Thus α is inverted into $\alpha' = \{X' \in \mathbb{D} \, : \, d_P(O, X') = r\}$, which is a Euclidean circle. Therefore α is also a Euclidean circle.

For the 4th postulate, we shall define angles just as they are defined in Euclidean geometry. That is we use the Euclidean tangent lines to the hyperbolic lines (Euclidean circular arcs) in the Poincaré model to determine angles. Hence, the angle determined by two hyperbolic lines will be the angle made by their Euclidean tangents.

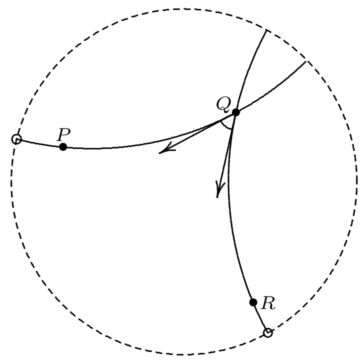


Figure 10.4: Angles in the Poincaré model

Since angles are defined in the Euclidean sense, the 4th postulate is automatically true.

For the 5th postulate, consider a line ℓ and a point P not on ℓ. Let X and Y be the intersection points of ℓ with the boundary circle. Then there are two Euclidean circular arcs, one through P and X and the other through P and Y, both are orthogonal to the boundary circle. So these are two hyperbolic lines through P parallel to ℓ. In fact, any hyperbolic line through P within the two sectors adjacent to the sector containing ℓ does not intersect ℓ, hence parallel to ℓ. Thus there are more than 1 line through P parallel to ℓ. The hyperbolic lines PX and PY are called the *limiting parallels* to ℓ at P. See Figure 10.5.

Let PN be a perpendicular to ℓ at N, and H a point on PY. Then $\angle NPH (= \theta)$ is called the *angle of parallelism* for ℓ at P. This angle has the following property. Let ℓ' be a hyperbolic line through P and H' a point on ℓ' situated on the same side of PN as Y. If $\angle NPH' \geq \theta$, then ℓ' does not intersect ℓ. If $\angle NPH' < \theta$, then ℓ' intersects ℓ.

Recall that there are two limiting parallels, one on each side of the common perpendicular PN.

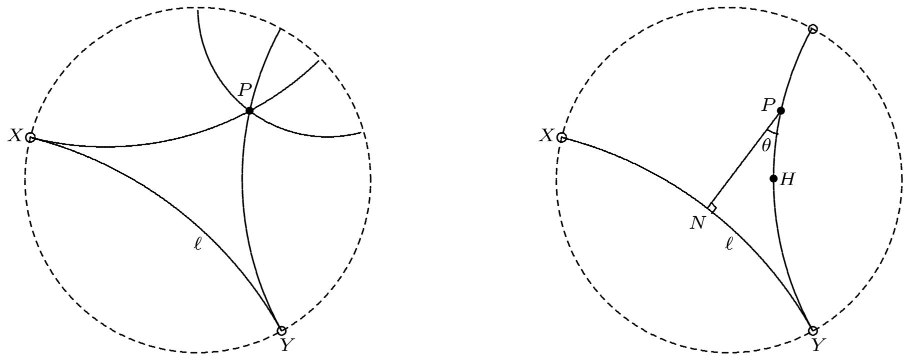


Figure 10.5: Limiting parallel lines and perpendiculars in the Poincaré model

They are called the *right* and *left limiting parallels* to ℓ at P respectively. Let PY' be the reflection of PY about the common perpendicular PN. Since reflection preserves parallelism, PY' must separate intersecting lines from parallels. Thus $PY' = PX$. Also reflections preserve angle, so $\angle YPN = \angle XPN = \theta$.

Exercise 10.1 Let P be a point in the Poincaré disk and ℓ a hyperbolic line. Show that there is a unique hyperbolic perpendicular from P onto ℓ.

[Hint: Invert at one of the boundary points of ℓ.]

Exercise 10.2 Show that if a point A is located at a distance $r < 1$ from the centre O of the Poincaré disk, its hyperbolic distance from O is given by

$$d_P(O, A) = \ln\left(\frac{1+r}{1-r}\right).$$

Theorem 10.3 *The angle of parallelism is always acute.*

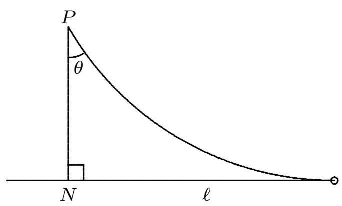


Figure 10.6: Limiting parallel in the Poincaré model

That is $0 < \theta < 90°$.
Proof. This follows from the definition of limiting parallel and proposition I.27.

Theorem 10.4 (Lobachevsky) *Given a point P with a hyperbolic distance d from a hyperbolic line ℓ, the angle of parallelism, θ, for ℓ at P is given by*

$$e^{-d} = \tan(\frac{\theta}{2}).$$

Proof. Recall that inversion preserves the cross-ratio and angles between circles. Thus it preserves the hyperbolic distance and angle. Therefore we can invert the configuration into a standard one of which the formula can be verified easily. If we invert in a circle centred at the second point of intersection between the circles defined by the arcs PN and PY, then $P'N'$ and $P'Y'$ are straight lines. Also ℓ' will be a circular arc orthogonal to $P'N'$ and tangent to $P'Y'$. The unit circle is inverted into a circle orthogonal to $P'N'$ and $P'Y'$, hence it is a circle centred at P' and passes through Y'. See Figure 10.7.

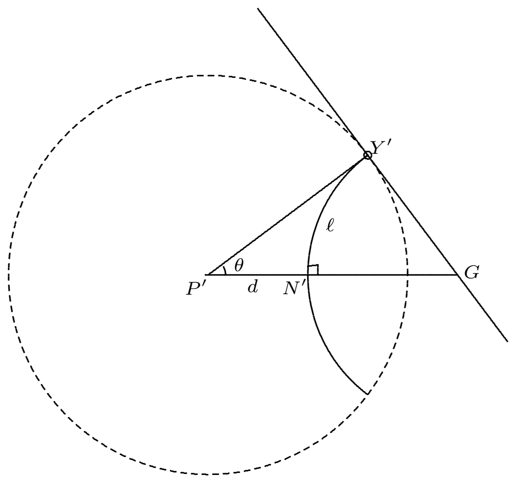


Figure 10.7: The angle of parallelism depends only on the perpendicular distance

Consider this circle centered at P' with radius $P'Y'$. By choosing a suitable radius for the circle of inversion, we may take $P'Y' = 1$. By a suitable translation, we may take P' to be the origin and thus it is the centre of the unit disk. Let G be the intersection of the tangent at Y' with the line $P'N'$. Now $N'G = Y'G = \tan\theta$ and $P'G = \sec\theta$. Thus $P'N' = \sec\theta - \tan\theta = \frac{1-\sin\theta}{\cos\theta}$. Therefore, $d = d_P(P,N) = d_P(P',N') = \ln(\frac{1+P'N'}{1-P'N'})$. That is

$$
\begin{aligned}
e^{-d} &= \frac{1-P'N'}{1+P'N'} = \frac{\cos\theta+\sin\theta-1}{\cos\theta-\sin\theta+1} \\
&= \frac{\cos\theta+\sin\theta-1}{\cos\theta-\sin\theta+1}\,\frac{\cos\theta+\sin\theta+1}{\cos\theta+\sin\theta+1} \\
&= \frac{2\cos\theta\sin\theta}{2\cos^2\theta+2\cos\theta} = \frac{\sin\theta}{\cos\theta+1} \\
&= \frac{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}{2\cos^2(\frac{\theta}{2})} = \tan(\frac{\theta}{2}).
\end{aligned}
$$

10.2 The Klein model

In the Klein model, points are again in the unit disk. However lines and angles are defined differently. A hyperbolic line (or Klein line) in this model will be any chord of the boundary circle (minus its points on the boundary circle).

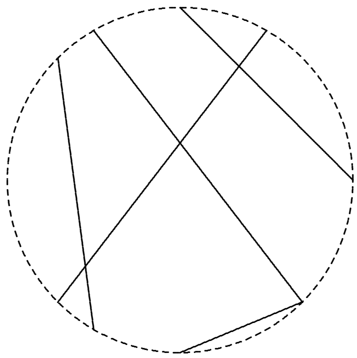


Figure 10.8: Hyperbolic lines in the Klein model

Definition 10.4 *The hyperbolic distance from P to Q in the Klein model is defined to be*

$$d_K(P,Q) = \frac{1}{2}\left|\ln\left(\frac{PS \cdot QR}{PR \cdot QS}\right)\right|,$$

where R and S are the points where the hyperbolic line (chord of the circle) through P and Q meets the boundary circle.

Note the similarity of this definition to the definition of distance in the Poincaré Model. In fact the two models are *isomorphic*. That is there is a one-to-one map between the models that preserves lines and angles and also preserves the distance functions.
As in the Poincaré Model, a circle is defined as the set of all points having a fixed hyperbolic distance from a centre point. The locus of a hyperbolic circle in the Klein model is a Euclidean circle. Again, the hyperbolic centres are not the same as the Euclidean centres.
For the 5th Postulate, we see that given a line and a point not on the line, there are many parallels (non-intersecting lines) to the given line through the point.
The notion of perpendicularity is slightly more subtle in the Klein model. We cannot use the usual notion of measuring angle in the Euclidean sense as in the Poincaré model. Otherwise, the angle sum of a triangle is $180°$, which is an equivalent condition for Euclidean geometry.

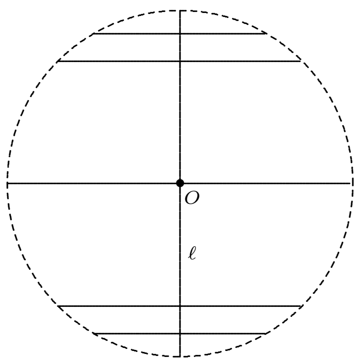


Figure 10.9: Hyperbolic perpendiculars to a diameter in the Klein model

For a hyperbolic line which is a diameter in the Klein model, the lines perpendicular to it are naturally taken as the hyperbolic lines which are chords perpendicular to it in the Euclidean sense. The observation is that all these perpendiculars pass through the inverse point (the point at infinity) of the midpoint O of this hyperbolic line ℓ. This generalizes to a notion of perpendicularity for other hyperbolic lines.

Definition 10.5 *The pole of the chord AB in a circle is the inverse point of the midpoint of AB with respect to this circle.*

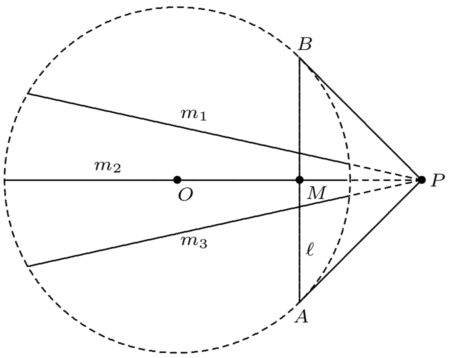


Figure 10.10: The pole of a line in the Klein model

Note that the pole P of AB is also the intersection of the tangents to the circle at A and B.

Definition 10.6 *A line m is perpendicular to a line ℓ in the Klein model if the Euclidean line for m passes through the pole P of ℓ. See Figure 10.10.*

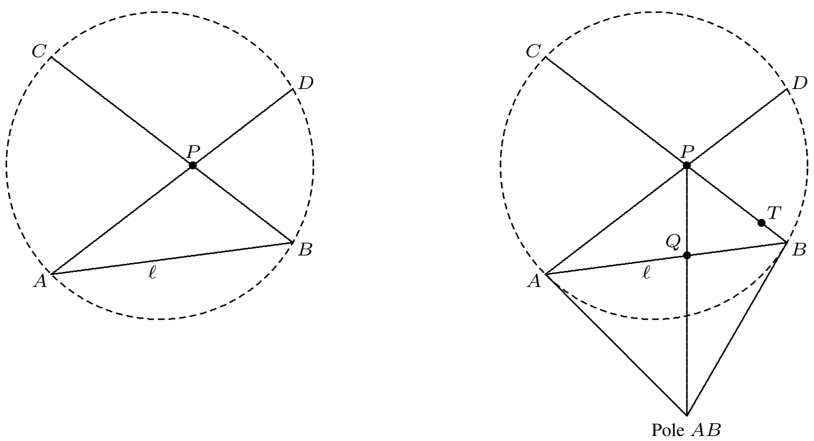


Figure 10.11: Limiting parallels and Perpendiculars in the Klein model

Let P be a point not on the line ℓ (defined by the chord AB). There are two chords AD and BC passing through P. These two parallels of ℓ divide the set of all lines through P into two subsets:

those that intersect ℓ and those that are parallel to ℓ. These special parallels (AD and BC) are the *limiting parallels* to ℓ at P.

From P drop a perpendicular to ℓ at Q. Consider the hyperbolic angle $\angle QPT$, where T is a point on the hyperbolic ray from P to B. This angle is the *angle of parallelism* for ℓ at P. Here the hyperbolic angle in the Klein model is defined by means of an isomorphism between the Klein model and the Poincarè model.

The correspondence between the Poincaré model and the Klein model is given by the map $f : \mathbb{D} \longrightarrow \mathbb{D}$ defined as

$$f(x, y) = (\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}).$$

To obtain this map, first consider the stereographic projection ζ from the north pole of the unit sphere onto the equatorial plane. The map ζ : Unit sphere in $\mathbb{R}^3 \setminus \{(0, 0, 1)\} \longrightarrow \mathbb{R}^2$ is a bijection and both ζ and its inverse mapping ζ^{-1} are continuous. The inverse mapping ζ^{-1} maps the unit disk $\mathbb{D}$ of the xy-plane bijectively onto the southern hemisphere. By projecting the points on the southern hemisphere back to the unit disk $\mathbb{D}$ of the xy-plane, we get the map f. More precisely, f can be obtained as follow.

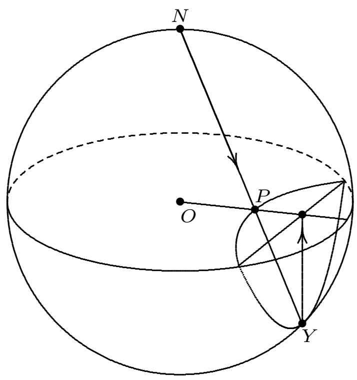


Figure 10.12: The map f from the Poincaré model to the Klein model

Let (x, y) be a point on $\mathbb{D}$. Regard it as a point in $\mathbb{R}^3$, we write it as $(x, y, 0)$. Then the coordinates of a point on the line joining the north pole $N = (0, 0, 1)$ and this point $P = (x, y, 0)$ is given by $\lambda(x, y, 0) + (1 - \lambda)(0, 0, 1) = (\lambda x, \lambda y, 1 - \lambda)$ for some λ. The ray NP meets the southern hemisphere at the point Y. As Y is on the unit sphere, we have $(\lambda x)^2 + (\lambda y)^2 + (1 - \lambda)^2 = 1$. Solving for λ, we get $\lambda = \frac{2}{1+x^2+y^2}$. Thus

$$Y = (\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, 1 - \frac{2}{1 + x^2 + y^2}).$$

Projecting back onto the xy-plane, we forget the z-coordinate of Y. Therefore,

$$f(x, y) = (\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}).$$

Let O be the centre of $\mathbb{D}$ and let P be a point $\mathbb{D}$. Consider a circular arc ℓ passing through P orthogonal to the boundary circle of $\mathbb{D}$ at its endpoints R and S. Thus ℓ is a hyperbolic line passing through P in the Poincaré model. Let the Euclidean line OP intersect the chord RS at Q. The chord RS defines a hyperbolic line κ passing through Q in the Klein model.

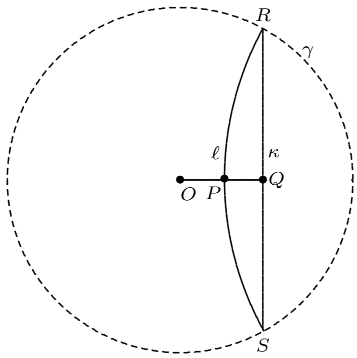


Figure 10.13: The map f maps a Poincaré line to a Klein line

Theorem 10.5 *The map f maps ℓ onto κ.*

Proof. Let the equation of the circular arc ℓ be $x^2 + y^2 + 2hx + 2gy + 1 = 0$. Note that the constant term is 1 because ℓ is orthogonal to the boundary circle γ of $\mathbb{D}$. The Klein line κ is the radical axis of ℓ and γ. It's equation is $hx + gy + 1 = 0$. Let (x, y) be a point on ℓ, and let $(X, Y) = f(x, y)$. Thus

$$X = \frac{2x}{1 + x^2 + y^2}, \ Y = \frac{2y}{1 + x^2 + y^2}.$$

Since (x, y) lies on ℓ, we can replace the term $1 + x^2 + y^2$ by $-2hx - 2gy$. That is

$$X = \frac{-x}{hx + gy}, \ Y = \frac{-y}{hx + gy}.$$

One can check easily that (X, Y) satisfies the equation $hX + gY + 1 = 0$. Therefore, (X, Y) lies on κ. Note that the points $(0, 0)$, (x, y) and (X, Y) are collinear.

Alternatively, one can prove this by using the fact that the stereographic projection preserves angles. Note that ℓ is orthogonal to γ at the two endpoints R and S. Under the inverse of the stereographic projection, it is mapped to a circle on the unit sphere and is orthogonal to γ at R and S. So it must be a vertical circle on the unit sphere through R and S. Thus its vertical projection onto the equatorial unit disc is the chord RS which is the Klein line κ.

Thus f can be defined as follow. Take any Poincaré line ℓ through P which is a circular arc orthogonal to the boundary of $\mathbb{D}$ with endpoints R and S. The chord RS is a Klein line κ . The ray OP intersects κ at Q. Then $f(P) = Q$. This is well-defined independent of ℓ by theorem 9.16.

The next exercise shows that f preserves distances (an isometry) of the two models.

Exercise 10.3 Show that for any point A in $\mathbb{D}$,

$$d_K(f(O), f(A)) = d_P(O, A),$$

where O is the centre of $\mathbb{D}$.

Exercise 10.4 Suppose in the Klein model ℓ_1 and ℓ_2 are two parallel lines which do not meet at the boundary. Show that there is a unique common perpendicular between them.

10.3 Upper half plane model

In this model, we take the set of all points to be the upper half plane of $\mathbb{R}^2$. That is

$$\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \ : \ y > 0\}.$$

Note that the x-axis is not included in $\mathbb{H}$. A hyperbolic line is either a Euclidean line perpendicular to the x-axis or a Euclidean semicircle with its centre on the x-axis.

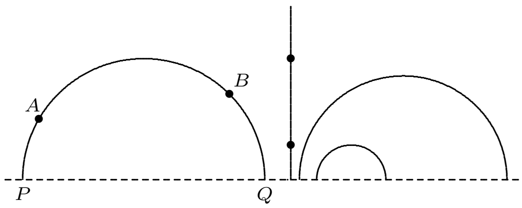


Figure 10.14: Hyperbolic lines in the upper half plane model

Exercise 10.5 Show that for any two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ in $\mathbb{H}$ with $a_1 \neq b_1$, there exists a unique semicircle (i.e. a hyperbolic line) with centre on the x-axis passing through them.

Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two points on a hyperbolic line ℓ in $\mathbb{H}$. Define the hyperbolic distance d_H as follow. If $a_1 = b_1$, then $d_H(A, B) = |\ln(a_2/b_2)|$. If $a_1 \neq b_1$, then

$$d_H(A, B) = |\ln\{AB, QP\}|.$$

As in the other models, a hyperbolic circle consists of all points whose hyperbolic distance from a given point is a positive constant. The locus of a hyperbolic circle in the upper half plane model is a Euclidean circle. In the upper half plane model, angles are measured in the usual Euclidean sense like the Poincaré model.

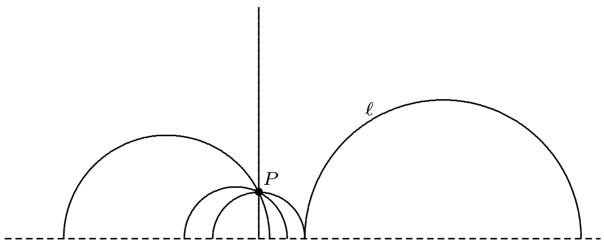


Figure 10.15: Limiting parallel lines in the upper half plane model

Given a point P not on a hyperbolic line ℓ, there exist more than one semi-circle with centres on the x-axis passing through P but not intersecting ℓ. Thus the hyperbolic postulate is satisfied. The two semi-circles through P tangent to ℓ at one of its end points are the *limiting parallels* to ℓ at P.

Exercise 10.6 Given a point P not on a hyperbolic line ℓ in $\mathbb{H}$, construct the perpendicular from P onto ℓ.

All the three models can be shown to be isomorphic to each other. That is there is a bijective map between any two of them preserving lines, angles and distances in the models. For example, if we regard the Poincaré disk as the unit disk in the complex plane and upper half plane as the set of all points on the complex plane with positive imaginary part, then the so-called Möbius transformation

$$g(z) = -i\frac{z+i}{z-i}$$

is an isomorphism from the Poincaré model to the upper half plane model.
Note that $g(0) = i, g(i) = \infty$ and $\operatorname{Im} g(z) = \frac{1-|z|^2}{|z-i|^2}$. Thus the unit circle $|z| = 1$ is mapped onto the real axis.

Exercise 10.7 (**A generalized Butterfly theorem**) In the Klein model, A, X, Z, B, W, Y are ideal points on the boundary circle in this order. The hyperbolic line AB intersects the hyperbolic lines XY, XW, ZY and ZW at the points C, M, N and D respectively. Let P be the midpoint of the Euclidean segment AB. See figure 10.16.
(a) Prove that $d_K(C, M) = d_K(N, D)$.
(b) Deduce that if the Euclidean segments MP and PN are of equal length, then the Euclidean segments CP and PD are also of equal length. See exercise 8.12.

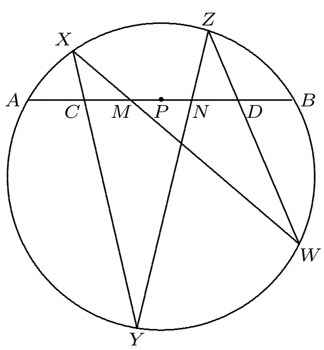


Figure 10.16: $d_K(C, M) = d_K(N, D)$

Exercise 10.8 (**Butterfly theorem revisited**) In the Klein model, A, X, Z, B, W, Y are ideal points on the boundary circle in this order. The hyperbolic line AB intersects the hyperbolic lines XY and ZW at the points M and N respectively. Suppose the hyperbolic lines AB, XW and YZ are concurrent at a point O. Let N_1 and N_2 be the feet of hyperbolic perpendiculars from O onto the hyperbolic lines XY and ZW respectively. See figure 10.17.
(a) Prove that $d_K(O, N_1) = d_K(O, N_2)$.
(b) Prove that the hyperbolic triangles OMN_1 and ONN_2 are congruent.
(c) Deduce that if O is the midpoint of the Euclidean segment AB, then the Euclidean segments OM and ON are of equal length.

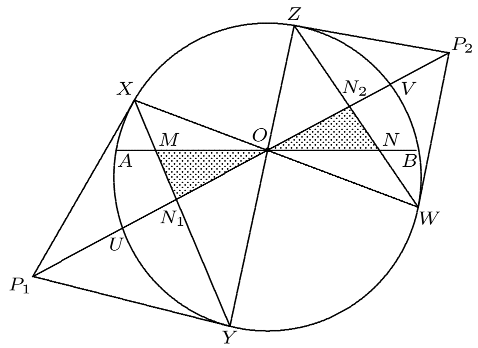


Figure 10.17: The hyperbolic triangles OMN_1 and ONN_2 are congruent.

Exercise 10.9 *Prove that in the Poincaré model, the angle sum of a hyperbolic triangle is less than* $180°$.

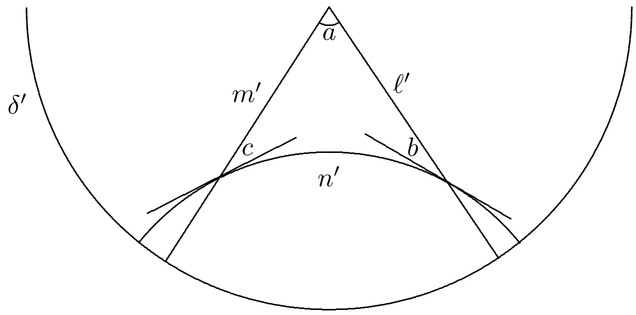


Figure 10.18: The angle sum of a hyperbolic triangle is less than $180°$.

[Hint: Let δ be the boundary circle of the unit disk, and ℓ, m, n be the sides of a hyperbolic triangle. Invert with respect to a circle centred at the intersection point of ℓ and m outside δ. Let δ', ℓ', m', n' be the respective inverses. Then ℓ' and m' are straight lines orthogonal to δ'. Thus they are the diameters of δ'. Since angles are preserved by inversion, the angle sum of the original triangle is the angle sum of the triangle formed by ℓ', m', n', i.e., the angles is $a + b + c$ and this is clearly less than $180°$.]

10.4 The isomorphism between the Poincaré model and the Klein model

Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be the map given by $f(x, y) = (\frac{2x}{1+x^2+y^2}, \frac{2x}{1+x^2+y^2})$. It maps the Poincaré disk bijectively onto the Klein disk. Also $f^{-1}(x, y) = \left(\frac{x(1-\sqrt{1-x^2-y^2})}{x^2+y^2}, \frac{y(1-\sqrt{1-x^2-y^2})}{x^2+y^2}\right)$.

Geometrically, f can be defined as follow. Take any Poincaré line ℓ through P which is a circular arc orthogonal to the boundary γ of $\mathbb{D}$ with endpoints R and S. The chord RS is a Klein line κ .

The ray OP intersects κ at Q. Then $f(P) = Q$. This is well-defined independent of ℓ by theorem 9.16. In fact, if ℓ' is another Poincaré line through P, then Q is the radical center of ℓ, ℓ', γ. We shall show f is an isometry.

Theorem 10.6 *Let O be the center of $\mathbb{D}$. Let $P \in \mathbb{D}$ and $Q = f(P)$. Let the circular arc through P centered at K be orthogonal to the boundary γ of $\mathbb{D}$ at R and S. The ray OP intersects the chord RS at Q. Let $\angle SOP = \alpha$ and $\angle POR = \beta$ and let $\angle OSP = \angle PRS = \theta$ and $\angle ORP = \angle PSR = \phi$. Then*

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin^2\theta}{\sin^2\phi}.$$

Proof. Since the arc RS is orthogonal to the boundary of $\mathbb{D}$, OR and OS are tangent to the arc RS at R and S respectively. We have $\angle PKS = 2\angle PRS = 2\theta$ and $\angle PKR = 2\angle PSR = 2\phi$.
By Ceva's theorem applied to the cevians OP, RP and SP, we have

$$\frac{\sin\angle SOP}{\sin\angle POR}\cdot\frac{\sin\angle ORP}{\sin\angle PRS}\cdot\frac{\sin\angle RSP}{\sin\angle PSO} = 1.$$

That is

$$\frac{\sin\alpha}{\sin\beta}\cdot\frac{\sin\phi}{\sin\theta}\cdot\frac{\sin\phi}{\sin\theta} = 1,$$

or equivalently,

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin^2\theta}{\sin^2\phi}. \tag{10.1}$$

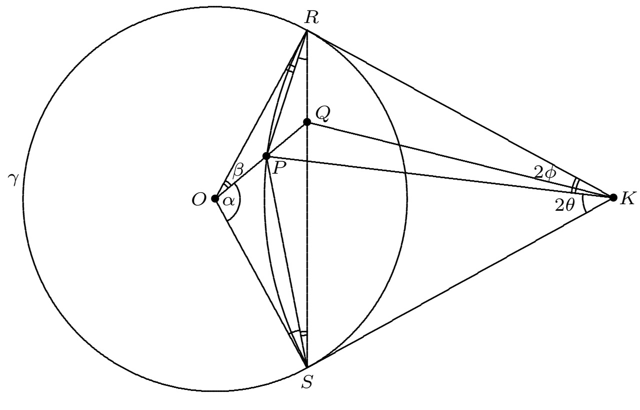


Figure 10.19: The map f maps a Poincaré line to a Klein line and $f(P) = Q$

Theorem 10.7 *Let $P_1, P_2 \in \mathbb{D}$ and $Q_1 = f(P_1)$, $Q_2 = f(P_2)$. Then $d_K(f(P_1), f(P_2)) = d_P(P_1, P_2)$.*

Proof. Let $\angle SOP_i = \alpha_i$, $\angle P_iOR = \beta_i$, $\angle P_iRS = \theta_i$ and $\angle P_iSR = \phi_i$, $i = 1, 2$
Let X be any point on the circle containing the arc RS. We may compute cross ratio of 4 points on the arc RS by means of the cross ratio of the rays from X to these 4 points. The notation

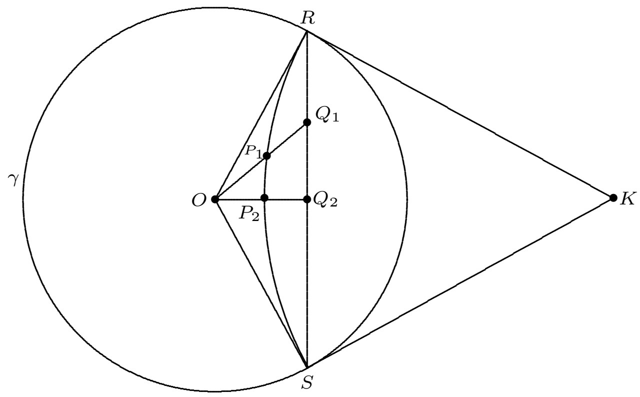


Figure 10.20: $d_K(f(P_1), f(P_2)) = d_P(P_1, P_2)$

$\{AB, CD\}_X$ denotes the cross ratio of the 4 rays: XA, XB, XC, XD in this order. Thus by (10.1), we have

$$\{P_1P_2, SR\} = \{P_1P_2, SR\}_X = \frac{\sin\theta_1 \sin\phi_2}{\sin\phi_1 \sin\theta_2} = \left(\frac{\sin\alpha_1}{\sin\beta_1}\frac{\sin\beta_2}{\sin\alpha_2}\right)^{\frac{1}{2}} = \{Q_1Q_2, SR\}_O^{\frac{1}{2}} = \{Q_1Q_2, SR\}^{\frac{1}{2}}.$$

Therefore, $d_P(P_1, P_2) = \ln\{P_1P_2, SR\} = \frac{1}{2}\ln\{Q_1Q_2, SR\} = d_K(Q_1, Q_2)$.
That is $d_K(f(P_1), f(P_2)) = d_P(P_1, P_2)$.

Next we prove the triangle inequality in the Klein model. By means of the above isomorphism f, we deduce the triangle inequality in the Poincaré model.

Lemma 10.8 *Let R, R', P, Q, S', S be 6 points on a line in this order. Then*
(a) $\{PQ, SR\} > 1$.
(b) $\{PQ, S'R'\} > \{PQ, SR\}$.
(c) $\ln\{PQ, S'R'\} > \ln\{PQ, SR\}$.

Proof. (a) $\{PQ, SR\} = \frac{PS\cdot QR}{QS\cdot PR} > 1$ as $PS > QS$ and $QR > PR$.

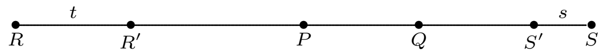


Figure 10.21: $\ln\{PQ, S'R'\} > \ln\{PQ, SR\}$

(b) Let's prove $\{PQ, S'R'\} > \{PQ, S'R\}$.
Let $RR' = t > 0$. Then $\frac{QR'}{PR'} = \frac{QR-t}{PR-t} > \frac{QR}{PR}$ as this is equivalent to $QR > PR$.
Thus $\{PQ, S'R'\} = \frac{PS'\cdot QR'}{QS'\cdot PR'} > \frac{PS'\cdot QR}{QS'\cdot PR} = \{PQ, S'R\}$.

Next we prove $\{PQ, S'R\} > \{PQ, SR\}$.

Let $S'S = s > 0$. Then $\frac{PS'}{QS'} = \frac{PS-s}{QS-s} > \frac{PS}{QS}$ as this is equivalent to $PS > QS$.
Thus $\{PQ, S'R\} = \frac{PS' \cdot QR}{QS' \cdot PR} > \frac{PS \cdot QR}{QS \cdot PR} = \{PQ, SR\}$.
Combining, we have $\{PQ, S'R'\} > \{PQ, SR\}$.
(c) follows from (a) and (b) as $\ln$ is an increasing function and $\ln(x) > 0$ for $x > 1$.

Theorem 10.9 *(Triangle Inequality) Let ABC be a triangle in the Klein disk. Then $d_K(B, A) + d_K(A, C) > d_K(A, C)$.*

Proof. Consider the following configuration.

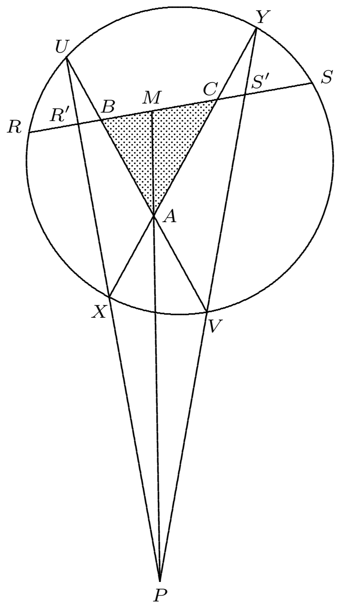


Figure 10.22: $d_K(B, A) + d_K(A, C) > d_K(A, C)$

By lemma 10.8, we have

$$\{BA, VU\} = \{BA, VU\}_P = \{BM, S'R'\} > \{BM, SR\},$$
$$\{AC, YX\} = \{AC, YX\}_P = \{MC, S'R'\} > \{MC, SR\}.$$

Thus $d_K(B, A) > d_K(B, M)$ and $d_K(A, C) > d_K(M, C)$.
Therefore, $d_K(B, A) + d_K(A, C) > d_K(B, M) + d_K(M, C) = d_K(A, C)$.

Theorem 10.10 *(Triangle Inequality) Let ABC be a triangle in the Poincaré disk. Then $d_P(B, A) + d_P(A, C) > d_P(A, C)$.*

Proof. Let $A' = f(A), B' = f(B), C' = f(C)$. Thus $A'B'C'$ is the corresponding triangle in the Klein disk. By the triangle inequality for the Klein model, we have $d_K(B', A') + d_K(A', C') > d_K(A', C')$. By the above result, we have $d_P(B, A) + d_P(A, C) > d_P(A, C)$.

Chapter 11

Basic Results of Hyperbolic Geometry

In this chapter, we shall explore some basic properties of hyperbolic geometry. We shall prove that the angle sum of any hyperbolic triangle is less than $180°$. This is in fact equivalent to the hyperbolic postulate. The crux of the proof is the concept of the so-called Saccheri quadrilateral. It follows that rectangles do not exist in hyperbolic geometry. Recall that two segments AB and PQ in hyperbolic space such as in the Poincaré model are congruent if and only if they have the same hyperbolic length. We will derive our results without reference to any model. Basically, we only assume Euclid's first 4 axioms, Pasch's axiom, a distance function along hyperbolic lines, a continuous function of angle measure and of course the hyperbolic postulate.

In this chapter, the notation PQ denotes the hyperbolic segment as well as its hyperbolic length. Also $PQ = AB$ will mean they have the same hyperbolic length and they are congruent.

11.1 Parallels in hyperbolic geometry

As we saw in the Poincaré Model, if ℓ is a hyperbolic line and P a point not on ℓ, there are always two limiting parallel lines through P. This is true in any model of hyperbolic geometry. In fact, this result is a consequence of the hyperbolic postulate, the continuity of the angle measure and Proposition 27.

Theorem 11.1 (Fundamental theorem of parallels in hyperbolic geometry) *Let ℓ be a hyperbolic line and P a point not on ℓ. Then there are exactly 2 lines m and n (the left and right limiting parallels) through P parallel to ℓ satisfying the following properties.*
(a) Any line through P within the angle between m or n and the perpendicular from P to ℓ must intersect ℓ while all other lines through P are parallel to ℓ.
(b) The limiting parallels m and n make equal acute angles (the angle of parallelism) with the perpendicular from P to ℓ.

Proof. Let N be the foot of the perpendicular from P onto ℓ. Consider all angles at P with the side PN. The set of these angles are divided into those angles $\angle NPA$ where the line PA intersects ℓ

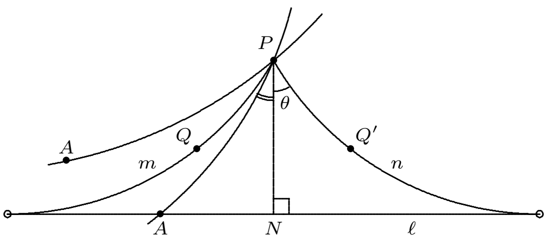


Figure 11.1: Parallels in hyperbolic geometry

and those where PA does not intersect ℓ. By the continuity of angle measure, there is an angle that separates the angles where the line PA intersects ℓ from those where it does not. Let $\angle QPN$ be this angle. Let PQ' be the reflection of PQ across PN. Since reflections preserve parallelism, PQ' must separate intersecting lines with ℓ from those parallel to ℓ. Also reflections preserve angle so that $\angle QPN = \angle Q'PN$. This proves (a).

To prove both these angles are acute, it suffices to prove neither can be a right angle. If $\angle NPQ = 90^\circ$, then the left and right limiting parallels PQ and PQ' coincide to one single line ℓ'. By Euclid's proposition 27, ℓ' is parallel to ℓ. By the hyperbolic postulate, there must be another line m through P parallel to ℓ. But m has to lie within one of the two right angles $\angle NPQ$ or $\angle NPQ'$. This contradicts the fact that $\angle NPQ$ and NPQ' separate intersecting and non-intersecting lines. Clearly the angle of parallelism $\angle NPQ$ cannot be obtuse. Thus it is always an acute angle.

Exercise 11.1 Let ℓ be a hyperbolic line. For any point P, define the reflection P' of P across ℓ as follow. Drop a perpendicular n from P onto ℓ meeting ℓ at the point N. Let P' be the point on n such that P and P' are on opposite sides of ℓ and $P'N = PN$. Prove the following.
(a) $(P')' = P$.
(b) For any hyperbolic segment AB, $A'B' = AB$, that is $A'B'$ is congruent to AB.
(c) For any hyperbolic triangle ABC, $\triangle A'B'C'$ is congruent to $\triangle ABC$.

Theorem 11.2 *Let ℓ be a hyperbolic line and P a point not on ℓ. Let n be a hyperbolic line which is right-limiting parallel to ℓ through P. Then for any point P' on n, n is right-limiting parallel to ℓ through P'.*

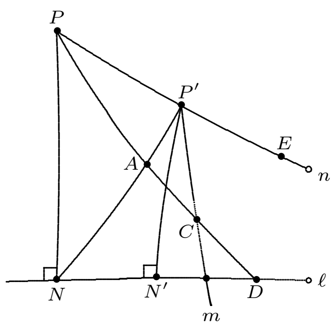


Figure 11.2: Each point on n is right-limiting parallel to ℓ

Proof. There are two cases to consider depending on which side of the line PN the point P' lies. Let's consider the case where P' lies on the right-hand side of the line PN. That is the ray PP' is right-limiting parallel to ℓ. Let $PN, P'N'$ be perpendiculars from P and P' onto ℓ at N and N' respectively. Note that PN is parallel to $P'N'$ by proposition 27. Thus all points on the line $P'N'$ lie on the same side of the line PN. Let E be a point on the extension of PP'. Let m be a line through P' within the angle $\angle N'P'E$, and C a point on m on the same side of ℓ as P'. We must show m intersects ℓ.

Since C is a point in the interior of $\angle NPE$, the ray PC must intersect ℓ at some point D. By Pasch's axiom, the ray PC also intersects the interior of the segment NP' at some point A since the ray PC does not intersect either of the other two sides of $\triangle PNP'$.

In $\triangle AND$, the line m intersects the side AD at C but it does not intersect AN as A and C are on opposite sides of the line $P'N'$. By Pasch's axiom, m must intersect ND and thus intersects ℓ.

Remark 11.1 The same result holds for left-limiting parallels.

Theorem 11.3 *If m is right-limiting parallel to ℓ, then ℓ is right-limiting parallel to m.*

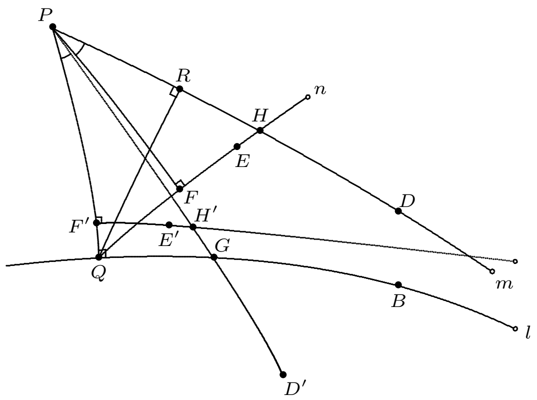


Figure 11.3: m are ℓ are limiting parallel to each other

Proof. Let P be a point on m. Drop a perpendicular from P to ℓ at Q. Fix a point D on m and on the right of P and a point B on ℓ and on the right of Q. Thus B and D are on the same side of the line PQ. Drop a perpendicular from Q to m at R. As the angle of parallelism $\angle QPD$ is acute, the point R must be on the right of P. To show that ℓ is right-limiting parallel to m, we have to show that any ray QE interior to $\angle BQR$ must intersect m. We also denote the line QE by n. Our goal is to show n intersects m.

Drop a perpendicular from P to n at F. Since $\angle PQE < \angle PQB = 90°$, the point F must lie on the same side of the line PQ as E. In the right-angled triangle PQF, we have $PQ > PF$. Note that if $\angle FPQ > \angle DPQ$, then by Pasch's axiom, m must intersect n, and we are done. So we may assume F is interior to $\angle DPQ$.

Now let's rotate the segments PF, FE and PD about P by the angle $\theta = \angle FPQ$. Since $PQ > PF$, F will rotate to a point F' on PQ and the line FE will rotate to a line $F'E'$ parallel to ℓ as both the angles at Q and F' are $90°$. Also the line PD will rotate to a line PD' that is interior to $\angle QPR$ (the angle of parallelism) and so it intersects ℓ at some point G.

Since the line $F'E'$ intersects $\triangle PQG$ and does not intersect QG, it must intersect PG at some point H' by Pasch's axiom. Now Rotating back about P through the angle $-\theta$ shows that n intersects m at a point H.

Theorem 11.4 *If m and n are right-limiting parallel to ℓ, then m and n are right-limiting to each other.*

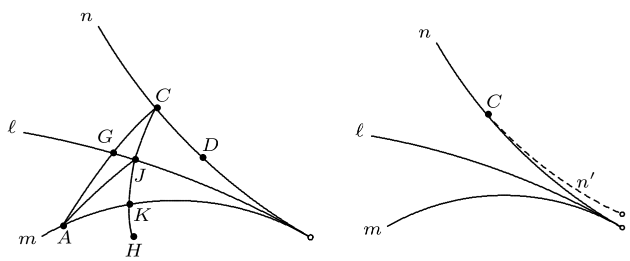


Figure 11.4: Limiting parallel is a transitive relation

Proof. There are two cases to consider. For the first case, m and n are on opposite sides of ℓ. Let A and C be points on m and n respectively. Then m and n are right-limiting parallel to ℓ at points A and C respectively. Note that the line AC intersects ℓ at some point G as A and C are on opposite sides of ℓ. Let CH be any ray interior to $\angle ACD$, where D is a point on n to the right of C. We must show CH intersect m. As n is right-limiting parallel to ℓ, the ray CH must intersect ℓ at some point J. Since right-limiting parallel is symmetric and m is right-limiting parallel to ℓ, the line ℓ is right-limiting parallel to m at any point along ℓ by Theorems 11.3 and 11.2. Therefore, the ray CJ intersects m at some point K. Since n and m do not intersect as they are on opposite sides of ℓ, and for any rays CH interior to $\angle ACD$ the ray CH intersects m. Thus n is right-limiting parallel to m. For the second case, m and n are on the same sides of ℓ. Let C be a point on n. Let n' through C be right-limiting parallel to m. By the first part of this proof, n' is right-limiting parallel to ℓ. Since n is also right-limiting parallel to ℓ, we must have $n' = n$ and n is right-limiting parallel to m.

11.2 Saccheri quadrilaterals

Definition 11.1 *A Saccheri quadrilateral is a quadrilateral $ABCD$ such that AB forms the base, AD and BC the sides such that $AD = BC$, and the angles at A and B are right angles. We shall refer to the $\angle C$ and $\angle D$ as the summit angles, CD as the summit and AB the base.*

Theorem 11.5 *The summit angles of a Saccheri quadrilateral are equal.*

Proof. Join AC and BD. Then $\triangle DAB$ is congruent to $\triangle CBA$ by (SAS). Thus $DB = CA$. That is the two hyperbolic segments DB and CA are congruent. Now $\triangle ACD$ is congruent to $\triangle BDC$ by (SSS). Therefore, $\angle C = \angle D$.

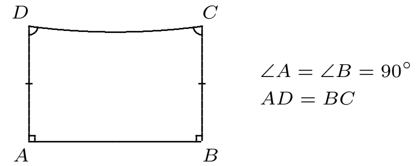


Figure 11.5: Saccheri quadrilateral

Exercise 11.2 Prove that the summit of a Saccheri quadrilateral is parallel to the base.

Theorem 11.6 *The summit angles of a Saccheri quadrilateral are acute.*

Proof. Consider a Saccheri quadrilateral $ABEF$ in the Poincaré model. Point D is one of the end points (called an Ω point) of the hyperbolic line BA. The hyperbolic lines ED and FD are right limiting parallels to BA from points E and F respectively. Note that ED and FD are right limiting parallels to each other. Since $EB = FA$, we have $\angle BED = \angle AFD$. Thus $\angle BED + \angle DEC = \angle BEF = \angle AFE$. As $\angle AFE + \angle AFD + \angle DFC = 180°$, we have $(\angle BED + \angle DEC) + \angle BED + \angle DFC = 180°$. By the Exterior Angle Theorem applied to $\triangle DEF$, we have $\angle DEC < \angle DFC$. Then $2\angle BED + 2\angle DEC < 180°$. This implies $\angle BED + \angle DEC < 90°$. Therefore, the summit angles of the Saccheri quadrilateral are acute.

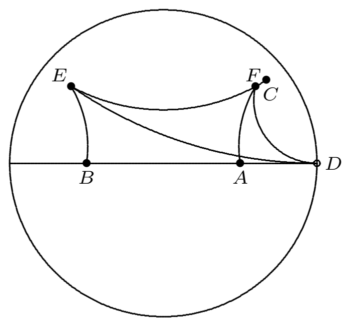


Figure 11.6: Summit angles of a Saccheri quadrilateral are acute

Exercise 11.3 Let m and ℓ be right limiting parallel lines meeting at the *"Ω point"* C. Let A be a point on ℓ and B a point on m. Show that for the triangle ABC, the exterior angle theorem holds.

Theorem 11.7 *The angle sum of any hyperbolic triangle is less than* $180°$.

Proof. Let ABC be a hyperbolic triangle. Points D and E are midpoints of the segments AB and AC, respectively. Segments AF, BG, and CH are drawn perpendicular to the line DE. Let's suppose $\angle ADE$ and $\angle AED$ are both acute.

By the exterior angle theorem, the point F must be within DE, G is on the side of AB opposite to F, and H is on the side of AC opposite to F. [If one of $\angle ADE$ or $\angle AED$ is obtuse, the other is acute. For example, if $\angle ADE$ is obtuse, then F is outside $\triangle ABC$ and G is inside $\triangle ABC$, and $\angle AED$ is acute.] As $\angle BDG = \angle ADF$, $\angle BGD = \angle AFD = 90°$ and $BD = AD$. we have $\triangle BDG$ is

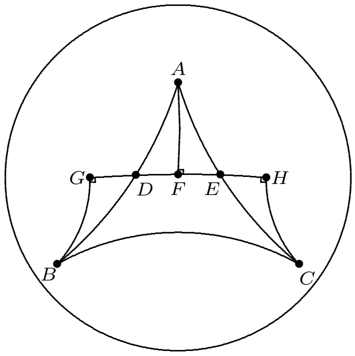


Figure 11.7: An equivalent Saccheri quadrilateral

congruent to $\triangle ADF$ by (AAS). Using a similar argument, $\triangle CEH$ is congruent to $\triangle AEF$. Thus $\angle GBD = \angle FAD$ and $\angle FAE = \angle HCE$. Hence $\angle B + \angle A + \angle C = \angle GBC + \angle HCB$. Also $BG = AF = CH$. Therefore $GHCB$ is a Saccheri quadrilateral. Consequently, the angle sum of $\triangle ABC$ is the same as the sum of the summit angles of the Saccheri quadrilateral $GHCB$. Since the summit angles of a Saccheri quadrilateral are acute, the angle sum of $\triangle ABC$ is less than $180°$.

Theorem 11.8 *The sum of the acute angles of a right hyperbolic triangle is less than* $90°$.

Theorem 11.9 *The sum of the interior angles of a convex hyperbolic polygon is less than* $(n-2) \times 180°$.

Theorem 11.10 *The angle sum of a quadrilateral is less than* $360°$.

Theorem 11.11 *Rectangles do not exist in hyperbolic space.*

Exercise 11.4 Prove that two Saccheri quadrilaterals with congruent summits and congruent summit angles must be congruent. That is the bases must be congruent and the sides must be congruent.

[Hint: If not, construct a rectangle from the quadrilateral with the longer sides.]

11.3 Lambert quadrilaterals

Definition 11.2 *A Lambert quadrilateral is a quadrilateral having three right angles.*

Theorem 11.12 *Let $ABCD$ be a Saccheri quadrilateral with summit CD, and let E and F be the midpoints of AB and CD respectively. Then $\angle AEF$ and $\angle EFD$ are right angles. Thus $AEFD$ and $EBCF$ are Lambert quadrilaterals.*

Proof. Join AF and BF. Then $\triangle ADF$ is congruent to $\triangle BCF$ by (SAS). Thus $AF = BF$. It follows that $\triangle AEF$ is congruent to $\triangle BEF$ by (SSS). Therefore, $\angle AEF = \angle BEF = 90°$. Similarly by joining DE and EC, we can prove that $\angle EFD = \angle EFC = 90°$.

Theorem 11.13 *In a Lambert quadrilateral, the fourth angle must be acute.*

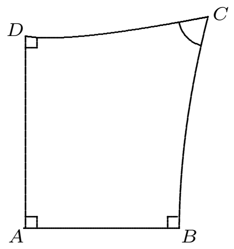


Figure 11.8: A Lambert quadrilateral

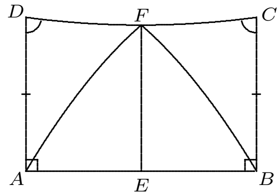


Figure 11.9: Two Lambert quadrilaterals inside a Saccheri quadrilateral

Proof. We can embed a given Lambert quadrilateral in a Saccheri quadrilateral and we know that the summit angles of a Saccheri quadrilateral are acute.

Theorem 11.14 *In a Lambert quadrilateral, the sides adjoining the acute angle are greater than the opposite sides.*

Proof. Let $ABCD$ be a Lambert quadrilateral in which $\angle C$ is acute. We shall prove $BC > AD$. Suppose $BC < AD$. Then we can mark off a point E on the extension of BC such that $BE = AD$.

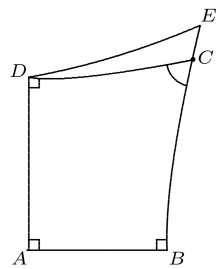


Figure 11.10: Sides adjoining the acute angle are longer

Then $ABED$ is a Saccheri quadrilateral with summit DE. Thus $\angle ADE = \angle BED$ and both are acute. Since A and E lie on opposite sides of CD, we must have $\angle ADE$ contains $\angle ADC = 90°$, so that $\angle ADE$ is not acute, which is a contradiction. If $AD = BC$, then $ABCD$ is a Saccheri quadrilateral. But then the summit angle D is not acute which is again a contradiction. Thus $BC > AD$. Similarly $DC > AB$.

Definition 11.3 *The shortest distance between two parallel lines is measured along a common perpendicular between the lines (if there exists one).*

Theorem 11.15 *Let D and C be points on the parallel lines ℓ_1 and ℓ_2 respectively. Let AB be a common perpendicular of ℓ_1 and ℓ_2, where $A \in \ell_1$ and $B \in \ell_2$, with $A \neq D$, $C \neq B$. Then $CD > AB$.*

Proof. Drop a perpendicular CD' from C onto ℓ_1. Then $ABCD'$ is a Lambert quadrilateral. Thus $\angle BCD'$ is acute. Since $\angle CD'D = 90° > \angle CDD'$, we have $CD > CD'$. For the Lambert quadrilateral $ABCD'$, $CD' > AB$. Therefore, $CD > AB$.

Theorem 11.16 *Parallel lines cannot have more than one common perpendicular.*

Proof. Let AB be perpendicular to AD and BC. Assume that a second line CD is also perpendicular to both AD and BC. Then the angle sum of the quadrilateral $ABCD$ is 360° which is a contradiction in hyperbolic geometry.

11.4 Triangles in hyperbolic geometry

Definition 11.4 *The defect of a hyperbolic triangle ABC with degree angle sum is* $180° - \angle A - \angle B - \angle C$.

Definition 11.5 *The defect of a hyperbolic quadrilateral $ABCD$ with degree angle sum is* $360° - \angle A - \angle B - \angle C - \angle D$.

Theorem 11.17 *Given a triangle ABC and a line ℓ intersecting the sides AB and AC at points D and E respectively, the defect of $\triangle ABC$ is equal to the sum of defects of $\triangle AED$ and the quadrilateral $EDBC$.*

We can use this result to prove one of the most amazing facts about triangles in hyperbolic geometry - similar triangles are congruent!

Theorem 11.18 *If two triangles have corresponding angles congruent, then the triangles are congruent.*

Proof. Let ABC and DEF be two hyperbolic triangles such that $\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$. If any pair of sides between them is congruent, then by (AAS) the triangles are congruent. So either a pair of sides in $\triangle ABC$ is larger than the corresponding pair in $\triangle DEF$ or smaller than the corresponding pair. Without loss of generality, we can assume that $AB > DE$ and $AC > DF$.

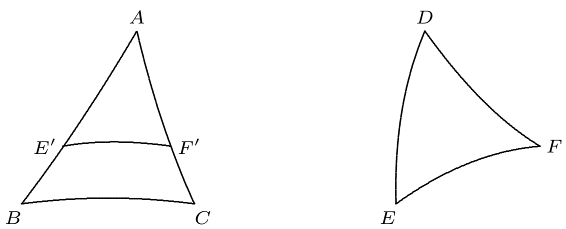


Figure 11.11: Similar triangles are congruent

Then we can find points E' on AB, and F' on AC so that $AE' = DE$ and $AF' = DF$. By (SAS), the triangles $AE'F'$ are DEF are congruent, and thus have the same defect.
Since $\triangle DEF$ and $\triangle ABC$ have the same defect, we have $\triangle AE'F'$ and ABC have the same defect. Therefore the defect of the quadrilateral $E'F'CB$ is zero, which is impossible.

Definition 11.6 *Two figures in hyperbolic geometry are equivalent if the figures can be subdivided into a finite number of pieces so that pairs of corresponding pieces are congruent.*

By using the notion of equivalence, we are able to base all Euclidean area calculations on the simple figure of a rectangle. But this is not possible in hyperbolic geometry as rectangles do not exist in hyperbolic geometry. Instead, area can be defined as a function satisfying the following axioms.

Area Axiom I. *If A, B, C are distinct and not collinear, then the area of $\triangle ABC$ is positive.*
Area Axiom II. *The area of equivalent sets must be the same.*
Area Axiom III. *The area of the union of disjoint sets is the sum of the separate areas.*

Theorem 11.19 *If two triangles ABC and $A'B'C'$ have two sides congruent and the same defect, then they are equivalent and hence have the same area.*

Proof. Suppose $BC = B'C'$. The triangle ABC is equivalent to the Saccheri quadrilateral $BCHG$ with summit BC. The sum of the two summit angles equal to the angle sum of $\triangle ABC$, and thus the summit angles are each half of the angle sum of $\triangle ABC$.

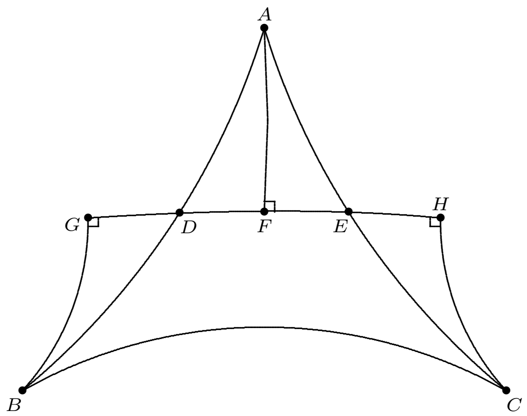


Figure 11.12: Angle sum of a triangle

Similarly the triangle $A'B'C'$ is equivalent to a Saccheri quadrilateral with summit $B'C'$. Now $BC = B'C'$ and the summit angles for both Saccheri quadrilaterals are equal because the two triangles have the same defect which means they have same angle sum. Hence the two Saccheri quadrilaterals are congruent. Therefore, the triangles ABC and $A'B'C'$ are equivalent and they have the same area.

Theorem 11.20 *If two triangles ABC and $A'B'C'$ have the same defect, then they are equivalent and hence have the same area.*

Proof. If one side of $\triangle ABC$ is congruent to a side of $\triangle A'B'C'$, then the result follows from the previous theorem.

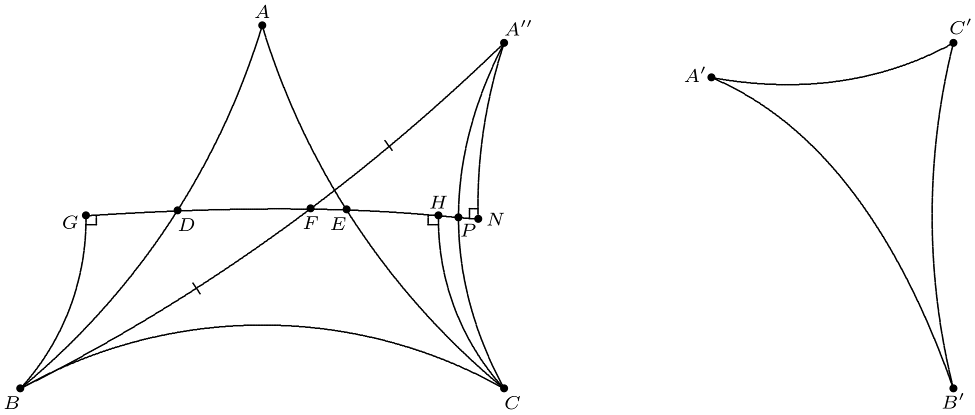


Figure 11.13: Two equivalent hyperbolic triangles

Suppose $A'B' > AB$. First construct the Saccheri quadrilateral $BCHG$ of $\triangle ABC$ with BC as the summit. Mark off a point F on DE such that $BF = \frac{1}{2}B'A'$. Then $F \neq D$ because D is the midpoint of AB. Extend BF to a point A'' such that $FA'' = FB$. Thus $BA'' = B'A'$. Join $A''C$ and let it intersect the line DE at P. Let $A''N$ be the perpendicular from A'' onto the line DE. Then by (AAS), $\triangle BGF$ is congruent to $\triangle A''NF$ so that $BG = A''N$. As $CH = BG$, we thus have $CH = A''N$. Hence by (AAS), $\triangle A''NP$ is congruent to $\triangle CHP$. This shows that P is the midpoint of $A''C$. Therefore $BCHG$ is a Saccheri quadrilateral for $\triangle A''BC$ with summit BC.
The defect of $\triangle A''BC$ equals to the defect of $\triangle ABC$ since each of their angle sums equals to the sum of the summit angles of the Saccheri quadrilateral $BCHG$. So $\triangle ABC$ and $\triangle A''BC$ are equivalent. On the other hand the defect of $\triangle A'B'C'$ equals to the defect of $\triangle ABC$ and so it also equals to the defect of $\triangle A''BC$. Now $\triangle A'B'C'$ and $A''BC$ have the sides $A'B'$ and $A''B$ congruent. Thus they are equivalent, and hence they have the same area. This completes the proof of the theorem.
Conversely, if two given triangles are equivalent, then they can be subdivided into sub-triangles with corresponding pairs congruent. Thus for each pair, the defect will be the same. As the defect of each of the original triangles is the sum of the defects of their sub-triangles, the two given triangles have the same defect. Combining with the last theorem, we have the following result.

Theorem 11.21 *Two triangles have the same defect if and only if they are equivalent, and thus they have the same area.*

Remark 11.2 In fact if we use a suitable "*metric*" say on the Poincaré model, the area of a triangle can be shown to be equal to the defect of the triangle.

Let $P(x, y)$ be a point in the Poincaré disk which is at a Euclidean distance $\rho = \sqrt{x^2 + y^2}$ from the centre O. Then its hyperbolic distance from the centre O is given by $r = \ln \frac{1+\rho}{1-\rho}$. Differentiating r

with respect to ρ along the ray OP, we get $\frac{dr}{d\rho} = \frac{2}{1-\rho^2}$. Thus

$$dr = \frac{2}{1-\rho^2}d\rho,$$

and the hyperbolic arclength differential is

$$ds = \frac{2\sqrt{dx^2+dy^2}}{1-x^2-y^2}.$$

Therefore, it is natural to define the hyperbolic area differential as

$$dA = \frac{4}{(1-\rho^2)^2}dxdy.$$

For a region R in the Poincaré disk, define the hyperbolic area of R to be

$$\mathbf{Area}(R) = \iint_R \frac{4}{(1-\rho^2)^2}\, dxdy,$$

if this double integral exists. Changing to polar coordinates, we have

$$\mathbf{Area}(R) = \iint_R \frac{4}{(1-\rho^2)^2}\rho\, d\rho d\theta.$$

By making the substitution $r = \ln\frac{1+\rho}{1-\rho}$, we can express this as

$$\mathbf{Area}(R) = \iint_R \sinh r\, drd\theta.$$

Here r is the hyperbolic distance from the point to the centre.

Example 11.1 Let $A = (0,0), B = (\frac{2\sqrt{3}}{3}-\frac{1}{2}, \frac{\sqrt{3}}{6}), C = (\frac{\sqrt{3}}{3}, 0)$. The hyperbolic line BC is given by $x^2+y^2-\frac{4\sqrt{3}}{3}x+1=0$.

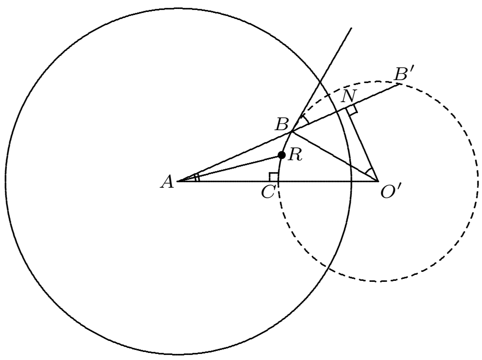


Figure 11.14: Area of a hyperbolic right triangle

We have $\angle A = \arctan(\frac{\frac{\sqrt{3}}{6}}{\frac{2\sqrt{3}}{3}-\frac{1}{2}}) = \tan^{-1}(\frac{\sqrt{3}+4}{13})$, the Euclidean length $AB = \frac{1}{3}\sqrt{15-6\sqrt{3}}$, and the Euclidean length $BN = \frac{\sqrt{3}-1}{\sqrt{15-6\sqrt{3}}}$. From this, we obtain $\tan(B) = \tan(\angle BO'N) = \sqrt{3}-1$. That is $\angle B = \tan^{-1}(\sqrt{3}-1)$.

Defect of $\triangle ABC$ in radian = $\pi - \angle A - \angle B - \angle C = \frac{\pi}{2} - \tan^{-1}(\sqrt{3}-1) - \tan^{-1}(\frac{\sqrt{3}+4}{13}) = 0.5236.$

Next let's find the hyperbolic area of $\triangle ABC$. Let $R = (\rho\cos\theta, \rho\sin\theta)$ be a point on the hyperbolic segment BC. The parametric equation satisfied by R in polar coordinates is given by

$$\rho = \frac{2}{\sqrt{3}}\cos\theta - \frac{1}{3}\sqrt{12\cos^2\theta - 9},$$

where $0 \le \theta \le \angle A$. This is obtained by substituting $x = \rho\cos\theta$, $y = \rho\sin\theta$ into $x^2 + y^2 - \frac{4\sqrt{3}}{3}x + 1 = 0$, and solving for ρ.

$$\begin{aligned}
\text{Area of } \triangle ABC &= \iint_{\triangle ABC} \frac{4\rho}{(1-\rho^2)^2}\, d\rho d\theta \\
&= \int_0^{\angle A}\int_0^{\rho(\theta)} \frac{4\rho}{(1-\rho^2)^2}\, d\rho d\theta \\
&= \int_0^{\angle A} \left[\frac{2}{1-\rho^2}\right]_0^{\rho(\theta)} d\theta \\
&= \int_0^{\angle A} \left[\frac{2}{1-\rho(\theta)^2} - 2\right] d\theta \\
&= \int_0^{\tan^{-1}(\frac{\sqrt{3}+4}{13})} \left[\frac{2}{1-\rho(\theta)^2} - 2\right] d\theta \\
&= \int_0^{\tan^{-1}(\frac{\sqrt{3}+4}{13})} \left[\frac{2}{1-(\frac{2}{\sqrt{3}}\cos\theta - \frac{1}{3}\sqrt{12\cos^2\theta-9})^2} - 2\right] d\theta \\
&= \int_0^{\tan^{-1}(\frac{\sqrt{3}+4}{13})} \frac{-9 + 24\cos^2\theta - 4\sqrt{3}\cos\theta\sqrt{12\cos^2\theta - 9}}{9 - 12\cos^2\theta + 2\sqrt{3}\cos\theta\sqrt{12\cos^2\theta-9}}\, d\theta \\
&= \int_0^{\tan^{-1}(\frac{\sqrt{3}+4}{13})} -1 + \frac{2\sqrt{3}\cos\theta(2\sqrt{3}\cos\theta - \sqrt{12\cos^2\theta-9})}{(2\sqrt{3}\cos\theta - \sqrt{12\cos^2\theta-9})\sqrt{12\cos^2\theta-9}}\, d\theta \\
&= \int_0^{\tan^{-1}(\frac{\sqrt{3}+4}{13})} -1 + \frac{2\sqrt{3}\cos\theta}{\sqrt{12\cos^2\theta-9}}\, d\theta \\
&= \int_0^{\tan^{-1}(\frac{\sqrt{3}+4}{13})} -1 + \frac{2\cos\theta}{\sqrt{1-4\sin^2\theta}}\, d\theta \\
&= \left[-\theta + \sin^{-1}(2\sin\theta)\right]_0^{\tan^{-1}(\frac{\sqrt{3}+4}{13})} \\
&= -\tan^{-1}(\frac{\sqrt{3}+4}{13}) + \sin^{-1}\left(\frac{\sqrt{3}+4}{\sqrt{47+2\sqrt{3}}}\right) \\
&= -\tan^{-1}(\frac{\sqrt{3}+4}{13}) + \tan^{-1}\left(\frac{4+\sqrt{3}}{3\sqrt{3}-1}\right) \\
&= -\tan^{-1}(\frac{\sqrt{3}+4}{13}) + \frac{\pi}{2} - \tan^{-1}\left(\frac{3\sqrt{3}-1}{4+\sqrt{3}}\right) \\
&= \frac{\pi}{2} - \tan^{-1}(\sqrt{3}-1) - \tan^{-1}(\frac{\sqrt{3}+4}{13}) = 0.5236.
\end{aligned}$$

Thus hyperbolic area of $\triangle ABC$ = defect of $\triangle ABC$.

Exercise 11.5 Prove that the hyperbolic circumference of a circle with hyperbolic radius r is equal to $2\pi\sinh r$.

Exercise 11.6 Prove that the hyperbolic area of a circle with hyperbolic radius r is equal to $4\pi \sinh^2 \frac{r}{2}$.

Exercise 11.7 Let A be a point in the Poincaré disk with centre O, and ℓ a hyperbolic line through A with endpoints P and Q on the boundary circle. Let the Euclidean lines OA and PQ intersect at the point A_1. Show that $d_P(O, A_1) = 2d_P(O, A)$.

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