# Notes on Barycentric Homogeneous Coordinates

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## 1 Barycentric Homogeneous Coordinates

Let $ABC$ be a triangle on the plane. For any point $P$, the ratio of the (signed) areas

$$[PBC] : [PCA] : [PAB]$$

is called the *barycentric coordinates* or *areal coordinates* of $P$.
Here $[PBC]$ is the signed area of the triangle $PBC$. It is positive, negative or zero according to both $P$ and $A$ lie on the same side, opposite side, or on the line $BC$. Generally, we use $(x : y : z)$ to denote the barycentric coordinates of a point $P$. The barycentric coordinates of a point are homogeneous. That is $(x : y : z) = (\lambda x : \lambda y : \lambda z)$ for any nonzero real number $\lambda$. If $x+y+z = 1$, then $(x : y : z)$ is called the *normalized barycentric coordinates* of the point $P$. For example, $A = (1 : 0 : 0), B = (0 : 1 : 0), C = (0 : 0 : 1)$. The triangle $ABC$ is called the *reference triangle* of the barycentric homogeneous coordinate system.

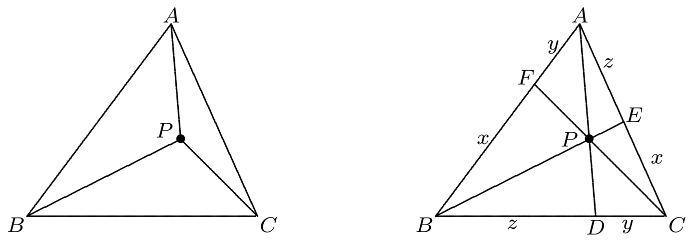



Fig. 1: Barycentric coordinates

**Theorem 1.** *Let $[PBC] = x$, $[PCA] = y$, $[PAB] = z$ and $[ABC] = 1$ so that $x + y + z = 1$. Let the extensions of the $AP, BP, CP$ meet the sides $BC, CA, AB$ at $D, E, F$ respectively. Then*

*(a)* $CE : EA = x : z$, *etc.*
*(b)* $AP : PD = (y + z) : x$.
*(c)* $\boldsymbol{BP} = z\boldsymbol{BC} + x\boldsymbol{BA}$.
*(d) If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are the position vectors of the points $A, B, C$ respectively, then $\boldsymbol{p} = x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c}$. Here $\boldsymbol{a} = \boldsymbol{OA}$ is the position vector of $A$ with respect to a fixed origin $O$, etc.*
*(e) The normalized barycentric coordinates $(x : y : z)$ of the point $P$ is unique.*

**Proof**. (a) $CE : EA = [PBC] : [PAB] = x : z$.
(b) Let $[PDC] = \alpha, [PBD] = \beta$. Then $\frac{AP}{PD} = \frac{y}{\alpha}$ and $\frac{AP}{PD} = \frac{z}{\beta}$. Thus

$$\frac{AP}{PD} = \frac{y+z}{\alpha+\beta} = \frac{y+z}{x}.$$

(c) follows from (b).
(d) $\boldsymbol{p} = \boldsymbol{b} + \boldsymbol{BP} = \boldsymbol{b} + z\boldsymbol{BC} + x\boldsymbol{BA} = \boldsymbol{b} + z(\boldsymbol{c} - \boldsymbol{b}) + x(\boldsymbol{a} - \boldsymbol{b}) = x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c}$.
(e) follows from (d).

### 1.1 Ratio Formula

Let $P_1 = (x_1 : y_1 : z_1)$ and $P_2 = (x_2 : y_2 : z_2)$ with $x_1 + y_1 + z_1 = 1$ and $x_2 + y_2 + z_2 = 1$. If $P$ divides $P_1P_2$ in the ratio $P_1P : PP_2 = \beta : \alpha$, then the point $P$ has barycentric coordinates $(\alpha x_1 + \beta x_2 : \alpha y_1 + \beta y_2 : \alpha z_1 + \beta z_2)$.

## 1.2 Common points

| | |
|---|---|
| $A$ | $(1:0:0)$ |
| centroid | $(1:1:1)$ |
| incentre | $(a:b:c)$ |
| symmedian point | $(a^2:b^2:c^2)$ |
| $A$- excentre | $(-a:b:c)$ |
| orthocentre | $(\tan A:\tan B:\tan C)=(S_BS_C:S_CS_A:S_AS_B)$ |
| circumcentre | $(\sin 2A:\sin 2B:\sin 2C)=(a^2S_A:b^2S_B:c^2S_C)$ |
| Nagel point | $(s-a:s-b:s-c)$ |
| Gergonne point | $((s-b)(s-c):(s-c)(s-a):(s-a)(s-b))$ |
| Isogonal conjugate of $(x:y:z)$ | $(a^2/x:b^2/y:c^2/z)=(a^2yz:b^2zx:c^2xy)$ |
| Isotomic conjugate of $(x:y:z)$ | $(1/x:1/y:1/z)$ |

Here $a=BC, b=CA, c=AB$, $s=\frac{1}{2}(a+b+c)$, and $S_A=\frac{1}{2}(b^2+c^2-a^2)$ etc.

# 2 Lines

## 2.1 Equation of a line

If $P=(x:y:z)$ is a point on the line joining $P_1=(x_1:y_1:z_1)$ and $P_2=(x_2:y_2:z_2)$, then $(x:y:z)=(\alpha x_1+\beta x_2:\alpha y_1+\beta y_2:\alpha z_1+\beta z_2)$, for some $\alpha,\beta$. (The normalizing factors for $P, P_1, P_2$ can be absorbed into $\alpha,\beta$.) Thus

$$\begin{pmatrix} -1 & \alpha & \beta \end{pmatrix}\begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}.$$

This implies that $\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=0$. Expanding the determinant about the first row, we have

$$\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}x-\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}y+\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}z=0.$$

That is $(x:y:z)$ satisfies the homogeneous linear equation

$$(y_1z_2-y_2z_1)x-(x_1z_2-x_2z_1)y+(x_1y_2-x_2y_1)z=0.$$

Conversely, any point $P=(x:y:z)$ satisfies this equation can be shown to lie on the line $P_1P_2$.

We may compute the coefficients of this line by the following determinant:

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=\left[\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}:-\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}:\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right],$$

and represent the line using the 3 coefficients as homogeneous coordinates enclosed in a square bracket.

The 2 representations $px + qy + rz = 0$ and $[p : q : r]$ are equivalent and will be used interchangeably.

For example, the line $BC$ is $x = 0$ or $[1 : 0 : 0]$.

## 2.2 Intersection of two lines

Let $\ell_1 : p_1x + q_1y + r_1z = 0$ and $\ell_2 : p_2x + q_2y + r_2z = 0$ be 2 lines. The intersection point is the simultaneous solution to the two equations, which is

$$(q_1r_2 - q_2r_1 : -p_1r_2 + p_2r_1 : p_1q_2 - p_2q_1).$$

We may facilitate the computation by means of the following determinant:

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = \left( \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} : - \begin{vmatrix} p_1 & r_1 \\ p_2 & r_2 \end{vmatrix} : \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \right),$$

## 2.3 Collinearity

Three points $P_1 = (x_1 : y_1 : z_1)$, $P_2 = (x_2 : y_2 : z_2)$ and $P_3 = (x_3 : y_3 : z_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

## 2.4 Concurrence

Three lines $\ell_1 = [p_1 : q_1 : r_1]$, $\ell_2 = [p_2 : q_2 : r_2]$ and $\ell_3 = [p_3 : q_3 : r_3]$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0.$$

## 2.5 Common lines

| | |
|---|---|
| $BC$ | $[1 : 0 : 0]$ |
| internal bisector from $A$ | $[0 : -c : b]$ |
| external bisector from $A$ | $[0 : c : b]$ |
| median from $A$ | $[0 : -1 : 1]$ |
| perpendicular from $A$ | $[0 : -S_B : S_C]$ |
| perpendicular bisector of $BC$ | $[c^2 - b^2 : -a^2 : a^2]$ |
| Euler line | $[S_A(S_B - S_C) : S_B(S_C - S_A) : S_C(S_A - S_B)]$ |
| Gergonne line | $[s - a : s - b : s - c]$ |

## 3 Area

If $P, Q, R$ are points with normalized barycentric coordinates $(x_1 : y_1 : z_1)$, $(x_2 : y_2 : z_2)$, $(x_3 : y_3 : z_3)$ respectively, then $[PQR] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [ABC]$.

**Proof**. Let $\delta$ be the determinant $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$. Since the sum of the entries in each row of the determinant is 1, we have

$$\delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} = (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1).$$

Similarly, $\delta = (y_2 - y_1)(z_3 - z_1) - (z_2 - z_1)(y_3 - y_1)$, and

$$\delta = (z_2 - z_1)(x_3 - x_1) - (x_2 - x_1)(z_3 - z_1).$$

Let $\boldsymbol{n}$ be the unit normal vector to the $x-y$ plane along the positive $z$ direction. Let $O$ be the origin. Then

$$\begin{aligned} & \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a} \\ = & \ (|\boldsymbol{a}||\boldsymbol{b}| \sin \measuredangle AOB + |\boldsymbol{b}||\boldsymbol{c}| \sin \measuredangle BOC + |\boldsymbol{c}||\boldsymbol{a}| \sin \measuredangle COA)\mathbf{n} \\ = & \ 2([OAB] + [OBC] + [OCA])\boldsymbol{n} \\ = & \ 2[ABC]\boldsymbol{n}. \end{aligned}$$

Here $\measuredangle AOB$ is the signed angle measured from the vector $\boldsymbol{OA}$ to $\boldsymbol{OB}$.

Let $\boldsymbol{x} = x_1\boldsymbol{a} + y_1\boldsymbol{b} + z_1\boldsymbol{c}$, $\boldsymbol{y} = x_2\boldsymbol{a} + y_2\boldsymbol{b} + z_2\boldsymbol{c}$, $\boldsymbol{z} = x_3\boldsymbol{a} + y_3\boldsymbol{b} + z_3\boldsymbol{c}$. Then

$$\begin{aligned} & 2[XYZ]\mathbf{n} \\ = & \ XY \cdot XZ \sin \measuredangle YXZ \ \mathbf{n} \\ = & \ \boldsymbol{XY} \times \boldsymbol{XZ} \\ = & \ ((x_2 - x_1)\boldsymbol{a} + (y_2 - y_1)\boldsymbol{b} + (z_2 - z_1)\boldsymbol{c}) \times ((x_3 - x_1)\boldsymbol{a} + (y_3 - y_1)\boldsymbol{b} + (z_3 - z_1)\boldsymbol{c}) \\ = & \ ((x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)) \, \boldsymbol{a} \times \boldsymbol{b} \\ & \ ((y_2 - y_1)(z_3 - z_1) - (z_2 - z_1)(y_3 - y_1)) \, \boldsymbol{b} \times \boldsymbol{c} \\ & \ ((z_2 - z_1)(x_3 - x_1) - (x_2 - x_1)(z_3 - z_1)) \, \boldsymbol{c} \times \boldsymbol{a} \\ = & \ \delta(\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a}) \\ = & \ 2\delta[ABC]\boldsymbol{n}. \end{aligned}$$

Consequently, $[XYZ] = \delta[ABC]$.

## 4 Distances

### 4.1 Displacement vectors

Let $P = (p_1 : p_2 : p_3)$ and $Q = (q_1 : q_2 : q_3)$ be two points in normalized barycentric coordinates. The displacement vector is the vector $\boldsymbol{PQ} = (q_1 - p_1 : q_2 - p_2 : q_3 - p_3)$. Expressed in the terms of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, we have $\boldsymbol{PQ} = (q_1 - p_1)\boldsymbol{a} + (q_2 - p_2)\boldsymbol{b} + (q_3 - p_3)\boldsymbol{c}$. Note that if $\boldsymbol{PQ} = (x : y : z)$ is a displacement vector, then $x + y + z = 0$.

## 4.2 Inner product of two displacement vectors

Let $\boldsymbol{PQ} = (x_1 : y_1 : z_1)$ and $\boldsymbol{EF} = (x_2 : y_2 : z_2)$ be two displacement vectors. Then

$$\boldsymbol{PQ} \cdot \boldsymbol{EF} = -\frac{1}{2}\left(a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2)\right).$$

**Proof**. Take $B$ be the origin so that $\boldsymbol{b} = \boldsymbol{0}$, $|\boldsymbol{a}| = c$, $|\boldsymbol{c}| = a$. Let $U$ and $V$ be the points such that $\boldsymbol{BU} = \boldsymbol{PQ}$ and $\boldsymbol{BV} = \boldsymbol{EF}$ as free vectors respectively. Note that $\boldsymbol{PQ} \cdot \boldsymbol{EF} = \boldsymbol{BU} \cdot \boldsymbol{BV}$. Let $U = (\lambda_1 : \mu_1 : \delta_1)$ and $V = (\lambda_2 : \mu_2 : \delta_2)$ in normalized barycentric coordinates. Then $\boldsymbol{BU} = (\lambda_1 : \mu_1 - 1 : \delta_1) = \lambda_1\boldsymbol{a} + \delta_1\boldsymbol{c}$. Since $\boldsymbol{BU} = \boldsymbol{PQ}$, we have $\lambda_1 = x_1, \mu_1 - 1 = y_1, \delta - 1 = z_1$. Thus $U = (x_1 : y_1 + 1 : z_1)$. Similarly, $V = (x_2 : y_2 + 1 : z_2)$. Therefore, $\boldsymbol{BU} = x_1\boldsymbol{a} + z_1\boldsymbol{c}$ and $\boldsymbol{BV} = x_2\boldsymbol{a} + z_2\boldsymbol{c}$. Then

$\boldsymbol{PQ} \cdot \boldsymbol{EF} = \boldsymbol{BU} \cdot \boldsymbol{BV} = (x_1\boldsymbol{a} + z_1\boldsymbol{c}) \cdot (x_2\boldsymbol{a} + z_2\boldsymbol{c}) = x_1x_2|\boldsymbol{a}|^2 + z_1z_2|\boldsymbol{c}|^2 + (x_1z_2 + z_1x_2)\boldsymbol{a} \cdot \boldsymbol{c} = x_1x_2c^2 + z_1z_2a^2 + (x_1z_2 + z_1x_2)\boldsymbol{a} \cdot \boldsymbol{c} = x_1x_2c^2 + z_1z_2a^2 + (x_1z_2 + z_1x_2)\frac{1}{2}(c^2 + a^2 - b^2) = \frac{1}{2}\left(a^2(2z_1z_2 + x_1z_2 + z_1x_2) - b^2(x_1z_2 + z_1x_2) + c^2(2x_1x_2 + x_1z_2 + z_1x_2)\right)$.

Note that

$2z_1z_2 + x_1z_2 + z_1x_2 = 2z_1z_2 + (-y_1 - z_1)z_2 + z_1(-y_2 - z_2) = -(z_1y_2 + y_1z_2)$.

$2x_1x_2 + x_1z_2 + z_1x_2 = 2x_1x_2 + x_1(-x_2 - y_2) + (-x_1 - y_1)x_2 = -(y_1x_2 + x_1y_2)$.

Therefore, $\boldsymbol{PQ} \cdot \boldsymbol{EF} = -\frac{1}{2}\left(a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2)\right)$.

## 4.3 The length of a displacement vector

If $\boldsymbol{PQ} = (x : y : z)$, where $x + y + z = 0$, is a displacement vector, then $PQ^2 = -(a^2yz + b^2zx + c^2xy)$.

## 4.4 Perpendicular displacement vectors

(a) $\boldsymbol{PQ} \perp \boldsymbol{EF}$ if and only if $0 = a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2)$.

(b) Let $(l : m : n)$ be a displacement vector. A displacement vector perpendicular to $(l : m : n)$ is given by

$$(a^2(n - m) + (c^2 - b^2)l : b^2(l - n) + (a^2 - c^2)m : c^2(m - l) + (b^2 - a^2)n).$$

**Example 4.1.** *A displacement vector perpendicular to* $\boldsymbol{AI} = (-(b + c) : b : c)$ *is* $(b - c : -b : c)$.

**Example 4.2.** *Let $I$ and $G$ be the incentre and the centroid of a triangle $ABC$ respectively. Then $\boldsymbol{IG}$ is perpendicular to $\boldsymbol{BC}$ if and only if $b = c$ or $b + c = 3a$.*

**Example 4.3.** *In a triangle $ABC$, the incircle touches the side $AC$ at $E$ and $AB$ at $F$. The line through $B$ and the incentre $I$ intersects the line $EF$ at $P$. Then $P = (\frac{a}{2c} : \frac{c-a}{2c} : \frac{1}{2})$, and $\boldsymbol{PB}$ is perpendicular to $\boldsymbol{PC}$.*

## 4.5 Conway's Notation

Denote $S_A = \frac{1}{2}(b^2 + c^2 - a^2)$, $S_B = \frac{1}{2}(c^2 + a^2 - b^2)$ and $S_C = \frac{1}{2}(a^2 + b^2 - c^2)$. Let $S$ be twice the area of the triangle $ABC$.

**Lemma 1.**

*(i)* $S_A + S_B = c^2, S_B + S_C = a^2, S_C + S_A = b^2$.

*(ii)* $S_A - S_B = b^2 - a^2$, $S_B - S_C = c^2 - b^2$, $S_C - S_A = a^2 - c^2$.

*(iii)* $S_A S_B + S_B S_C + S_C S_A = S^2$.

*(iv)* $a^2 S_A + b^2 S_B + c^2 S_C = 2S^2$.

*(v)* $b^2c^2 - S_A^2 = S^2 = S_B S_C + a^2 S_A$.

**Proof**. Let's prove (v). By (i) (ii) and (iii), $b^2c^2 - S_A^2 = b^2(S_A + S_B) - S_A(b^2 - S_C) = b^2 S_B + S_A S_C = (S_C + S_A)S_B + S_A S_C = S^2$.

## 4.6 Examples

**Example 4.4.** *The perpendicular bisector of $BC$ has the equation $a^2(z - y) + (c^2 - b^2)x = 0$, or $[c^2 - b^2 : -a^2 : a^2]$.*

**Solution**. The midpoint $M$ of $BC$ is $(0 : \frac{1}{2} : \frac{1}{2})$. Let $X = (x : y : z)$ be a point in normalized barycentric coordinates on the perpendicular bisector of $BC$. Then $XM = (-x : \frac{1}{2} - y : \frac{1}{2} - z)$ and $BC = (0 : -1 : 1)$. By the condition on perpendicular displacement vectors, we have $a^2(z - y) + (c^2 - b^2)x = 0$.

**Example 4.5.** *The barycentric coordinates of the foot of perpendicular from a point $P$ with normalized barycentric coordinates $(x_0 : y_0 : z_0)$*

*onto $BC$ is*

$$(0 : (a^2 + b^2 - c^2)x_0 + 2a^2y_0 : (a^2 - b^2 + c^2)x_0 + 2a^2z_0) = (0 : \frac{S_C}{a^2}x_0 + y_0, \frac{S_B}{a^2}x_0 + z_0).$$

*onto $CA$ is $((b^2 - c^2 + a^2)y_0 + 2b^2x_0 : 0 : (b^2 + c^2 - a^2)y_0 + 2b^2z_0)$,*

*onto $AB$ is $((c^2 + a^2 - b^2)z_0 + 2c^2x_0 : (c^2 - a^2 + b^2)z_0 + 2c^2y_0 : 0)$.*

**Example 4.6.** *The barycentric coordinates of the foot of perpendicular from $A$ onto $BC$ is $(0 : a^2 + b^2 - c^2 : c^2 + a^2 - b^2) = (0 : S_C : S_B)$.*

**Example 4.7.** *The reflection of $P$ with normalized barycentric coordinates $(x_0 : y_0 : z_0)$ across the line $BC$ is the point $(-a^2x_0 : (a^2 + b^2 - c^2)x_0 + a^2y_0 : (a^2 - b^2 + c^2)x_0 + a^2z_0)$, or*

$$(-x_0 : \frac{2S_C}{a^2}x_0 + y_0, \frac{2S_B}{a^2}x_0 + z_0).$$

**Example 4.8.** *The reflection of the line $[\alpha, \beta, \gamma]$ across the line $BC$ is the line*

$$[\frac{2}{a^2}(S_B\gamma + S_C\beta) - \alpha, \beta, \gamma].$$

# 5 Circles I

## 5.1 Parallel Axis Theorem

If $O$ is the circumcentre of the reference triangle $ABC$ and $P$ has normalized barycentric coordinates $(x:y:z)$, then

$$R^2 - OP^2 = xAP^2 + yBP^2 + zCP^2 = a^2yz + b^2zx + c^2xy.$$

**Proof**. Take the circumcentre $O$ be the origin so that $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}| = R$. We have $\boldsymbol{p} = x\boldsymbol{a}+y\boldsymbol{b}+z\boldsymbol{c}$ with $x+y+z=1$. As $\boldsymbol{AP} = \boldsymbol{p}-\boldsymbol{a}$, we have $AP^2 = |\boldsymbol{AP}|^2 = |\boldsymbol{p}-\boldsymbol{a}|^2 = |\boldsymbol{p}|^2+|\boldsymbol{a}|^2-2\,\boldsymbol{p}\cdot\boldsymbol{a} = OP^2+R^2-2\,\boldsymbol{p}\cdot\boldsymbol{a}$. Similarly, $BP^2 = OP^2+R^2-2\,\boldsymbol{p}\cdot\boldsymbol{b}$ and $CP^2 = OP^2+R^2-2\,\boldsymbol{p}\cdot\boldsymbol{c}$. Therefore, $xAP^2+yBP^2+zCP^2 = (x+y+z)OP^2+(x+y+z)R^2-2\,\boldsymbol{p}\cdot(x\boldsymbol{a}+y\boldsymbol{b}+z\boldsymbol{c}) = OP^2+R^2-2\,\boldsymbol{p}\cdot\boldsymbol{p} = OP^2+R^2-2\,OP^2 = R^2-OP^2$.

Since $\boldsymbol{AP} = \boldsymbol{p}-\boldsymbol{a} = (x-1)\boldsymbol{a}+y\boldsymbol{b}+z\boldsymbol{c}$, we have $AP^2 = -a^2yz-b^2z(x-1)-c^2(x-1)y = -(a^2yz+b^2zx+c^2xy)+zb^2+yc^2$. Similarly, $BP^2 = -(a^2yz+b^2zx+c^2xy)+xc^2+za^2$ and $CP^2 = -(a^2yz+b^2zx+c^2xy)+ya^2+xb^2$. Therefore, $xAP^2+yBP^2+zCP^2 = -(x+y+z)(a^2yz+b^2zx+c^2xy)+x(zb^2+yc^2)+y(xc^2+za^2)+z(ya^2+xb^2) = a^2yz+b^2zx+c^2xy$.

## 5.2 Power of a point with respect to the circumcircle

The power of $P$ with respect to the circumcircle of $ABC$ is $-(a^2yz+b^2zx+c^2xy)$, where $P$ has normalized barycentric coordinates $(x:y:z)$.

## 5.3 Circumcircle

The equation of the circumcircle of the reference triangle $ABC$ is $a^2yz+b^2zx+c^2xy=0$.

## 5.4 Power of a point with respect to a circle

Let $\omega$ be a circle and $P'$ a point having normalized barycentric coordinates $(x:y:z)$. Then the negative of the power of $P'$ with respect to $\omega$ can be expressed in the form

$$a^2yz+b^2zx+c^2xy+(x+y+z)(px+qy+rz),$$

where $p, q, r$ are constants depending only on $\omega$.

**Proof**. Every circle $\omega$ is homothetic to the circumcircle of the reference triangle by a homothety, say $h$, with a center of similitude $S = (u:v:w)$ (in normalized barycentric coordinates) and similitude ratio $k$. As the point $P'$ has normalized barycentric coordinates $(x:y:z)$, we have $P \equiv h(P') = kP'+(1-k)S = k(x+tu(x+y+z) : y+tv(x+y+z) : z+tw(x+y+z))$, where $t = \frac{1-k}{k}$, is the normalized barycentric coordinates of $P$.

Let $R$ and $R'$ be the radii of the circumcircle of the reference triangle and $\omega$ respectively. Then $R = kR'$. Let the circumcentre of the reference triangle be $O$ and the center of $\omega$ be $O'$. Then $O = h(O')$. Also $PO = kP'O'$.

The power of $P'$ with respect to $\omega = P'O'^2 - R'^2 = k^{-2}(PO^2-R^2) = k^{-2}\times$ the power of $P$ with respect to the circumcircle.

The negative of the power of $P$ with respect to the circumcircle is obtained by substituting normalized barycentric coordinates of $P$ into the equation of the circumcircle. Therefore, the negative of the power of $P'$ with respect to $\omega$ is equal to

$$\begin{aligned}
& \sum_{cyclic} a^2(y+tv(x+y+z))(z+tw(x+y+z)) \\
= & \sum_{cyclic} a^2(yz+t(wy+vz)(x+y+z)+t^2vw(x+y+z)^2) \\
= & (a^2yz+b^2zx+c^2xy)+t(x+y+z)\sum_{cyclic} a^2(wy+vz)+ \\
& t^2(x+y+z)^2(a^2vw+b^2wu+c^2uv) \\
= & (a^2yz+b^2zx+c^2xy)+t(x+y+z)\sum_{cyclic}\left(b^2w+c^2v+t(a^2vw+b^2wu+c^2uv)\right)x \\
= & a^2yz+b^2zx+c^2xy+(x+y+z)(px+qy+rz).
\end{aligned}$$

## 5.5 Equation of a circle

The general equation of a circle is $a^2yz+b^2zx+c^2xy+(x+y+z)(px+qy+rz)=0$.

Note that the powers of the points $A, B, C$ with respect to the circle are $-p, -q, -r$ respectively. For example, the powers of $A, B, C$ with respect to the incircle are $(s-a)^2, (s-b)^2, (s-c)^2$ respectively. Therefore, the equation of the incircle is

$$a^2yz+b^2zx+c^2xy-(x+y+z)\left((s-a)^2x+(s-b)^2y+(s-c)^2z\right)=0.$$

| | |
|---|---|
| circumcircle | $a^2yz+b^2zx+c^2xy=0$ |
| incircle | $a^2yz+b^2zx+c^2xy-(x+y+z)\left((s-a)^2x+(s-b)^2y+(s-c)^2z\right)=0$ |
| $A$-excircle | $a^2yz+b^2zx+c^2xy-(x+y+z)\left(s^2x+(s-c)^2y+(s-b)^2z\right)=0$ |
| nine-point circle | $a^2yz+b^2zx+c^2xy$ <br> $-\frac{1}{4}(x+y+z)\left((-a^2+b^2+c^2)x+(a^2-b^2+c^2)y+(a^2+b^2-c^2)z\right)=0$ |
| circle centred at $A$ with radius $\rho$ | $a^2yz+b^2zx+c^2xy+(x+y+z)\left(\rho^2x+(\rho^2-c^2)y+(\rho^2-b^2)z\right)=0$ |
| circle through the 3 excentres of $ABC$ | $a^2yz+b^2zx+c^2xy+(x+y+z)(bcx+cay+abz)=0$ |
| mixtilinear incircle opposite to $A$ | $a^2yz+b^2zx+c^2xy-\frac{b^2c^2}{s^2}(x+y+z)(x+(s/b-1)^2x+(s/c-1)^2z)=0$ <br> point of tangency with the circumcircle is $(-a:\frac{b^2}{s-b}:\frac{c^2}{s-c})$ |

**Example 5.1.** *The centroid lies on the incircle if and only if* $5(a^2+b^2+c^2)=6(ab+bc+ca)$.

**Example 5.2.** *The Nagel point lies on the incircle if and only if* $(a+b-3c)(b+c-3a)(c+a-3b)=0$.

## 5.6 Radical axis

The line $px+qy+rz=0$ is the radical axis of $\omega$ and the circumcircle whenever $\omega$ is not concentric with the circumcircle.

For example, the radical axis of the circumcircle and the nine-point circle is the line

$$(b^2+c^2-a^2)x+(c^2+a^2-b^2)y+(a^2+b^2-c^2)z=0.$$

### 5.7 The arc $BC$

(a) The midpoint of the arc $BC$ not containing $A$ has barycentric coordinates $(-a^2 : b(b+c) : c(b+c))$.

(b) The midpoint of the arc $BC$ containing $A$ has barycentric coordinates $(a^2 : -b(b-c) : c(b-c))$.

(c) The median through $A$ meets the circumcircle at the point with barycentric coordinates $(a^2 : b^2+c^2 : b^2+c^2)$.

## 6 Circles II

### 6.1 The Feuerbach point

Consider the difference of the coefficients of $x$ in the equations of the incircle and nine-point circle, we have $(s-a)^2 - \frac{1}{4}(-a^2+b^2+c^2) = \frac{1}{2}(a-b)(a-c)$. Thus subtracting the equations of the incircle and the nine-point circle, we get

$$\ell : (a-b)(a-c)x + (b-c)(b-a)y + (c-a)(c-b) = 0,$$

which can be written as

$$\frac{x}{b-c} + \frac{y}{c-a} + \frac{z}{a-b} = 0.$$

Observe that there are two points $P = ((b-c)^2 : (c-a)^2 : (a-b)^2)$ and $Q = (a(b-c)^2 : b(c-a)^2 : c(a-b)^2)$ on $\ell$. Using these two points, we can parameterize $\ell$, except $P$, as

$$(x : y : z) = \left((a+t)(b-c)^2 : (b+t)(c-a)^2 : (c+t)(a-b)^2\right).$$

The centre $N$ of the nine-point circle is the midpoint of $OH$, its barycentric coordinates is given by

$$N = (a^2(b^2+c^2) - (b^2-c^2)^2 : b^2(c^2+a^2) - (c^2-a^2)^2 : c^2(a^2+b^2) - (a^2-b^2)^2).$$

Also $I = (a : b : c)$. Thus the line $IN$ has parametric equation:

$$\begin{aligned}(x : y : z) = \big(&a^2(b^2+c^2) - (b^2-c^2)^2 + ak : \\ &b^2(c^2+a^2) - (c^2-a^2)^2 + bk : c^2(a^2+b^2) - (a^2-b^2)^2 + ck\big).\end{aligned}$$

If we take $k = -2abc$, then $x = a^2(b^2+c^2) - (b^2-c^2)^2 - 2a^2bc = a^2(b-c)^2 - (b^2-c^2)^2 = (b-c)^2(a^2-(b+c)^2) = (a+b+c)(b-c)^2(a-b-c) = -4s(b-c)^2(s-a)$. Similarly we get the expressions for $y$ and $z$ by cyclically permuting $a, b, c$. Thus we have a point

$$((b-c)^2(s-a) : (c-a)^2(s-b) : (a-b)^2(s-c))$$

on the line $IN$.

If we take $t = -s$ in $\ell$, we obtain the same point $F$. Thus $F$ is the intersection of the line $IN$ and $\ell$. $F$ is the *Feuerbach point* of the triangle $ABC$.

We can verify by direct substitution that the barycentric coordinates of the Feuerbach point satisfy the equation of the nine-point circle. It follows that the nine-point circle and the incircle are tangent at the Feuerbach point.

## 6.2 Excircles

The equation of the common tangent $\ell_A$ of the nine-point circle and the $A$-excircle is

$$\frac{x}{b-c}+\frac{y}{c+a}-\frac{z}{a+b}=0.$$

There are two points $P=((b-c)^2:(c+a)^2:(a+b)^2)$ and $Q=(-a(b-c)^2:b(c+a)^2:c(a+b)^2)$ on $\ell_A$. We parameterize $\ell_A$ as

$$(x:y:z)=\left((-a+t)(b-c)^2:(b+t)(c+a)^2:(c+t)(a+b)^2\right).$$

Recall that $I_A=(-a:b:c)$. Thus the line $I_AN$ has parametric equation:

$$\begin{aligned}(x:y:z)=\big(&a^2(b^2+c^2)-(b^2-c^2)^2-ak:\\ &b^2(c^2+a^2)-(c^2-a^2)^2+bk:c^2(a^2+b^2)-(a^2-b^2)^2+ck\big).\end{aligned}$$

Taking $k=2abc$, we get $x=a^2(b^2+c^2)-(b^2-c^2)^2-2a^2bc=-4s(b-c)^2(s-a)$, $y=b^2(c^2+a^2)-(c^2-a^2)^2+2ab^2c=4(c+a)^2(s-a)(s-c)$, $z=c^2(a^2+b^2)-(a^2-b^2)^2+2abc^2=4(a+b)^2(s-a)(s-b)$.

Thus we have the point

$$F_A=(-s(b-c)^2:(s-c)(c+a)^2:(s-b)(a+b)^2)$$

on $I_AN$. On the other hand, if we let $t=a-s$ in $\ell_A$, we get the same point. Therefore $F_A$ is the intersection point of $\ell_A$ and $I_AN$.

We can verify by direct substitution that the barycentric coordinates of the point $F_A$ satisfy the equation of the nine-point circle. It follows that the nine-point circle and the $A$-excircle are tangent at $F_A$. Similarly, the nine-point circle is tangent to the other 2 excircles. The points of tangency are

$$F_B=((s-c)(b+c)^2:-s(c-a)^2:(s-a)(a+b)^2),$$

$$F_C=((s-b)(b+c)^2:(s-a)(c+a)^2:-s(a-b)^2).$$

## 6.3 Pedal triangle

Let $P$ be a point with normalized barycentric coordinates $(x_0:y_0:z_0)$. The determinant formed by the normalized barycentric coordinates of the foot of perpendiculars from $P$ onto the sides of $ABC$ is

$$\frac{1}{8a^2b^2c^2}\begin{vmatrix}0 & (a^2+b^2-c^2)x_0+2a^2y_0 & (a^2-b^2+c^2)x_0+2a^2z_0\\ (b^2-c^2+a^2)y_0+2b^2x_0 & 0 & (b^2+c^2-a^2)y_0+2b^2z_0\\ (c^2+a^2-b^2)z_0+2c^2x_0 & (c^2-a^2+b^2)z_0+2c^2y_0 & 0\end{vmatrix}$$

$$\begin{aligned}&=\tfrac{1}{4a^2b^2c^2}(b+c-a)(c+a-b)(a+b-c)(a+b+c)(x_0+y_0+z_0)(a^2y_0z_0+b^2z_0x_0+c^2x_0y_0)\\ &=\tfrac{4}{a^2b^2c^2}s(s-a)(s-b)(s-c)(a^2y_0z_0+b^2z_0x_0+c^2x_0y_0)\\ &=\left(\tfrac{2[ABC]}{abc}\right)^2(a^2y_0z_0+b^2z_0x_0+c^2x_0y_0).\\ &=\tfrac{1}{4R^2}(a^2y_0z_0+b^2z_0x_0+c^2x_0y_0).\end{aligned}$$

This proves Simson's theorem that $P$ lies on the circumcircle if and only if the 3 feet of perpendiculars are collinear. Also the area of the pedal triangle of $P$ is $\frac{[ABC]}{4R^2}(a^2y_0z_0+b^2z_0x_0+c^2x_0y_0)$.

## 6.4 Equation of the circle with center $(\alpha : \beta : \gamma)$ and radius $\rho$

Let the normalized barycentric coordinates of the centre $O$ be $(\alpha : \beta : \gamma)$. Let $P = (x : y : z)$, where $x + y + z = 1$, be a point on the circle. Then $\boldsymbol{OP} = (x - \alpha : y - \beta : z - \gamma)$. Thus the equation of the circle is

$$a^2(y-\beta)(z-\gamma) + b^2(x-\alpha)(z-\gamma) + c^2(x-\alpha)(y-\beta) = -\rho^2.$$

Expanding, we have

$$\begin{aligned} & a^2yz + b^2zx + c^2xy - ((b^2\gamma + c^2\beta)x + (c^2\alpha + a^2\gamma)y + (a^2\beta + b^2\alpha)z) \\ & \qquad + a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta + \rho^2 = 0 \\ \Leftrightarrow\ & a^2yz + b^2zx + c^2xy - ((b^2\gamma + c^2\beta)x + (c^2\alpha + a^2\gamma)y + (a^2\beta + b^2\alpha)z) \\ & \qquad + (a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta + \rho^2)(x+y+z) = 0. \end{aligned}$$

The coefficient of $x$ is

$$\begin{aligned} & -(b^2\gamma + c^2\beta) + (a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta + \rho^2) \\ = \ & -b^2\gamma(\alpha+\beta+\gamma) - c^2\beta(\alpha+\beta+\gamma) + (a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta + \rho^2) \\ = \ & -(b^2\gamma^2 + 2S_A\beta\gamma + c^2\beta^2 - \rho^2). \end{aligned}$$

Thus the equation is $a^2yz + b^2zx + c^2xy - \sum_{cyclic} (b^2\gamma^2 + 2S_A\beta\gamma + c^2\beta^2 - \rho^2)x = 0.$

Therefore, the general equation of the circle with centre $(\alpha : \beta : \gamma)$ and radius $\rho$ in homogeneous form is

$$a^2yz + b^2zx + c^2xy - (x+y+z)\sum_{cyclic}\left(\frac{b^2\gamma^2 + 2S_A\beta\gamma + c^2\beta^2}{(\alpha+\beta+\gamma)^2} - \rho^2\right)x = 0.$$

On the other hand, the centre of the circle $a^2yz + b^2zx + c^2xy + (x+y+z)(px+qy+rz) = 0$ is

$$(a^2S_A - S_B(r-p) + S_C(p-q) : b^2S_B - S_C(p-q) + S_A(q-r) : c^2S_C - S_A(q-r) + S_B(r-p)),$$

and its radius is given by

$$\rho^2 = \frac{1}{4S^2}\left((abc)^2 + 2(a^2S_Ap + b^2S_Bq + c^2S_Cr) + S_A(q-r)^2 + S_B(r-p)^2 + S_C(p-q)^2\right).$$

## 6.5 Concentric circles

The equation of the circle concentric with the circumcircle with radius $kR$ is

$$a^2yz + b^2zx + c^2xy + (x+y+z)(\tau x + \tau y + \tau z) = 0,$$

where $\tau = (k^2-1)a^2b^2c^2(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4)^{-1}$.

**Proof**. Let the normalized barycentric coordinates of the circumcentre be $(\alpha : \beta : \gamma)$. Thus the equation of the circle concentric with the circumcircle with radius $kR$ is

$$a^2(y-\beta)(z-\gamma) + b^2(x-\alpha)(z-\gamma) + c^2(x-\alpha)(y-\beta) = -(kR)^2,$$

where $(x : y : z)$ is the normalized barycentric coordinates of a point on the circle. The left hand side is equal to $a^2yz + b^2zx + c^2xy - R^2$. Thus the above equation can be written as $a^2yz + b^2zx + c^2xy + (k^2 - 1)R^2 = 0$. Note that $R = \frac{abc}{2S}$, where $S$ is the twice the area of the triangle $ABC$. Thus $R^2 = a^2b^2c^2(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4)^{-1}$. Using $x + y + z = 1$, we may write the equation as

$$a^2yz + b^2zx + c^2xy + (x + y + z)(\tau x + \tau y + \tau z) = 0.$$

## 6.6 The equation of the tangent to a circle

Let $\omega$ be the circle with equation

$$a^2yz + b^2zx + c^2xy + (x + y + z)(px + qy + rz) = 0.$$

Let $P$ be a point on $\omega$ with homogeneous barycentric coordinates $(x_0 : y_0 : z_0)$. The equation of the tangent to $\omega$ at $P$ is given by

$$\begin{aligned}&a^2(y_0z + yz_0) + b^2(z_0x + zx_0) + c^2(x_0y + xy_0)\\&+(x_0 + y_0 + z_0)(px + qy + rz) + (x + y + z)(px_0 + qy_0 + rz_0) = 0.\end{aligned}$$

That is

$$\begin{aligned}[2px_0 + (c^2 + p + q)y_0 + (b^2 + r + p)z_0 &: (c^2 + p + q)x_0 + 2qy_0 + (a^2 + q + r)z_0\\&: (b^2 + r + p)x_0 + (a^2 + q + r)y_0 + 2rz_0].\end{aligned}$$

**Proof**. Let the parametric equation of the tangent line to $\omega$ at $P$ be $x = x_0 + \alpha t, y = y_0 + \beta t, z = z_0 + \gamma t$, where $(\alpha : \beta : \gamma)$ is a displacement vector along the direction of the tangent. Note that $\alpha + \beta + \gamma = 0$. Substituting $(x_0 + \alpha t : y_0 + \beta t : z_0 + \gamma t)$ into the equation of the circle and using the fact that $(x_0 : y_0 : z_0)$ satisfies the equation of the circle, we get a quadratic equation in $t$.

$$\begin{aligned}(a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta)t^2 + [&a^2(y_0\gamma + \beta z_0) + b^2(z_0\alpha + \gamma x_0) + c^2(x_0\beta + \alpha y_0)\\&+(x_0 + y_0 + z_0)(p\alpha + q\beta + r\gamma)]t = 0.\end{aligned}$$

Since the tangent line intersects $\omega$ only at the point $P$, this quadratic equation has a double root $t = 0$, or equivalently, the coefficient of $t$ is 0. Thus

$$a^2(y_0\gamma + \beta z_0) + b^2(z_0\alpha + \gamma x_0) + c^2(x_0\beta + \alpha y_0) + (x_0 + y_0 + z_0)(p\alpha + q\beta + r\gamma) = 0.$$

To find a linear equation satisfied by $x, y, z$, consider

$$\begin{aligned}&a^2(y_0z + yz_0) + b^2(z_0x + zx_0) + c^2(x_0y + xy_0)\\&\quad+(x_0 + y_0 + z_0)(px + qy + rz) + (x + y + z)(px_0 + qy_0 + rz_0)\\&= a^2(y_0(z_0 + \gamma t) + (y_0 + \beta t)z_0)) + b^2(z_0(x_0 + \alpha t) + (z_0 + \gamma t)x_0) + c^2(x_0(y_0 + \beta t) + (x_0 + \alpha t)y_0)\\&\quad+(x_0 + y_0 + z_0)(p(x_0 + \alpha t) + q(y_0 + \beta t) + r(z_0 + \gamma t))\\&\quad+(x_0 + y_0 + z_0 + (\alpha + \beta + \gamma)t)(px_0 + qy_0 + rz_0)\\&= 2a^2y_0z_0 + ta^2(y_0\gamma + \beta z_0) + 2b^2z_0x_0 + tb^2(z_0\alpha + \gamma x_0) + 2c^2x_0y_0 + tc^2(x_0\beta + \alpha y_0)\\&\quad+(x_0 + y_0 + z_0)(px_0 + qy_0 + rz_0 + t(p\alpha + q\beta + r\gamma))\\&\quad+(x_0 + y_0 + z_0)(px_0 + qy_0 + rz_0)\\&= 2(a^2y_0z_0 + b^2z_0x_0 + c^2x_0y_0) + t(a^2(y_0\gamma + \beta z_0) + b^2(z_0\alpha + \gamma x_0) + c^2(x_0\beta + \alpha y_0))\end{aligned}$$

$$+t(x_0+y_0+z_0)(p\alpha+q\beta+r\gamma))+2(x_0+y_0+z_0)(px_0+qy_0+rz_0)$$
$$=0.$$

The last line is by the above relation and the fact that $(x_0 : y_0 : z_0)$ satisfies the equation of the circle.

**Example 6.1.** *The equation of the tangent to the circumcircle at* $A$ *is* $b^2z + c^2y = 0$.

## 6.7 The line at infinity and the circumcircle

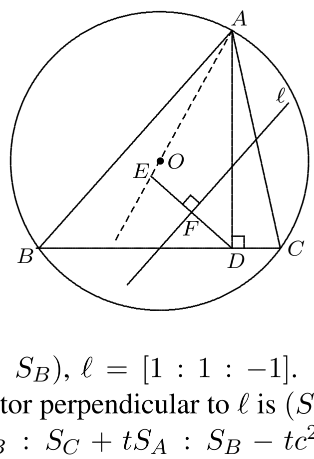
Recall that if $(x : y : z)$ is a displacement vector which is a difference of two points $A$ and $B$ in normalized barycentric coordinates, then $x + y + z = 0$. If we identify all displacement vectors along $AB$ using homogeneous coordinates so that $(x : y : z) = (kx : ky : kz)$ for any $k \neq 0$, then the displacement vector $(x : y : z)$ represents either direction of the line $AB$. A displacement vector in homogeneous coordinates is also called the *infinite point* of the line $AB$. See [1]. The set of all infinite points constitutes a line called the *line at infinity* which has the equation $x+y+z = 0$.

In the setting of the usual plane geometry, the isogonal conjugate of a point on the circumcircle is not defined. More precisely, if $P$ is a point on the circumcircle of the triangle $ABC$, then the reflection of the line $AP$ about the bisector of $\angle A$, the reflection of the line $BP$ about the bisector of $\angle B$ and the reflection of the line $CP$ about the bisector of $\angle C$ are all parallel. Thus they meet at an infinite point. In other words, if $P = (x : y : z)$ is a point on circumcircle of the triangle $ABC$, then its isogonal conjugate $(a^2yz : b^2zx : c^2xy)$ lies on the line at infinity. Thus $a^2yz + b^2zx + c^2xy = 0$, which is the equation of the circumcircle.

# 7 Worked Examples

1. [CentroAmerican 2017]. Let $ABC$ be a triangle and $D$ be the foot of the altitude from $A$. Let $l$ be the line that passes through the midpoints of $BC$ and $AC$. $E$ is the reflection of $D$ over $l$. Prove that the circumcentre of the triangle $ABC$ lies on the line $AE$.

   **Solution**.




It is known that $D = (0 : S_C : S_B)$, $\ell = [1 : 1 : -1]$. A displacement vector along $\ell$ is $(-1 : 1 : 0)$. A displacement vector perpendicular to $\ell$ is $(S_B : S_A : -c^2)$. Thus a parametric equation of the line $DE$ is $(tS_B : S_C + tS_A : S_B - tc^2)$. To find the intersection point $F$ between $\ell$ and $DE$, we substitute the parametric equation of $DE$ into the equation of $\ell$. Solving for $t$, we get $t = \frac{S_B-S_C}{S_A+S_B+c^2} = \frac{S_B-S_C}{2c^2} = \frac{c^2-b^2}{2c^2}$.

Therefore, $F = (\frac{c^2-b^2}{2c^2}S_B : S_C + \frac{c^2-b^2}{2c^2}S_A : S_B - \frac{c^2-b^2}{2}) = ((c^2-b^2)S_B : b^2c^2 + c^2S_C - b^2S_A : c^2(2S_B - c^2 + b^2)) = ((c^2-b^2)S_B : b^2S_B + c^2S_C : c^2S_B + c^2S_C)$.

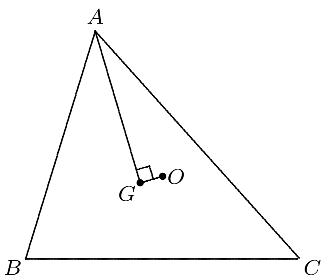
Here we use the relations $S_A + S_B = c^2$ etc. The normalized barycentric coordinates of $F$ and $D$ are

$$F = \frac{1}{2a^2c^2}((c^2-b^2)S_B : b^2S_B + c^2S_C : c^2S_B + c^2S_C), \;\; D = \frac{1}{a^2}(0 : S_C : S_B).$$

Thus we can compute $E$ as $2F - D$. That is $E = \frac{1}{a^2c^2}((c^2-b^2)S_B : b^2S_B + c^2S_C : c^2S_B + c^2S_C) - \frac{1}{a^2c^2}(0 : c^2S_C : c^2S_B) = ((c^2-b^2)S_B : b^2S_B : c^2S_C)$. Since $A = (1:0:0)$ and $O = (a^2S_A : b^2S_B : c^2S_C)$. Clearly the determinant formed by these 3 points is zero. Consequently, $A, O, E$ are collinear.

2. Let $O$ and $G$ be the circumcentre and the centroid of a triangle $ABC$. Prove that $GA$ is perpendicular to $OG$ if and only if $b^2 + c^2 = 2a^2$.

**Solution**.




We have $\boldsymbol{GA} = \frac{1}{3}(-2 : 1 : 1)$ and

$\boldsymbol{OG} = (\frac{1}{3} - \frac{a^2S_A}{a^2S_A+b^2S_B+c^2S_C}, \frac{1}{3} - \frac{b^2S_B}{a^2S_A+b^2S_B+c^2S_C}, \frac{1}{3} - \frac{c^2S_C}{a^2S_A+b^2S_B+c^2S_C}) = \frac{1}{6S^2}(-2a^2S_A + b^2S_B + c^2S_C : a^2S_A - 2b^2S_B + c^2S_C : a^2S_A + b^2S_B - 2c^2S_C)$.

Then $\boldsymbol{GA} \cdot \boldsymbol{OG} = -\frac{1}{36S^2}(a^2(2a^2S_A - b^2S_B - c^2S_C) + b^2(-4a^2S_A - b^2S_B + 5c^2S_C) + c^2(-4a^2S_A + 5b^2S_B - c^2S_C))$.

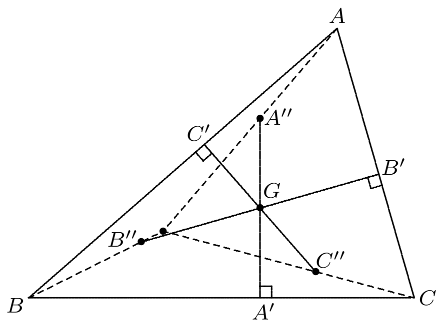
Therefore,

$$\begin{aligned}
&-36S^2\boldsymbol{GA}\cdot\boldsymbol{OG}\\
&= 2a^2(a^2-2b^2-2c^2)S_A - b^2(a^2+b^2-5c^2)S_B - c^2(a^2+c^2-5b^2)S_C\\
&= 2a^2(a^2-2b^2-2c^2)S_A - 2b^2S_BS_C + 4b^2c^2S_B - 2c^2S_BS_C + 4b^2c^2S_C\\
&= -2a^2(2b^2+2c^2-a^2)S_A - 2(b^2+c^2)S_BS_C + 4b^2c^2(S_B+S_C)\\
&= -2a^2(b^2+c^2-a^2)S_A - 2a^2(b^2+c^2)S_A - 2(b^2+c^2)S_BS_C + 4a^2b^2c^2\\
&= -4a^2S_A^2 - 2(b^2+c^2)(S_BS_C + a^2S_A) + 4a^2b^2c^2\\
&= 4a^2(b^2c^2 - S_A^2) - 2(b^2+c^2)(S_BS_C + a^2S_A)\\
&= 4a^2S^2 - 2(b^2+c^2)S^2\\
&= 2(2a^2-b^2-c^2)S^2.
\end{aligned}$$

That is $\boldsymbol{GA} \cdot \boldsymbol{OG} = -\frac{1}{18}(2a^2 - b^2 - c^2)$. Consequently, $\boldsymbol{GA} \cdot \boldsymbol{OG} = 0$ if and only if $2a^2 - b^2 - c^2 = 0$.

3. [Donova Mathematical Olympiad 2010]. Given a triangle $ABC$, let $A', B', C'$ be the perpendicular feet dropped from the centroid $G$ of the triangle $ABC$ onto the sides $BC, CA, AB$ respectively. Reflect $A', B', C'$ through $G$ to $A'', B'', C''$ respectively. Prove that the lines $AA'', BB'', CC''$ are concurrent.

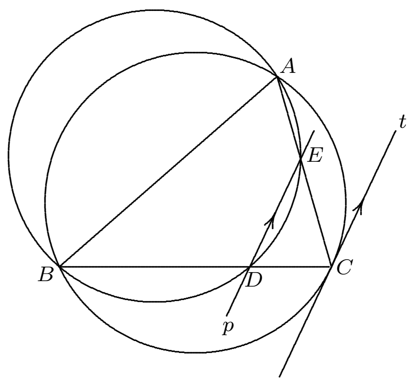
   **Solution**.




   Direct computation gives $A'' = (2a^2 : S_B : S_C), B'' = (S_A : 2b^2 : S_C), C'' = (S_A : S_B : 2c^2)$. Thus $AA'' = [0 : -S_C : S_B]$, $BB'' = [S_C : 0 : -S_A]$, $CC'' = [-S_B : S_A : 0]$. The determinant formed by these 3 lines is clearly zero. Thus $AA'', BB'', CC''$ are concurrent.

4. [JBMO Shortlist 2015]. Around the triangle $ABC$ the circle is circumscribed, and at the vertex $C$ tangent $t$ to this circle is drawn. The line $p$, which is parallel to this tangent intersects the lines $BC$ and $AC$ at the points $D$ and $E$, respectively. Prove that the points $A, B, D, E$ belong to the same circle.

   **Solution**.




   Let $D = (0 : 1-\alpha : \alpha)$. The tangent $t$ to the circumcircle of $ABC$ is $[b^2 : a^2 : 0]$. The points $(0 : 0 : 1)$ and $(-a^2 : b^2 : 0)$ lie on $t$. Therefore a displacement vector along $t$ is $(-\frac{a^2}{b^2-a^2} : \frac{b^2}{b^2-a^2} : -1) = (a^2 : -b^2 : a^2 - b^2)$. We can parametrize $p$ by $(0 : 1-\alpha : \alpha) + s(a^2 : -b^2 : a^2 - b^2) = (sa^2 : 1 - \alpha - sb^2 : \alpha + s(a^2 - b^2))$. Substituting this into the line $AC = [0 : 1 : 0]$, we have $1 - \alpha - sb^2 = 0$ so that $s = \frac{1-\alpha}{b^2}$. Thus

$$E = (\frac{(1-\alpha)a^2}{b^2} : 0 : \alpha + \frac{1-\alpha}{b^2}(a^2 - b^2)) = ((1-\alpha)a^2 : 0 : b^2 - a^2(1-\alpha)).$$

By substituting the coordinates of $A, B, D$ into the general equation of a circle, the equation of the circumcircle of $ABD$ is found to be

$$a^2yz + b^2zx + c^2xy - (x + y + z)a^2(1 - \alpha)z = 0.$$

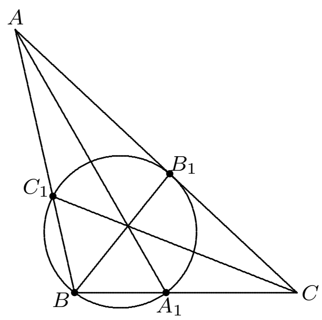
Check that $E$ satisfies this equation. Thus $A, B, D, E$ are concyclic.

**Remark**. The result follows easily from alternate segment theorem.

5. [Mongolia 2000]. The bisectors of $\angle A, \angle B, \angle C$ of a triangle $ABC$ intersect its sides at points $A_1, B_1, C_1$. Prove that $B, A_1, B_1, C_1$ are concyclic if and only if

$$\frac{BC}{AC + AB} = \frac{AC}{AB + BC} - \frac{AB}{BC + AC}.$$

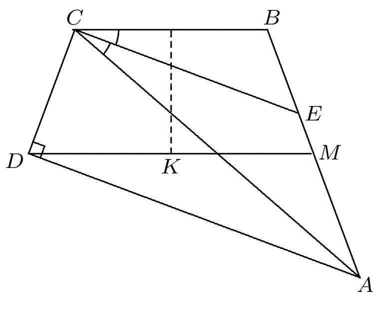
**Solution**.




First $A_1 = (0 : b : c), B_1 = (a : 0 : c), C_1 = (a : b : 0)$. Substituting the coordinates of these 3 points into the general equation of a circle, the equation of the circumcircle of $A_1B_1C_1$ is $a^2yz + b^2zx + c^2xy + (x + y + z)(px + qy + rz) = 0$, where $p = \frac{bc}{2}(\frac{a}{b+c} - \frac{b}{c+a} - \frac{c}{a+b})$, $q = \frac{ca}{2}(\frac{b}{c+a} - \frac{c}{a+b} - \frac{a}{b+c})$, and $r = \frac{ab}{2}(\frac{c}{a+b} - \frac{a}{b+c} - \frac{b}{c+a})$. Note that $B = (0 : 1 : 0)$ lies on this circle if and only if $q = 0$, which gives $\frac{a}{b+c} = \frac{b}{c+a} - \frac{c}{a+b}$.

6. [Benelux 2017]. In the convex quadrilateral $ABCD$ we have $\angle B = \angle C$ and $\angle D = 90°$. Suppose that $AB = 2CD$. Prove that the angle bisector of $\angle ACB$ is perpendicular to $CD$.

**Solution**.



Let $M$ be the midpoint of $AB$ and $CE$ the bisector of $\angle C$. Since $MBCD$ is an isosceles trapezium, $D$ is the reflection of $M$ across the perpendicular bisector of $BC$. First note that $\angle B = \angle C$ are obtuse angles. If $\angle B = \angle C < 90°$, then $\angle CDM > 90°$. Since $\angle D = 90°$, this contradicts the given condition that $ABCD$ is a convex quadrilateral.

Take $ABC$ be the reference triangle. Then $M = \frac{1}{2}(1 : 1 : 0)$, $E = (\frac{a}{a+b} : \frac{b}{a+b} : 0)$, and the perpendicular bisector of $BC$ is $[c^2 - b^2 : -a^2 : a^2]$. We can parametrize $MD$ as $(1 : 1 : 0) + t(0 : -1 : 1) = (1 : 1 - t : t)$. Substituting this into the equation of the perpendicular bisector of $BC$, we have $(c^2 - b^2) - a^2(1 - t) + a^2t = 0$. Solving for $t$, we get $t = \frac{a^2+b^2-c^2}{2a^2}$. Thus the midpoint of $DM$ is $K = \frac{1}{2}(1 : 1 - \frac{a^2+b^2-c^2}{2a^2} : \frac{a^2+b^2-c^2}{2a^2})$. Thus $D = 2K - M = (\frac{1}{2} : \frac{1}{2} - \frac{a^2+b^2-c^2}{2a^2} : \frac{a^2+b^2-c^2}{2a^2})$.

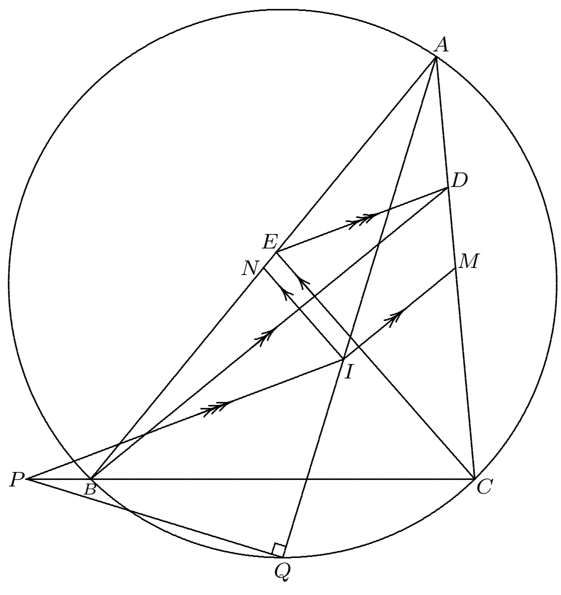
Then $\boldsymbol{DC} = (-\frac{1}{2} : -\frac{1}{2} + \frac{a^2+b^2-c^2}{2a^2} : 1 - \frac{a^2+b^2-c^2}{2a^2}) = (-a^2 : b^2 - c^2 : c^2 + a^2 - b^2)$. Similarly, $\boldsymbol{DA} = (a^2 : b^2 - c^2 : -a^2 - b^2 + c^2)$. Since $\angle ADC = 90°$, we have $0 = \boldsymbol{DA} \cdot \boldsymbol{DC} = 2a^2(ab + b^2 - c^2)(ab - b^2 + c^2)$.

Since $\angle B$ is obtuse, we have $b > c$ so that $ab + b^2 - c^2 \neq 0$. Therefore, we must have $ab - b^2 + c^2 = 0$ or $b^2 - c^2 = ab$. Thus $\boldsymbol{DA} = (a^2 : ab : -a^2 - ab) = (a : b : -a - b)$.

Note that $\boldsymbol{CE} = (\frac{a}{a+b} : \frac{b}{a+b} : -1) = (a : b : -a + b)$. Consequently, $DA$ is parallel to $CE$, or equivalently, $CE$ is perpendicular to $CD$.

7. [China 2010]. In a triangle $ABC$, $AB > AC$, $I$ is its incentre, $M$ is the midpoint of $AC$ and $N$ is the midpoint of $AB$. The line through $B$ parallel to $IM$ meets $AC$ at $D$, and the line through $C$ parallel to $IN$ meets $AB$ at $E$. The line through $I$ parallel to $DE$ meets the line $BC$ at $P$. If $Q$ is the foot of the perpendicular from $P$ onto the line $AI$, prove that $Q$ lies on the circumcircle of $ABC$.

**Solution**.




First $\boldsymbol{IM} = (\frac{1}{2} : 0 : \frac{1}{2}) - (\frac{a}{a+b+c} : \frac{b}{a+b+c} : \frac{c}{a+b+c}) = (b + c - a : -2b : a + b - c)$. We can parametrize $BD$ as $(0 : 1 : 0) + t(b + c - a : -2b : a + b - c)$. Substituting this into the line $AC = [0 : 1 : 0]$ and solving for $t$, we have $t = \frac{1}{2b}$. Thus $D = (\frac{b+c-a}{2b} : 0 : \frac{a+b-c}{2b})$. Similarly, $E = (\frac{b+c-a}{2c} : \frac{a-b+c}{2c} : 0)$. Therefore, $\boldsymbol{DE} = (\frac{b+c-a}{2c} - \frac{b+c-a}{2b} : \frac{a-b+c}{2c} : -\frac{a+b-c}{2b}) = ((b-c)(b+c-a) : b(c+a-b) : -c(a+b-c))$. We can parametrize $IP$ as $(a : b : c) + t((b-c)(b+c-a) : b(c+a-b) : -c(a+b-c))$. To find $P$, we substitute this into the line $BC = [1 : 0 : 0]$ and solve for $t$. We have $t = -\frac{a}{(b-c)(b+c-a)}$. Therefore, $P = (0 : b - \frac{ab(c+a-b)}{(b-c)(b+c-a)} : c + \frac{ac(a+b-c)}{(b-c)(b+c-a)}) = (0 : -b(c^2+a^2-b^2) : c(a^2+b^2-c^2)) = (0 : -bS_B : cS_C)$.

A displacement vector perpendicular to $AI$ is given by $(b-c : -b : c)$. Thus we can parametrize $PQ$ as $(0 : -bS_B : cS_C) + t(b-c : -b : c)$.

To find $Q$, we substitute this into the line $AI = [0 : -c : b]$ and solve for $t$. We have $t = -\frac{a^2}{2}$. Therefore, $Q = (-\frac{a^2(b-c)}{2} : -b(S_B - \frac{a^2}{2}) : c(S_C - \frac{a^2}{2})) = (-\frac{a^2(b-c)}{2} : -\frac{b(c^2-b^2)}{2} : \frac{c(b^2-c^2)}{2}) = (-a^2 : b(b+c) : c(b+c))$, which lies on the circumcircle. In fact, $Q$ is the midpoint of the arc not containing $A$.

## 8 Exercises

1. Prove that in any triangle $ABC$, the centroid $G$, the incentre $I$ and the Nagel point $N$ are collinear.

2. [Newton's line]. Let $ABCD$ be a quadrilateral. Let $H, I, G, J, E, F$ be the midpoints of $AB, BC, CD, DA, BD, CA$ respectively. Let $IJ$ intersect $HG$ at $M$, $AB$ intersect $CD$ at $U$, $BC$ intersect $AD$ at $V$. Let $N$ be the midpoint of $UV$. Prove that $E, F, M, N$ are collinear.

3. Prove that in any triangle the 3 lines each of which joins the midpoint of a side to the midpoint of the altitude to that side are concurrent.

4. In a triangle $ABC$, $\angle A = 90°$, the bisector of $\angle B$ meets the altitude $AD$ at the point $E$, and the bisector of $\angle CAD$ meets the side $CD$ at $F$. The line through $F$ perpendicular to $BC$ intersects $AC$ at $G$. Prove that $B, E, G$ are collinear.

5. In a triangle $ABC$, $M$ is the midpoint of $BC$ and $D$ is the point on $BC$ such that $AD$ bisects $\angle BAC$. The line through $B$ perpendicular to $AD$ intersects $AD$ at $E$ and $AM$ at $G$. Prove that $GD$ is parallel to $AB$.

6. In an acute-angled triangle $ABC$, $N$ is a point on the altitude $AM$. The line $CN$, $BN$ meet $AB$ and $AC$ respectively at $F$ and $E$. Prove that $\angle EMN = \angle FMN$.

7. [Pascal's theorem]. Let $A, F, B, D, C, E$ be six points on a circle in this order. Let $AF$ intersect $CD$ at $P$, $FB$ intersect $EC$ at $Q$ and $BD$ intersect $AE$ at $R$. Prove that $P, Q, R$ are collinear.

8. In a triangle $ABC$, $\angle A \neq 90°$, $M$ is the midpoint of $BC$ and $H$ is the orthocentre. The feet of the perpendiculars from $H$ onto the internal and external bisectors of $\angle BAC$ are $N$ and $L$ respectively. Prove that $M, N, L$ are collinear.

9. Let $ABC$ be an acute-angled triangle with orthocenter $H$. The circle with diameter $AH$ intersects the circumcircle of the triangle $ABC$ at the point $N$ distinct from $A$. Prove that the line $NH$ bisects the segment $BC$.

10. Let $ABC$ be an acute-angled triangle with incentre $I$. The circle with diameter $AI$ intersects the circumcircle of the triangle $ABC$ at the point $N$ distinct from $A$. Let the incircle of the triangle $ABC$ touch the side $BC$ at $D$. Prove that the line $ND$ bisects the arc $BC$ not containing $A$.

11. Let $ABC$ be a triangle with circumcentre $O$. Points $E, F$ lie on $CA, AB$ respectively. The line $EF$ cuts the circumcircles of $AEB$ and $AFC$ again at $M, N$ respectively. Prove that $OM = ON$.

12. [IMO 2017]. Let $R$ and $S$ be different points on a circle $\Omega$ such that $RS$ is not a diameter. Let $\ell$ be the tangent line to $\Omega$ at $R$. Point $T$ is such that $S$ is the midpoint of the line segment $RT$. Point $J$ is chosen on the shorter arc $RS$ of $\Omega$ so that the circumcircle $\Gamma$ of triangle $JST$ intersects $\ell$ at two distinct points. Let $A$ be the common point of $\Gamma$ and $\ell$ that is closer to $R$. Line $AJ$ meets $\Omega$ again at $K$. Prove that the line $KT$ is tangent to $\Gamma$.

13. [IMO 2016]. Triangle $BCF$ has a right angle at $B$. Let $A$ be the point on line $CF$ such that $FA = FB$ and $F$ lies between $A$ and $C$. Point $D$ is chosen so that $DA = DC$ and $AC$ is the bisector of $\angle DAB$. Point $E$ is chosen so that $EA = ED$ and $AD$ is the bisector of $\angle EAC$. Let $M$ be the midpoint of $CF$. Let $X$ be the point such that $AMXE$ is a parallelogram. Prove that $BD, FX$ and $ME$ are concurrent.

14. [IMO 2014]. Let $P$ and $Q$ be on segment $BC$ of an acute triangle $ABC$ such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let $M$ and $N$ be the points on $AP$ and $AQ$, respectively, such that $P$ is the midpoint of $AM$ and $Q$ is the midpoint of $AN$. Prove that the intersection of $BM$ and $CN$ is on the circumference of triangle $ABC$.

15. [IMO 2012]. Given triangle $ABC$ the point $J$ is the centre of the excircle opposite the vertex $A$. This excircle is tangent to the side $BC$ at $M$, and to the lines $AB$ and $AC$ at $K$ and $L$, respectively. The lines $LM$ and $BJ$ meet at $F$, and the lines $KM$ and $CJ$ meet at $G$. Let $S$ be the point of intersection of the lines $AF$ and $BC$, and let $T$ be the point of intersection of the lines $AG$ and $BC$. Prove that $M$ is the midpoint of $ST$.

16. [IMO 2010]. Given a triangle $ABC$, with $I$ as its incentre and $\Gamma$ as its circumcircle, $AI$ intersects $\Gamma$ again at $D$. Let $E$ be a point on the arc $BDC$, and $F$ a point on the segment $BC$, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If $G$ is the midpoint of $IF$, prove that the meeting point of the lines $EI$ and $DG$ lies on $\Gamma$.

17. [APMO 2017]. Let $ABC$ be a triangle with $AB < AC$. Let $D$ be the intersection point of the internal bisector of angle $BAC$ and the circumcircle of $ABC$. Let $Z$ be the intersection point of the perpendicular bisector of $AC$ with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment $AB$ lies on the circumcircle of triangle $ADZ$.

18. [IMO 2016 Shortlist]. Let $ABC$ be a triangle with $AB = AC \neq BC$ and let $I$ be its incentre. The line $BI$ meets $AC$ at $D$, and the line through $D$ perpendicular to $AC$ meets $AI$ at $E$. Prove that the reflection of $I$ in $AC$ lies on the circumcircle of triangle $BDE$.

19. [Nordic 2017]. Let $M$ and $N$ be the midpoints of the sides $AC$ and $AB$, respectively, of an acute triangle $ABC$, $AB \neq AC$. Let $\omega_B$ be the circle centered at $M$ passing through $B$, and let $\omega_C$ be the circle centered at $N$ passing through $C$. Let the point $D$ be such that $ABCD$ is an isosceles trapezoid with $AD$ parallel to $BC$. Assume that $\omega_B$ and $\omega_C$ intersect in two distinct points $P$ and $Q$. Show that $D$ lies on the line $PQ$.

20. [China 2017]. In the non-isosceles triangle $ABC$, $D$ is the midpoint of side $BC$, $E$ is the midpoint of side $CA$, $F$ is the midpoint of side $AB$. The line (different from line $BC$) that is tangent to the inscribed circle of triangle $ABC$ and passing through point $D$ intersect line $EF$ at $X$. Define $Y, Z$ similarly. Prove that $X, Y, Z$ are collinear.

# 9 Hints

1. Check that the determinant formed by the 3 points is 0.

2. Take $ABC$ be the reference triangle and $D = (u : v : 1 - u - v)$. Find the barycentric coordinates of $E, F, M, N$.

3. Let $A_1$ be the midpoint of the altitude from $A$ onto $BC$, and $A_2$ the midpoint of $BC$. Show that $A_1A_2 = [\tan C - \tan B : \tan B + \tan C : -\tan B - \tan C]$.

4. Following the condition of the question, show that $G = (a : 0 : c)$.

5. A displacement vector perpendicular to $\boldsymbol{AD}$ is $(-b + c : b : -c)$. Using this, show that $E = (b - c : b : c)$ and $G = (b - c : c : c)$.

6. Let $\ell$ be the line through $A$ parallel to $BC$. Let the extensions of $MF$ and $ME$ meet $\ell$ at $P$ and $Q$ respectively. Then $\ell = [0 : 1 : 1]$ and $M = (0 : S_C : S_B)$. Let $N = (a^2(1-t) : S_Ct : S_Bt)$, for some $t$. Find the barycentric coordinates of $P, Q$, and show that $A$ is the midpoint of $PQ$.

7. Take $ABC$ be the reference triangle. Let $D = (d_1 : d_2 : d_3), E = (e_1 : e_2 : e_3), F = (f_1 : f_2 : f_3)$. Then show that $P = (d_1f_2 : d_2f_2 : d_2f_3), Q = (e_1f_1 : e_2f_1 : e_1f_3), R = (d_1e_3 : d_3e_2 : d_3e_3)$. Check that the determinant formed by the 3 points is 0.

8. A displacement vector for $\boldsymbol{HL}(= \boldsymbol{NA})$ is $(b + c : -b : -c)$. Then parametrize $HL$ as $(S_BS_C : S_CS_A : S_AS_B) + t(b + c : -b : -c)$. Using this, show that $L = (2bcS_BS_C + c(b+c)S_CS_A + b(b+c)S_AS_B : -bS_A(bS_B - cS_C) : cS_A(bS_B - cS_C))$. Similarly, show that $N = (2bcS_BS_C + c(b-c)S_CS_A - b(b-c)S_AS_B : bS_A(bS_B + cS_C) : cS_A(bS_B + cS_C))$.

9. Let $M$ be the midpoint of $BC$. Take $N$ to be the point on the circumcircle of $ABC$ such that $\angle ANM = 90°$. Try to show $N, H, M$ are collinear. Let $N = (x : y : z)$ in normalized barycentric coordinates. The displacement vectors $\boldsymbol{AN} = (x - 1 : y : z)$ and $\boldsymbol{MN} = (x : y - \frac{1}{2} : z - \frac{1}{2})$ are perpendicular. Thus $a^2(y(z-\frac{1}{2}) + (y-\frac{1}{2})z) + b^2((x-1)(z-\frac{1}{2}) + zx) + c^2((x-1)(y-\frac{1}{2}) + xy) = 0$. Using $a^2yz + b^2zx + c^2xy = 0$, this can be simplified to $S_By + S_Cz = 0$. Take $y = S_C$, $z = -S_B$ and substitute this into $a^2yz + b^2zx + c^2xy = 0$ to get $x = \frac{a^2S_BS_C}{(b^2-c^2)S_A}$. Therefore $N = (\frac{a^2S_BS_C}{(b^2-c^2)S_A} : S_C : -S_B) = (a^2S_BS_C : (b^2 - c^2)S_CS_A : -(b^2 - c^2)S_AS_B)$.

10. Compute the barycentric coordinates of $N$. Show that $N = (a^2(s - b)(s - c) : b(b - c)(s - a)(s - c) : -c(b - c)(s - a)(s - b))$.

11. Let $E = (1 - t : 0 : t)$ and $F = (1 - s : s : 0)$. Show that the equation of the circumcircle of $AEB$ is $a^2yz + b^2zx + c^2xy - (x + y + z)(1 - t)b^2z = 0$. Parametrize the line $EF$ as $(1 - t : 0 : t) + \alpha(t - s : s : -t)$. Substituting this into the equation of the circumcircle of

$AEB$ to get $\alpha = \frac{a^2st+b^2t(t-s)+c^2s(1-t)}{a^2st+b^2t(t-s)-c^2s(t-s)}$. Denote this value of $\alpha$ by $\alpha_M$, and the corresponding barycentric coordinates of $M$ by $(x_M : y_M : z_M)$. The power of $M$ with respect to the circumcircle of $ABC$ is $-(a^2y_Mz_M + b^2z_Mx_M + c^2x_My_M) = -(x_M + y_M + z_M)(1 - t)b^2z_M = -(1-t)b^2z_M$. To compute the power, we only need the explicit value of $z_M$. Then show that $M$ and $N$ have the same power with respect to the circumcircle of $ABC$.

12. Let $RSJ$ be the reference triangle, where $R = (1:0:0), S = (0:1:0), J = (0:0:1)$. Thus $\Omega : a^2yz + b^2zx + c^2xy = 0$ and $\Gamma :\ a^2yz + b^2zx + c^2xy - 2c^2x(x+y+z) = 0$. The tangent $\ell$ at $R$ is $b^2z+c^2y = 0$. We may parametrize $\ell$ by $x = 1, y = -b^2t, z = c^2t$. Thus $A = (1 : -b^2t : c^2t)$, for some $t$. Since $A$ lies on $\Gamma$, we have $-a^2b^2c^2+b^2c^2t-c^2b^2t-2c^2(1-b^2t+c^2t) = 0$, or equivalently, $a^2b^2t^2+2(c^2-b^2)t+2 = 0$. Line $AJ = [b^2t : 1 : 0]$. The tangent to $\Gamma$ at $T$ is $a^2(2z)+b^2(-z)+c^2(-y+2x)-2c^2x+(x+y+z)(2c^2) = 0$; which can be simplified to $2c^2x+c^2y+(2a^2-b^2+2c^2)z = 0$. That is $[2c^2 : c^2 : (2a^2-b^2+2c^2)]$. Compute the barycentric coordinates of $K$ and show that it lies on $\Omega$ using the relation $a^2b^2t^2 + 2(c^2 - b^2)t + 2 = 0$.

13. Let $FBC$ be the reference triangle, where $F = (1:0:0), B = (0:1:0), C = (0:0:1)$. Note that $a^2 = b^2 - c^2$. Since $F$ divides $CA$ in the ratio $b : c$, we have $A = (1 + \frac{c}{b} : 0 : -\frac{c}{b})$. The midpoint of $AC$ is $N = (\frac{b+c}{2b} : 0 : \frac{b-c}{2b})$. Next obtain the displacement vector $(c^2 - b^2 : b^2 : -c^2)$ which is perpendicular to $\boldsymbol{CA}$. Then parametrize the perpendicular bisector of $CA$ by $x = \frac{b+c}{2b} + t(c^2 - b^2), y = tb^2, z = \frac{b-c}{2b} - tc^2$. This is the line $DN$, whose intersection with $AB$ is $D'$. Then $D' = (\frac{(2c-b)(b+c)}{2bc} : \frac{b}{2c} : \frac{b-2c}{2b})$. Hence $D = 2N - D' = (\frac{b+c}{2c} : \frac{-b}{2c} : \frac{1}{2})$. The midpoint of $AD$ is $(\frac{(b+c)(b+2c)}{4bc} : \frac{-b}{4c} : \frac{b-2c}{4b})$. A displacement vector perpendicular to $\boldsymbol{AD}$ is $((b+c)(b-2c) : -b^2 : bc+2c^2)$. Since $E$ lies on the perpendicular bisector of $AD$, we may take $E = (\frac{(b+c)(b+2c)}{4bc} + t(b+c)(b-2c) : \frac{-b}{4c} - tb^2 : \frac{b-2c}{4b} + t(bc+2c^2))$, for some $t$. As $DE$ is parallel to $CA$, the second coordinate of the displacement vector $\boldsymbol{DE}$ must be 0. Therefore, $\frac{-b}{4c} - tb^2 + \frac{b}{2c} = 0$ giving $t = \frac{1}{4bc}$. Consequently, $E = (\frac{b+2c}{2c} : \frac{-b}{2c} : 0)$. Lastly, $X = (\frac{bc-2c^2+b^2}{2bc} : \frac{-b}{2c} : \frac{b+2c}{2b})$. From this, we find that $BD = [c : 0 : -(b+c)]$, $FX = [0 : 2c^2+bc : b^2]$ and $ME = [b : b+2c : -b]$.

14. Note that the triangles $ABC, PBA, QAC$ are all similar. The point of intersection of the lines $BM$ and $CN$ is $(-a^2 : 2b^2 : 2c^2)$.

15. Show that $S = (0 : \frac{sa-(s-b)c}{a(s-c)} : -\frac{c}{a})$ and $T = (0 : -\frac{b}{a} : \frac{as-b(s-c)}{a(s-b})$.

16. Let $AE$ intersect $BC$ at $F'$. Then $F'$ is the isogonal conjugate of $F$. Let $F = (0 : 1-\alpha : \alpha)$. Then $F' = (0 : \alpha b^2 : (1-\alpha)c^2)$. Then show that $E = (\alpha(1-\alpha)a^2 : -\alpha b^2 : -(1-\alpha)c^2)$, $EI = [bc(\alpha b - (1-\alpha)c) : ca(1-\alpha)(c+\alpha a) : -ab\alpha((1-\alpha)a+b)]$ and $GD = [-(b+c)(\alpha b-(1-\alpha)c) : -a(c+\alpha a) : a((1-\alpha)a+b)]$. Then $X = (a(c+\alpha a)((1-\alpha)a+b) : -b(\alpha b-(1-\alpha)c)((1-\alpha)a+b) : c(c+\alpha a)(\alpha b-(1-\alpha)c)$; and show that it lies on $\Gamma$.

17. Let $M$ be the midpoint of $AB$. Then the circumcircle of $AMD$ has the equation $a^2yz+b^2zx+c^2xy - \frac{c^2}{2}(x+y+z)(y-\frac{b}{c}z) = 0$. Show that $Z$ lies on this circle.

18. Let $ABC$ be the reference triangle with $b = c \neq a$. Direct calculation gives $E = (a(-a^2-ab+4b^2) : 2b^3 : 2b^3)$ and $I' = (a(a+b) : -b^2 : 3b^2 - a^2)$. Also the equation of the circumcircle of $BDE$ is $a^2yz + b^2zx + c^2xy + \frac{b^2}{(a^2-b^2)(a+2b)}(x+y+z)(2b^3x - a^2(a+b)z) = 0$.

19. The reflection of $A$ about the perpendicular bisector of $BC$ is $D = (a^2 : c^2 - b^2 : b^2 - c^2)$. Let $L = \sqrt{2c^2 + 2a^2 - b^2}$ and $K = \sqrt{2a^2 + 2b^2 - c^2}$. Then the lengths of the medians from $B$ and $C$ are $L/2$ and $K/2$ respectively. The circle $\omega_B$ centred at the midpoint of $CA$ with radius $L/2$ passes through the points $B = (0 : 1 : 0)$, $(b + L : 0 : b - L)$ and $(b - L : 0 : b + L)$. Substituting these points into the general equation of a circle, we find that the equation of $\omega_B$ equal to $a^2yz + b^2zx + c^2xy + (x + y + z)(S_Bx + S_Bz) = 0$. Similarly, the equation of $\omega_C$ equal to $a^2yz + b^2zx + c^2xy + (x + y + z)(S_Cx + S_Cy) = 0$. Then show that $D$ lies on the radical axis $PQ$ of $\omega_B$ and $\omega_C$.

20. Let $A_1$ be the point of tangency of the incircle with the side $BC$, and let $A_1A_2$ be a diameter of the incircle. Let the other tangent from $D$ to the incircle meet the incircle at $A_3$. (That is $A_3 \neq A_1$). Thus $\angle A_1A_3A_2 = 90°$. The extension of $A_2A_3$ meets the side $BC$ at a point $A_4$ such that $D$ is the midpoint of $A_1A_4$. This means that $A_4$ is the point of tangency of the $A$-excircle with the side $BC$. That is $A_4 = (0 : s - b : s - c)$. Also it is well-known that $A, A_2, A_4$ are collinear. Thus $A, A_1, A_2, A_4$ are collinear. Use this information to find $A_3$. Show that $A_3 = (\frac{(b-c)^2}{(s-a)} : s - b : s - c)$. Similarly, $B_3 = (s - a : \frac{(c-a)^2}{s-b} : s - c)$ and $C_3 = (s - a : s - b : \frac{(a-b)^2}{s-c})$. From this, show that $X = (b - c : s - c : b - s)$, $Y = (c - s : c - a : s - a), Z = (s - b : a - s : a - b)$.

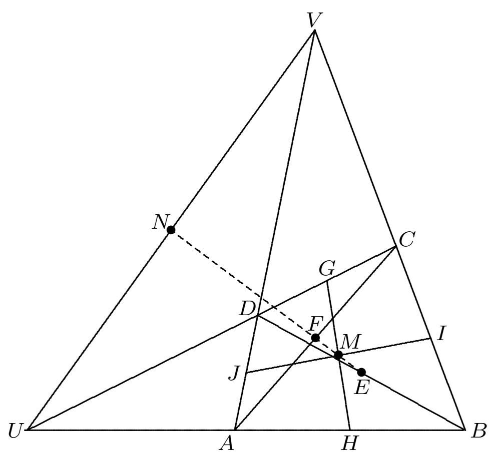
# 10 Solutions

1. Prove that in any triangle $ABC$, the centroid $G$, the incentre $I$ and the Nagel point $N$ are collinear.

   **Solution**. This is because $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ s-a & s-b & s-c \end{vmatrix} = 0$. In fact $G$ divides the segment $IN$ in the ratio 1:2.

2. [Newton's line]. Let $ABCD$ be a quadrilateral. Let $H, I, G, J, E, F$ be the midpoints of $AB, BC, CD, DA, BD, CA$ respectively. Let $IJ$ intersect $HG$ at $M$, $AB$ intersect $CD$ at $U$, $BC$ intersect $AD$ at $V$. Let $N$ be the midpoint of $UV$. Prove that $E, F, M, N$ are collinear.

   **Solution**.



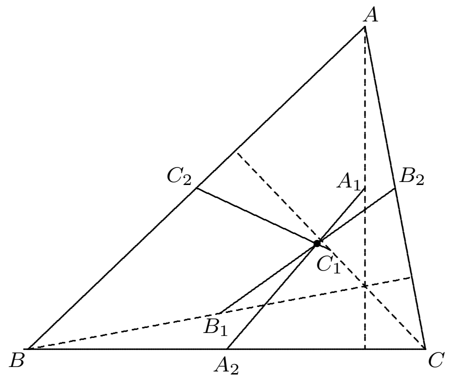

   Let $A = (1:0:0), B = (0:1:0), C = (0:0:1)$ and $D = (u:v:1-u-v)$. Then $H = (1:1:0), G = (u:v:2-u-v), I = (0:1:1), J = (1+u:v:1-u-v)$. Direct computation gives $F = (1:0:1)$, $E = (u:1+v:1-u-v)$ and $M = (u+1:v+1:2-u-v)$.

   The determinant $\begin{vmatrix} 1 & 0 & 1 \\ u & 1+v & 1-u-v \\ u+1 & v+1 & 2-u-v \end{vmatrix} = 0$, since the third row is the sum of the first two rows. Therefore, $E, F, M$ are collinear.

   Similarly, we can show that $N = (u-u^2 : v+v^2 : w-w^2) = (u(1-u) : v(1+v) : (u+v)(1-u-v))$, and the determinant $\begin{vmatrix} 1 & 0 & 1 \\ u & 1+v & 1-u-v \\ u(1-u) & v(1+v) & (u+v)(1-u-v) \end{vmatrix} = 0$. Therefore, $E, F, N$ are collinear.

3. Prove that in any triangle the 3 lines each of which joins the midpoint of a side to the midpoint of the altitude to that side are concurrent.

   **Solution**.

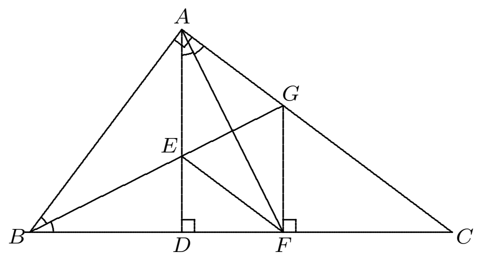



Take $A = (1:0:0), B = (0:1:0), C = (0:0:1)$. Let $A_1, B_1$ and $C_1$ be the midpoints of the altitudes from $A$ onto $BC$, from $B$ onto $CA$ and from $C$ onto $AB$ respectively. If $A_2, B_2$ and $C_2$ are the midpoints of the sides $BC$, $CA$ and $AB$ respectively, we find that $A_1A_2 = [\tan C - \tan B : \tan B + \tan C : -\tan B - \tan C]$, $B_1B_2 = [-\tan A - \tan C : \tan A - \tan C : \tan A + \tan C]$, and $C_3C_2 = [\tan A + \tan B : -\tan A - \tan B : \tan B - \tan A]$.

The determinant $\begin{vmatrix} \tan C - \tan B & \tan B + \tan C & -\tan B - \tan C \\ -\tan A - \tan C & \tan A - \tan C & \tan A + \tan C \\ \tan A + \tan B & -\tan A - \tan B & \tan B - \tan A \end{vmatrix} = 0$, since the sum of the 3 rows is the zero row. Therefore, $A_1A_2$, $B_1B_2$ and $C_1C_2$ are concurrent.

4. In a triangle $ABC$, $\angle A = 90°$, the bisector of $\angle B$ meets the altitude $AD$ at the point $E$, and the bisector of $\angle CAD$ meets the side $CD$ at $F$. The line through $F$ perpendicular to $BC$ intersects $AC$ at $G$. Prove that $B, E, G$ are collinear.

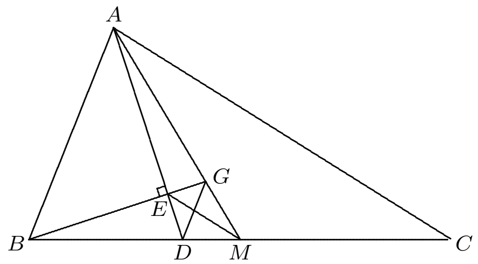
**Solution.**




Let $ABC$ be the reference triangle. Note that $a^2 = b^2 + c^2$. Using the angle bisector theorem, direct computation gives $D = (0 : b^2 : c^2)$, $E = (ca : b^2 : c^2)$, $F = (0 : b^2 : c(a+c))$ and $G = (a : 0 : c)$. As $\begin{vmatrix} 0 & 1 & 0 \\ ca & b^2 & c^2 \\ a & 0 & c \end{vmatrix} = ac^2 - ac^2 = 0$, the points $B, E, G$ are collinear.

5. In a triangle $ABC$, $M$ is the midpoint of $BC$ and $D$ is the point on $BC$ such that $AD$ bisects $\angle BAC$. The line through $B$ perpendicular to $AD$ intersects $AD$ at $E$ and $AM$ at $G$. Prove that $GD$ is parallel to $AB$.

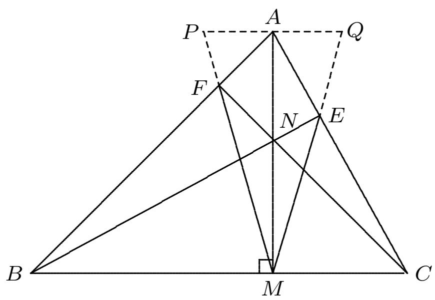
**Solution.**




Let $ABC$ be the reference triangle. Then $M = (0 : 1 : 1)$ and $D = (0 : b : c)$. The displacement vector $\boldsymbol{AD} = (-1 : \frac{b}{b+c} : \frac{c}{b+c}) = (b+c : -b : -c)$. A displacement vector perpendicular to $\boldsymbol{AD}$ is $(-b+c : b : -c)$. We may write $E = (0 : 1 : 0) + t(-b+c : b : -c) = ((-b+c)t : 1+bt : -ct)$ for some $t$. The equation of the line $AD$ is $-cy + bz = 0$. Substituting the coordinates of $E$ into the equation of the line $AD$ and solving for $t$, we obtain $t = -\frac{1}{2b}$. Therefore, $E = (\frac{b-c}{2b} : \frac{1}{2} : \frac{c}{2b}) = (b-c : b : c)$. From this we obtain: line $BE = [c : 0 : -b+c]$ and line $AM = [0 : -1 : 1]$. Thus $G = (b-c : c : c)$. Hence, $\boldsymbol{GD} = (-b+c : b-c : 0) = (-1 : 1 : 0) = \boldsymbol{AB}$ which means $GD$ is parallel to $AB$.

6. In an acute-angled triangle $ABC$, $N$ is a point on the altitude $AM$. The line $CN$, $BN$ meet $AB$ and $AC$ respectively at $F$ and $E$. Prove that $\angle EMN = \angle FMN$.

   **Solution**.




Let $\ell$ be the line through $A$ parallel to $BC$. Let the extensions of $MF$ and $ME$ meet $\ell$ at $P$ and $Q$ respectively. We have $\ell = [0 : 1 : 1]$ and $M = (0 : S_C : S_B)$. Let $N = (a^2(1-t) : S_C t : S_B t)$, for some $t$. Then $BN = [tS_B : 0 : -a^2(1-t)]$, $E = (a^2(1-t) : 0 : tS_B)$, $ME = [tS_BS_C : a^2(1-t)S_B : -a^2(1-t)S_C]$. Thus $Q = (-a^2(1-t)(S_B+S_C) : tS_BS_C : -tS_BS_C) = (2a^4(1-t) : -tS_BS_C : tS_BS_C)$.

Similarly, $CN = [-tS_C : a^2(1-t) : 0]$, $F = (a^2(1-t) : tS_C : 0)$, $MF = [tS_BS_C : -a^2(1-t)S_B : a^2(1-t)S_C]$, and $P = (2a^4(1-t) : tS_BS_C : -tS_BS_C)$. Consequently, the midpoint of $PQ$ is $A$. It follows that $\angle EMN = \angle FMN$.

7. [Pascal's theorem]. Let $A, F, B, D, C, E$ be six points on a circle in this order. Let $AF$ intersect $CD$ at $P$, $FB$ intersect $EC$ at $Q$ and $BD$ intersect $AE$ at $R$. Prove that $P, Q, R$ are collinear.

**Solution**. Let $ABC$ be the reference triangle and $\omega$ its circumcircle. Let $D = (d_1 : d_2 : d_3), E = (e_1 : e_2 : e_3), F = (f_1 : f_2 : f_3)$. Direct computations give $P = (d_1f_2 : d_2f_2 : d_2f_3), Q = (e_1f_1 : e_2f_1 : e_1f_3), R = (d_1e_3 : d_3e_2 : d_3e_3)$. Then $\begin{vmatrix} d_1f_2 & d_2f_2 & d_2f_3 \\ e_1f_1 & e_2f_1 & e_1f_3 \\ d_1e_3 & d_3e_2 & d_3e_3 \end{vmatrix} =$
$d_1d_3e_2e_3f_1f_2 + d_1d_2e_1e_3f_2f_3 + d_2d_3e_1e_2f_1f_3 - d_1d_2e_2e_3f_1f_3 - d_1d_3e_1e_2f_2f_3 - d_2d_3e_1e_3f_1f_2$
Using the relations $a^2d_2d_3 + b^2d_3d_1 + c^2d_1d_2 = 0, a^2e_2e_3 + b^2e_3e_1 + c^2e_1e_2 = 0, a^2f_2f_3 + b^2f_3f_1 + c^2f_1f_2 = 0$, we can eliminate the terms $d_2d_3, e_2e_3, f_2f_3$ and show that the above determinant has value 0. Thus $P, Q, R$ are collinear.

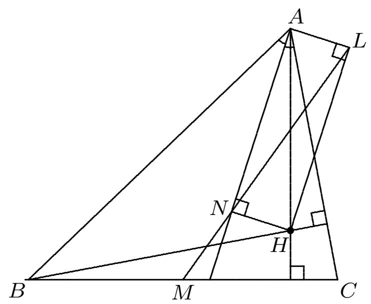
Alternatively, we may rewrite the expression of the above determinant as $\begin{vmatrix} d_2d_3 & d_3d_1 & d_1d_2 \\ f_2f_3 & f_3f_1 & f_1f_2 \\ e_2e_3 & e_3e_1 & e_1e_2 \end{vmatrix}$, which is equal to

$$\frac{1}{a^2}\begin{vmatrix} a^2d_2d_3 & d_3d_1 & d_1d_2 \\ a^2f_2f_3 & f_3f_1 & f_1f_2 \\ a^2e_2e_3 & e_3e_1 & e_1e_2 \end{vmatrix} = \frac{1}{a^2}\begin{vmatrix} -b^2d_3d_1 - c^2d_1d_2 & d_3d_1 & d_1d_2 \\ -b^2f_3f_1 - c^2f_1f_2 & f_3f_1 & f_1f_2 \\ -b^2e_3e_1 - c^2e_1e_2 & e_3e_1 & e_1e_2 \end{vmatrix} = 0,$$

as the first column is a linear combination of the last two columns.

8. In a triangle $ABC$, $\angle A \neq 90°$, $M$ is the midpoint of $BC$ and $H$ is the orthocentre. The feet of the perpendiculars from $H$ onto the internal and external bisectors of $\angle BAC$ are $N$ and $L$ respectively. Prove that $M, N, L$ are collinear.

**Solution**.




A displacement vector for $\boldsymbol{HL}(= \boldsymbol{NA})$ is $(b + c : -b : -c)$. We can parametrize $HL$ as $(S_BS_C : S_CS_A : S_AS_B) + t(b+c : -b : -c)$. The external bisector of $\angle A$ is $AL = [0 : c : b]$. Substituting the parametric equation of $HL$ into the equation of $AL$, we get $t = \frac{S_A(bS_B+cS_C)}{2bc}$. From this, we obtain

$L = (2bcS_BS_C + c(b + c)S_CS_A + b(b + c)S_AS_B : -bS_A(bS_B - cS_C) : cS_A(bS_B - cS_C))$.

Similarly, we get

$N = (2bcS_BS_C + c(b - c)S_CS_A - b(b - c)S_AS_B : bS_A(bS_B + cS_C) : cS_A(bS_B + cS_C))$.

We also know $M = (0 : 1 : 1)$.

To evaluate the following determinant, we add row 2 to row 3, and −row 3 to row 2.

$$\begin{vmatrix} 0 & 1 & 1 \\ 2bcS_BS_C + c(b+c)S_CS_A + b(b+c)S_AS_B & -bS_A(bS_B - cS_C) & cS_A(bS_B - cS_C) \\ 2bcS_BS_C + c(b-c)S_CS_A - b(b-c)S_AS_B & bS_A(bS_B + cS_C) & cS_A(bS_B + cS_C)) \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ c^2S_CS_A + b^2S_AS_B & -b^2S_AS_B & -c^2S_AS_C \\ 4bcS_BS_C + 2bcS_CS_A + 2bcS_AS_B & 2bcS_CS_A & 2bcS_AS_B \end{vmatrix}$$

$$= 2bcS_A \begin{vmatrix} 0 & 1 & 1 \\ c^2S_C + b^2S_B & -b^2S_B & -c^2S_C \\ 2S_BS_C + S_A(S_C + S_B) & S_CS_A & S_AS_B \end{vmatrix} \text{(add (−col 2 + col 3) to col 1)}$$

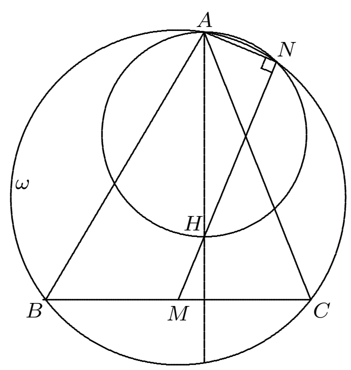
$$= 2bcS_A \begin{vmatrix} 0 & 1 & 1 \\ 2b^2S_B & -b^2S_B & -c^2S_C \\ 2S_BS_C + 2S_AS_B & S_CS_A & S_AS_B \end{vmatrix}$$

$$= 2bcS_A \begin{vmatrix} 0 & 1 & 1 \\ 2b^2S_B & -b^2S_B & -c^2S_C \\ 2b^2S_B & S_CS_A & S_AS_B \end{vmatrix} = 4b^3cS_AS_B \begin{vmatrix} 0 & 1 & 1 \\ 1 & -b^2S_B & -c^2S_C \\ 1 & S_CS_A & S_AS_B \end{vmatrix}$$

$= 4b^3cS_AS_B(-S_AS_B - c^2S_C + S_CS_A + b^2S_B)$
$= 4b^3cS_AS_B(S_A(S_C - S_B) - c^2S_C + b^2S_B)$
$= 4b^3cS_AS_B((b^2 - c^2)S_A - c^2S_C + b^2S_B)$
$= 4b^3cS_AS_B(b^2(S_A + S_B) - c^2(S_C + S_A))$
$= 4b^3cS_AS_B(b^2c^2 - c^2b^2) = 0$. Thus $M, N, L$ are collinear.

9. Let $ABC$ be an acute-angled triangle with orthocenter $H$. The circle with diameter $AH$ intersects the circumcircle of the triangle $ABC$ at the point $N$ distinct from $A$. Prove that the line $NH$ bisects the segment $BC$.

**Solution.**




Let $ABC$ be the reference triangle and $M = (0 : \frac{1}{2} : \frac{1}{2})$ the midpoint of $BC$. Take $N$ to be the point on the circumcircle of $ABC$ such that $\angle ANM = 90°$. Then we show $N, H, M$ are collinear.

Let $N = (x : y : z)$ in normalized barycentric coordinates. The displacement vectors $\boldsymbol{AN} = (x-1 : y : z)$ and $\boldsymbol{MN} = (x : y - \frac{1}{2} : z - \frac{1}{2})$ are perpendicular. Thus

$$a^2(y(z-\frac{1}{2}) + (y-\frac{1}{2})z) + b^2((x-1)(z-\frac{1}{2}) + zx) + c^2((x-1)(y-\frac{1}{2}) + xy) = 0.$$

Using $a^2yz + b^2zx + c^2xy = 0$, this can be simplified to $(a^2-b^2+c^2)y + (a^2+b^2-c^2)z = 0$. Let $S_A = \frac{1}{2}(b^2+c^2-a^2)$, etc. We may write this as $S_By + S_Cz = 0$. Take $y = S_C$ and $z = -S_B$. Then $S_By + S_Cz = 0$. We substitute this into $a^2yz + b^2zx + c^2xy = 0$ to find $x$.

That is $0 = -a^2S_BS_C - b^2S_Bx + c^2xS_C = -a^2S_BS_C - x(b^2S_B - c^2S_C) = -a^2S_BS_C - x(c^2-b^2)S_A$. Thus $x = \frac{a^2S_BS_C}{(b^2-c^2)S_A}$. Therefore

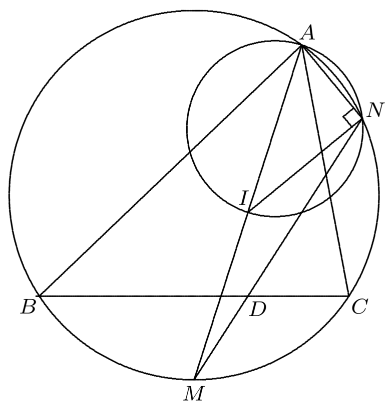
$$N = (\frac{a^2S_BS_C}{(b^2-c^2)S_A} : S_C : -S_B) = (a^2S_BS_C : (b^2-c^2)S_CS_A : -(b^2-c^2)S_AS_B).$$

Since $H = (S_BS_C : S_CS_A : S_AS_B)$, we check that

$$\begin{vmatrix} 0 & 1 & 1 \\ a^2S_BS_C & (b^2-c^2)S_CS_A & -(b^2-c^2)S_AS_B \\ S_BS_C & S_CS_A & S_AS_B \end{vmatrix} = 0.$$

Thus $N, H, M$ are collinear.

10. Let $ABC$ be an acute-angled triangle with incentre $I$. The circle with diameter $AI$ intersects the circumcircle of the triangle $ABC$ at the point $N$ distinct from $A$. Let the incircle of the triangle $ABC$ touch the side $BC$ at $D$. Prove that the line $ND$ bisects the arc $BC$ not containing $A$.




**Solution**. Let $ABC$ be the reference triangle. Let $N = (u : v : w)$ with $u + v + w = 1$. As $N$ lies on the circumcircle, we have $a^2vw + b^2wu + c^2uv = 0$. Then $\boldsymbol{AN} = (u-1 : v : w)$ and $\boldsymbol{IN} = (u - \frac{a}{2s} : v - \frac{b}{2s} : w - \frac{c}{2s})$. Since $\angle ANI = 90°$, we have

$$0 = a^2(v(w-\frac{c}{2s})+(v-\frac{b}{2s})w)+b^2((u-1)(w-\frac{c}{2s})+(u-\frac{a}{2s})w)+c^2((u-1)(v-\frac{b}{2s})+(u-\frac{a}{2s})v).$$

Using the relations $a^2vw + b^2wu + c^2uv = 0$ and $u + v + w = 1$, this can be simplified to $c(s-b)v+b(s-c)w = 0$. Thus we may take $N = (u' : b(s-c) : -c(s-b))$. Substituting this into the equation of the circumcircle, we get $-a^2bc(s-b)(s-c)-b^2c(s-b)u'+c^2b(s-c)u' = 0$. Solving for $u'$, we get $u' = \frac{a^2(s-b)(s-c)}{(b-c)(s-a)}$.

That is

$$N = (a^2(s-b)(s-c) : b(b-c)(s-a)(s-c) : -c(b-c)(s-a)(s-b)).$$

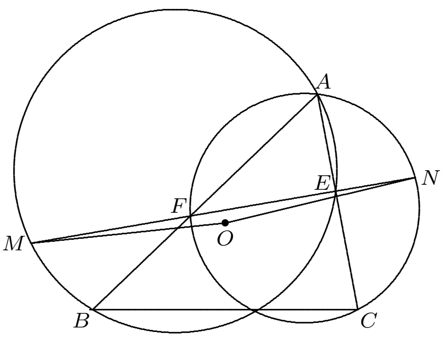
Let $M$ be the midpoint of the arc $BC$ not containing $A$. We know that $D = (0 : s-c : s-b)$ and $M = (-a^2 : b(b+c) : c(b+c))$. We check that

$$\begin{vmatrix} a^2(s-b)(s-c) & b(b-c)(s-a)(s-c) & -c(b-c)(s-a)(s-b) \\ 0 & s-c & s-b \\ -a^2 & b(b+c) & c(b+c) \end{vmatrix} = 0.$$

Thus $N, D, M$ are collinear.

11. Let $ABC$ be a triangle with circumcentre $O$. Points $E, F$ lie on $CA, AB$ respectively. The line $EF$ cuts the circumcircles of $AEB$ and $AFC$ again at $M, N$ respectively. Prove that $OM = ON$.

**Solution**.




Let $E = (1-t : 0 : t)$ and $F = (1-s : s : 0)$. Substituting the coordinates of the points $A, E, B$ into the general equation of a circle, the equation of the circumcircle of $AEB$ is found to be

$$a^2yz + b^2zx + c^2xy - (x+y+z)(1-t)b^2z = 0.$$

Similarly, the equation of the circumcircle of $AFC$ is

$$a^2yz + b^2zx + c^2xy - (x+y+z)(1-s)c^2y = 0.$$

The displacement vector $\boldsymbol{EF} = (t-s : s : -t)$. Thus the line $EF$ can be parametrized as the normalized barycentric coordinates: $(1-t : 0 : t) + \alpha(t-s : s : -t) = ((1-t) + \alpha(t-s) :$

$\alpha s : t - \alpha t)$. To find $M$, we substitute this into the equation of the circumcircle of $AEB$. We get $\alpha = \frac{a^2st+b^2t(t-s)+c^2s(1-t)}{a^2st+b^2t(t-s)-c^2s(t-s)}$. Let's denote this value of $\alpha$ by $\alpha_M$, and the corresponding barycentric coordinates of $M$ by $(x_M : y_M : z_M)$.

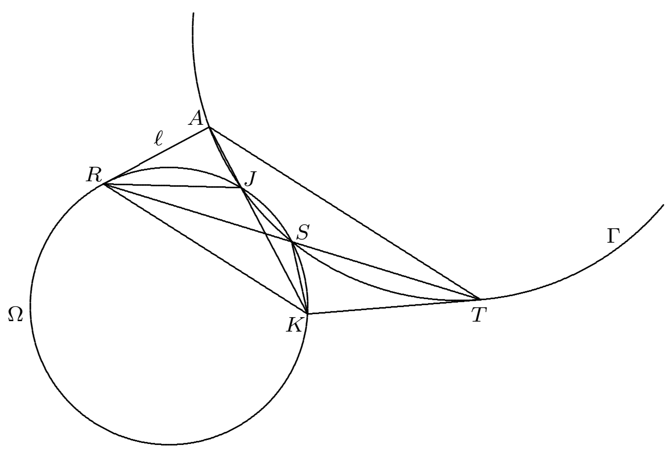
The power of $M$ with respect to the circumcircle of $ABC$ is $-(a^2y_Mz_M + b^2z_Mx_M + c^2x_My_M)$. Since $M$ lies on the circumcircle of $AEB$, we have $-(a^2y_Mz_M + b^2z_Mx_M + c^2x_My_M) = -(x_M + y_M + z_M)(1-t)b^2z_M = -(1-t)b^2z_M$.

To compute the power, we only need the explicit value of $z_M$. Here $z_M = t(1-\alpha_M) = \frac{-c^2ts(1-s)}{a^2st+b^2t(t-s)-c^2s(t-s)}$. Therefore the power of $M$ with respect to the circumcircle of $ABC$ is $\frac{b^2c^2ts(1-t)(1-s)}{a^2st+b^2t(t-s)-c^2s(t-s)}$.

Similarly, we can parametrize the line $EF$ as $(1-s : s : 0) + \beta(t-s : s : -t) = ((1-s) + \beta(t-s) : s + \beta s : -\beta t)$. Using this, we find that $\beta_N = \frac{-a^2st-b^2t(1-s)+c^2s(t-s)}{a^2st+b^2t(t-s)-c^2s(t-s)}$, which is the value of the parameter $\beta$ corresponding to the point $N = (x_N : y_N : z_N)$. Also $y_N = s(1+\beta_N) = \frac{-b^2ts(1-t)}{a^2st+b^2t(t-s)-c^2s(t-s)}$. Thus, the power of $N$ with respect to the circumcircle of $ABC$ is $-(1-s)c^2y_N = \frac{b^2c^2ts(1-t)(1-s)}{a^2st+b^2t(t-s)-c^2s(t-s)}$. Since $M$ and $N$ have the same power with respect to the circumcircle of $ABC$, this implies $OM = ON$.

12. [IMO 2017]. Let $R$ and $S$ be different points on a circle $\Omega$ such that $RS$ is not a diameter. Let $\ell$ be the tangent line to $\Omega$ at $R$. Point $T$ is such that $S$ is the midpoint of the line segment $RT$. Point $J$ is chosen on the shorter arc $RS$ of $\Omega$ so that the circumcircle $\Gamma$ of triangle $JST$ intersects $\ell$ at two distinct points. Let $A$ be the common point of $\Gamma$ and $\ell$ that is closer to $R$. Line $AJ$ meets $\Omega$ again at $K$. Prove that the line $KT$ is tangent to $\Gamma$.

**Solution**.




Let $RSJ$ be the reference triangle, where $R = (1:0:0), S = (0:1:0), J = (0:0:1)$. Thus $\Omega : a^2yz + b^2zx + c^2xy = 0$. Since $S$ is the midpoint of $RT$, we have $T = (-1:2:0)$.

Substituting the coordinates of the 3 points $S, J, T$ into the general equation of a circle, we obtain the equation of $\Gamma$.

$$\Gamma: \ a^2yz + b^2zx + c^2xy - 2c^2x(x+y+z) = 0.$$

The tangent $\ell$ at $R$ is $b^2z + c^2y = 0$. We may parametrize $\ell$ by $x = 1, y = -b^2t, z = c^2t$. Thus $A = (1 : -b^2t : c^2t)$, for some $t$.

Since $A$ lies on $\Gamma$, we have $-a^2b^2c^2 + b^2c^2t - c^2b^2t - 2c^2(1 - b^2t + c^2t) = 0$, or equivalently,

$$a^2b^2t^2 + 2(c^2 - b^2)t + 2 = 0. \tag{1.1}$$

Line $AJ = [b^2t : 1 : 0]$.

The tangent to $\Gamma$ at $T$ is $a^2(2z)+b^2(-z)+c^2(-y+2x)-2c^2x+(x+y+z)(2c^2) = 0$; which can be simplified to $2c^2x + c^2y + (2a^2 - b^2 + 2c^2)z = 0$. That is $[2c^2 : c^2 : (2a^2 - b^2 + 2c^2)]$.

Thus $K = \begin{vmatrix} b^2t & 1 & 0 \\ 2c^2 & c^2 & 2a^2 - b^2 + 2c^2 \end{vmatrix} = (2a^2-b^2+2c^2 : -tb^2(2a^2-b^2+2c^2) : b^2c^2t-2c^2)$

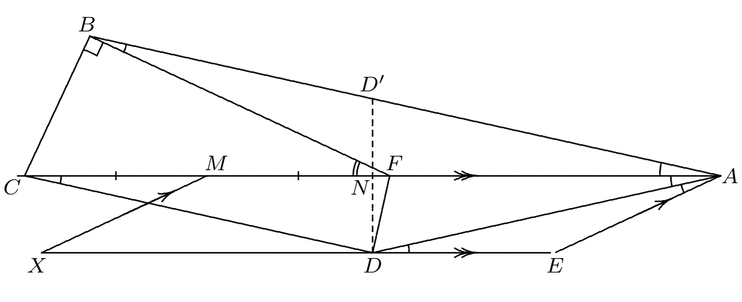
Substituting the coordinates of $K$ into the equation of $\Omega$, we get

$$\begin{aligned} &-a^2tb^2(2a^2-b^2+2c^2)(b^2c^2t-2c^2)+b^2(2a^2-b^2+2c^2)(b^2c^2t-2c^2)-tc^2b^2(2a^2-b^2+2c^2)^2 \\ &= b^2(2a^2 - b^2 + 2c^2)(-a^2t(b^2c^2t - 2c^2) + (b^2c^2t - 2c^2) - tc^2(2a^2 - b^2 + 2c^2)) \\ &= b^2(2a^2 - b^2 + 2c^2)(-a^2b^2c^2t^2 - 2c^2(c^2 - b^2)t - 2c^2) \\ &= -b^2c^2(2a^2 - b^2 + 2c^2)(a^2b^2t^2 + 2(c^2 - b^2)t + 2) = 0, \quad \text{by (1.1).} \end{aligned}$$

Thus $K$ lies on $\Omega$. This also shows that if $A'$ is the other intersection point between $\ell$ and $\Gamma$, and $A'J$ meets $\Omega$ at $K'$, then $K'T$ is also tangent to $\Gamma$ and $K', K, T$ are collinear.

13. [IMO 2016]. Triangle $BCF$ has a right angle at $B$. Let $A$ be the point on line $CF$ such that $FA = FB$ and $F$ lies between $A$ and $C$. Point $D$ is chosen so that $DA = DC$ and $AC$ is the bisector of $\angle DAB$. Point $E$ is chosen so that $EA = ED$ and $AD$ is the bisector of $\angle EAC$. Let $M$ be the midpoint of $CF$. Let $X$ be the point such that $AMXE$ is a parallelogram. Prove that $BD, FX$ and $ME$ are concurrent.

    **Solution**. Let $FBC$ be the reference triangle, where $F = (1 : 0 : 0), B = (0 : 1 : 0), C = (0 : 0 : 1)$. Note that $a^2 = b^2 - c^2$. Since $F$ divides $CA$ in the ratio $b : c$, we have $A = (1 + \frac{c}{b} : 0 : -\frac{c}{b})$.




The midpoint of $AC$ is $N = (\frac{b+c}{2b} : 0 : \frac{b-c}{2b})$.

The equation of the line $AB$ is $cx + (b + c)z = 0$, or $[c : 0 : b + c]$.

The displacement vector $\boldsymbol{CA} = (\frac{b+c}{b} : 0 : \frac{-b-c}{b}) = (b+c : 0 : -b-c)$.

A displacement vector perpendicular to $\boldsymbol{CA}$ is given by
$(-a^2(b+c) + (c^2-b^2)(b+c) : 2b^2(b+c) : -c^2(b+c) - (b^2-a^2)(b+c))$.
Dividing by a factor of $(b+c)$, we may take this displacement vector to be
$(-a^2+c^2-b^2 : 2b^2 : -c^2-b^2+a^2) = (2(c^2-b^2) : 2b^2 : -2c^2)$, by the relation $a^2 = b^2-c^2$.
Therefore, the displacement vector $(c^2-b^2 : b^2 : -c^2)$ is perpendicular to $\boldsymbol{CA}$.
We can parametrize the perpendicular bisector of $CA$ by $x = \frac{b+c}{2b} + t(c^2-b^2), y = tb^2, z = \frac{b-c}{2b} - tc^2$. This is the line $DN$, whose intersection with $AB$ is $D'$. To find $D'$, we substitute this parametric equation of $DN$ into the equation of $AB$. Thus

$$c(\frac{b+c}{2b} + t(c^2-b^2)) + (b+c)(\frac{b-c}{2b} - tc^2) = 0.$$

Solving for $t$, we have $t = \frac{1}{2bc}$.
Thus $D' = (\frac{b+c}{2b} + \frac{1}{2bc}(c^2-b^2) : \frac{1}{2bc}b^2 : \frac{b-c}{2b} - \frac{1}{2bc}c^2) = (\frac{(2c-b)(b+c)}{2bc} : \frac{b}{2c} : \frac{b-2c}{2b})$.
Hence $D = 2N - D' = (\frac{b+c}{2c} : \frac{-b}{2c} : \frac{1}{2})$. The midpoint of $AD$ is $(\frac{(b+c)(b+2c)}{4bc} : \frac{-b}{4c} : \frac{b-2c}{4b})$.
A displacement vector perpendicular to $\boldsymbol{AD}$ is $((b+c)(b-2c) : -b^2 : bc+2c^2)$.
Since $E$ lies on the perpendicular bisector of $AD$, we may take

$$E = (\frac{(b+c)(b+2c)}{4bc} + t(b+c)(b-2c) : \frac{-b}{4c} - tb^2 : \frac{b-2c}{4b} + t(bc+2c^2)),$$

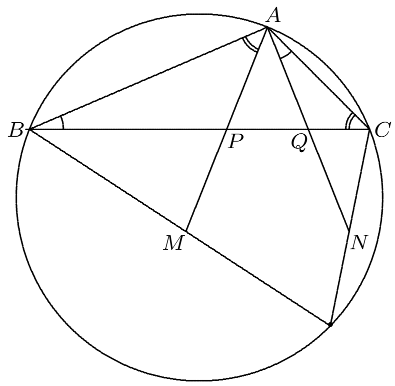
for some $t$.
As the $DE$ is parallel to $CA$, the second coordinate of the displacement vector $\boldsymbol{DE}$ must be 0. Therefore, $\frac{-b}{4c} - tb^2 + \frac{b}{2c} = 0$ giving $t = \frac{1}{4bc}$. Consequently, $E = (\frac{b+2c}{2c} : \frac{-b}{2c} : 0)$. [Since the last coordinate of $E$ is 0, $E$ in fact lies on the line $BF$.]
Since $AMXE$ is a parallelogram, $M - A = X - E$. Thus $X = (\frac{bc-2c^2+b^2}{2bc} : \frac{-b}{2c} : \frac{b+2c}{2b})$.
From this, we find that

$BD = [c : 0 : -(b+c)]$, $FX = [0 : 2c^2+bc : b^2]$ and $ME = [b : b+2c : -b]$.

The determinant $\begin{vmatrix} c & 0 & -(b+c) \\ 0 & 2c^2+bc & b^2 \\ b & b+2c & -b \end{vmatrix}$ can be checked easily equal to 0. Consequently, $BD, FX$ and $ME$ are concurrent.

14. [IMO 2014]. Let $P$ and $Q$ be on segment $BC$ of an acute triangle $ABC$ such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let $M$ and $N$ be the points on $AP$ and $AQ$, respectively, such that $P$ is the midpoint of $AM$ and $Q$ is the midpoint of $AN$. Prove that the intersection of $BM$ and $CN$ is on the circumference of triangle $ABC$.

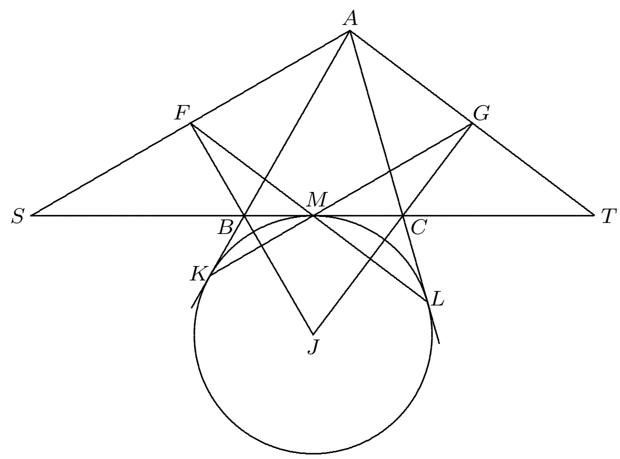
**Solution**.




First, the triangles $ABC, PBA, QAC$ are all similar. Thus $PB = c^2/a$. Then we have $P = (0 : a^2 - c^2 : c^2)$, so $M = (-a^2 : 2(a^2 - c^2) : 2c^2)$. Similarly $N = (-a^2 : 2b^2 : 2(a^2 - c^2))$. The lines $BM$ and $CN$ have equations $2c^2x + a^2z = 0$ and $2b^2x + a^2y = 0$ respectively. Thus the point of intersection of the lines $BM$ and $CN$ is $(-a^2 : 2b^2 : 2c^2)$ which clearly lies on the circumcircle.

15. [IMO 2012]. Given triangle $ABC$ the point $J$ is the centre of the excircle opposite the vertex $A$. This excircle is tangent to the side $BC$ at $M$, and to the lines $AB$ and $AC$ at $K$ and $L$, respectively. The lines $LM$ and $BJ$ meet at $F$, and the lines $KM$ and $CJ$ meet at $G$. Let $S$ be the point of intersection of the lines $AF$ and $BC$, and let $T$ be the point of intersection of the lines $AG$ and $BC$. Prove that $M$ is the midpoint of $ST$.

**Solution**.




Let $ABC$ be the reference triangle. We have $J = (-a : b : c), K = (s - c : -s : 0), L = (s - b : 0 : -s), M = (0 : s - b : s - c) = (0 : \frac{s-b}{2a} : \frac{s-c}{2a})$.

Thus $BJ = [c : 0 : a]$, and $ML = \begin{vmatrix} 0 & s-b & s-c \\ s-b & 0 & -s \end{vmatrix} = [-s(s-b) : (s-b)(s-c) : -(s-b)^2] = [-s : s-c : -(s-b)]$.

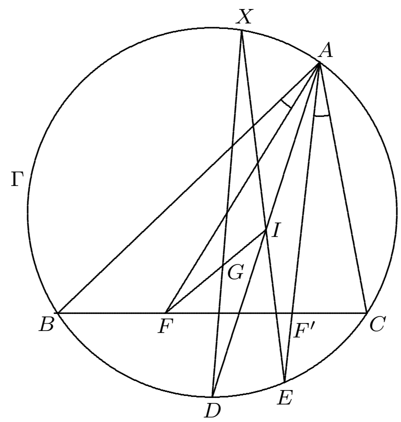
Then $F = \begin{vmatrix} c & 0 & sa \\ -s & s-c & -(s-b) \end{vmatrix} = (-a(s-c) : s(c-a) - bc : c(s-c))$. From this, we get $AF = [0 : -c(s-c) : s(c-a) - bc]$ and $S = (0 : sa - (s-b)c : -c(s-c))$.

Similarly, by switching the roles of $b$ and $c$ and interchanging the positions of the second and third coordinates, we have $T = (0 : -b(s-b) : sa - (s-c)b)$.

Normalizing, we have $S = (0 : \frac{sa-(s-b)c}{a(s-c)} : -\frac{c}{a})$, and $T = (0 : -\frac{b}{a} : \frac{as-b(s-c)}{a(s-b})$. It can be checked that $(S+T)/2 = M$ so that $M$ is the midpoint of $ST$.

16. [IMO 2010]. Given a triangle $ABC$, with $I$ as its incentre and $\Gamma$ as its circumcircle, $AI$ intersects $\Gamma$ again at $D$. Let $E$ be a point on the arc $BDC$, and $F$ a point on the segment $BC$, such that $\angle BAF = \angle CAE < \dfrac{1}{2}\angle BAC$. If $G$ is the midpoint of $IF$, prove that the meeting point of the lines $EI$ and $DG$ lies on $\Gamma$.

**Solution**.




Let $AE$ intersect $BC$ at $F'$. Then $F'$ is the isogonal conjugate of $F$. Let $F = (0 : 1-\alpha : \alpha)$. Then $F' = (0 : \alpha b^2 : (1-\alpha)c^2)$. The line $AF'$ can be parametrized as $(1-t : \frac{\alpha b^2 t}{\alpha b^2+(1-\alpha)c^2} : \frac{(1-\alpha)c^2 t}{\alpha b^2+(1-\alpha)c^2})$. Substituting into the equation of the circumcircle and solving for $t$, we have $t = 0$, or $t = \frac{\alpha b^2+(1-\alpha)c^2}{-\alpha(1-\alpha)a^2+\alpha b^2+(1-\alpha)c^2}$. Here $t = 0$ corresponds to the point $A$. The other value of $t$ corresponds to $E$. From this, we find that

$$E = (\alpha(1-\alpha)a^2 : -\alpha b^2 : -(1-\alpha)c^2).$$

It is well-known that $D = (-a^2 : b(b+c) : c(b+c))$ and that $I = (a : b : c)$.

Then

$$EI = [bc(\alpha b - (1-\alpha)c) : ca(1-\alpha)(c+\alpha a) : -ab\alpha((1-\alpha)a+b)].$$

As $G$ is the midpoint of $IF$, we have $G = (a : b + (1-\alpha)(a+b+c) : c + \alpha(a+b+c))$.

Thus

$$GD = [-(b+c)(\alpha b - (1-\alpha)c) : -a(c+\alpha a) : a((1-\alpha)a+b)].$$

Let $X$ be the intersection of $EI$ and $DG$. We have

$$X = a(c+\alpha a)((1-\alpha)a+b) : -b(\alpha b-(1-\alpha)c)((1-\alpha)a+b) : c(c+\alpha a)(\alpha b-(1-\alpha)c).$$

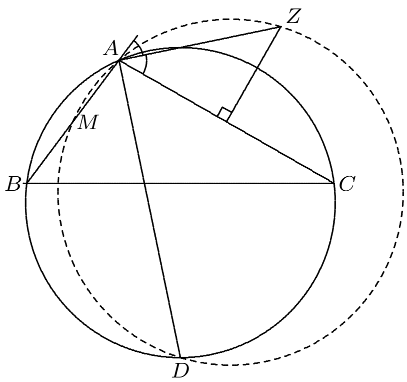
We may write it as

$$X = (apq : -bqr : crp),\ \text{where } p = c + \alpha a, q = (1-\alpha)a + b, r = \alpha b - (1-\alpha)c.$$

Substituting $X$ into the equation of the circumcircle, we have
$a^2(-bcpqr^2) + b^2(cap^2qr) + c^2(-abpq^2r) = -abcpqr(ar - bp + cq) = 0$. Therefore, $X$ lies on the circumcircle.

17. [APMO 2017]. Let $ABC$ be a triangle with $AB < AC$. Let $D$ be the intersection point of the internal bisector of angle $BAC$ and the circumcircle of $ABC$. Let $Z$ be the intersection point of the perpendicular bisector of $AC$ with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment $AB$ lies on the circumcircle of triangle $ADZ$.

**Solution**.



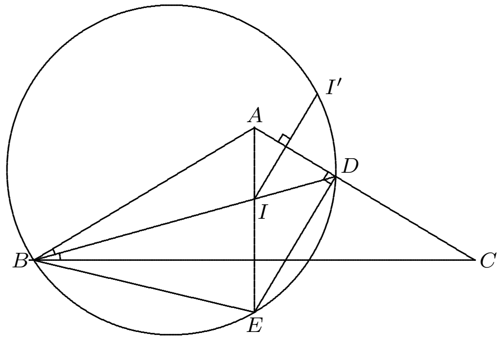

Let $ABC$ be the reference triangle. We have $A = (1:0:0), M = (1:1:0)$ and $D = (-a^2 : b(b+c) : c(b+c))$. Substituting these points into the general equation of a circle, we find that the circumcircle of $AMD$ has the equation

$$a^2yz + b^2zx + c^2xy - \frac{c^2}{2}(x+y+z)(y - \frac{b}{c}z) = 0.$$

The perpendicular bisector of $CA$ is $[b^2 : a^2 - c^2 : -b^2]$ and the line $AZ$ is $[0 : c : b]$. From this, we obtain the intersection $Z = (bc + a^2 - c^2 : -b^2 : bc)$. It can be verified directly that $Z$ satisfies the equation of the circumcircle of $AMD$.

18. [IMO 2016 Shortlist]. Let $ABC$ be a triangle with $AB = AC \neq BC$ and let $I$ be its incentre. The line $BI$ meets $AC$ at $D$, and the line through $D$ perpendicular to $AC$ meets $AI$ at $E$. Prove that the reflection of $I$ in $AC$ lies on the circumcircle of triangle $BDE$.

**Solution**.



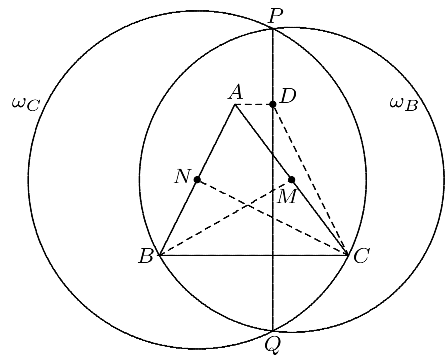

Let $ABC$ be the reference triangle with $b = c \neq a$. We have $D = (a : 0 : b)$ and $I = (a : b : b)$. Line $AC = [0 : 1 : 0]$, and a displacement vector perpendicular to $AC$ is $(a^2 : -2b^2 : 2b^2 - a^2)$. We can parametrize the line $DE$ as $(a : 0 : b) + t(a^2 : -2b^2 : 2b^2 - a^2)$. Line $AI = [0 : -1, 1]$, or $y = z$. Solving this with the parametric equation of $DE$, we have $t(-2b^2) = b + (2b^2 - a^2)t$. From this, we have $t = b/(a^2 - 4b^2)$. From this we have $E = (a(-a^2 - ab + 4b^2) : 2b^3 : 2b^3)$. Also $I' = (a(a+b) : -b^2 : 3b^2 - a^2)$. The equation of the circumcircle of $BDE$ is

$$a^2yz + b^2zx + c^2xy + \frac{b^2}{(a^2 - b^2)(a + 2b)}(x + y + z)(2b^3x - a^2(a + b)z) = 0.$$

We can verify directly that the coordinates of $I'$ satisfies this equation. Thus $I'$ lies on the circumcircle of triangle $BDE$.

19. [Nordic 2017]. Let $M$ and $N$ be the midpoints of the sides $AC$ and $AB$, respectively, of an acute triangle $ABC$, $AB \neq AC$. Let $\omega_B$ be the circle centered at $M$ passing through $B$, and let $\omega_C$ be the circle centered at $N$ passing through $C$. Let the point $D$ be such that $ABCD$ is an isosceles trapezoid with $AD$ parallel to $BC$. Assume that $\omega_B$ and $\omega_C$ intersect in two distinct points $P$ and $Q$. Show that $D$ lies on the line $PQ$.

**Solution**.

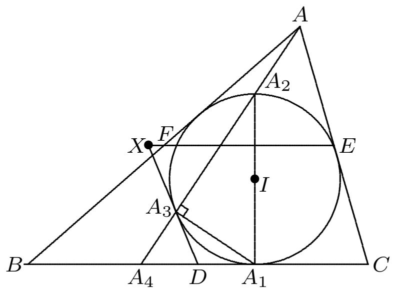



The reflection of $A$ about the perpendicular bisector of $BC$ is $D = (a^2 : c^2 - b^2 : b^2 - c^2)$. Let $L = \sqrt{2c^2 + 2a^2 - b^2}$ and $K = \sqrt{2a^2 + 2b^2 - c^2}$. Then the lengths of the medians from $B$ and $C$ are $L/2$ and $K/2$ respectively. The circle $\omega_B$ centred at the midpoint of $CA$ with radius $L/2$ passes through the points $B = (0 : 1 : 0)$, $(b + L : 0 : b - L)$ and $(b - L : 0 : b + L)$. Substituting these points into the general equation of a circle, we find that the equation of $\omega_B$ equal to $a^2yz + b^2zx + c^2xy + (x + y + z)(S_Bx + S_Bz) = 0$. The circle

$\omega_C$ centred at the midpoint of $AB$ with radius $K/2$ passes through the points $C = (0:0:1)$, $(c+K : c-K : 0)$ and $(c-K : c+K : 0)$. From this we get the equation of $\omega_C$ equal to $a^2yz + b^2zx + c^2xy + (x+y+z)(S_Cx + S_Cy) = 0$. Subtracting the equations of $\omega_B$ and $\omega_C$, we get the equation of the radical axis $PQ$ of $\omega_B$ and $\omega_C$, which is equal to $(S_B - S_C)x - S_Cy + S_Bz = 0$. That is $(c^2-b^2)x - S_Cy + S_Bz = 0$. Substituting the coordinates of $D$ into the left hand side of this equation, we get $(c^2-b^2)a^2 - S_C(c^2-b^2) + S_B(b^2-c^2) = 0$. Thus $D$ lies on the radical axis $PQ$ of $\omega_B$ and $\omega_C$.

20. [China 2017]. In the non-isosceles triangle $ABC$, $D$ is the midpoint of side $BC$, $E$ is the midpoint of side $CA$, $F$ is the midpoint of side $AB$. The line (different from line $BC$) that is tangent to the inscribed circle of triangle $ABC$ and passing through point $D$ intersect line $EF$ at $X$. Define $Y, Z$ similarly. Prove that $X, Y, Z$ are collinear.

**Solution**.




Let $A_1$ be the point of tangency of the incircle with the side $BC$, and let $A_1A_2$ be a diameter of the incircle. Let the other tangent from $D$ to the incircle meet the incircle at $A_3$. (That is $A_3 \neq A_1$). Thus $\angle A_1A_3A_2 = 90°$. The extension of $A_2A_3$ meets the side $BC$ at a point $A_4$ such that $D$ is the midpoint of $A_1A_4$. This means that $A_4$ is the point of tangency of the $A$-excircle with the side $BC$. That is $A_4 = (0 : s-b : s-c)$. Also it is well-known that $A, A_2, A_4$ are collinear. Thus $A, A_1, A_2, A_4$ are collinear. We use this information to find $A_3$.

First $\boldsymbol{AA}_4 = (-a : s-b : s-c)$. A displacement vector perpendicular to $\boldsymbol{AA}_4$ is given by $(2a(b-c)s : -ab^2 - b^2(s-c) + (a^2-c^2)(s-b) : c^2a + c^2(s-b) + (b^2-a^2)(s-c)) = (2a(b-c)s : s(a^2-2ab-(b-c)^2) : -s(a^2-2ac-(b-c)^2) = (2a(b-c) : a^2-2ab-(b-c)^2 : -a^2+2ac+(b-c)^2)$.

Also $A_1 = (0 : s-c : s-b)$. Then we can parametrize the line $A_1A_3$ as

$$(2a(b-c)t : s-c+t(a^2-2ab-(b-c)^2) : s-b+(-a^2+2ac+(b-c)^2)).$$

Substituting this into the line $AA_4 = [0 : -(s-c) : s-b]$, we get $t = (-b+c)/(a^2-ab-ac-2b^2+4bc-2c^2) = \frac{2(b-c)}{a(s-a)+4(b-c)^2}$. From this we get

$A_3 = (-4(b-c)^2 : (a-b)^2 - c^2 : (c-a)^2 - b^2) = (-4(b-c)^2 : -4(s-b)(s-a) : -4(s-c)(s-a)) = (\frac{(b-c)^2}{(s-a)} : s-b : s-c)$. Similarly, $B_3 = (s-a : \frac{(c-a)^2}{s-b} : s-c)$ and $C_3 = (s-a : s-b : \frac{(a-b)^2}{s-c})$.

From this we get $DA_3 = [s-a : b-c : c-b]$. Since $EF = [-1 : 1 : 1]$, we obtain $X = (b-c : s-c : b-s)$. Similarly, by cyclically permuting $a, b, c$ and the positions of the coordinates, we obtain $Y = (c-s : c-a : s-a)$, $Z = (s-b : a-s : a-b)$.

Now $\begin{vmatrix} b-c & s-c & b-s \\ c-s & c-a & s-a \\ s-b & a-s & a-b \end{vmatrix} = 0$ as the sum of the three rows is the zero row. Consequently, $X, Y, Z$ are collinear.

## 11 References

[1] Paul Yiu, “*Introduction to the Geometry of the Triangle*”.