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INVARIANT DISTRIBUTIONS OF CLASSICAL GROUPS

CHEN-BO ZHU

CONTENTS

0.1. Introduction and main results. The problem of identifying G-invariant tempered distributions for the action of a classical group G on a vector space has been around for some time. In particular, the case of $G = O(p, q)$ acting on $\mathbb{R}^{p+q}$ was studied by many people, among them Methee, de Rham, Gärding, Tengstrand, Gelfand, and Shilov, in [Me], [dR], [Ga], [Te], [GS], etc. In this article, we consider the following.

Let G be a classical group of one of the following types:

$$O(p, q), \quad U(p, q), \quad Sp(p, q), \quad Sp(2m, \mathbb{R}), \quad O^*(2m), \tag{0.1.1}$$

and let V be its standard module, namely,

$$\mathbb{R}^{p+q}, \quad \mathbb{C}^{p+q}, \quad \mathbb{H}^{p+q}, \quad \mathbb{R}^{2m}, \quad \mathbb{H}^{m}. \tag{0.1.2}$$

The action of G on V induces an (linear) action of G on $L^2(V^k)$ given by

$$(g \cdot f)(v_1, v_2, \ldots, v_k) = f(g^{-1}v_1, g^{-1}v_2, \ldots, g^{-1}v_k) \tag{0.1.3}$$

where $f \in L^2(V^k)$, $v_i \in V$, $1 \leqslant i \leqslant k$.

Let $\mathscr{S}(V^k)$ be the Schwarz space of rapidly decreasing functions on V^k and $\mathscr{S}^*(V^k)$

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be its continuous dual. In usual terminology $\mathscr{S}^*(V^k)$ is called the space of tempered distributions on V^k. See [Tr].

G acts on $\mathscr{S}(V^k)$ by the restriction of (0.1.3). Thus, we have the induced action of G on $\mathscr{S}^*(V^k)$:

$$(g \cdot A)(f) = A(g^{-1} \cdot f), \qquad A \in \mathscr{S}^*(V^k), \quad f \in \mathscr{S}(V^k). \tag{0.1.4}$$

Let $W = V^k \oplus V^k$. We can define a real symplectic form $\langle\ ,\ \rangle$ on W (§1.1) such that our original V^k is a maximal isotropic subspace (a polarization). Also, there exists a classical group G' such that G and G' are mutual centralizers in $Sp = Sp(W)$, the isometry group of $\langle\ ,\ \rangle$. In other words, G and G' form a reductive dual pair in Sp ([H1]).

We list below five corresponding dual pairs:

$$\begin{aligned} &(1)\ (O(p, q), Sp(2k, \mathbb{R})) \subseteq Sp(2(p+q)k, \mathbb{R}),\\ &(2)\ (U(p, q), U(k, k)) \subseteq Sp(4(p+q)k, \mathbb{R}),\\ &(3)\ (Sp(p, q), O^*(4k)) \subseteq Sp(8(p+q)k, \mathbb{R}),\\ &(4)\ (Sp(2m, \mathbb{R}), O(k, k)) \subseteq Sp(4mk, \mathbb{R}),\\ &(5)\ (O^*(2m), Sp(k, k)) \subseteq Sp(8mk, \mathbb{R}). \end{aligned} \tag{0.1.5}$$

Let $\widetilde{Sp}$ be the unique double cover of Sp and π: $\widetilde{Sp} \mapsto Sp$ be the projection map. For a subgroup B of Sp, let $\widetilde{B} = \pi^{-1}(B)$.

Since V^k is a polarization of the symplectic form, the oscillator representation of $\widetilde{Sp}$, denoted by ω, has a Schrödinger realization in $L^2(V^k)$ ([Ge]). By twisting $\omega|_{\widetilde{G}}$ with a character of $\widetilde{G}$, the resulting action can be made to factor through the linear G-action on $L^2(V^k)$ given by (0.1.3). Below, we shall always twist $\omega|_{\widetilde{G}}$ by such a character. Thus, we shall be concerned with G instead of $\widetilde{G}$.

We denote the induced action of $\widetilde{Sp}$ on $\mathscr{S}^*(V^k)$ still by the same symbol ω for reasons which shall be explained at the beginning of §2.1. But this slight abuse of notation shall not cause any confusion.

Since $\widetilde{G}'$ commutes with G, $\mathscr{S}^*(V^k)^G$, the space of G-invariant tempered distributions is a $\widetilde{G}'$ module under ω. Let $\widetilde{K}'$ be a maximal compact subgroup of $\widetilde{G}'$. The set of isomorphism classes of irreducible finite-dimensional representations of $\widetilde{K}'$ is denoted by $\hat{\widetilde{K}}'$ and likewise for any compact group.

THEOREM I. (a) *$\mathscr{S}^*(V^k)^G$ is the closed span of the set $\{\omega(g')\delta \mid g' \in \widetilde{G}'\}$, where δ is the Dirac distribution at the origin of V^k. In fact, it is the closed span of the set $\{\omega(k')\delta \mid k' \in \widetilde{K}'\}$.*

(b) *The multiplicity of τ in $\mathscr{S}^*(V^k)^G$ is at most one for any $\tau \in \hat{\widetilde{K}}'$. All the $\widetilde{K}'$-types which do occur in $\mathscr{S}^*(V^k)^G$ can be explicitly described. In fact, we shall exhibit a*

G-invariant tempered distribution with one of those $\tilde{K}'$-types by proving that the projection of the Dirac distribution to such a $\tilde{K}'$-type is nonzero.

We pause to describe $\omega|_{\tilde{G}}$. Let P' be the subgroup of G' which stabilizes the first factor of $W = V^k \oplus V^k$. Then G' is generated by P' and an element of order four. For simplicity we shall only give explicit formulas of $\omega|_{\tilde{P}'}$ and the action of that element of order four for the case $G = O(p, q)$, $G' = Sp(2k, \mathbb{R})$. We shall use the following notation throughout this article. For two integers a and b, $M_{a,b}$ denotes the space of matrices of order $a \times b$. A superscript t denotes transpose, and tr X denotes the trace of a square matrix X. I_k denotes the identity matrix of order $k \times k$, and $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, $J_k = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$. Then, obviously, V^k can be identified with $M_{p+q,k}(\mathbb{R})$, and under this identification $O(p, q)$ acts on $M_{p+q,k}(\mathbb{R})$ via matrix multiplication on the left.

We have $P' \cong MN$, where

$$(0.1.6)\qquad \begin{aligned} M &= \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix}, a \in Gl(k, \mathbb{R}) \right\}, \\ N &= \left\{ n(b) = \begin{pmatrix} I_l & b \\ 0 & I_l \end{pmatrix} \middle| b = b^t \in M_{k,k}(\mathbb{R}) \right\}, \end{aligned}$$

and $G' = Sp(2k, \mathbb{R})$ is generated by P' and the element J_k.

Let χ_0 be the following character of $\tilde{M} = \{(m(a), \varepsilon) | a \in Gl(k, \mathbb{R}), \varepsilon = \pm 1\}$.

$$\chi_0(m(a), \varepsilon) = \varepsilon \cdot \begin{cases} 1, & \text{if } \det(a) > 0, \\ i, & \text{if } \det(a) < 0. \end{cases}$$

χ_0 is of order four. We then have

$$(0.1.8)\qquad \begin{aligned} &[\omega(m(a), \varepsilon)f](x) = \chi_0(m(a), \varepsilon)^\alpha |\det a|^{(p+q)/2} f(xa), \qquad a \in Gl(k, \mathbb{R}), \\ &[\omega(n(b))](x) = e^{i/2\, tr(I_{p,q} x b x^t)} f(x), \quad f \in L^2(M_{p+q,k}(\mathbb{R})), \quad x \in M_{p+q,k}(\mathbb{R}), \\ &[\omega(J_k)f](x) = v_0 \left(\frac{1}{2\pi}\right)^{(p+q)k/2} \int_{M_{p+q,k}(\mathbb{R})} e^{itr(x^t I_{p,q} y)} f(y)\, dy. \end{aligned}$$

Hence, $\alpha \equiv p - q \pmod 4$, and v_0 is an eighth root of unity. See [KV], [Ge], etc.

For explicit formulas in the other four cases, we refer the reader to [KV], [Ge], and [GK2].

In order to state our second main result, we shall develop some more notations. Throughout this article, if a Lie group is denoted by a capital letter, its Lie algebra is denoted by the corresponding lower case German letter, and vice versa. Also, for a compact subgroup E of Sp, the set of $\tilde{E}$-types occurring in the oscillator representation ω is denoted by $R(\tilde{E}, \omega)$.

Let U be a maximal compact subgroup of Sp. We have $U \cong U(l)$, the unitary group in l variables where $2l = \dim_{\mathbb{R}} W$. Let $\mathscr{P}$ be the $(\mathfrak{sp}, \tilde{U})$-module of ω, namely, the derived $(\mathfrak{sp}, \tilde{U})$-module on the space of $\tilde{U}$-finite vectors of ω, and let

$$\mathscr{P} = \sum_{\nu \in R(\tilde{U}, \omega)} \omega_\nu \tag{0.1.9}$$

be the $\tilde{U}$-isotypic decomposition.

Let $C^{-\infty}(V^k) = \omega^{-\infty}$ be the space of formal linear combinations

$$\sum_{\nu \in R(\tilde{U}, \omega)} f_\nu, \qquad f_\nu \in \omega_\nu. \tag{0.1.10}$$

We shall refer to $\omega^{-\infty}$ as the space of formal vectors of ω, f_ν's as the $\tilde{U}$ Fourier components of the formal vector $\sum_{\nu \in R(\tilde{U}, \omega)} f_\nu$. Thus, $C^{-\infty}(V^k)$ is the space of generalized functions on V^k with $\tilde{U}$ Fourier components. Moreover, $\mathscr{S}(V^k)$ (resp. $\mathscr{S}^*(V^k)$) can be characterized as the subspace of $C^{-\infty}(V^k)$ consisting of those elements such that their $\tilde{U}$ Fourier components decay rapidly (resp. grow at most polynomially). See (2.1.2) and (2.1.3).

Because of the simple way $\tilde{P}'$ acts in the oscillator representation (see 0.1.8), we see that for any $\phi \in \mathscr{S}(V^k)$, the function

$$g' \mapsto \omega(g')\phi(0), \qquad g' \in G' \tag{0.1.11}$$

is in the space of the induced representation $\mathrm{Ind}_{\tilde{P}'}^{\tilde{G}'}(\chi)$ for a character χ of $\tilde{P}'$ trivial on $\tilde{N}$. See §2.3 for details. We denote this natural map by $\lambda\colon \mathscr{S}(V^k) \to \mathrm{Ind}_{\tilde{P}'}^{\tilde{G}'}(\chi)$.

Since $\tilde{G}' = \tilde{K}'\tilde{P}'$, we can define the spaces $C^{-\infty}(\mathrm{Ind}_{\tilde{P}'}^{\tilde{G}'}(\chi))$, $\mathscr{S}(\mathrm{Ind}_{\tilde{P}'}^{\tilde{G}'}(\chi))$, $\mathscr{S}^*(\mathrm{Ind}_{\tilde{P}'}^{\tilde{G}'}(\chi))$ according to the growth behavior of their $\tilde{K}'$ Fourier components (§2.3).

We extend $\lambda\colon \mathscr{S}(V^k) \to \mathrm{Ind}_{\tilde{P}'}^{\tilde{G}'}(\chi)$ to $\lambda\colon C^{-\infty}(V^k) \to C^{-\infty}(\mathrm{Ind}_{\tilde{P}'}^{\tilde{G}'}(\chi))$ by linearity.

THEOREM II. *$\lambda|_{\mathscr{S}^*(V^k)^G}\colon \mathscr{S}^*(V^k)^G \to C^{-\infty}(\mathrm{Ind}_{\tilde{P}'}^{\tilde{G}'}(\chi))$ is injective. Possibly except for $G = Sp(p, q)$, we have*

$$\lambda|_{\mathscr{S}^*(V^k)^G}\colon \mathscr{S}^*(V^k)^G \hookrightarrow \mathscr{S}^*(\mathrm{Ind}_{\tilde{P}'}^{\tilde{G}'}(\chi)),$$

and it is a topological embedding with closed image.

Remark 0.1.12. The complete description of $\mathscr{S}^*(\mathbb{R}^{p+q})^{O(p,q)}$, i.e., the case $G = O(p, q)$, $k = 1$, was obtained by various people (Methee [Me] in the Lorentzian case, Tengstrand [Te], etc.), though the results were not stated in our form. Here, it is worthwhile to mention that certain structure results of $\mathscr{S}^*(\mathbb{R}^{p+q})^{O(p,q)}$ can be effectively used to give an elegant treatment for the fundamental solution of the indefinite Laplacian ([dR], [HT]). For $G = O(p, q)$ and arbitrary k, the description of $\mathcal{S}^*(V^k)^G$ as the closed span of the set $\{\omega(g')\delta | g' \in \tilde{G}'\}$ is due to Kudla and Rallis [KR]. Our method is very different from theirs, and it is quite "canonical" from the point of view of invariant theory.

We want to emphasize that there is another realization of the oscillator representation called the Fock model (§1.2). We shall exclusively work in this model. Basically, there are two parts in our approach: uniqueness and existence. In the uniqueness part we use extensively the fine structure of reductive dual pairs in the Fock model as developed by Howe in [H2], in particular, "seesaw" dual pairs ([Ku]) and "diamond" dual pairs (§1.3). What we actually prove is that, for any $\tau \in \hat{\tilde{K}}'$, the multiplicity of τ in $(\omega^{-\infty})^{(\mathfrak{g}, K)}$ is at most one, where K is a maximal compact subgroup of G. In the process we also single out all the possible $\tilde{K}'$-types which can have a $(\mathfrak{g}, K)$-invariant in $\omega^{-\infty}$. In the existence part we show that the projections of the Dirac distribution to those "possible" $\tilde{K}'$-types are always nonzero. Except for the case $G = Sp(p, q)$ where we turn to a general result about multiplicity-free actions ([Zhu2]), we accomplish this task by explicitly computing in the Fock model the inner product of the Dirac distribution with some lowest highest-weight vectors. This computation is made possible by a critical use of the famous Capelli identity in classical invariant theory ([W1]). Then the explicit inner product formulas are used to derive some growth estimate for the $\tilde{K}'$ Fourier components of the Dirac distribution, which implies Theorem II.

One feature of our approach is that the proof for various cases is parallel, and therefore, in order to bring out the ideas clearly, we shall only give a complete proof of Theorem I and Theorem II for $G = O(p, q)$ (§2.1, §2.2, §2.3). The adjustments needed for the other four cases are sketched in §3.1, §3.2.

1.1. Preliminaries: reductive dual pairs. Let F be one of the three division algebras over $\mathbb{R}$; i.e., $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ as usual. Recall that, as a real vector space, $\mathbb{H}$ has a standard basis consisting of the four elements $1, i, j, k$ with rules for multiplication:

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j.$$

F has a standard involution $\natural$, namely,

$$\natural = \begin{cases} \text{identity}, F = \mathbb{R}; \\ \text{complex conjugation}, F = \mathbb{C}; \\ \text{quaternionic conjugation}, F = \mathbb{H}. \end{cases}$$

We shall use a subscript L (resp. R) to denote a left (resp. right) vector space over F. Let $V = F_L^{p+q}$ be equipped with the nondegenerate $\natural$-hermitian form $(\ ,\ )_1$

$$(z, w)_1 = z^t I_{p,q} w^\natural, \qquad z = \begin{bmatrix} z_1 \\ \vdots \\ z_{p+q} \end{bmatrix}, \qquad w = \begin{bmatrix} w_1 \\ \vdots \\ w_{p+q} \end{bmatrix} \in V.$$

Let G be the isometry group of $(\ ,\ )_1$; i.e.,

$$G = O(p, q), \qquad \text{when } F = \mathbb{R}. \tag{1.1.1}$$

$$G = U(p, q), \qquad \text{when } F = \mathbb{C}. \tag{1.1.2}$$

$$G = Sp(p, q), \qquad \text{when } F = \mathbb{H}. \tag{1.1.3}$$

We also introduce two other series of classical groups. We use the same notations for convenience.

Let $V = \mathbb{R}^{2m}$ be equipped with the symplectic form $(\ ,\ )_1$

$$(x, y)_1 = x^t J_m y, \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_{2m} \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_{2m} \end{bmatrix} \in \mathbb{R}^{2m}$$

where $J_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, as usual. Let G be its isometry group; i.e.,

$$G = Sp(2m, \mathbb{R}). \tag{1.1.4}$$

Next, let $V = \mathbb{H}_L^m$ be equipped with the $\natural$-skew-hermitian form $(\ ,\ )_1$

$$(h, h')_1 = h^t (jI_m) h'^{\natural}, \qquad h = \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix}, \qquad h' = \begin{bmatrix} h'_1 \\ \vdots \\ h'_m \end{bmatrix} \in \mathbb{H}^m.$$

Let G be its isometry group. We identify $\mathbb{H}^m$ with $\mathbb{C}^{2m}$ by the rule

$$\mathbb{H}^m \ni h = \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} \leftrightarrow z = \begin{bmatrix} z_1 \\ \vdots \\ z_{2m} \end{bmatrix} \in \mathbb{C}^{2m}, \qquad \text{if } h_l = z_l + z_{m+l} j,\ 1 \leqslant l \leqslant m.$$

Then, it is easy to see that G is isomorphic to the subgroup of $GL(V, \mathbb{C})$ preserving the two forms: $z^t z$ and $z^t J_m \bar{z}$. In other words,

$$G = O^*(2m). \tag{1.1.5}$$

See [He1] for this description of $O^*(2m)$.

Definition 1.1.6. Let $(W, \langle\ ,\ \rangle)$ be a real symplectic vector space. A pair of subgroups (G, G') of the symplectic group $Sp(W, \langle\ ,\ \rangle)$ is called a *reductive dual pair* if

(i) G' is the centralizer of G in $Sp(W)$ and vice versa, and
(ii) both G and G' act (absolutely) reductively on W.

Below we introduce five reductive dual pairs listed in (0.1.5).

Let $W = V^k \oplus V^k$, where V is one of the five vector spaces equipped with a $\natural$-hermitian or $\natural$-skew-hermitian form $(\ ,\ )_1$ specified above. Consider the isomorphisms

$$W \cong V \otimes_F F_R^k \oplus V \otimes_F F_R^k \cong V \otimes F_R^{2k}.$$

Here, the right vector space structure of F_R^k (resp. F_R^{2k}) is obtained by composing the standard left vector space structure of F^k (resp. F^{2k}) with the involution $\natural$.

F_L^{2k} has a $\natural$-skew-hermitian form $(\ ,\ )_2$:

$$((u, v), (u', v'))_2 = uv'^{\natural t} - vu'^{\natural t},$$

and a $\natural$-hermitian form, again denoted by $(\ ,\ )_2$:

$$((u, v), (u', v'))_2 = uv'^{\natural t} + vu'^{\natural t}$$

where $u = (u_1, u_2, \dots, u_k) \in F_L^k$, etc. For the first three series of classical groups, we take $(\ ,\ )_2$ to be $\natural$-skew-hermitian, and for the last two we take $(\ ,\ )_2$ to be $\natural$-hermitian.

Let ν be the reduced trace map from F to $\mathbb{R}$, namely, $\nu(\lambda) = \frac{1}{2}(\lambda + \lambda^\natural)$, $\lambda \in F$. Define $\langle\ ,\ \rangle = \nu((\ ,\ )_1 \otimes (\ ,\ )_2^\natural)$; i.e.,

$$\langle z \otimes z', w \otimes w' \rangle = \nu((z, w)_1 (z', w')_2^\natural), \qquad z, w \in V, \qquad z', w' \in F_R^{2k}.$$

Notice that, although the tensor product of forms over F does not make sense for $F = \mathbb{H}$, when you take the reduced trace, you do get a well-defined $\mathbb{R}$-bilinear form. In fact, a straightforward computation yields that $\langle\ ,\ \rangle$ is a real symplectic form on W ([H4]). We denote the corresponding symplectic group by $Sp = Sp(W)$.

Let G' be the isometry group of $(\ ,\ )_2$. Obviously,

$$(1.1.7) \qquad G' \cong \begin{cases} Sp(2k, \mathbb{R}), & \text{if } G = O(p, q) \\ U(k, k), & \text{if } G = U(p, q) \\ O^*(4k), & \text{if } G = Sp(p, q) \\ O(k, k), & \text{if } G = Sp(2m, \mathbb{R}) \\ Sp(k, k), & \text{if } G = O^*(2m). \end{cases}$$

From [H1] we have the following proposition.

PROPOSITION 1.1.8. *G and G′ form a reductive dual pair in Sp.*

1.2. Preliminaries: the Fock model. For the following discussions we refer the reader to [Ge], [Ba], [Ca], [Fo].

Fix a symplectic vector space $(W, \langle\ ,\ \rangle)$, $\dim W = 2l$, and a *complete polarization* $W = X \oplus Y$; i.e., both X and Y are maximal isotropic with respect to $\langle\ ,\ \rangle$. We may select a basis $\{e_j\}_{j=1}^l$ for X, a basis $\{f_j\}_{j=1}^l$ for Y such that

$$\langle e_i, e_j\rangle = 0, \qquad \langle f_i, f_j\rangle = 0, \qquad \langle e_i, f_j\rangle = \delta_{ij}.$$

$\{e_i, f_i\}_{i=1}^l$ is called a standard symplectic basis for the symplectic form $\langle\ ,\ \rangle$. Taking the coordinates

$$w = \sum_{i=1}^{l} (x_i e_i + y_i f_i), \qquad w \in W \tag{1.2.1}$$

identifies W with $\mathbb{R}^{2l}$ and the symplectic group $Sp(W)$ with $Sp(2l, \mathbb{R})$, the isometry group of the symplectic form on $\mathbb{R}^{2l}$ given by

$$\langle (x, y), (x', y')\rangle = (x, y) J_l \begin{pmatrix} x' \\ y' \end{pmatrix}$$

where $x = (x_1, \ldots, x_l)$, $y = (y_1, \ldots, y_l) \in \mathbb{R}^l$, etc.

$\widetilde{Sp}(W)$ acts on $L^2(X)$ via the *Schrödinger realization* of the oscillator representation ω. (See [Ge, Introduction].) As usual, let $\mathfrak{sp}$ be the Lie algebra of $Sp = Sp(W)$. Then $\omega(\mathfrak{sp})$ consists of the differential operators of total degree 2, i.e.,

$$x_i x_j, \qquad \frac{\partial^2}{\partial x_i\, \partial x_j}, \qquad \frac{1}{2}\left(x_i \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} x_i\right)$$

where $\{x_j\}$'s are as in (1.2.1).

Now introduce a complex structure on W by setting

$$z_j = x_j + i y_j, \qquad 1 \leqslant j \leqslant l. \tag{1.2.2}$$

Let

$$z = (z_1, \ldots, z_l) = (x_1 + i y_1, \ldots, x_l + i y_l)$$

with

$$\bar{z} = (\bar{z}_1, \ldots, \bar{z}_l) = (x_1 - i y_1, \ldots, x_l - i y_l).$$

Set

$$z \cdot z' = \sum_{j=1}^{l} z_j z'_j$$

so that

$$|z|^2 = z \cdot \bar{z}.$$

From [Ba], [Ca], we know that ω may also be realized in the Hilbert space

$$\mathscr{F} = \left\{ f(z) \text{ holomorphic on } W: \left(\frac{i}{2\sqrt{\pi}}\right)^l \int_W |f(z)|^2 e^{-|z|^2} \, dz_1 \, d\bar{z}_1 \dots dz_l \, d\bar{z}_l < \infty \right\}$$

with the inner product

$$(f_1, f_2) = \left(\frac{i}{2\sqrt{\pi}}\right)^l \int_W f_1(z) \bar{f}_2(z) e^{-|z|^2} \, dz_1 \, d\bar{z}_1 \dots dz_l \, d\bar{z}_l, \qquad f_1, f_2 \in \mathscr{F}.$$

This is called the *Fock realization.* A straightforward calculation gives

$$\left(\frac{\partial}{\partial z_j}\right)^* = z_j, \qquad 1 \leqslant j \leqslant l \tag{1.2.3}$$

where A^* is the adjoint operator of A acting on the Hilbert space $\mathscr{F}$.

It is possible to write down the explicit isomorphism between the Schrödinger model and the Fock model. (See [Fo].) For our purpose we need the following properties of this isomorphism:

$$L^2(X) \ni e^{-\sum_{j=1}^l x_j^2/2} \mapsto 1 \in \mathscr{F}$$

and

$$x_j \to \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z_j} + z_j\right), \qquad \frac{\partial}{\partial x_j} \to \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z_j} - z_j\right). \tag{1.2.4}$$

The following is an immediate consequence.

Let δ be the Dirac distribution at the origin of X. Since δ satisfies

$$x_j \delta = 0,$$

it must have the form

$$\delta = e^{-\sum_{j=1}^l z_j^2/2} \qquad \text{(up to a scalar)} \tag{1.2.5}$$

in the Fock model.

Let U be the isometry group of $z \cdot \bar{z}'$. It is isomorphic to the unitary group in l variables, and it is a maximal compact subgroup of Sp. Now $\tilde{U}$ can be identified with $\{(g, \lambda) | g \in U, \lambda^2 = \det g\}$. In the Fock model, $\tilde{U}$ acts in a simple manner:

$$\omega(g, \lambda)f(z') = \lambda^{-1}f(g^{-1}z'), \qquad \tilde{g} = (g, \lambda) \in \tilde{U}. \tag{1.2.6}$$

We remark that (1.2.6) is often abbreviated as

$$\omega(g)f(z') = (\det g)^{-1/2}f(g^{-1}z'). \tag{1.2.7}$$

1.3. Preliminaries: structure of dual pairs in the Fock model. We now review some results we will need from [H2]. We shall state everything in the Fock model ω of the oscillator representation of $\widetilde{Sp} = \widetilde{Sp}(2l, \mathbb{R})$, sometimes without explicitly mentioning ω.

Choose a maximal compact subgroup U of Sp as in §1.2. In the Fock model, $\mathscr{P}$, the space of $\tilde{U}$-finite vectors in ω is isomorphic to $\mathscr{P}_l$, the space of polynomials on $\mathbb{C}^l$ with complex coordinates $z_1, \ldots, z_l$ as in (1.2.2).

Using this identification, we have

$$\omega(\mathfrak{sp}_{\mathbb{C}}) = \mathfrak{sp}^{(1,1)} \oplus \mathfrak{sp}^{(2,0)} \oplus \mathfrak{sp}^{(0,2)} \tag{1.3.1}$$

where

$$\begin{aligned} \mathfrak{sp}^{(1,1)} &= \text{span of } \left\{\frac{1}{2}\left(z_i\frac{\partial}{\partial z_j} + \frac{\partial}{\partial z_j}z_i\right)\right\}, \\ \mathfrak{sp}^{(2,0)} &= \text{span of } \{z_i z_j\}, \\ \mathfrak{sp}^{(0,2)} &= \text{span of } \left\{\frac{\partial^2}{\partial z_i \partial z_j}\right\}. \end{aligned} \tag{1.3.2}$$

Here and afterward, a subscript $\mathbb{C}$ denotes complexification.

Let

$$\mathfrak{sp} = \mathfrak{u} \oplus \mathfrak{q} \tag{1.3.3}$$

be the Cartan decomposition of $\mathfrak{sp}$ with this choice of U. We then have

$$\omega(\mathfrak{u}_{\mathbb{C}}) = \mathfrak{sp}^{(1,1)}, \qquad \omega(\mathfrak{q}_{\mathbb{C}}) = \mathfrak{sp}^{(2,0)} \oplus \mathfrak{sp}^{(0,2)}. \tag{1.3.4}$$

Consider a reductive dual pair $(G, G') \subseteq Sp$. We may assume G and G' are embedded in Sp in such a way that the Cartan decomposition (1.3.3) of $\mathfrak{sp}$ also induces Cartan decompositions of $\mathfrak{g}$ and $\mathfrak{g}'$. Thus,

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} \oplus \mathfrak{p}, & \mathfrak{k} &= \mathfrak{u} \cap \mathfrak{g}, & \mathfrak{p} &= \mathfrak{q} \cap \mathfrak{g}, \\ \mathfrak{g}' &= \mathfrak{k} \oplus \mathfrak{p}', & \mathfrak{k}' &= \mathfrak{u} \cap \mathfrak{g}', & \mathfrak{p}' &= \mathfrak{q} \cap \mathfrak{g}'. \end{aligned} \tag{1.3.5}$$

FACT 1.3.6. *$\mathfrak{k}$ and $\mathfrak{k}$ are themselves members of reductive dual pairs* $(\mathfrak{k}, \mathfrak{m}')$ *and* $(\mathfrak{m}, \mathfrak{k}')$ ([H2, p. 539]):

$$\begin{array}{cc} \mathfrak{k} & \mathfrak{k}' \\ \cap\!\!| & \cap\!\!| \\ \mathfrak{m} & \mathfrak{m}'. \end{array}$$

The pairs of Lie algebras placed in the opposite position of the above diagram are reductive dual pairs. Therefore, they are called *"seesaw" dual pairs*. See [Ku].

We have the inclusions

$$\begin{array}{cc} \mathfrak{k} & \mathfrak{k}' \\ \cap\!\!| & \cap\!\!| \\ \mathfrak{g} & \mathfrak{g}' \\ \cap\!\!| & \cap\!\!| \\ \mathfrak{m} & \mathfrak{m}'. \end{array} \tag{1.3.7}$$

Since $\mathfrak{k} \subseteq \mathfrak{u}$ (resp. $\mathfrak{k}' \subseteq \mathfrak{u}$), ad $\mathfrak{k}$ (resp. ad $\mathfrak{k}'$) preserves the decomposition (1.3.1), and since $\mathfrak{m}'$ (resp. $\mathfrak{m}$) is the full centralizer of $\mathfrak{k}$ (resp. $\mathfrak{k}'$) in $\mathfrak{sp}$, we have

$$\begin{aligned} &\mathfrak{m}'_{\mathbb{C}} = \mathfrak{m}'^{(1,1)} \oplus \mathfrak{m}'^{(2,0)} \oplus \mathfrak{m}'^{(0,2)}, \qquad \mathfrak{m}'^{(i,j)} = \mathfrak{m}'_{\mathbb{C}} \cap \mathfrak{sp}^{(i,j)}, \\ &\mathfrak{m}_{\mathbb{C}} = \mathfrak{m}^{(1,1)} \oplus \mathfrak{m}^{(2,0)} \oplus \mathfrak{m}^{(0,2)}, \qquad \mathfrak{m}^{(i,j)} = \mathfrak{m}_{\mathbb{C}} \cap \mathfrak{sp}^{(i,j)}. \end{aligned} \tag{1.3.8}$$

Let

$$\mathfrak{m}_0^{(1,1)} = \mathfrak{m}^{(1,1)} \cap \mathfrak{sp}, \qquad \mathfrak{m}_0'^{(1,1)} = \mathfrak{m}'^{(1,1)} \cap \mathfrak{sp}. \tag{1.3.9}$$

Clearly, $\mathfrak{m}_0^{(1,1)}$ (resp. $\mathfrak{m}_0'^{(1,1)}$) is the Lie algebra of a maximal compact subgroup of M (resp. M').

FACT 1.3.10. $(\mathfrak{m}_0^{(1,1)}, \mathfrak{m}_0'^{(1,1)})$ *is a reductive dual pair in* $\mathfrak{sp}$ ([H2], p. 540).

Thus, we can expand (1.3.7) to

$$\begin{array}{ccccc} & \mathfrak{m}_0^{(1,1)} & & & \\ \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\supseteq$} & & \\ \mathfrak{k} & & \mathfrak{m} & & \\ \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\supseteq$} & & \\ & \mathfrak{g} & & & \end{array} \qquad \begin{array}{ccc} & \mathfrak{m}_0'^{(1,1)} & \\ \rotatebox{45}{$\supseteq$} & & \rotatebox{-45}{$\subseteq$} \\ \mathfrak{m}' & & \mathfrak{k}'. \\ \rotatebox{-45}{$\supseteq$} & & \rotatebox{45}{$\subseteq$} \\ & \mathfrak{g}' & \end{array} \tag{1.3.11}$$

The pairs of Lie algebras similarly placed in the two diamonds are reductive dual pairs. Therefore, they are called *"diamond" dual pairs*.

Lastly, we have the following *embedding property*.

FACT 1.3.12 ([H2, p. 540])

$$\mathfrak{m}^{(2,0)} \oplus \mathfrak{m}^{(0,2)} = \mathfrak{p}_{\mathbb{C}} \oplus \mathfrak{m}^{(0,2)} = \mathfrak{m}^{(2,0)} \oplus \mathfrak{p}_{\mathbb{C}}.$$

List of "diamond" dual pairs ([H2])

Case 1. $G = O(p, q)$, $G' = Sp(2k, \mathbb{R})$.

(1.3.13)

$$\begin{tikzcd}[column sep=small]
 & U(p) \times U(q) \arrow[dr, phantom, sloped, "\subseteq"] & \\
O(p) \times O(q) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & U(p, q) \\
 & O(p, q) \arrow[ur, phantom, sloped, "\subseteq"] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & U(k) \times U(k) \arrow[dr, phantom, sloped, "\supseteq"] & \\
Sp(2k, \mathbb{R}) \times Sp(2k, \mathbb{R}) \arrow[ur, phantom, sloped, "\supseteq"] \arrow[dr, phantom, sloped, "\supseteq"] & & U(k) \\
 & Sp(2k, \mathbb{R}) \arrow[ur, phantom, sloped, "\supseteq"] &
\end{tikzcd}$$

Case 2. $G = U(p, q)$, $G' = U(k, k)$.

(1.3.14)

$$\begin{tikzcd}[column sep=small]
 & U(p) \times U(q) \times U(p) \times U(q) \arrow[dr, phantom, sloped, "\subseteq"] & \\
U(p) \times U(q) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & U(p, q) \times U(p, q) \\
 & U(p, q) \arrow[ur, phantom, sloped, "\subseteq"] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & U(k) \times U(k) \times U(k) \times U(k) \arrow[dr, phantom, sloped, "\supseteq"] & \\
U(k, k) \times U(k, k) \arrow[ur, phantom, sloped, "\supseteq"] \arrow[dr, phantom, sloped, "\supseteq"] & & U(k) \times U(k) \\
 & U(k, k) \arrow[ur, phantom, sloped, "\supseteq"] &
\end{tikzcd}$$

Case 3. $G = Sp(p, q)$, $G' = O^*(4k)$.

(1.3.15)

$$\begin{tikzcd}[column sep=small]
 & U(2p) \times U(2q) \arrow[dr, phantom, sloped, "\subseteq"] & \\
Sp(p) \times Sp(q) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & U(2p, 2q) \\
 & Sp(p, q) \arrow[ur, phantom, sloped, "\subseteq"] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & U(2k) \times U(2k) \arrow[dr, phantom, sloped, "\supseteq"] & \\
O^*(4k) \times O^*(4k) \arrow[ur, phantom, sloped, "\supseteq"] \arrow[dr, phantom, sloped, "\supseteq"] & & U(2k) \\
 & O^*(4k) \arrow[ur, phantom, sloped, "\supseteq"] &
\end{tikzcd}$$

Case 4. $G = Sp(2m, \mathbb{R})$, $G' = O(k, k)$.

(1.3.16)

$$\begin{tikzcd}[column sep=small]
 & U(m) \times U(m) \arrow[dr, phantom, sloped, "\subseteq"] & \\
U(m) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & Sp(2m, \mathbb{R}) \times Sp(2m, \mathbb{R}) \\
 & Sp(2m, \mathbb{R}) \arrow[ur, phantom, sloped, "\subseteq"] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & U(k) \times U(k) \arrow[dr, phantom, sloped, "\supseteq"] & \\
U(k, k) \arrow[ur, phantom, sloped, "\supseteq"] \arrow[dr, phantom, sloped, "\supseteq"] & & O(k) \times O(k) \\
 & O(k, k) \arrow[ur, phantom, sloped, "\supseteq"] &
\end{tikzcd}$$

Case 5. $G = O^*(2m)$, $G' = Sp(k, k)$.

(1.3.17)

$$\begin{tikzcd}[column sep=small]
 & U(m) \times U(m) \arrow[dr, phantom, sloped, "\subseteq"] & \\
U(m) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & O^*(2m) \times O^*(2m) \\
 & O^*(2m) \arrow[ur, phantom, sloped, "\subseteq"] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & U(2k) \times U(2k) \arrow[dr, phantom, sloped, "\supseteq"] & \\
U(2k, 2k) \arrow[ur, phantom, sloped, "\supseteq"] \arrow[dr, phantom, sloped, "\supseteq"] & & Sp(k) \times Sp(k) \\
 & Sp(k, k) \arrow[ur, phantom, sloped, "\supseteq"] &
\end{tikzcd}$$

Let us define the space of K'-pluriharmonics.

$$\mathscr{H}(K') = \{P \in \mathscr{P} \mid X \cdot P = 0 \text{ for all } X \in \mathfrak{m}^{(0,2)}\}. \tag{1.3.18}$$

(By [H1], $\mathfrak{m}^{(0,2)}$ consists of all $\mathfrak{k}'$-invariant differential operators of second order in $\mathscr{P}$. Therefore, we have the name K'-pluriharmonics for $\mathscr{H}(K')$.)

Let

$$\mathscr{P} = \sum_{\tau \in R(\tilde{K}', \omega)} \mathscr{P}_\tau \tag{1.3.19}$$

be the isotypic decomposition of $\mathscr{P}$ as a $\tilde{K}'$-module.

The following theorem is a generalization of the classical theory of spherical harmonics.

THEOREM 1.3.20. (See [H2], p. 542.)

(a) *The joint action of* $\mathfrak{m} \times \tilde{K}'$ *on* $\mathscr{P}_\tau$ *is irreducible for each* $\tau \in R(\tilde{K}', \omega)$.

(b) $\mathscr{H}(K')_\tau = \mathscr{P}_\tau \cap \mathscr{H}(K')$ *consists precisely of the space of polynomials of lowest degree* $d(\tau)$ *in* $\mathscr{P}_\tau$.

(c) *One has* $\mathscr{P}_\tau = \mathscr{U}(\mathfrak{m}^{(2,0)}) \cdot \mathscr{H}(K')_\tau$, *where* $\mathscr{U}(\mathfrak{m}^{(2,0)})$ *is the universal enveloping algebra of* $\mathfrak{m}^{(2,0)}$.

(d) *The group* $\tilde{K}'$ *and the Lie algebra* $\mathfrak{m}^{(1,1)}$ *generate mutual commutants in* $\mathscr{H}(K')$. *Equivalently, each* $\mathscr{H}(K')_\tau$ *is irreducible under the joint action of* $\mathfrak{m}^{(1,1)} \times \tilde{K}'$, *and, if we write* $\mathscr{H}(K')_\tau \cong \rho(\tau) \otimes \tau$ *for* $\tau \in R(\tilde{K}', \omega)$, *then* τ *determines* $\rho(\tau)$ *and vice versa, so that* $\tau \mapsto \rho(\tau)$ *is an injection from* $R(\tilde{K}', \omega)$ *into* $R(\mathfrak{m}^{(1,1)}, \omega)$, *defined similarly.*

Below, we give the description of $R(\tilde{K}', \omega)$, the correspondence $\tau \mapsto \rho(\tau)$ for $G = O(p, q)$, $G' = Sp(2k, \mathbb{R})$. See [KV] for the other cases. In the present case we have $U \cong U((p+q)k)$, $K' = U(k)$, $M = U(p, q)$, $M^{(1,1)} = U(p) \times U(q)$.

The $\tilde{U}$-finite vectors of ω form a space isomorphic to $\mathscr{P} = \mathscr{P}_{p+q,k}$, the space of polynomials on $M_{p+q,k}(\mathbb{C})$. Moreover, if we write the coordinates in terms of a $(p+q) \times k$ matrix $\begin{pmatrix} Z \\ W \end{pmatrix}$

$$\begin{pmatrix} Z \\ W \end{pmatrix} = \begin{bmatrix} z_{11} & \cdots & z_{1k} \\ \cdots & \cdots & \cdots \\ z_{p1} & \cdots & z_{pk} \\ w_{11} & \cdots & w_{1k} \\ \cdots & \cdots & \cdots \\ w_{q1} & \cdots & w_{qk} \end{bmatrix}, \tag{1.3.21}$$

then $\tilde{U}(p) \times \tilde{U}(q) \times \tilde{U}(k)$ acts by

$$(\tilde{g}_1, \tilde{g}_2, \tilde{D}) \circ \begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} a^k d^{-p} g_1 Z D^{-1} \\ b^{-k} d^q (g_2^t)^{-1} W D^t \end{pmatrix} \tag{1.3.22}$$

where $\tilde{g}_1 = (g_1, a) \in \tilde{U}(p)$ with $a^2 = \det g_1$, $\tilde{g}_2 = (g_2, b) \in \tilde{U}(q)$ with $b^2 = \det g_2$, and $\tilde{D} = (D, d) \in \tilde{U}(k)$ with $d^2 = \det D$. (See 1.2.6 of Preliminaries.)

$\mathfrak{m}^{(0,2)}$, being the space of $\mathfrak{k}'$-invariant differential operators of second order in $\mathscr{P}$, is spanned by

$$\Delta_{i,j} = \sum_{\nu=1}^{k} \frac{\partial^2}{\partial z_{i\nu} \partial w_{j\nu}}, \qquad 1 \leqslant i \leqslant p, \qquad 1 \leqslant j \leqslant q,$$

and, therefore, $\mathscr{H}(K')$ is the space of polynomials f on $M_{p,k}(\mathbb{C}) \times M_{q,k}(\mathbb{C})$ satisfying

$$(\Delta_{i,j} f)(z, w) = 0. \tag{1.3.23}$$

We denote by $\mathfrak{b}_k^+, \mathfrak{b}_k^-$ the upper and lower triangular Borel subalgebras of $\mathfrak{gl}(k, \mathbb{C})$, i.e., the Lie algebras of the upper and lower triangular matrices of order $k \times k$, etc. With respect to $\mathfrak{b}_p^- \times \mathfrak{b}_q^+ \times \mathfrak{b}_k^+$ the simultaneous highest weight vectors of $\tilde{U}(p) \times \tilde{U}(q)$ and $\tilde{U}(k)$ in $\mathscr{H}(K')$ are of form ([KV])

$$d_1(Z)^{\alpha_1} d_2(Z)^{\alpha_2} \ldots d_t(Z)^{\alpha_t} \tilde{d}_1(W)^{\beta_1} \tilde{d}_2(W)^{\beta_2} \ldots \tilde{d}_s(W)^{\beta_s} \tag{1.3.24}$$

where

$$d_i(Z) = \det \begin{bmatrix} z_{11} & \ldots & z_{1i} \\ \ldots & \ldots & \ldots \\ z_{i1} & \ldots & z_{ii} \end{bmatrix}, \qquad 1 \leqslant i \leqslant \min(p, k),$$

$$\tilde{d}_j(W) = \det \begin{bmatrix} w_{1,k-j+1} & \ldots & w_{1,k} \\ \ldots & \ldots & \ldots \\ w_{j,k-j+1} & \ldots & w_{j,k} \end{bmatrix}, \qquad 1 \leqslant j \leqslant \min(q, k),$$

α_i, β_j are nonnegative integers.

Parametrizing the irreducible representations of $\tilde{U}(p) \times \tilde{U}(q)$ (resp. $\tilde{U}(k)$) by their highest weights with respect to $\mathfrak{b}_p^+ \times \mathfrak{b}_q^+$ (resp. $\mathfrak{b}_k^+$), we have the following theorem.

THEOREM 1.3.25 ([KV]).

(a) $$R(\tilde{K}', \omega) = \left\{ \tau = \left(\frac{p-q}{2}, \ldots, \frac{p-q}{2} \right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0, -b_s, -b_{s-1}, \ldots, -b_1) \right\};$$

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t > 0, \quad b_1 \geqslant b_2 \geqslant \cdots \geqslant b_s > 0,$$

$$t \leqslant \min(k, p), \quad s \leqslant \min(k, q).$$

(b) *If* $\tau \in R(\tilde{K}', \omega)$ *and if*

$$\tau = \left(\frac{p-q}{2}, \dots, \frac{p-q}{2}\right) + (a_1, a_2, \dots, a_t, 0, \dots, 0, -b_s, -b_{s-1}, \dots, -b_1),$$

then the representation $\rho(\tau)$ *is equal to* $\rho_1(\tau) \otimes \rho_2(\tau)$, *where*

$$\rho_1(\tau) = \left(-\frac{k}{2}, \dots, -\frac{k}{2}\right) + (0, \dots, 0, -a_t, -a_{t-1}, \dots, -a_1),$$

$$\rho_2(\tau) = \left(\frac{k}{2}, \dots, \frac{k}{2}\right) + (b_1, b_2, \dots, b_s, \dots, 0, \dots, 0).$$

2.1. $\tilde{K}'$–multiplicity-one property: $G = O(p, q)$ case. We begin with some discussions about formal vectors (see 0.1.10 for the definition) and tempered distributions.

Let $Sp = Sp(W)$ be a real symplectic group and let U be a maximal compact subgroup of Sp. Let ω be the oscillator representation of $\widetilde{Sp}$, $\mathscr{P}$ be the space of $\tilde{U}$-finite vectors of ω, and $\omega^{-\infty}$ be the space of formal vectors of ω, as in the Introduction.

Recall the Fock model of ω. Let $\{z_1, \dots, z_l\}$ be the complex coordinates in this model. (See 1.2.2.) Then the monomials $\left\{\frac{1}{(\sqrt{\pi})^{l/2}(k_1! \dots k_l!)^{1/2}} z_1^{k_1} \dots z_l^{k_l}\right\}$ form an orthonormal basis of $\mathscr{P}$. Recall also that the Schrödinger model of ω is realized in $L^2(X)$, where X is a factor of the complete polarization $W = X \oplus Y$. Let $\{\phi_{k_1, \dots, k_l}\}$ correspond to $\left\{\frac{1}{(\sqrt{\pi})^{l/2}(k_1! \dots k_l!)^{1/2}} z_1^{k_1} \dots z_l^{k_l}\right\}$ under the isomorphism from the Schrödinger model to the Fock model. Then, in fact, $\{\phi_{k_1, \dots, k_l}\}$'s are the so-called normalized Hermite functions ([H5]). They are inside the Schwarz space $\mathscr{S}(X)$, and they form a basis of $\mathscr{P}$ in the Schrödinger model. Since $\mathscr{P}$ is dense in $\mathscr{S}(X)$, we see that a tempered distribution is determined by its values on $\phi_{k_1, \dots, k_l}$. This fact allows us to represent a tempered distribution A by the formal vector

$$\sum \lambda_{k_1, \dots, k_l} \phi_{k_1, \dots, k_l}, \qquad \text{where } \lambda_{k_1, \dots, k_l} = A(\phi_{k_1, \dots, k_l}). \tag{2.1.1}$$

Thus, we have

$$\mathscr{P} \subset \mathscr{S}^*(X) \subset \omega^{-\infty}.$$

Moreover, if we let $\{x_1, \dots, x_l\}$ be the real coordinates in X (see 1.2.1), then we have $\mathscr{C}\phi_{k_1, \dots, k_l} = \sum_{i=1}^l (k_i + \frac{1}{2})\phi_{k_1, \dots, k_l}$, where $\mathscr{C}$ is the Hermite operator $\sum_{i=1}^l x_i^2 - \sum_{i=1}^l \frac{\partial^2}{\partial x_i^2}$. From that it is quite easy to see that a formal vector $\sum_{k_1, \dots, k_l} \lambda_{k_1, \dots, k_l} \phi_{k_1, \dots, k_l}$

is of Schwarz class if and only if

$$\sum_{k_1,\ldots,k_l} |\lambda_{k_1,\ldots,k_l}|(1 + k_1^2 + \cdots k_l^2)^r < \infty \tag{2.1.2}$$

for any positive integer r ([H5]). Similarly, $\sum_{k_1,\ldots,k_l} \lambda_{k_1,\ldots,k_l}\phi_{k_1,\ldots,k_l}$ is a tempered distribution if and only if

$$|\lambda_{k_1,\ldots,k_l}| \leqslant (1 + k_1^2 + \cdots k_l^2)^N \tag{2.1.3}$$

for some N.

Let us look at the reductive dual pair $(G, G') = (O(p, q), Sp(2k, \mathbb{R})) \subseteq Sp(2l, \mathbb{R})$, $l = (p + q)k$. We shall write Sp instead of $Sp(2l, \mathbb{R})$. Let ω be the (twisted) oscillator representation of $\widetilde{Sp}$ associated to the above dual pair. (See the Introduction.) Recall that $K = O(p) \times O(q)$ is a maximal compact subgroup of G and that $K' = U(k)$ is a maximal compact subgroup of G'. We may choose a maximal compact subgroup U of Sp in such a way that (see 1.3.5)

$$K = U \cap G \qquad \text{and} \qquad K' = U \cap G'. \tag{2.1.4}$$

We have the $(\mathfrak{sp}, \widetilde{U})$ module $\omega^{-\infty}$. For $\tau \in \hat{\tilde{K}}'$ define

$$M((\omega^{-\infty})_\tau^{(\mathfrak{g},K)}) \overset{\text{def}}{=} \text{multiplicity of } \tau \text{ in } (\omega^{-\infty})^{(\mathfrak{g},K)} \overset{\text{def}}{=} \dim \operatorname{Hom}_{\tilde{K}'}(\tau, (\omega^{-\infty})^{(\mathfrak{g},K)}).$$

The purpose of the remaining part of this section is to prove $M((\omega^{-\infty})_\tau^{(\mathfrak{g},K)}) \leqslant 1$. For the compact case $G = O(p)$ we use the following "seesaw" dual pairs (see 1.3.6) and explore the fact that $O(p) \subseteq U(p)$ is a compact spherical pair, i.e., $\dim \pi^{O(p)} \leqslant 1$, for $\pi \in \hat{U}(p)$.

$$\begin{array}{cc} O(p) & U(k) \\ \cap & \cap \\ U(p) & Sp(2k, \mathbb{R}) \end{array} \tag{2.1.5}$$

For the noncompact case $G = O(p, q)$ we use the following "diamond" dual pairs (see 1.3.13) and prove a reduction result which essentially says the following: A holomorphic representation of $U(p, q)$ has an $O(p, q)$ invariant only if its lowest $U(p) \times U(q)$-type has an $O(p) \times O(q)$ invariant.

$$\begin{tikzcd}[column sep=small] & U(p)\times U(q) \arrow[dl, phantom, sloped, "\supseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & \\ O(p)\times O(q) \arrow[dr, phantom, sloped, "\supseteq"] & & U(p,q) \arrow[dl, phantom, sloped, "\supseteq"] \\ & O(p,q) & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & U(k)\times U(k) \arrow[dl, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\supseteq"] & \\ Sp(2k,\mathbb{R})\times Sp(2k,\mathbb{R}) \arrow[dr, phantom, sloped, "\supseteq"] & & U(k) \arrow[dl, phantom, sloped, "\subseteq"] \\ & Sp(2k,\mathbb{R}) & \end{tikzcd} \tag{2.1.6}$$

We shall deal with the case $G = O(p)$, $G' = Sp(2k, \mathbb{R})$, first.

PROPOSITION 2.1.7. $M((\omega^{-\infty})_\tau^{O(p)}) \leqslant 1$ *for* $\tau \in \hat{\tilde{U}}(k)$. *It is equal to one if and only if* τ *has highest weight*

$$\left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}\right) + (a_1, a_2, \ldots, a_t, 0, 0, \ldots, 0),$$

$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t > 0$, *all even integers* a_j *for* $j \leqslant t \leqslant \min(p, k)$.

Proof. Recall the "seesaw" dual pairs (2.1.5). For $\tau \in \hat{\tilde{U}}(k)$ let $\mathscr{P}_\tau$ be the τ-isotypic component of $\mathscr{P}$ as before. By the standard result of Howe ([H2]) or [KV], we have

$$\mathscr{P}_\tau \cong \sigma(\tau) \otimes \tau$$

where $\sigma(\tau) \in \hat{U}(p)$. Thus,

$$\mathscr{P}_\tau^{O(p)} \cong \sigma(\tau)^{O(p)} \otimes \tau .$$

Observe that the injection

$$\mathrm{Hom}_{\tilde{U}(k)}(\tau, \mathscr{P}) \hookrightarrow \mathrm{Hom}_{\tilde{U}(k)}(\tau, \omega^{-\infty})$$

induced by $\mathscr{P} \hookrightarrow \omega^{-\infty}$ is an isomorphism. Therefore,

$$\begin{aligned} M((\omega^{-\infty})_\tau^{O(p)}) &= \dim \mathrm{Hom}_{\tilde{U}(k)}(\tau, (\omega^{-\infty})^{O(p)}) = \dim \mathrm{Hom}_{\tilde{U}(k)}(\tau, \mathscr{P}^{O(p)}) \\ &= \dim \sigma(\tau)^{O(p)}. \end{aligned}$$

Observe also that the pair $O(p) \subseteq U(p)$ is a compact symmetric pair with θ as the involutive automorphism:

$$O(p) \subseteq U(p), \qquad \theta(X) = \overline{X}.$$

By a classical result due to Cartan (see [He2]), it is a compact spherical pair. Thus,

$$\text{(2.1.8)} \quad M((\omega^{-\infty})_\tau^{O(p)}) \leqslant 1, \quad \text{and} \quad M((\omega^{-\infty})_\tau^{O(p)}) = 1 \quad \text{if and only if} \quad \sigma(\tau)^{O(p)} \neq 0.$$

From the theory of models of representations for the classical groups ([BGG]) or from classical invariant theory ([H1]) (see also [He2]), we know $\sigma(\tau)^{O(p)} \neq 0$ if and only if the highest weight of $\sigma(\tau)$ is of the form

$$\text{(2.1.9)} \qquad D = (a_1, a_2, \ldots, a_p), \qquad a_i \in 2\mathbb{Z}, \qquad 1 \leqslant i \leqslant p.$$

Let us describe $\sigma(\tau)$ in the Fock model of ω. This is more or less a special case of

Theorem (1.3.25) for $q = 0$, except that our $\omega|_{\tilde{O}(p)}$ here is twisted by a character of $\tilde{O}(p)$. For that reason we still give some details.

The $\tilde{U}$-finite vectors of ω form a space isomorphic to $\mathscr{P}_{p,k}$, the space of polynomials on

$$Z = \begin{bmatrix} z_{11} & \cdots & z_{1k} \\ \cdots & \cdots & \cdots \\ z_{p1} & \cdots & z_{pk} \end{bmatrix}.$$

Moreover, $\tilde{U}(p) \times \tilde{U}(k)$ acts by

$$(\tilde{g}, \tilde{D}) \circ Z = a^k d^{-p} g Z D^{-1}$$

where $\tilde{g} = (g, a) \in \tilde{U}(p)$ with $a^2 = \det g$ and $\tilde{D} = (D, d) \in \tilde{U}(k)$ with $d^2 = \det D$. From [KR] we know that the action of $\tilde{U}(p)|_{\tilde{O}(p)} \times \tilde{U}(k)$

$$(\tilde{g}, \tilde{D}) \circ Z = d^{-p} g Z D^{-1} \tag{2.1.10}$$

descends to $O(p)$. We also know from Theorem (1.3.25) that, if $\tau \in \hat{\tilde{U}}(k)$ has the highest weight

$$\left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}\right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0),$$

then the representation $\sigma(\tau) \in \hat{\tilde{U}}(p)$ has the highest weight

$$\left(-\frac{k}{2}, -\frac{k}{2}, \ldots, -\frac{k}{2}\right) + (0, \ldots, 0, -a_t, -a_{t-1}, \ldots, -a_1).$$

Thus, it is the representation of $\tilde{U}(p)$ with the highest weight $(0, \ldots, 0, -a_t, -a_{t-1}, \ldots, -a_1)$ which descends to the linear $O(p)$ action.

Combining the above with (2.1.8) and (2.1.9), Proposition (2.1.7) follows.

Now we deal with the noncompact case: $G = O(p, q)$, $G' = Sp(2k, \mathbb{R})$. Recall $K = O(p) \times O(q)$, $K' = U(k)$, $M = U(p, q)$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$, $\mathfrak{m}_{\mathbb{C}} = \mathfrak{m}^{(1,1)} \oplus \mathfrak{m}^{(2,0)} \oplus \mathfrak{m}^{(0,2)}$ is the Harish-Chandra decomposition of $\mathfrak{m}_{\mathbb{C}}$ as in (1.3.8). For other notations in the following, we refer the reader back to §1.3.

Recall also the K'-isotypic decomposition of $\mathscr{P}$

$$\mathscr{P} = \sum_{\tau \in R(\tilde{K}', \omega)} \mathscr{P}_\tau$$

where $R(\tilde{K}', \omega)$ is defined as in §0.1, namely, as the set of the $\tilde{K}'$-types occurring in the oscillator representation ω associated to the dual pair $(G, G') \subseteq Sp$.

Fix $\tau \in R(\tilde{K}', \omega)$. By part (c) of Theorem (1.3.20)

$$\mathscr{P}_\tau = \mathscr{U}(\mathfrak{m}^{(2,0)}) \cdot \mathscr{H}(K')_\tau .$$

Since all of the operators in $\mathfrak{m}^{(2,0)}$ raise the degree by 2, we have the following (unique) decomposition for a vector $v \in \mathscr{P}_\tau$:

$$v = v_d + v_{d+2} + v_{d+4} + \cdots \tag{2.1.12}$$

where $d = d(\tau)$, the lowest degree in $\mathscr{P}_\tau$, and $\deg v_{d+2i} = d + 2i$, $i \geqslant 0$.

We denote by $P \colon v \mapsto v_d$, for $v \in \mathscr{P}_\tau$.

Since K commutes with $\tilde{K}'$, it preserves $\mathscr{P}_\tau$, and, since $K \subseteq U$, it preserves the degree. Therefore, the decomposition (2.1.12) is K-equivariant. In particular, if v is $(\mathfrak{g}, K)$-invariant, v_{d+2i} is K-invariant for any i.

PROPOSITION 2.1.13. *$P \colon (\omega^{-\infty})_\tau^{(\mathfrak{g},K)} = \mathscr{P}_\tau^{(\mathfrak{g},K)} \to \mathscr{H}(K')_\tau^K$ is injective for $\tau \in R(\tilde{K}', \omega)$.*

Proof. Suppose $v \in \mathscr{P}_\tau^{(\mathfrak{g},K)}$ and $v_d = 0$. Let i be the smallest nonzero integer such that $v_{d+2i} \neq 0$. We shall arrive at a contradiction so that $v = 0$, and the proposition will follow.

For any $X \in \mathfrak{p}_{\mathbb{C}}$ we have $X \cdot v = 0$ by the $\mathfrak{g}$-invariance assumption. Since we choose K, K', and U so that (2.1.4) is satisfied, we have the "embedding" property (see 1.3.12)

$$\mathfrak{m}^{(2,0)} \oplus \mathfrak{m}^{(0,2)} = \mathfrak{p}_{\mathbb{C}} \oplus \mathfrak{m}^{(0,2)} = \mathfrak{m}^{(2,0)} \oplus \mathfrak{p}_{\mathbb{C}} . \tag{2.1.14}$$

Thus, we can write (uniquely)

$$X = L_X + R_X, \qquad L_X \in \mathfrak{m}^{(0,2)}, \qquad R_X \in \mathfrak{m}^{(2,0)}.$$

Now

$$\begin{aligned} 0 = X \cdot v &= (L_X + R_X) \cdot (v_{d+2i} + v_{d+2i+2} + \cdots) \\ &= L_X v_{d+2i} + R_X v_{d+2i} + L_X v_{d+2i+2} + R_X v_{d+2i+2} + \cdots . \end{aligned}$$

Since L_X lowers the degree by 2, R_X raises the degree by 2; the above implies

$$L_X v_{d+2i} = 0 .$$

Since $\{L_X\}_{X \in \mathfrak{p}_{\mathbb{C}}}$ spans $\mathfrak{m}^{(0,2)}$ by (2.1.14), we see that v_{d+2i} is K'-pluriharmonic, and therefore, it has to be of the lowest degree d, a contradiction. □

By Theorem (1.3.20), $\mathscr{H}(K')_\tau \cong \rho(\tau) \otimes \tau$, where $\rho(\tau) \in \hat{U}(p) \times \hat{U}(q)$. So we have $\mathscr{H}(K')_\tau^K = \mathscr{H}(K')_\tau^{O(p) \times O(q)} \cong \rho(\tau)^{O(p) \times O(q)} \otimes \tau$. Since the pair $O(p) \times O(q) \subseteq U(p) \times U(q)$ is the product of compact spherical pairs $O(p) \subseteq U(p)$ and $O(q) \subseteq U(q)$, the dimension of $\rho(\tau)^{O(p)\times O(q)}$ is at most one. It is equal to one precisely when the highest weights of $\rho(\tau)$ are "even" as in (2.1.9). Using the explicit description of $\rho(\tau)$ in Theorem 1.3.25 and adjusting for the fact that we are using the twisted oscillator representation (see [KR] and the proof of Proposition 2.1.7), we obtain the following theorem.

THEOREM 2.1.15. *Let $(G, G') = (O(p, q), Sp(2k, \mathbb{R}))$ be the reductive dual pair in $Sp(2(p+q)k, \mathbb{R})$ and ω be the (twisted) oscillator representation of $\widetilde{S}p(2(p+q)k, \mathbb{R})$. Then $M((\omega^{-\infty})_\tau^{(\mathfrak{g},K)}) \leqslant 1$ for any $\tau \in \hat{\tilde{U}}(k)$. It is equal to one only if the highest weight of τ is from the set R_0*

$$\left(\frac{p-q}{2}, \frac{p-q}{2}, \ldots, \frac{p-q}{2}\right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0, -b_s, \ldots, -b_2, -b_1),$$

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t > 0, \quad b_1 \geqslant b_2 \geqslant \cdots \geqslant b_s > 0, \quad t \leqslant \min(k, p), \quad s \leqslant \min(k, q),$$

a_i, b_j are all even integers.

Remark 2.1.16. The injective map P in Proposition (2.1.13) has to be bijective, for the multiplicity of τ in $\mathscr{H}(K')^K$ is at most one. Thus, we can replace the word "only if" in Theorem (2.1.15) by "if and only if". We caution the reader that, even with this improved version, we are still not sure whether there are $O(p, q)$-invariant tempered distributions with the $\tilde{U}(k)$-types specified in the above theorem. At this point we have only proved that there are $(\mathfrak{o}(p, q), O(p) \times O(q))$-invariant formal vectors with these $\tilde{U}(k)$-types.

2.2. Existence of invariant distributions: $G = O(p, q)$ case. In this section we shall exhibit $O(p, q)$-invariant distributions with appropriate $\tilde{U}(k)$-types by showing that the projections of the Dirac distribution onto those $\tilde{U}(k)$-types are nonzero. Our strategy is as follows: For $G = O(p)$ we first use a variant of procedure (1.2.5) to give an explicit formula of the Dirac distribution δ as a formal vector in the Fock model. We then explicitly compute the inner product of δ with the simultaneous $\tilde{U}(p) \times \tilde{U}(k)$ highest-weight vectors. For $G = O(p, q)$ the desired result is an easy consequence of these inner product formulas together with the explicit description of $U(k)$-pluriharmonics and functorial properties of the Dirac distribution.

Let $(G, G') = (O(p), Sp(2k, \mathbb{R})) \subseteq Sp(2l, \mathbb{R})$, $l = pk$. Let $(x_{ij})_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant k}$ be the real coordinates of $V^k \cong M_{p,k}(\mathbb{R})$ and $Z = (z_{ij})_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant k}$ be the complex coordinates in the Fock model of the oscillator representation of $\widetilde{S}p(2l, \mathbb{R})$, as in §2.1.

Since the isomorphism of the Schrödinger model with the Fock model is such that

$$x_{ij} \to \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z_{ij}} + z_{ij}\right), \qquad \frac{\partial}{\partial x_{ij}} \to \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z_{ij}} - z_{ij}\right) \tag{2.2.1}$$

and since the Dirac distribution at the origin of V^k satisfies

$$x_{ij}\delta = 0,$$

then it must have the form

(2.2.2) $$\delta = e^{-\sum_{i,j} z_{ij}^2/2} \qquad \text{(up to a scalar)}$$

in the Fock model. See (1.2.4) and (1.2.5).

Recall the "seesaw" dual pairs (1.3.6)

$$\begin{array}{cc} O(p) & U(k) \\ \cap\!| & \cap\!| \\ U(p) & Sp(2k, \mathbb{R}). \end{array}$$

We know that all the simultaneous highest-weight vectors of $\tilde{U}(p) \times \tilde{U}(k)$ are of the form

(2.2.3) $$d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t}$$

where

$$d_n = d_n(Z) = \det \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \cdots & \cdots & \cdots \\ z_{n1} & \cdots & z_{nn} \end{bmatrix},$$

$\alpha_1, \alpha_2, \dots, \alpha_t$ are nonnegative integers, $n \leqslant t \leqslant \min(p, k)$. See (1.3.24). Let

(2.2.4) $$\partial_n = \det \begin{bmatrix} \frac{\partial}{\partial z_{11}} & \frac{\partial}{\partial z_{12}} & \cdots & \frac{\partial}{\partial z_{1n}} \\ \frac{\partial}{\partial z_{21}} & \frac{\partial}{\partial z_{22}} & \cdots & \frac{\partial}{\partial z_{2n}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial}{\partial z_{n1}} & \frac{\partial}{\partial z_{n2}} & \cdots & \frac{\partial}{\partial z_{nn}} \end{bmatrix}.$$

Therefore, since $\dfrac{\partial}{\partial z_{ij}} \delta = -z_{ij}\delta$, we see that

(2.2.5) $$\begin{aligned} \partial_n \delta &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{\partial}{\partial z_{\sigma(1)1}} \frac{\partial}{\partial z_{\sigma(2)2}} \cdots \frac{\partial}{\partial z_{\sigma(n)n}} \delta \\ &= (-1)^n \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) z_{\sigma(1)1} z_{\sigma(2)2} \cdots z_{\sigma(n)n} \delta = (-1)^n d_n \delta. \end{aligned}$$

Let us now recall the *Capelli identity* ([W1], [H1]). In our context it asserts

$$(2.2.6)\qquad d_t\partial_t = \det(E^t_{ij} + \delta_{ij}(t-i))_{1\leqslant i\leqslant t,\,1\leqslant j\leqslant t}$$

$$= \det\begin{bmatrix} E^t_{11} + (t-1) & E^t_{12} & \dots & E^t_{1t} \\ E^t_{21} & E^t_{22} + (t-2) & \dots & E^t_{2t} \\ \dots & \dots & \dots & \dots \\ E^t_{t1} & E^t_{t2} & \dots & E^t_{tt} \end{bmatrix}$$

where

$$E^t_{ij} = \sum_{\mu=1}^{t} z_{\mu i}\frac{\partial}{\partial z_{\mu j}},$$

and $\det(A_{ij})$, for a matrix $\{A_{ij},\ 1 \leqslant i, j \leqslant n\}$ of noncommuting variables, is defined to be

$$\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)A_{\sigma(1)1}A_{\sigma(2)2}\dots A_{\sigma(n)n}.$$

LEMMA 2.2.7. $E^t_{ij}(d_1^{\alpha_1}d_2^{\alpha_2}\dots d_t^{\alpha_t}) = 0,\ 1 \leqslant i < j \leqslant t.$

Proof. $Gl(t, \mathbb{C})$ acts on $Z_t = \begin{bmatrix} z_{11} & \cdot & z_{1t} \\ \cdot & \cdot & \cdot \\ z_{t1} & \cdot & z_{tt} \end{bmatrix}$ by right translation with the derived action of $\mathfrak{gl}(t, \mathbb{C})$ given by $e^t_{ij} \to E^t_{ij}$, where e^t_{ij} is the $t \times t$ matrix with one at the (i, j) entry and zeros elsewhere. The lemma follows by observing that $d_1^{\alpha_1}d_2^{\alpha_2}\dots d_t^{\alpha_t}$ is invariant under the upper triangular matrices with ones in the diagonal.

Applying the Capelli identity to $d_1^{\alpha_1}d_2^{\alpha_2}\dots d_t^{\alpha_t}$, we get

$$\begin{aligned} d_t\partial_t(d_1^{\alpha_1}d_2^{\alpha_2}\dots d_t^{\alpha_t}) &= \det(E^t_{ij} + \delta_{ij}(t-i))d_1^{\alpha_1}d_2^{\alpha_2}\dots d_t^{\alpha_t} \qquad \text{(by 2.2.6)} \\ &= \prod_{i=1}^{t}(E^t_{ii} + (t-i))d_1^{\alpha_1}d_2^{\alpha_2}\dots d_t^{\alpha_t} \qquad \text{(by 2.2.7)} \\ &= B(\alpha_1, \alpha_2, \dots, \alpha_t)d_1^{\alpha_1}d_2^{\alpha_2}\dots d_t^{\alpha_t} \end{aligned}$$

where

$$(2.2.8)\qquad B(\alpha_1, \alpha_2, \dots, \alpha_t) = \prod_{i=1}^{t}\left\{\sum_{s\geqslant i}^{t}[\alpha_s + (t-i)]\right\}.$$

In other words,

$$(2.2.9)\qquad \partial_t(d_1^{\alpha_1}d_2^{\alpha_2}\dots d_t^{\alpha_t}) = B(\alpha_1, \alpha_2, \dots, \alpha_t)d_1^{\alpha_1}d_2^{\alpha_2}\dots d_t^{\alpha_t - 1}.$$

Since in the Fock model, we have $\left(\frac{\partial}{\partial z_{ij}}\right)^* = z_{ij}$ (see 1.2.3), we have

(2.2.10) $$d_n^* = \partial_n.$$

We compute the inner product

(2.2.11)
$$\begin{aligned}
&(\delta, d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t})\\
&= (\delta, d_t \cdot d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t - 1})\\
&= (\partial_t \delta, d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t - 1}) \qquad \text{(by 2.2.10)}\\
&= (-1)^t (d_t \delta, d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t - 1}) \qquad \text{(by 2.2.5)}\\
&= (-1)^t (\delta, \partial_t (d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t - 1})) \qquad \text{(by 2.2.10)}\\
&= (-1)^t B(\alpha_1, \alpha_2, \dots, \alpha_t - 1)(\delta, d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t - 2}) \qquad \text{(by 2.2.9)}.
\end{aligned}$$

By the above recursion formula we obtain the following proposition.

PROPOSITION 2.2.12. *Let* $\phi_\alpha = d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t}$ *for* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_t)$, $\alpha_i \geqslant 0$, $t \leqslant \min(p, k)$. *Then*

$$(\delta, \phi_\alpha) \neq 0$$

if and only if $\alpha_i \in 2\mathbb{Z}^+$, $1 \leqslant i \leqslant t$. *Moreover,*

(2.2.13) $$(\delta, \phi_\alpha) = \left[\prod_{1 \leqslant i \leqslant t} \prod_{\left\{\substack{1 \leqslant c_i \leqslant \alpha_i - 1\\ c_i \text{ odd}}\right.} (-1)^i B(\alpha_1, \alpha_2, \dots, \alpha_{i-1}, c_i)\right] (1, 1)$$

if $\alpha_1, \alpha_2, \dots, \alpha_t$ *are all even.*

For Theorem II we need a formula for $(\phi_\alpha, \phi_\alpha)$. We derive it by similar means.

PROPOSITION 2.2.14.

$$(\phi_\alpha, \phi_\alpha) = \left(\prod_{1 \leqslant i \leqslant t} \prod_{1 \leqslant c_i \leqslant \alpha_i} B(\alpha_1, \alpha_2, \dots, \alpha_{i-1}, c_i)\right)(1, 1).$$

Proof. We have

$$\begin{aligned}
(\phi_\alpha, \phi_\alpha) &= (d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t}, d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t})\\
&= (\partial_t (d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t}), d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t - 1})\\
&= (B(\alpha_1, \alpha_2, \dots, \alpha_t) d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t - 1}, d_1^{\alpha_1} d_2^{\alpha_2} \dots d_t^{\alpha_t - 1}).
\end{aligned}$$

By induction we get the desired formula for $(\phi_\alpha, \phi_\alpha)$.

Now assume $G = O(p, q)$. Let R_0 be the set of $\tau \in \hat{\tilde{U}}(k)$ specified in Theorem (2.1.15).

THEOREM 2.2.15. *The Dirac distribution at the origin of $V^k = (\mathbb{R}^{p+q})^k$ has a nonzero projection to the τ-type for any $\tau \in R_0$.*

Proof. Recall again the two diamonds of reductive dual pairs (1.3.13)

$$\begin{tikzcd}[column sep=small] & U(p)\times U(q) & \\ O(p)\times O(q) \arrow[ur, phantom, sloped, "\subseteq"] & & U(p,q) \arrow[ul, phantom, sloped, "\supseteq"] \\ & O(p,q) \arrow[ul, phantom, sloped, "\supseteq"] \arrow[ur, phantom, sloped, "\subseteq"] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & U(k)\times U(k) & \\ Sp(2k,\mathbb{R})\times Sp(2k,\mathbb{R}) \arrow[ur, phantom, sloped, "\supseteq"] & & U(k). \arrow[ul, phantom, sloped, "\subseteq"] \\ & Sp(2k,\mathbb{R}) \arrow[ul, phantom, sloped, "\subseteq"] \arrow[ur, phantom, sloped, "\supseteq"] & \end{tikzcd}$$

From Theorem (1.3.25) we know that, if $\tau \in R_0 \subseteq \hat{\tilde{U}}(k)$ has the highest weight

$$\left(\frac{p-q}{2}, \frac{p-q}{2}, \ldots, \frac{p-q}{2}\right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0, -b_s, \ldots, -b_2, -b_1),$$

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t > 0, \quad b_1 \geqslant b_2 \geqslant \cdots \geqslant b_s > 0, \quad t \leqslant \min(k, p), \quad s \leqslant \min(k, q),$$

a_i, b_j all even, then the simultaneous highest-weight vector of $\mathscr{H}(K')_\tau$ (the $K' = U(k)$-pluriharmonics of type τ) under the joint action of $\tilde{U}(p) \times \tilde{U}(q) \times \tilde{U}(k)$ is

$$v_\tau = v_{\tau_p} \otimes v_{\tau_q}$$

where

$$v_{\tau_p} = d_1(Z)^{a_1-a_2} d_2(Z)^{a_2-a_3} \ldots d_t(Z)^{a_t}$$

and

$$v_{\tau_q} = \tilde{d}_1(W)^{b_1-b_2} \tilde{d}_2(W)^{b_2-b_3} \ldots \tilde{d}_s(W)^{b_s}.$$

See §1.3 for the notations.

Let δ_p (resp. δ_q) be the Dirac distribution at the origin of $(\mathbb{R}^p)^k$ (resp. $(\mathbb{R}^q)^k$), the direct sum of k-copies of $\mathbb{R}^p$ (resp. $\mathbb{R}^q$) for which $\mathbb{R}^{p+q} = \mathbb{R}^p \oplus \mathbb{R}^q$. Since $a_1, \ldots, a_t$, $b_1, \ldots, b_s$ are all even integers, we have $(\delta_p, v_{\tau_q}) \neq 0$, $(\delta_q, v_{\tau_q}) \neq 0$ by the inner product formula (2.2.13).

Now the Dirac distribution δ at the origin of $(\mathbb{R}^{p+q})^k$ can be expressed as

$$\delta = \delta_p \otimes \delta_q. \tag{2.2.16}$$

We have

$$(\delta, v_\tau) = (\delta_p, v_{\tau_p})(\delta_q, v_{\tau_q}) \neq 0. \quad \square \tag{2.2.17}$$

Combining Theorem (2.1.15) with our present Theorem (2.2.15), Theorem I follows for $G = O(p, q)$.

2.3. A topological embedding. In this section we show Theorem II for $G = O(p, q)$.

Recall the dual pair $(G, G') = (O(p, q), Sp(2k, \mathbb{R})) \subseteq Sp(2(p+q)k, \mathbb{R})$. Recall also the Siegal parabolic subgroup P' of G'. $P' \cong MN$, where M, N are given in (0.1.6).

For $\alpha \in \mathbb{Z}/4\mathbb{Z}$, $s \in \mathbb{C}$, let $\chi^\alpha(s)$ be the following character of $\tilde{M}$:

$$\chi^\alpha(s)(m(g), \varepsilon) = |\det(g)|^s \chi_0(m(g), \varepsilon)^\alpha$$

where χ_0 is as in (0.1.7). We extend this to a character of $\tilde{P}'$ by letting $\tilde{N}$ act trivially and define $I^\alpha(s) = \mathrm{Ind}_{\tilde{P}'}^{\tilde{G}'} \chi^\alpha(s)$ to be the representation of $\tilde{G}'$ induced from $\chi^\alpha(s)$, i.e.,

$$I^\alpha(s) = \{f\colon \tilde{G}' \to \mathbb{C} \mid f(g'p') = \chi^\alpha(s)(p')f(g'),\ g' \in \tilde{G}',\ p' \in \tilde{P}'\}. \tag{2.3.1}$$

G' acts on $I^\alpha(s)$ by left translation.

From the explicit formula (0.1.8) of the oscillator representation, we see that the image of the natural map

$$\mathscr{S}(V^k) \ni \phi \mapsto \omega(g')\phi(0)$$

lies in $I^{\alpha_0}(s_0)$ for $s_0 = \dfrac{p+q}{2}$, $\alpha_0 \equiv p - q \pmod 4$. We denote this map by

$$\lambda\colon \mathscr{S}(V^k) \to I^{\alpha_0}(s_0). \tag{2.3.2}$$

Recall $C^{-\infty}(V^k) = \omega^{-\infty}$, the space of formal vectors of the oscillator representation. (See the Introduction.) We would like to extend the domain of the map λ from $\mathscr{S}(V^k)$ to $C^{-\infty}(V^k)$. To do this we need to define the space $C^{-\infty}(I^{\alpha_0}(s_0))$.

Recall that $K' = U(k)$ is a maximal compact subgroup of G'. Since $G' = K'P'$ and $K' \cap P' \cong O(k)$, we see that restriction to $\tilde{K}'$ yields an isomorphism: $I^{\alpha_0}(s_0) \cong \mathrm{Ind}_{\tilde{O}(k)}^{\tilde{U}(k)}(\chi^{\alpha_0})$, where χ^{α_0} is the character of $\tilde{U}(k)$ whose differential on the maximal torus is given by the weight $\dfrac{p-q}{2}(1, 1, \ldots, 1)$. Obviously, we have $\mathrm{Ind}_{\tilde{O}(k)}^{\tilde{U}(k)}(\chi^{\alpha_0}) \cong \mathrm{Ind}_{O(k)}^{U(k)} 1 \otimes \chi^{\alpha_0} \cong L^2(U(k)/O(k)) \otimes \chi^{\alpha_0}$, where 1 is the trivial representation of $\tilde{U}(k)$, $L^2(U(k)/O(k))$ is the $U(k)$ module by left translation. Thus,

$$I^{\alpha_0}(s_0) \cong L^2(U(k)/O(k)) \otimes \chi^{\alpha_0}. \tag{2.3.3}$$

This isomorphism allows us to identify $I^{\alpha_0}(s_0)$ with the space of functions on $U(k)/O(k)$.

Let

$$L^2(U(k)/O(k)) = \sum_{\tau \in R} L^2(U(k)/O(k))_\tau \tag{2.3.4}$$

be the $U(k)$-isotypic decomposition, where R is the set of $U(k)$-types occurring in $L^2(U(k)/O(k))$. By (2.1.9), $\tau \in R$ if and only if the highest weight of τ is "even". But we shall not need this fact.

Let $C^{-\infty}(U(k)/O(k))$ be the space of formal linear combinations

$$\sum_{\tau \in R} f_\tau, \qquad f_\tau \in L^2(U(k)/O(k))_\tau . \tag{2.3.5}$$

We define $\mathscr{S}(U(k)/O(k))$ to be the subspace of $C^{-\infty}(U(k)/O(k))$ consisting of those elements such that its $U(k)$ Fourier component f_τ decreases rapidly; namely,

$$\sum_{\tau \in R} (1 + \|\tau\|^2)^r \|f_\tau\|_2^2 < \infty \qquad \text{for any positive integer } r. \tag{2.3.6}$$

(See [Wa1] for the definition of $\|\tau\|$, etc.) We also define $\mathscr{S}^*(U(k)/O(k))$ to be the subspace of $C^{-\infty}(U(k)/O(k))$ such that

$$\|f_\tau\|_2^2 < (1 + \|\tau\|^2)^N \qquad \text{for some positive integer } N. \tag{2.3.7}$$

It is well known that these characterizations completely determine the topologies of $\mathscr{S}(U(k)/O(k))$ and $\mathscr{S}^*(U(k)/O(k))$.

We now transport everything back to $I^{\alpha_0}(s_0)$ by using the isomorphism in (2.3.3) and define the corresponding spaces $C^{-\infty}(I^{\alpha_0}(s_0))$, $\mathscr{S}(I^{\alpha_0}(s_0))$, $\mathscr{S}^*(I^{\alpha_0}(s_0))$.

We extend $\lambda\colon \mathscr{S}(V^k) \to I^{\alpha_0}(s_0)$ to $\lambda\colon C^{-\infty}(V^k) \to C^{-\infty}(I^{\alpha_0}(s_0))$ by linearity.

THEOREM 2.3.8. *$\lambda|_{\mathscr{S}^*(V^k)^{O(p,q)}}$ is injective. Moreover,*

$$\lambda\colon \mathscr{S}^*(V^k)^{O(p,q)} \hookrightarrow \mathscr{S}^*(I^{\alpha_0}(s_0)),$$

and it is a topological embedding with closed image.

Proof. Injectivity. For $A \in \mathscr{S}^*(V^k)^{O(p,q)}$ let

$$A = \sum_{\tau \in R_0 \subseteq \hat{\tilde{U}}(k)} A_\tau$$

be its isotypic decomposition into $\tilde{U}(k)$-types. (R_0 is given in Theorem 2.1.15.) Also, let

$$\delta = \sum_{\tau \in R_0 \subseteq \hat{\tilde{U}}(k)} \delta_\tau .$$

By Theorem (2.2.15), $\delta_\tau \neq 0$ for $\tau \in R_0$.

Suppose that $A \neq 0$. We can pick some $\tau \in R_0 \subseteq \hat{\tilde{U}}(k)$ such that $A_\tau \neq 0$. Now for $k' \in \tilde{U}(k)$ we have

$$\lambda(A_\tau)(k') = \omega(k')A_\tau(0) = (\delta, \omega(k')A_\tau) = (\delta_\tau, \omega(k')A_\tau). \tag{2.3.9}$$

See convention (2.1.1) for the equality $\omega(k')A_\tau(0) = (\delta, \omega(k')A_\tau)$.

Since $S^*(V^k)_\tau^{O(p,q)}$ is multiplicity one, $\{\omega(k')A_\tau | k' \in \tilde{U}(k)\}$ spans $S^*(V^k)_\tau^{O(p,q)}$. Since $0 \neq \delta_\tau \in S^*(V^k)_\tau^{O(p,q)}$, expression (2.3.9) cannot be identically zero. This proves that $\lambda|_{\mathscr{S}^*(V^k)^{O(p,q)}}$ is injective.

Topological embedding. For $G = O(p)$ let us prove the following growth estimate for the $\tilde{U}(k)$ Fourier components of the Dirac distribution:

$$\frac{(\phi_\alpha, \phi_\alpha)}{P(\alpha)}(1, 1) \leqslant |(\delta, \phi_\alpha)|^2 \leqslant (\phi_\alpha, \phi_\alpha)(1, 1) \tag{2.3.10}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t)$ is "even" as in (2.1.9), P is some polynomial. See Proposition (2.2.12) for the notations here.

From (2.2.13) and (2.2.14) we have

$$(\delta, \phi_\alpha) = \left(\prod_{1 \leqslant i \leqslant t} \prod_{\substack{1 \leqslant c_i \leqslant \alpha_i - 1 \\ c_i \text{ odd}}} (-1)^i B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, c_i)\right)(1, 1),$$

$$(\phi_\alpha, \phi_\alpha) = \left(\prod_{1 \leqslant i \leqslant t} \prod_{1 \leqslant c_i \leqslant \alpha_i} B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, c_i)\right)(1, 1).$$

Therefore,

$$\frac{|(\delta, \phi_\alpha)|^2}{(\phi_\alpha, \phi_\alpha)(1, 1)} = \prod_{1 \leqslant i \leqslant t} \prod_{\substack{1 \leqslant c_i \leqslant \alpha_i - 1 \\ c_i \text{ odd}}} \frac{B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, c_i)}{B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, c_i + 1)}.$$

Since

$$B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, x) \leqslant B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, y), \qquad 0 \leqslant x \leqslant y, \tag{2.3.12}$$

by the explicit formula (2.2.8) of the function B, we obtain the second equality in (2.3.10).

On the other hand, we have

$$\prod_{\substack{1 \leqslant c_i \leqslant \alpha_i - 1 \\ c_i \text{ odd}}} \frac{B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, c_i)}{B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, c_i + 1)}$$

$$= \frac{B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, 1) B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, 3) \ldots B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i - 1)}{B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, 2) B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, 4) \ldots B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i)}$$

$$\geqslant \frac{B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, 1)}{B(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i)},$$

again by inequality (2.3.12). Clearly, $B(\alpha_1, \ldots, \alpha_i)$ is a polynomial of $\alpha_1, \ldots, \alpha_i$; so we arrive at the first inequality in (2.3.10).

A consequence of the above estimate is the following: Let τ_α be the irreducible representation of $\tilde{U}(k)$ generated by the highest-weight vector ϕ_α. Since $|(\delta, \phi_\alpha)|^2 = |(\delta_{\tau_\alpha}, \phi_\alpha)|^2 \leqslant (\delta_{\tau_\alpha}, \delta_{\tau_\alpha})(\phi_\alpha, \phi_\alpha)$ by the Schwarz inequality, we have $(\delta_{\tau_\alpha}, \delta_{\tau_\alpha}) \geqslant \dfrac{(1, 1)}{P(\alpha)}$. On the other hand, since δ is tempered, we have $(\delta_\tau, \delta_\tau) \leqslant (1 + \|\tau\|^2)^N$ for some N. (See 2.1.3.) Thus,

$$\frac{(1, 1)}{P(\alpha)} \leqslant \|\delta_{\tau_\alpha}\|^2 \leqslant (1 + \|\tau_\alpha\|^2)^N. \tag{2.3.13}$$

We continue the proof of Theorem (2.3.8) for $G = O(p)$. Let $A \in \mathscr{S}^*(V^k)^{O(p)}$. From (2.3.9), $\lambda(A_\tau)(k') = (\delta_\tau, \omega(k')A_\tau)$. Thus,

$$\|\lambda(A_\tau)\|_2^2 = \int_{\tilde{U}(k)} |(\delta_\tau, \omega(k')A_\tau)|^2 \, dk' = \frac{1}{\dim(\tau)} \|\delta_\tau\|^2 \|A_\tau\|^2 \tag{2.3.14}$$

by the Schur orthogonality relation ([Wa2]), where $\dim(\tau)$ is the dimension of τ. Notice that we have used the multiplicity-one property of $\mathscr{S}^*(V^k)_\tau^{O(p)}$, namely, $\mathscr{S}^*(V^k)_\tau^{O(p)}$ is an irreducible representation of $\tilde{U}(k)$ via ω. Also, by the Weyl dimension formula, $\dim(\tau)$ is a polynomial of τ, meaning a polynomial of the highest weight of τ.

Combining the above with (2.3.13), we conclude that

(i) $\lambda|_{\mathscr{S}^*(V^k)_\tau^{O(p)}}: \mathscr{S}^*(V^k)_\tau^{O(p)} \to \lambda(\mathscr{S}^*(V^k)_\tau^{O(p)})$ is a partial isometry, and

(ii) $\|A_\tau\|^2$ is (at most) of polynomial growth if and only if $\|\lambda(A_\tau)\|_2^2$ is (at most) of polynomial growth.

For $G = O(p)$ Theorem (2.3.8) clearly follows from these two assertions. For $G = O(p, q)$ we use (2.2.17), and, therefore, we can obtain a similar growth estimate for the $\tilde{U}(k)$ Fourier components of the Dirac distribution as in (2.3.10). Then exactly the same argument as for $G = O(p)$ yields the desired result for $G = O(p, q)$. □

3.1. $\tilde{K}'$-multiplicity-one property: other four cases. Let $(G, G') \subseteq Sp$ be one of the four reductive dual pairs (2–5) in (0.1.5) and let ω be the (twisted) oscillator representation of $\tilde{S}p$. Recall that K (resp. K') is a maximal compact subgroup of G (resp. G') and that $M((\omega^{-\infty})_\tau^{(\mathfrak{g}, K)})$ is the multiplicity of τ in $(\omega^{-\infty})^{(\mathfrak{g}, K)}$, the $(\mathfrak{g}, K)$-invariants in the space of formal vectors of ω.

We now state the analog to Theorem (2.1.15). Our parametrizations for various irreducible representations are the same as in [KV].

THEOREM 3.1.1. *$M((\omega^{-\infty})_\tau^{(\mathfrak{g}, K)}) \leqslant 1$ for any $\tau \in \hat{\tilde{K}}'$. It is equal to one only if the following conditions are satisfied.*

(2) *$G = U(p, q)$, $G' = U(k, k)$, $\tau = \tau_1 \otimes \tau_2$, $\tau_1, \tau_2 \in \hat{\tilde{U}}(k)$, $\tau_2 \cong \tau_1^*$, and the highest*

weight of τ_1 *is*

$$\left(\frac{p-q}{2}, \frac{p-q}{2}, \dots, \frac{p-q}{2}\right) + (a_1, a_2, \dots, a_t, 0, \dots, 0, -b_s, \dots, -b_2, -b_1),$$

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t > 0, b_1 \geqslant b_2 \geqslant \cdots \geqslant b_s > 0,$$

$$t \leqslant \min(k, p), \quad s \leqslant \min(k, q).$$

(3) $G = Sp(p, q)$, $G' = O^*(4k)$, $\tau \in \hat{U}(2k)$, *the highest weight of* τ *is*

$$(p-q, p-q, \dots, p-q)$$

$$+ (a_1, a_1, a_2, a_2, \dots, a_t, a_t, 0, \dots, 0, -b_s, -b_s, \dots, -b_2, -b_2, -b_1, -b_1),$$

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t > 0, \quad b_1 \geqslant b_2 \geqslant \cdots \geqslant b_s > 0,$$

$$t \leqslant \min(k, p), \quad s \leqslant \min(k, q).$$

(4) $G = Sp(2m, \mathbb{R})$, $G' = O(k, k)$, $\tau = \tau_1 \otimes \tau_2$, τ_1, $\tau_2 \in \hat{\hat{O}}(k)$, $\tau_2 \cong \tau_1^*$, *and the highest weight of* τ_1 *is, if* k *is odd,* $k = 2r + 1$,

$$\left(\frac{m}{2}, \frac{m}{2}, \dots, \frac{m}{2}\right) + (a_1, a_2, \dots, a_t, 0, \dots, 0; \varepsilon), \qquad \varepsilon = (-1)^{a_1 + a_2 + \cdots + a_t},$$

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t > 0, \qquad t \leqslant \min(m, r),$$

or

$$\left(\frac{m}{2}, \frac{m}{2}, \dots, \frac{m}{2}\right) + (a_1, a_2, \dots, a_t, 0, \dots, 0; \varepsilon), \qquad \varepsilon = (-1)^{a_1 + a_2 + \cdots + a_t + 1},$$

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t > 0, \qquad 2r + 1 - m \leqslant t \leqslant \min(m, r),$$

or, if k *is even,* $k = 2r$,

$$\left(\frac{m}{2}, \frac{m}{2}, \dots, \frac{m}{2}\right) + (a_1, a_2, \dots, a_t, 0, \dots, 0)_+,$$

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t > 0, \qquad t \leqslant \min(m, r).$$

or

$$\left(\frac{m}{2}, \frac{m}{2}, \dots, \frac{m}{2}\right) + (a_1, a_2, \dots, a_t, 0, \dots, 0)_-,$$

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t > 0, \qquad 2r - m \leqslant t \leqslant \min(m, r).$$

(5) $G = O^*(2m)$, $G' = Sp(k, k)$, $\tau = \tau_1 \otimes \tau_2$, τ_1, $\tau_2 \in \hat{\hat{S}}p(k)$, $\tau_2 \cong \tau_1^*$, *and the highest weight of* τ_1 *is*

$$\left(\frac{m}{2}, \frac{m}{2}, \ldots, \frac{m}{2}\right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0),$$

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t > 0, \qquad t \leqslant \min(m, k).$$

We give ideas about the proof in the following. We refer the reader to [Zhu1] for details.

One first proves an analog of Proposition (2.1.7) for $G = U(p), Sp(p)$. To do this, one just needs to replace the "seesaw" dual pairs (1.3.6)

$$\begin{array}{cc} O(p) & U(k) \\ \cap\!\!| & \cap\!\!| \\ U(p) & Sp(2k, \mathbb{R}) \end{array}$$

with the "seesaw" dual pairs

$$\begin{array}{cc} U(p) & U(k) \times U(k) \\ \cap\!\!| & \cap\!\!| \\ U(p) \times U(p) & U(k, k) \end{array}$$

and

$$\begin{array}{cc} Sp(p) & U(2k) \\ \cap\!\!| & \cap\!\!| \\ U(2p) & O^*(4k), \end{array}$$

and then applies the corresponding result for the spherical pairs $U(p) \subseteq U(p) \times U(p)$, $Sp(p) \subseteq U(2p)$ from the theory of models of representations ([BGG]) or from classical invariant theory ([H1]), as follows.

An irreducible finite-dimensional representation of $U(p) \times U(p)$, $\pi_1 \otimes \pi_2$, has a $U(p)$-invariant if and only if π_2 is isomorphic to the contragradient of π_1. Moreover, the dimension of $U(p)$ invariants is one.

An irreducible finite-dimensional representation of $U(2p)$ has a $Sp(p)$-invariant if and only if its highest weight is of the form

$$(a_1, a_1, a_2, a_2, \ldots, a_p, a_p).$$

Moreover, the dimension of $Sp(p)$ invariants is one.

For the noncompact cases the reduction result (Proposition 2.1.13) is still valid; i.e., $P: (\omega^{-\infty})_\tau^{(\mathfrak{g},K)} = \mathscr{P}_\tau^{(\mathfrak{g},K)} \to \mathscr{H}(K')_\tau^K$ is injective for $\tau \in R(\tilde{K}', \omega)$, except that, in its proof, one uses the other four sets of diamond dual pairs instead (1.3.14–1.3.17).

Now in all cases except $G = Sp(p, q)$, we have explicit descriptions of pluriharmonics $\mathscr{H}(K')$ given in [VK]. For $G = Sp(p, q)$ the description is analogous; see [Zhu1]. Combining these descriptions with the result in the compact cases, the reduction theorem, everything follows. (See the proof of Theorem (2.1.15).)

3.2. Existence of invariant distributions: other four cases. Let $R(\tilde{K}', \omega)_0$ be the set of $\tau \in R(\tilde{K}', \omega)$ such that $\mathscr{H}(K')_\tau^K \neq 0$. All the highest weights of $\tau \in R(\tilde{K}', \omega)_0$ are listed in Theorem 3.1.1 for the four cases we are concerned with.

THEOREM 3.2.1. *The Dirac distribution at the origin of V^k has a nonzero projection to the τ-type for any $\tau \in R(\tilde{K}', \omega)_0$.*

The present theorem together with Theorem (3.1.1) imply Theorem I for $G = U(p, q)$, $Sp(p, q)$, $Sp(2m, \mathbb{R})$, $O^*(2m)$.

Sketch of proof of Theorem (3.2.1) *for $G = U(p, q)$, $Sp(2m, \mathbb{R})$, $O^*(2m)$.* One first proves analogs of the inner-product formulas (2.2.13), (2.2.14) for the dual pair $(U(p), U(k, k))$, again using the Capelli identity. Since their derivation is parallel to the case of the dual pair $(O(p), Sp(2k, \mathbb{R}))$, we state below the result without proof. (See [Zhu1] for details.)

In the Fock model the $\tilde{U}(2pk)$-finite vectors in ω form a space isomorphic to $\mathscr{P}_{p,k,\bar{p},\bar{k}}$, the space of polynomials on

$$(Q, \bar{Q}) = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1k} & \bar{q}_{11} & \bar{q}_{12} & \cdots & \bar{q}_{1k} \\ q_{21} & q_{22} & \cdots & q_{2k} & \bar{q}_{21} & \bar{q}_{22} & \cdots & \bar{q}_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{p1} & q_{p2} & \cdots & q_{pk} & \bar{q}_{p1} & \bar{q}_{p2} & \cdots & \bar{q}_{pk} \end{bmatrix}.$$

The Dirac distribution at the origin of V^k is of the form

$$\delta = \exp\left(-\frac{\sum_{ij} q_{ij}\bar{q}_{ij}}{2}\right) \qquad \text{(up to a scalar)}.$$

With respect to a suitable Borel subgroup, the simultaneous highest-weight vectors of $\tilde{U}(p) \times \tilde{U}(p) \times \tilde{U}(k) \times \tilde{U}(k)$ are of the form

$$d_1^{\alpha_1} d_2^{\alpha_2} \ldots d_t^{\alpha_t} \bar{d}_1^{\beta_1} \bar{d}_2^{\beta_2} \ldots \bar{d}_t^{\beta_t}$$

where

$$d_i = \det \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1i} \\ q_{21} & q_{22} & \cdots & q_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ q_{i1} & q_{i2} & \cdots & q_{ii} \end{bmatrix}, \qquad \bar{d}_i = \det \begin{bmatrix} \bar{q}_{11} & \bar{q}_{12} & \cdots & \bar{q}_{1i} \\ \bar{q}_{21} & \bar{q}_{22} & \cdots & \bar{q}_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{q}_{i1} & \bar{q}_{i2} & \cdots & \bar{q}_{ii} \end{bmatrix},$$

α_i, β_i are nonnegative integers for $i \leqslant t \leqslant \min(p, k)$.

Denote by $\psi_{\alpha,\beta} = d_1^{\alpha_1} d_2^{\alpha_2} \ldots d_t^{\alpha_t} \bar{d}_1^{\beta_1} \bar{d}_2^{\beta_2} \ldots \bar{d}_t^{\beta_t}$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_t)$. Then we have

$$(\delta, \psi_{\alpha,\beta}) \neq 0$$

if and only if $\alpha = \beta$. Moreover,

$$(\delta, \psi_{\alpha,\alpha}) = \prod_{1 \leqslant i \leqslant t} \prod_{1 \leqslant c_i \leqslant \alpha_i} (-2)^{c_1 + 2c_2 + \cdots + tc_t} B(c_1, c_2, \ldots, c_t)(1, 1),$$

$$(\psi_{\alpha,\alpha}, \psi_{\alpha,\alpha}) = \prod_{1 \leqslant i \leqslant t} \prod_{1 \leqslant c_i \leqslant \alpha_i} (2)^{2c_1 + 4c_2 + \cdots + 2tc_t} B(c_1, c_2, \ldots, c_t)^2(1, 1),$$

and, therefore,

$$|(\delta, \psi_{\alpha,\alpha})|^2 = (\psi_{\alpha,\alpha}, \psi_{\alpha,\alpha})(1, 1).$$

Combining the above inner-product formulas with the explicit descriptions of $\mathscr{H}(K')$ ([KV]), one proves Theorem (3.2.1) for $G = U(p, q)$, $Sp(2m, \mathbb{R})$, $O^*(2m)$. (See the proof of Theorem 2.2.15.)

Sketch of proof of Theorem (3.2.1) *for* $G = Sp(p, q)$. Unfortunately, the author was unable to obtain similar inner-product formulas for the pair $(Sp(p), O^*(4k))$. In this case we proceed as follows.

In the Fock model the $U(4pk)$-finite vectors in ω form a space isomorphic to $\mathscr{P}_{2p,2k}$, the space of polynomials on

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,2k-1} & x_{1,2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{p,1} & x_{p,2} & \cdots & x_{p,2k-1} & x_{p,2k} \\ y_{1,1} & y_{1,2} & \cdots & y_{1,2k-1} & y_{1,2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{p,1} & y_{p,2} & \cdots & y_{p,2k-1} & y_{p,2k} \end{bmatrix}.$$

The Dirac distribution at the origin of V^k has the form

$$\delta = \exp\left(-\frac{\sum_{1\leqslant j\leqslant k}\Lambda_{2j-1,2j}}{2}\right)$$

$$= \exp\left(-\frac{\sum_{1\leqslant i\leqslant p,1\leqslant j\leqslant k}(x_{i,2j-1}y_{i,2j}-y_{i,2j-1}x_{i,2j})}{2}\right) \qquad \text{(up to a scalar)}$$

where

$$\Lambda = \begin{pmatrix}X\\Y\end{pmatrix}^t J_p \begin{pmatrix}X\\Y\end{pmatrix} = X^tY - Y^tX,$$

a $2k \times 2k$ complex skew-symmetric matrix with rank less than or equal to $\min(2p, 2k)$. Let $Pf_n(\Lambda)$ be the nth principal Pfaffian of Λ, $1 \leqslant n \leqslant \min(p, k)$. (See [W1] for its definition.)

Observe that in the twisted oscillator representation, $U(2k)$, a maximal compact subgroup of $O^*(4k)$ acts by $\begin{pmatrix}X\\Y\end{pmatrix} \mapsto (\det g)^{-p}\begin{pmatrix}X\\Y\end{pmatrix}g^{-1}$. (See 1.2.6.) Therefore, $U(2k)$ acts on $\Lambda = \begin{pmatrix}X\\Y\end{pmatrix}^t J_p\begin{pmatrix}X\\Y\end{pmatrix}$ by

$$\Lambda = \begin{pmatrix}X\\Y\end{pmatrix}^t J_p \begin{pmatrix}X\\Y\end{pmatrix} \mapsto (\det g)^{-p}\left(\begin{pmatrix}X\\Y\end{pmatrix}g^{-1}\right)^t J_p\begin{pmatrix}X\\Y\end{pmatrix}g^{-1}$$

$$= (\det g)^{-p} g^{t-1}\Lambda g^{-1}, \qquad g \in U(2k).$$

This action is multiplicity-free ([Shi], [HU]).

For $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t$, $t \leqslant \min(p, k)$, let ρ_{2k}^D be the irreducible representation of Gl_{2k} generated by the highest-weight vector (with respect to $\mathfrak{b}_{2k}^+$, the upper Borel subgroup of Gl_{2k})

$$v_D = Pf_1(\Lambda)^{a_1-a_2}Pf_2(\Lambda)^{a_2-a_3}\ldots Pf_{t-1}(\Lambda)^{a_{t-1}-a_t}Pf_t(\Lambda)^{a_t}$$

with the highest weight $D = (p, p, \ldots, p) + (a_1, a_1, a_2, a_2, \ldots, a_t, a_t, 0, \ldots, 0)$. Let δ_D be the projection of $\delta = e^{-1/2\sum_{1\leqslant j\leqslant k}\Lambda_{2j-1,2j}}$ to the isotypic component of ρ_{2k}^D and

$$\delta_D = \lambda_D v_D + N_D$$

where N_D is a sum of weight vectors in this isotypic component with weights strictly less than that of v_D.

Let H be stabilizer in Gl_{2k} of the linear functional

$$\Lambda \mapsto -\frac{1}{2}\sum_{1\leqslant j\leqslant k}\Lambda_{2j-1,2j}.$$

Clearly, H is isomorphic to $Sp(2k, \mathbb{C})$. Since ρ_{2k}^{D} contains a $Sp(2k, \mathbb{C})$-invariant ([Shi], [H1]), we see from [Zhu2], Theorem 1′ that $\lambda_D \neq 0$. This implies Theorem (3.2.1) for $G = Sp(p)$.

For $G = Sp(p, q)$, Theorem (3.2.1) follows from the nonvanishing of λ_D in the above, the explicit description of $U(2k)$-pluriharmonics (see [Zhu1]) and functorial properties of the Dirac distribution. (See the proof of Theorem 2.2.15.) □

Finally, the topological embedding for the case of $G = U(p, q), Sp(2m, \mathbb{R}), O^*(2m)$ can be proved by following exactly the same argument as for $G = O(p, q)$ and using $|(\delta, \psi_{\alpha,\alpha})|^2 = (\psi_{\alpha,\alpha}, \psi_{\alpha,\alpha})(1, 1)$ instead of (2.3.10). The lack of inner-product formulas for $G = Sp(p)$ explains the possible exclusion of $Sp(p, q)$ in Theorem II.

References

[Ba] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform, I*, Comm. Pure Appl. Math. **14** (1961), 187–214.

[BGG] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Models of representations of Lie groups*, Selecta Math. Sov. **1** (1981), 121–142.

[Ca] P. Cartier, *Quantum mechanical commutation relations and θ-functions*, Proc. Sympos. Pure Math. **9** (1966), 361–383.

[dR] G. deRham, *Solution elementaire d'opereteurs differentiels du second ordre*, Ann. Inst. Fourier (Grenoble) **8** (1958), 338–366.

[EHW] T. Enright, R. Howe, and N. Wallach, "A classification of unitary highest weight modules" in *Representation Theory of Reductive Groups*, ed. P. C. Trombi, Progr. Math. **40**, Birkhäuser, Boston, 1983, 97–143.

[Fo] G. Folland, *Harmonic Analysis in Phase Space*, Ann. of Math. Stud. **122**, Princeton Univ. Press, Princeton, 1989.

[Ga] L. Gårding, "Distributions invariantes" in *Seminaire Leray, 1961–62.*

[Ge] S. Gelbart, "Examples of dual reductive pairs" in *Automorphic Forms, Representations, and L-functions*, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, 1979, 287–296.

[GK1] K. Gross and R. Kunze, *Bessel functions and representation theory, II*, J. Funct. Anal. **25** (1977), 1–49.

[GK2] ———, "Fourier Bessel transforms and holomorphic discrete series" in *Conference on Harmonic Analysis, College Park, Maryland, 1971*, Lecture Notes in Math. **266**, Springer, Berlin, 1972, 79–122.

[GS] I. M. Gelfand and G. E. Shilov, *Generalized Functions, Volume 1*, Academic, New York, 1964.

[H1] R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), 539–570.

[H2] ———, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), 535–552.

[H3] ———, "Dual pairs in physics: harmonic oscillators, photons, electrons, and singletons" in *Applications of Group Theory in Physics and Mathematical Physics*, Lectures in Appl. Math. **21**, Amer. Math. Soc., Providence, 1985, 179–206.

[H4] ———, "θ-series and invariant theory" in *Automorphic Forms, Representations, and L-functions*, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, 1979, 275–286.

[H5] ———, *Quantum mechanics and partial differential equations*, J. Funct. Anal. **38** (1980), 188–254.

[H6] ———, *Oscillator representation: algebraic and analytic preliminaries*, preprint.

[He1] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic, New York, 1962.

[He2] ———, *Groups and Geometric Analysis*, Academic, Orlando, Florida, 1984.

[HT] R. Howe and E. Tan, *Topics in representation theory of Lie groups*, lecture notes in preparation, Yale, 1989.

[HU] R. HOWE AND T. UMEDA, *The Capelli identity, the double commutatnt theorem, and multiplicity-free actions*, Math. Ann., to appear.

[Jo] K. JOHNSON, *On a ring of invariant polynomials on a Hermitian symmetric space*, J. Algebra **67** (1980), 72–81.

[KR] S. KUDLA AND S. RALLIS, *Degenerate principal series and invariant distributions*, Israel J. Math. **69** (1990), 25–45.

[Ku] S. KUDLA, "Seesaw dual reductive pairs" in *Automorphic Forms of Several Variables*, Progr. Math. **46**, Birkhäuser, Boston, 1983, 244–268.

[KV] M. KASHIWARA AND M. VERGNE, *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math. **44** (1978), 1–47.

[Me] P. METHEE, *Sur les distributions invariantes dans le groupe des rotations de Lorentz*, Comment. Math. Helv. **28** (1954), 224–269.

[Se] F. J. SERVEDIO, *Prehomogeneous vector spaces and varieties*, Trans. Amer. Math. Soc. **176** (1973), 421–444.

[Sha] D. SHALE, *Linear symmetries of free boson fields*, Trans. Amer. Math. Soc. **103** (1962), 149–167.

[Shi] G. SHIMURA, *On differential operators attached to certain representations of classical groups*, Invent. Math. **77** (1984), 463–488.

[Ta] M. TAYLOR, *Noncommutative Harmonic Analysis*, Math. Surveys Monographs **22**, Amer. Math. Soc., Providence, 1986.

[Te] A. TENGSTRAND, *Distributions invariant under an orthogonal group of arbitrary signature*, Math. Scand. **8** (1960), 201–218.

[Tr] F. TRÈVES, *Topological Vector Spaces, Distributions, and Kernels*, Academic, New York, 1967.

[VK] E. B. VINBERG AND B. N. KIMELFELD, *Homogeneous domains in flag manifolds and spherical subgroups of semi-simple Lie groups*, Functional Anal. Appl. **12** (1978), 12–19.

[Wa1] N. WALLACH, *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, New York, 1973.

[Wa2] ———, *Real Reductive Groups, I*, Academic, Boston, 1988.

[We] A. WEIL, *Sur certains groups d'operateurs unitaries*, Acta Math. **111** (1964), 143–211.

[W1] H. WEYL, *The Classical Groups*, Princeton Univ. Press, Princeton, 1946.

[Zhe] D. ZHELOBENKO, *Compact Lie Groups and Their Representations*, Transl. Math. Monographs **40**, Amer. Math. Soc., Providence, 1973.

[Zhu1] C. ZHU, *Two topics in harmonic analysis on reductive groups*, thesis, Yale Univ., 1990.

[Zhu2] ———, *A generic H-invariant in a multiplicity-free G-action*, Proc. Amer. Math. Soc., to appear.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742
CURRENT: DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, KENT RIDGE, SINGAPORE 0511, SINGAPORE; MATZHUCB%NUSVM.BITNET@CUNYVM.CUNY.EDU