Invariant Theory and Duality for Classical Groups over Finite Fields, with Applications to their Singular Representation Theory

by

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§0. Introduction: In his celebrated Acta paper [ ], André Weil constructed a certain (projective) unitary representation of a symplectic group over a local field.* This representation has many fascinating properties which have been brought gradually to light by the work of numerous authors (see the bibliography). It now appears that this representation is a central phenomenon linking apparently diverse topics, including classical invariant theory, the theories of theta and abelian functions and of automorphic forms, and quantum mechanics. Until now, this representation has enjoyed the rather ad hoc name "Weil representation". However, in view of the increasing evidence that it is a fundamental object, and because of its origins in physics and by analogy with the origin of the term "spin representation" for orthogonal groups, I am so bold as to attempt to rechristen it: in this paper we shall refer to this representation as the <u>oscillator representation</u>.

The goal of the paper is to give an overview of the more purely group theoretic aspects of the oscillator representation. We limit

* For the real field, this representation had been discussed previously by Shale [ ].

discussion to the case of finite fields for two reasons. On the one hand, most of the phenomena of the general case show up here, and they show up clearly, unobscured by the usual technical analytical smog overspreading the infinite field. On the other hand, this theory has direct and significant applications to the representation theory of the finite groups themselves. The philosophy of cusp forms, enunciated by Harish-Chandra [ ], has served as a fruitful organizing principle for studying the representation theory of reductive groups. In particular, it suggested the conjectures of MacDonald [ ], recently spectacularly verified by Deligne-Lusztig [ ] and by Kazhdan [ ]. The duality phenomenon associated with the oscillator representation also suggests an organizing principle for the representations of the classical groups. This new principle is almost orthogonal to the philosophy of cusp forms, and it may be hoped that combining the two will significantly clarify the problem of computing representations of the classical groups. We will expand this remark. Roughly speaking, one has a notion that "most" representations of a finite reductive group have a certain size, whereas some are smaller, or "singular". The philosophy of cusp forms seems to distinguish very effectively between the non-singular representations, but it does not cope well with singular representations. The duality principle, on the other hand, seems to be sensitive precisely to the degree of singularity of representations, but offers no particular insight into the non-singular representations. Thus the two principles complement each other very well, and should combine effectively.

In order to keep the account on a human scale (an attempt too rarely made in this theory) and to keep the narrative flowing, with focus on the main elements of structure, we do not give complete proofs. Some proofs are omitted, others sketched. Many of the omitted arguments can be found in the existing literature (see bibliography). A fuller account will follow at some time.

§1. The Heisenberg group.

The root and cause of this discussion is the canonical commutation relation of Heisenberg. This is embodied group theoretically in the Heisenberg group, which we now define. Let $F_q = F$ be our finite field. We assume the characteristic of F is odd. Let V be a vector space over F, and let $<\ ,\ >$ be a symplectic (non-degenerate, bilinear, anti-symmetric) form defined on V. The pair $(V, <,>)$ are said to form a symplectic vector space. Often we will mention only V, the form $<\ ,\ >$ being implicit. From $(V, <,>)$, we form $H = H(V)$, the <u>Heisenberg group attached to V</u> as follows. As set $H = V \oplus F$. The multiplication in H is given by

(1) $$(v,s)(v',s') = (v+v',\ s+s'+(\tfrac{1}{2}) < v,v' >).$$

One checks instantly that (1) indeed defines a group law on H, and moreover the following facts hold.

i) H is two step nilpotent.

ii) F is the center $Z(H)$ of H, and the commutator subgroup of H.

iii) A subspace $U \subseteq V$ which is isotropic for $< , >$ (i.e., on which $< , >$ is trivial) is an abelian subgroup of H. If U is maximal isotropic, then $U \oplus F$ is maximal abelian in H.

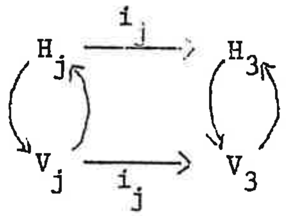
iv) If X and Y are two maximal isotropic subspaces of V, and $V = X \oplus Y$, then H is isomorphic to a semidirect product $X \times_s (Y \oplus F)$.

We see from b) that $H/Z(H)$ is isomorphic to V and from (1) that taking commutators in H induces the given form $< , >$ on this quotient. We will throughout this paper regard V as either a subset or a quotient of H according to convenience. When regarding elements of V as elements of H, we embed them in an ordered pair. Thus, if $v \in V$, then $(v,o) \in H$. We will generally write $Z(H)$ rather than F for the center of F to avoid confusion.

A maximal isotropic subspace of V will also sometimes be called a <u>polarizing subspace</u> or <u>polarization</u>. A pair (X,Y) of polarizing subspaces, which are complementary, so that $X \oplus Y = V$, as in iv) will be called a <u>complete polarization</u> of V. We denote the set of polarizing subspaces of V by Ω.

Suppose V_1 and V_2 are two vector spaces over F, equipped with two symplectic forms $< , >_1$ and $< , >_2$. Then we may form $V_3 = V_1 \oplus V_2$, and equip V_3 with the symplectic form $< , >_3$ such that $< , >_3$ restricts to $< , >_1$ or $< , >_2$ on V_1 or V_2 and $< v_1, v_2 >_3 = 0$ if $v_j \in V_j$. This V_3 is called the <u>orthogonal direct sum</u> of V_1 and V_2. Denote the inclusion of V_j in V_3 by i_j for $j = 1,2$.

Write $H_j = H(V_j)$ for $j = 1,2,3$. It is easy to see that there is a unique inclusion $i_j : H_j \to H_3$ for $j = 1,2$ such that



commutes, the vertical arrows being the canonical inclusion and quotient maps. The groups $i_j(H_j)$ are mutual centralizers in H_3, and the combined map

$$i_1 \times i_2 : H_1 \times H_2 \to H_3$$

is surjective. The kernel of $i_1 \times i_2$ is the antidiagonal $\{(t,-t)\} \subseteq F \times F \simeq Z(H) \times Z(H)$.

Again let V be a symplectic vector space and put $H = H(V)$. The unitary representations of H are the base on which our whole development builds, and we now describe them. Let χ be a non-trivial unitary character of $Z(H)$. Let $L^2(H,\chi)$ be the space of complex-valued functions f on H such that

(2) $$f(hz) = \chi^{-1}(z)f(h) \quad \text{for } h \in H \text{ and } z \in Z(H).$$

We normalize Haar measure on H so that $Z(H)$ has mass 1. We then regard $L^2(H,\chi)$ as a Hilbert space with the corresponding inner product.

(3) $$(f,g) = \int_H f(h)\bar{g}(h)dh, \quad \text{for } f,g \in L^2(H,\chi).$$

In (3), the function $\bar{g}$ is the complex-conjugate of g. We note that $L^2(H,\chi)$ is also an algebra under convolution, given by the usual formula.

(4) $\quad f * g(h) = \int_H f(x)g(x^{-1}h)dx, \quad \text{for} \quad f,g \in L^2(H,\chi)$

Since a given $f \in L^2(H,\chi)$ is determined by its restriction to V, we have an isomorphism $r_\chi = r: L^2(H,\chi) \to L^2(V)$. If we endow V with counting measure and $L^2(V)$ with the corresponding Hilbert space structure, then r is an isometry. Convolution in $L^2(H,\chi)$ is expressed in $L^2(V)$ by twisting the usual convolution with a cocycle

(5) $\quad r(f * g)(v) = \int_V r(f)(y)r(g)(v-y)\chi(\frac{<y,v>}{2})dy.$

The basic result on the representations of H is the following version of the Stone-vonNeuman theorem.

Theorem 1.1: a) For a non-trivial unitary character χ on $Z(H)$, there is, up to unitary equivalence, a unique irreducible unitary representation ρ_χ of H with the property that $\rho_\chi(z) = \chi(z)1$, where 1 here denotes the identity operator.

b) Let $\mathcal{H}(\rho_\chi) = \mathbf{\mathcal{H}}$ be the space of ρ_χ. Let $L^2(\mathcal{H})$ be the algebra of endomorphisms of $\mathcal{H}$. Regard $L^2(\mathcal{H})$ as a Hilbert space with respect to the normalized Hilbert-Schmidt inner product $(T,S) = (\dim \mathcal{H})^{-1}\mathrm{trace}(TS^*)$, where S^* is the adjoint of S. Then $\rho_\chi: L^2(H,\chi) \to L^2(\mathcal{H})$ is an isometric $*$-isomorphism. In particular the operators $\{\rho_\chi((v,o)), v \in V\}$ form an orthonormal basis for $L^2(\mathcal{H})$. Also $\dim\rho_\chi = \#(V)^{1/2}$, where $\#$ indicates cardinality. Also $\mathrm{trace}\rho_\chi(f) = (\dim\rho_\chi)f(1)$.

c) Let V_3 be the orthogonal direct sum of V_1 and V_2, and let ${}^j\rho_\chi$ for $j=1,2,3$ be the representation of $H_j = H(V_j)$ described in a). Then ${}^3\rho_\chi \circ (i_1 \times i_2) = {}^1\rho_\chi \otimes {}^2\rho_\chi$ (outer tensor product).

Here is the standard manner of realizing the representation ρ_χ. Let (X,Y) be a complete polarization of V. Let χ_Y be the extension of χ to $Y \oplus Z(H)$ which is trivial on Y. Then ρ_χ is realizable as the representation induced from χ_Y:

(6) $$\rho_\chi \simeq \mathrm{ind}_{Y \oplus Z(H)}^{H} \chi_Y$$

We can be more explicit. The space $\mathcal{H} = \mathcal{H}(Y,\chi)$ of this indirect representation consists of functions f on H such that

(7) $$f(hy) = \chi_Y^{-1}(y) f(h) \quad \text{for } h \in H, y \in Y \oplus Z(H)$$

and the action is given by left translation:

(8) $$\rho_\chi(h) f(h') = f(h^{-1}h').$$

From fact iv) above, we see the restriction map $r: \mathcal{H}(Y,\chi) \to L^2(X)$ is an isomorphism. Moreover for suitable normalizations of the natural inner products (we will take Haar measure on X to be counting measure), r is an isometry. Thus the space of ρ_χ may be taken to be $L^2(X)$. If this is done, we obtain the following explicit form for the action of ρ_χ.

(9) a) $\rho_\chi((x,o)) f(x') = f(x'-x)$ for $x,x' \in X$.

b) $\rho_\chi((y,o)) f(x') = \chi(\langle y,x' \rangle) f(x')$ for $y \in Y, x' \in X$.

c) $\rho_\chi((o,t)) f(x') = \chi(t) f(x')$ for $t \in F, x' \in X$.

The relization of ρ_χ on $L^2(X)$ by the formulas (9) is called the Schrödinger model of ρ_χ attached to (X,Y). By virtue of the existence of the Schrodinger model, we obtain the following key fact from Frobenius reciprocity.

Theorem 1.2: For every polarizing subspace $Y \subseteq V$, there is in $\mathcal{H}_\chi$ a vector q_Y, unique up to multiples, which is invariant under $\rho_\chi(Y)$.

Recall Ω is the set of polarizing subspaces of V. Theorem 1.2 gives a well defined map

(10) $$\mathcal{Q}:\Omega \to \mathbb{P}\mathcal{H}_\chi$$

where $\mathbb{P}\mathcal{H}_\chi$ is the projectivization of $\mathcal{H}_\chi$. Of course $\mathcal{Q}$ takes Y to the line through q_Y. The map $\mathcal{Q}$ is called the quantization map.

We may define in the standard manner an action σ_χ of $H \times H$ on $\mathcal{L}^2(\mathcal{H}_\chi)$ by the formula

(11) $$\sigma_\chi(h_1,h_2)(T) = \rho_\chi(h_1)T\rho_\chi(h_2^{-1}) \quad \text{for } h_1,h_2 \in H, \text{ and } T \in \mathcal{L}^2(\mathcal{H}_\chi).$$

Of course σ_χ is the outer tensor product $\rho_\chi \otimes \rho_\chi^*$, where ρ_χ^* is the contragredient of ρ_χ. Moreover, theorem 1.1 says σ_χ is isomorphic to the joint left and right actions on $L^2(H,\chi)$. It is important that we can think of σ_χ as a Schrödinger model for another Heisenberg group. Explicitly, given a symplectic vector space V, with form $\langle\, ,\, \rangle$, define V^- to be the same vector space, equipped with the form $-\langle\, ,\rangle$. Denote by $\tilde{V}$ the orthogonal direct sum $V \oplus V^-$. The form on $\tilde{V}$ is written $\widetilde{\langle\, ,\rangle}$. We call $\tilde{V}$ the double of V. We write $H(\tilde{V}) = \tilde{H}(V) = \tilde{H}$.

There are two injections i_1 and i_2 of H into $\tilde{H}$, given by

(12) $$i_1(v,t) = ((v,o),t) \quad \text{and} \quad i_2(v,t) = ((o,v),-t).$$

We see the following diagram commutes

(13) $$\begin{array}{ccccc} H \times H & & \xrightarrow{\;i_1 \times i_2\;} & & \tilde{H} \\ \updownarrow & & & & \updownarrow \\ V \times V & \xrightarrow{1 \times 1} & V \times V^- & \xrightarrow{1} & \tilde{V} \end{array}$$

By means of $i_1 \times i_2$, we may think of $\tilde{H}$ as a quotient of $H \times H$. It is also quite clear that $\sigma_\chi = \tilde{\rho}_\chi \circ (i_1 \times i_2)$, where $\tilde{\rho}_\chi$ is the irreducible representation of $\tilde{H}$ associated to χ. Thus the explicit action of $H \times H$ on $L^2(H,\chi)$ via right and left translation gives a realization of $\tilde{\rho}_\chi$. Let us identify this realization. Our conventions yield the following formula

(14) $\quad \tilde{\rho}_\chi(i_1(h_1) \times i_2(h_2))f(h) = f(h_1^{-1}hh_2) \quad$ for $h_1,h_2h \in H$.

In particular

(15) $\quad$ a) $\rho_\chi((v,v),o))f((v',o) = \chi(<v,v'>)f(v',o)$

$\qquad$ b) $\rho_\chi((v,-v),o))f(v',o) = f(v'-2v)$.

Let $\Delta^+ \subseteq V \oplus V^-$ be the diagonal: $\Delta^+ = \{(v,v):v \in V\}$. Similarly, let $\Delta^- = \{(v,-v):v \in V\}$ be the anti-diagonal. One easily sees that (Δ^-,Δ^+) form a complete polarization in $\tilde{V}$. Moreover the formulas (15) immediately imply the following fact.

Proposition 1.3: Let $d:\Delta^- \to V$ be the map given by $d(v,-v) = 2v$. Then recalling that $r:L^2(H,\chi) \to L^2(V)$ is the restriction map, we see that $d^* \circ r:L^2(H,\chi) \to L^2(\Delta^-)$ identifies the realization of $\tilde{\rho}_\chi$ on $L^2(H,\chi)$ with the Schrödinger model of $\tilde{\rho}_\chi$ attached to (Δ^-,Δ^+).

It will be important for us to have an explicit formula for the quantization map $\mathcal{Q}$ in the coördinates provided by a Schrödinger model. First we will give the well known coördinatization of Ω provided by a choice of complete polarization (X,Y) for V. Let π be the projection of V onto X with kernel Y. Let $Z \subseteq V$ be a third polarizing subspace.

Define a bilinear form $B = B_Z$ on Z by the formula $B(z_1,z_2) = <\pi(z_1),z_2>$ Using the isotropy of X, Y and Z, it may be checked that B is symmetric. Therefore B factors to $\pi(Z)$, and so may be regarded as a bilinear form on $\pi(Z)$. Thus to Z we may associate the pair $(\pi(Z),B_Z)$, and this pair consists of a subspace of X and a symmetric bilinear form on the subspace. It is not hard to check that the correspondence between $Z \in \Omega$ and the set of such pairs is bijective.

Now consider the Schrodinger model attached to (X,Y). We will compute the action of $\rho_X(Z)$ in this model. If $z \in Z$, write $z = \pi(z) + (1-\pi)z$. Therefore

(16) $$(z,o) = ((1-\pi)z,o)(\pi(z),o)(o(\tfrac{1}{2})B(z,z)) \quad \text{where } B = B_Z.$$

Using the formulas (9) we then see that

(17) $$\rho_X(z,o)f(x) = \chi(<(1-\pi)z,x>)\chi(\tfrac{1}{2})B(z,z))f(x-\pi(z)).$$

From (17) it is easy to check the truth of the following assertion.

Proposition 1.4: Given $Z \in \Omega$, let $(\pi(Z),B_Z)$ be its coördinatizing pair relative to the complete polarization (X,Y). Then in the Schrödinger model of ρ_X attached to (X,Y), the vector q_Z is given by

$$q_Z(x) = \chi(-\tfrac{1}{2}B_Z(x,x)) \quad \text{for } x \in \pi(Z)$$
$$q_Z(x) = 0 \quad \text{for } x \notin \pi(Z).$$

§2: The oscillator representation

Let $Sp = Sp(V) = Sp(V, <\ ,\ >)$ be the group of isometries of the symplectic form $<\ ,\ >$ on V. From the definition of $H(V)$, it is clear that for $g \in Sp$, the map $g:(v,t) \to (g(v),t)$ of H is an automorphism of H. Note that g preserves V and acts trivially on $Z(H)$. Conversely, one sees that any automorphism of H which preserves V and acts trivially on $Z(H)$ comes from Sp. Thus we may canonically identify Sp with this subgroup of $Aut(H)$. We will not distinguish notationally between the action of Sp on V and on H.

Since the action of Sp on H is trivial on $Z(H)$, it preserves each character of the center. Thus if we define $g^*(\rho_\chi)$ by the rule $g^*(\rho_\chi)(h) = \rho_\chi(g^{-1}(h))$, we see $g^*(\rho_\chi) \simeq \rho_\chi$ for any $g \in Sp$. There arises in the usual way a projective unitary representation of Sp. That is, for each $g \in Sp$, there is a unitary operator $\omega_\chi(g) \in L^2(H_\chi)$ such that

(1) $$\rho_\chi(g(h)) = \omega_\chi(g)\rho_\chi(h)\omega_\chi(g)^{-1}.$$

The $\omega_\chi(g)$ satisfy $\omega_\chi(gg') = c(g,g')\omega_\chi(g)\omega_\chi(g')$, where $c(g,g')$ is a scalar. In our case, a basic fact is that we may arrange for $c(g,g')$ to be 1.

Theorem 2.1: The $\omega_\chi(g)$ may be chosen so that $\omega_\chi : g \to \omega_\chi(g)$ is a representation. That is, so that $\omega_\chi(gg') = \omega_\chi(g)\omega_\chi(g')$ for all $g,g' \in Sp$. Moreover, the representation ω_χ is uniquely determined except in the case $\#(F) = 3$ and $\dim V = 2$.

The representation ω_χ defined (except in the one exceptional case) by theorem 2.1 will be called the <u>oscillator representation</u> of Sp. Of course it depends on χ. Let us clarify the extent of this dependence.

Let $Z(H)^\wedge$ denote the Pontrjagin dual of $Z(H)$. First note that since $Z(H) \simeq F$, and in particular may be viewed as a one-dimensional vector space over F, if we pick a non-trivial $\chi_1 \in Z(H)^\wedge$ then any other χ in $Z(H)^\wedge$ has the form

(2) $$\chi(z) = \chi_1(az)$$

for some $a \in F^x$, the multiplicative group of F. We write $\chi = \chi_a$ when (2) holds.

Next note that besides Sp, there are automorphisms of H which preserve V but do not act trivially on $Z(H)$. Specifically, let (X,Y) be a complete polarization of V. Let π be projection onto X along Y. For $a \in F^x$, define $\delta_a : H \to H$ by

(3) $$\delta_a(v,t) = ((a-1)\pi(v) + v, at).$$

It is immediate that δ_a is an automorphism of H. Further it is easy to see that

(4) $$\delta_a^*(\rho_{\chi_b}) = \rho_{\chi_{a^{-1}b}} .$$

Moreover δ_a normalizes Sp inside Ad(H), so it also acts on the ω_χ, and the analogue of (4) holds

(5) $$\delta_a^*(\omega_{\chi_b}) = \omega_{\chi_{a^{-1}b}} .$$

Thus the ρ_χ and ω_χ are all conjugate to one another under outer automorphisms. Something more is true. For $a \in F^x$, define $\sigma_a(v,t) = (av, a^2t)$. Then σ_a is again an automorphism of H, and σ_a not only normalizes, but centralizes the action of Sp. Thus, from the obvious relation $\sigma_a^*(\rho_{\chi_b}) = \rho_{\chi_{a^{-2}b}}$, we conclude

(6) $$\omega_{\chi_{a^2b}} \simeq \omega_{\chi_b} \qquad \text{for } a,b \in F^x.$$

On the other hand it will be seen below that ω_{χ_a} and ω_{χ_b} are non-isomorphic if $a^{-1}b$ is not a square in F^x. We summarize.

Proposition 2.2: The action $\omega_{\chi_b} \to \omega_{\chi_{ab}} = \delta_a^*(\omega_{\chi_b})$ defines a simply transitive action of F^x/F^{x2} on the isomorphism classes of oscillator representations. Moreover $\omega_{\chi_{-a}}$ is the contragredient of ω_{χ_a}.

Since $\#(F^x/F^{x2}) = 2$ for finite fields, we see there are precisely 2 isomorphism classes of oscillator representations. Further, if -1 is a square in F, that is, if $\#(F) \equiv 1 \pmod 4$, then each representation will be self-contragredient, and there will be two such, one of which may be taken to be ω_{χ_1}. The character of ω_{χ_1} will then be real. If $\#(F) \equiv 3 \pmod 4$ so that -1 is not a square, then $\omega_{\chi_{\pm 1}}$ represent the two isomorphism classes of oscillator representations, and are mutually contragredient.

In the light of this discussion it seems reasonable to normalize matters by fixing once and for all the character $\chi = \chi_1$, and to understand by "oscillator representation" the particular representation ω_χ.

We do so, and henceforth let χ be implicit, so this representation will be denoted simply by ω.

As is well known, the group F^x/F^{x2} may be thought of as a generating set for the Grothendieck-Witt semigroup of quadratic forms over F. Indeed, there is a strong connection between the oscillator representation and groups of forms, as we will no-- partially explain. To do so, we must show how to combine Weil representations. Let V_1 and V_2 be symplectic vector spaces, and let $V_3 = V_1 \oplus V_2$ be their orthogonal sum. We can define injections $i_j : Sp(V_j) \to Sp(V_3)$ in the obvious manner:

(7) $\quad i_1(g)(v_1,v_2) = (g(v_1),v_2)$ and $i_2(g')(v_1,v_2) = (v_1,g'(v_2))$

for $g \in Sp(V_1)$ and $g' \in Sp(V_2)$ and $v_j \in V_j$. We have the following immediate consequence of theorem 2.1 and theorem 1.1 c).

Proposition 2.3: If ${}^j\omega$ is the oscillator representation of $Sp(V_j)$, then ${}^3\omega \circ (i_1 \times i_2) = {}^1\omega \otimes {}^2\omega$ (outer tensor product).

On the basis of proposition 2.3, we may set up the following formal scheme which gives a framework in which to state our results on the representations of the finite classical groups. The scheme bears some similarities to definitions of L-functions. Let G be any group. Let V be a symplectic space over F. Let $\phi : G \to Sp(V)$ be a homomorphism. Then write $\omega(G,\phi) = \omega \circ \phi$. We call $\omega(G,\phi)$ the Weil representation of G associated to ϕ.

If $\phi_j : G \to Sp(V_j)$, for $j = 1,2$ are two homomorphisms into two symplectic groups, then we say ϕ_1 and ϕ_2 are equivalent if there is a symplectic isometry $T: V_1 \to V_2$ such that $\phi_2 \circ T = T \circ \phi_1$. Whether or not the ϕ_i are equivalent, we let $V_3 = V_1 \oplus V_2$, and define $\phi_1 + \phi_2 : G \to Sp(V_3)$ by the rule $\phi_1 + \phi_2 = \phi_1 \circ i_1 \times \phi_2 \circ i_2$, where $i_j : Sp(V_j) \to Sp(V_3)$ are the maps of (7). We let $RS(G,F) = RS(G)$ be the collection of all possible homomorphisms from G to various $SP(V)$, modulo the above notion of equivalence. With the addition just defined, $RS(G)$ is a commutative semigroup. With this language, we see that proposition 2.3 says that the correspondence $\phi \to \omega(G,\phi)$ for $\phi \in RS(G)$ defines a homomorphism from $RS(G)$ to the multiplicative semigroup of unitary representations of G, with tensor product as operation. In other words

(8) $$\omega(\phi_1 + \phi_2, G) = \omega(\phi_1, G) \otimes \omega(\phi_2, G).$$

Given our particular symplectic vector space $(V, \langle\ ,\ \rangle)$ we might in the above discussion incestuously take $G = Sp(V, \langle\ ,\ \rangle)$. If we do so, we note there is a distinguished subsemigroup of $RS(Sp(V, \langle\ ,\ \rangle))$. To define it, consider the symplectic vector space $(V, a\langle\ ,\ \rangle)$ where $a \in F^x$. Of course $Sp(V,\langle\ ,\ \rangle) = Sp(V, a\langle\ ,\ \rangle)$ as sets. However, if $a \notin F^2$, the symplectic similitude of V which takes $\langle\ ,\ \rangle$ to $a\langle\ ,\ \rangle$ will define a non-trivial outer automorphism of $Sp(V,\langle\ ,\ \rangle)$ so as an element of $RS(Sp(V,\langle\ ,\ \rangle))$, the identity map $1_a : Sp(V,\langle\ ,\ \rangle) \to Sp(V, a\langle\ ,\ \rangle)$ is equivalent to this automorphism of $Sp(V,\langle\ ,\ \rangle)$, and not the identity of $Sp(V,\langle\ ,\ \rangle)$. We will call the subsemigroup of $RS(Sp(V,< , >))$ generated by the identity maps 1_a the <u>linear subsemigroup of $RS(Sp(V,< , >)$</u>. We call the corresponding Weil representations <u>linear Weil representations</u>. Recalling that $\omega = \omega_{\chi_1}$, it is clear that $\omega \circ 1_a = \omega_{\chi_a}$, that is, is just another oscillator representation. We summarize this discussion.

Proposition 2.4: The linear subsemigroup of $RS(Sp)$ is naturally isomorphic to $\widetilde{WO}(F)$, the Grottiendieck-Witt semigroup of (isomorphism classes of) quadratic forms over F. Therefore the semigroup of representations of Sp generated by tensor products of oscillator representations is a homomorphic image of $\widetilde{WO}(F)$. (Actually, it is an isomorphic image.

Particularly interesting representations in this semigroup are the representations $\omega_\chi \otimes \omega_\chi^*$, the tensor product of an arbitrary oscillator representation with its adjoint. Of course, this is just the action of conjugation on $L^2(H_\chi)$ by $\omega_\chi(Sp)$. It was by this action that ω_χ was originally defined using equation (1). From (1) and theorem 1.1 b), and the system of maps

(9) $$L^2(V) \xleftarrow{r} L^2(H,\chi) \xrightarrow{\rho_\chi} L^2(H_\chi)$$

defined in §1, we arrive at the following conclusion.

Theorem 2.5: The representation $\omega_\chi \otimes \omega_\chi^*$ is isomorphic to the natural permutation action of Sp on $L^2(V)$. This isomorphism is accomplished by $r \circ \rho_\chi^{-1}$ in the system (9).

Corollary 2.5.1: If $G \subseteq Sp$, then the intertwining operators for $\omega_{\chi|G}$ are spanned by the images of the G-orbits in V under $\rho_\chi \circ r^{-1}$. In particular, the G-intertwining number of ω_χ is equal to the cardinality $\#(V/G)$ of the orbit space (of G acting on V).

Corollary 2.5.2: In particular, ω_χ consists of 2 irreducible components which are the eigenspaces of $\omega_\chi(-1)$. (Note that ± 1 is the center of Sp). These spaces have dimensions $\frac{1}{2}(\#V)^{1/2} \pm 1)$.

Corollary 2.5.3: We have $|\mathrm{trace}\ \omega_\chi(g)|^2 = \#(\ker(g-1)$ for any $g \in Sp$.

Remarks: a) Were we striving for logical nicety, we could have carried the discussion to this point without theorem 2.1, that is, treating ω_χ as a projective representation (which in fact it is for infinite fields). Then combining proposition 2.4 and theorem 2.5 with the known structure of $\widetilde{\mathcal{W}O}(F)$, we could have concluded that the obstruction to making ω_χ into an actual representation had order dividing 4 . But this obstruction also has order dividing $\dim \rho_\chi = \#(V)^{1/2}$. Since we have taken our field F to have odd characteristic, $\dim \rho_\chi$ and 4 are relatively prime, so the obstruction is trivial, and theorem 2.1 follows. In characteristic 2, this argument fails and theorem 2.1 also fails.

b) Again using the structure of $\widetilde{\mathcal{W}O}(F)$, and particularly the fact that the sum of 4 copies of any quadratic form over F is split, we see the values of the character of ω must be either real or pure imaginary. Of course, when $\#(F) \equiv 1 (\mathrm{mod}\ 4)$ we already knew the characters of distinct oscillator representations takes values only ± 1.

It is probably a good idea to get a somewhat concrete picture of what ω looks like, and, indeed, this will be useful for certain purposes. We can do so quite nicely, at least on a subgroup of Sp, by using a Schrödinger model. Let (X,Y) be a complete polarization for V. Let $P(Y) = P$ be the subgroup of Sp which preserves Y. Let $N = N(Y)$ be the subgroup of P which acts trivially on Y. Let $M = M(X,Y) = P(X) \cap P(Y)$. Then the following facts are well known.

i) $P(Y) = M(X,Y) \times_s N(Y)$ (semidirect product)

ii) The restriction map r_X, sending $m \in M$ to $m|_X$ is an isomorphism $r_X : M \to GL(X)$.

iii) $N(Y)$ acts simply transitively on the set of all maximal isotropic subspaces of V complementary to Y. Attaching $n(X)$ to $n \in N$ defines an isomorphism from $N(Y)$ to $S^2(X^*)$, the space of symmetric bilinear forms on X. Otherwise put, for every $n \in N$, the form $B_n(x,x') = \langle x, n(x') \rangle$ is a symmetric bilinear form on x, and $n \to B_n$ is the stated isomorphism.

Consider the Schrodinger model of ρ_χ attached to (X,Y). The space of this model is $L^2(X)$. It may be computed without difficulty that in this model, the elements of $P(Y)$ act as follows.

(10) $\omega_\chi(n)f(x) = \chi(-(1/2)B_n(x,x))f(x)$ for $n \in N$, $f \in L^2(X)$.

$\omega_\chi(m)f(x) = \mathrm{sgn}(m)f(m^{-1}(x))$ for $m \in M$.

Here sgn is the unique character of order 2 on M.

Remark: From (10) one can read off the eigenspaces for $\omega_\chi(-1)$. These are the even and the odd subspaces of functions on X. We may refer to the corresponding components of ω_χ as ω_χ^+ and ω_χ^- respectively. The smaller component ω_χ^- has dimension $1/2(^{\#}(X)-1)$, and is irreducible on P and multiplicity free on N. It has the smallest possible dimension for a non-trivial representation of Sp.

We will again drop the χ from ω.

Recall that Ω is the set of polarizing subspaces of V, and that theorem 1.2 gives us an embedding $\mathcal{Q}\Omega \to \mathbb{P}\mathcal{H}_\chi$. Of course Ω is a homogeneous space for Sp. In fact $\Omega \simeq Sp/P(Y)$ for $Y \in \Omega$. From the definition of q and formula (1) defining ω, we see that $\mathcal{Q}$ is equivariant for the permutation action of Sp on Ω and the action induced by ω on $\mathbb{P}\mathcal{H}_\chi$. The hyperplane section bundle induces a line bundle

$$\text{(11)} \qquad \begin{array}{c} h(\Omega) \\ \downarrow \\ \Omega \end{array}$$

Theorem 2.1 says $h(\Omega)$ is a homogeneous line bundle. If we observe that in the Schrödinger model for ρ_χ attached to (X,Y) the vector q_Y is just evaluation at 0 in X, then formulas (9) tell us that h arises from the principle bundle

$$\begin{array}{ccc} P(Y) & \to & Sp \\ & & \downarrow \\ & & \Omega \end{array}$$

by taking the character sgn of P(Y).

Evaluation (e.g., taking inner products with) on the vectors q_Y, $Y \in \Omega$ gives a map

(12) $$e: H_\chi \to C(h(\Omega)),$$

where $C(h(\Omega))$ is the space of sections of $h(\Omega)$. The fact that Q was equivariant makes e an intertwining map. Since the vectors $q(\Omega)$ are all even functions in the Schrödinger models, we have the following result.

Proposition 2.6: The intertwining map e of (12) defines an injection of ω^+ into the induced representation $\mathrm{ind}_P^{Sp}\,\mathrm{sgn}$.

As we saw in §1, the space $L^2(H_\chi)$, the operators on H_χ may themselves be viewed as a module for $\tilde{H}$, the double of H. Thus if $\tilde{\Omega}$ is the set of isotropic subspaces of $\tilde{V}$, then $Q(\tilde{\Omega})$ may be viewed as a certain set of operators on H_χ. (Strictly speaking $Q(\tilde{\Omega}) \subseteq \mathbb{P}L^2(H_\chi)$, but we may for purposes of this discussion choose suitable representatives in $L^2(H_\chi)$). It is important to us to know what operators belong to $Q(\tilde{\Omega})$, and we will now compute them.

Recall that by (9) we have the isomorphism $r \circ \rho_\chi^{-1}: L^2(H_\chi) \to L^2(V)$. Under this identification, multiplication in $L^2(H_\chi)$ becomes the convolution in $L^2(V)$ given by formula (5) of §1. Since (Δ^-, Δ^+) is a complete polarization of $\tilde{V}$, we may identify $W \in \tilde{\Omega}$ to a pair (U', B') where $U' \in \Delta^-$ and B' is a symmetric bilinear form on U'. Let d be the map of proposition 1.3, and let $(U,B) \equiv (d(U'), B' \circ d^{-1})$. Then according to proposition 1.4, we have the following formula for q_W on $L^2(V)$.

(13) $$q_W(v) = \begin{cases} 0 & \text{if } v \notin U \\ \chi(-1/2B(v,v)) & \text{if } v \in U. \end{cases}$$

Let U be an arbitrary subspace of V. To U we may attach the space $\delta(U) \in \tilde{\Omega}$ by the rule

(14) $$\delta(U) = \Delta^- \cap (U \oplus U) \oplus \Delta^+ \cap (U^\perp \oplus U^\perp),$$

where $U^\perp$ is the annihilator of U in V with respect to $\langle\ ,\ \rangle$. From this it is clear that the pair corresponding to $\delta(U)$ is $(U,0)$, where 0 means the zero bilinear form. Hence $q_{\delta(U)}$ is just the characteristic function of U. The following fact may be read off from the convolution formula (5) of §1.

Proposition 2.7: If $U \subseteq V$ is isotropic, then up to multiples $q_{\delta(U)}$ is orthogonal projection of H_χ onto the subspace of $\rho_\chi(U)$ fixed vectors. In particular, if $U = \{0\}$, then $q_{\delta(\{0\})}$ is the identity operator.

We know $\tilde{\Omega}$ is a homogeneous space for $\tilde{Sp} = Sp(\tilde{V}, \langle\ ,\ \rangle)$. Moreover, we have the two embeddings i_1 and i_2 of Sp into $\tilde{Sp}$, defined by (7). We will compute the action of $i_1(Sp) \times i_2(Sp)$ on $\tilde{\Omega}$. Take $W \in \tilde{\Omega}$. Write $\tilde{V} = V \oplus V^-$. Let $W_1 = W \cap V$ and $W_2 = W \cap V^-$. Let W_1' be the projection of W on V, and let W_2' be projection of W_2 on V'. Then it is easily seen that $W_1' \subseteq W_1^\perp$, where again W_1 is the annihilator of W_1 with respect to $\langle\ ,\ \rangle$. Similarly $W_2' \subseteq W_2^\perp$. Of course W_1 and W_2 are both isotropic. I claim $W_i' = W_i^\perp$ for $i = 1,2$. If not, then we could find a larger isotropic subspace in $W_i'^\perp$, contradicting maximality of W, so the claim is true. Finally we see that if $a,b,c,d \in V$ and (a,b) and (c,d) belong to W, then $0 = \langle (a,b), (c,d)\rangle^{\sim} = \langle a,c \rangle - \langle b,d \rangle$, so that W defines an isometry from W_1'/W_1 to W_2'/W_2 if both w_1' are considered subspaces of V. The following result is thus clear.

Proposition 2.8: To every $W \in \tilde{\Omega}$ may be associated a triple $t(W) = (W_1, W_2, s)$, where W_1 and W_2 are isotropic subspaces of V, with $\dim W_1 = \dim W_2$, and s is an isometry from $W_2^{\perp}/W_2$ to $W_1^{\perp}/W_1$. If $g_1, g_2 \in Sp$, then $t(i_1(g_1) \times i_2(g_2)(W)) = (g_1(W_1), g_2(W_2), g_1 s g_2^{-1})$. In particular the $i_1(Sp) \times i_2(Sp)$ orbit of W is determined by $\dim W_1$. Also the map $g \to i_1(g)(\Delta^+)$ embeds Sp bijectively in $\tilde{\Omega}$.

Note also that if $U \subseteq V$ is isotropic, then $t\delta(U) = (U,U,1)$, where 1 is here the identity from $U^{\perp}$ to itself, and δ is defined by (14). Thus any element of $\tilde{\Omega}$ has the form $i_1(g)\delta(U)$ for appropriate $g \in Sp$ and isotropic $U \subseteq V$. Similarly, any element of $\tilde{\Omega}$ also has the form $i_2(g)\delta(U)$.

Since it is evident that for $g \in Sp$, $\tilde{\omega}(i_1(g))$ is left multiplication by $\omega(g)$, and $\tilde{\omega}(i_2(g))$ is right multiplication by $\omega(g)^{-1}$, the first part of the next result is clear.

Theorem 2.9: a) Up to multiples, for $g \in Sp$ we have $\mathcal{Q}(i_1(g)(\Delta^+)) = \mathcal{Q}(i_2(g)\Delta^+) = \omega(g)$.

b) An arbitrary element of $\mathcal{Q}(\tilde{\Omega})$ may be expressed in either of the forms $\omega(g)P_U$ or $P_{U'}\omega(g')$, where $g,g' \in Sp$, and U,U' are isotropic subspaces of V, and P_U indicates projection onto the $\rho_\chi(U)$-fixed vectors.

c) $\mathcal{Q}(\tilde{\Omega})$ forms a multiplicative semigroup in $L^2(H_\chi)$, of which $\omega(Sp)$ constitute the invertible elements. That is $q_W q_{W'}$ again belongs to $\mathcal{Q}(\tilde{\Omega})$ for any $W, W' \in \tilde{\Omega}$.

Proof: It remains only to prove c). To do this, it suffices by b) to show that $P_U P_{U'}$ again belongs to $\mathcal{Q}(\tilde{\Omega})$ for any isotropic U, U' in V. Let $Z = U + U'$. Put $U_0 = U \cap Z^\perp$ and $U_0' = U' \cap Z^\perp$. Let $\{e_i\}_{i=1}^{\ell}$ be a basis for U modulo U_0. Let $\{f_i\}_{i=1}^{\ell}$ be a set in U' such that $< e_i, f_j > = \delta_{ij}$ (Kronecker's δ). Then the $\{f_i\}$ form a basis for U' modulo U_0'. Let V_i, for $1 \leq i \leq \ell$ be the span of e_i and f_i, and let $V_0 = (\sum_{i=1}^{\ell} V_i)^\perp$. Then we have direct sum decompositions $V = \bigoplus_{i=0}^{\ell} V_i$, and $U = \bigoplus_{i=0}^{\ell} U_i$, where $U_i = U \cap V_i$, and $U' = \bigoplus_{i=0}^{\ell} U_i'$. The decomposition of V is orthogonal. Therefore ρ_χ decomposes as a tensor product of the ${}^i\rho_\chi$ attached to the V_i, by theorem 1.1 c). Thus it is enough to prove it in the cases when either $\dim V = 2$ with $V = U \oplus U'$, or $U + U'$ is isotropic. In the latter case it is obvious that $P_U P_{U'} = P_{U+U'}$. In the former case, P_U has rank 1, so if $g \in Sp$ is such that $g(U') = U$, we see that $P_U P_{U'}$ is a multiple of $\omega(g) P_{U'}$. This proves the theorem.

Remarks: a) We note that $P_U P_{U'}$ is non-zero. In fact, if (U, U') form a complete polarization, the worst possible case, then trace $(P_U P_{U'}) = (\dim \rho_\chi)^{-1}$.

b) According to the theorem, the operators $\omega(g)$ for $g \in Sp$ correspond to functions on V of the form (13). We will compute the correspondence explicitly. Take $g \in Sp$. As we have noted, $\omega(g) = \mathcal{Q}(i_1(g)\Delta^+)$. Take $(g(v), v)$ in $i_1(g)\Delta^+$, and write

$$(15)\qquad (g(v),v) = \tfrac{1}{2}((g+1)v,(g+1)v) + \tfrac{1}{2}((g-1)v,-(g-1)v)$$

Then compute $\frac{1}{2}<(g-1)v,-(g-1)v),(g(v),v)>^{\sim} = \frac{1}{2}<(g-1)v,(g+1)v>$. Putting $u = (g-1)v$, we see that $\frac{1}{2}<(g-1)v,(g+1)v> = \frac{1}{2}<u,(\frac{g+1}{g-1})u>$. Thus according to (13), the function on V defined by $\omega(g)$ is

$$(16)\qquad \omega(g)(u) = \begin{cases} 0 & \text{if } u \notin (g-1)V \\ (\dim \rho_\chi)^{-1}\operatorname{trace} \omega(g)\chi(-\frac{1}{4}<u,(\frac{g+1}{g-1})u>) & \text{if } u \in (g-1)V \end{cases}$$

c) Note that $\frac{g+1}{g-1}$ is just the Cayley transform of V, at least if $g-1$ is invertible. Thus we have a purely geometric description of the Cayley transform. Note that the open cell of $\tilde{\Omega}$ consisting of polarizing subspaces of $\tilde{V}$ which are transverse to Δ^+ is canonically isomorphic to the space of symmetric bilinear forms on V, and this in turn is canonically isomorphic to the Lie algebra of Sp. If $g-1$ is non-singular, then $i_1(g)\Delta^+$ is transverse to Δ^+, so the Cayley transform maps the Zariski open set of Sp consisting of elements without fixed points to the Lie algebra of Sp.

d) It is possible to characterize those pairs (U,B) which correspond via (13) to spaces $W \in \tilde{\Omega}$ such that $t(W) = (W_1,W_2,s)$ with $W_i \neq \{0\}$. (These are the W such that q_W is not invertible). It may be computed that $W_1 = \{w \in U: B(u,w) = \frac{1}{2}<u,w>\}$.

§3: Reductive dual pairs.

Let S be any group, and let (G,G') be a pair of subgroups of S. We will say G and G' are <u>in duality in S</u>, or that they form a <u>dual pair</u> if G is the centralizer of G' in S and vice versa.

Let $(V,\langle\ ,\ \rangle)$ be our symplectic vector space, and consider a pair of subgroups (G,G') of $Sp = Sp(V,\langle\ ,\ \rangle)$. We will call (G,G') a reductive dual pair in Sp if they are in duality and if they each act (absolutely) reductively on V.

We will classify reductive dual pairs. Suppose $V = V_1 \oplus V_2$ is an orthogonal direct sum, and suppose (G,G') is a reductive dual pair in $Sp(V)$, such that V_1 and V_2 are invariant under both G and G'. Let G_i be the image in $Sp(V_i)$ of the restriction of G to V_i. Define G_i' the same way. Let $i_j:Sp(V_i) \to Sp(V)$ be the canonical injections. A moments thought shows that $G = i_1(G_1) \times i_2(G_2)$ and likewise for G'. In this situation we will say that (G,G') is a direct sum of the pairs (G_i,G_i'). If (G,G') is not the direct sum of smaller reductive dual pairs, we say (G,G') is irreducible. The usual considerations give

Proposition 3.1: Every reductive dual pair is the direct sum of irreducible pairs in an essentially unique way (i.e., up to renumbering of the subpairs.)

Therefore we concentrate on irreducible reductive dual pairs.

Proposition 3.2: Let (G,G') be an irreducible reductive dual pair in $Sp(V)$. Then one of two possibilities obtains.

I. $G\cdot G'$ acts irreducibly on V, and V consists of a single isotypic component (which is self-contragredient) for G or G'.

II. There exists a complete polarization (X,Y) invariant under $G\cdot G'$ and the images by restriction of G and G' in $GL(X)$ or $GL(Y)$ are in duality.

According to which possibility above holds, we will refer to (G,G') as being irreducible of type I or type II respectively. Pairs of type II are classically known, usually in the context of the theory of associative algebras, by virtue of the double commutant theorem. The pairs of type I come up classically also, in the theory of "semisimple algebras with involution," or the "multiplications of Riemann matrices" ([ ],[ ]). The main features separating the present approach from say [ ], is the primacy of Sp, and the lack of favoritism between G and G'. We list below the 3 types of reductive dual pairs occurring over finite fields. The first two are the basic examples of types I and II and are the only examples over algebraically closed fields. The third example, though of type I, would become type II after extension of scalars.

A: Let $(V_0, < , >_0)$ be a symplectic vector space. Let $(U_0, (,))$ be an orthogonal vector space. That is, $(,)_0$ is an inner product (a non-degenerate, symmetric bilinear form) on U_0. Put $V = V_0 \otimes U_0$. On V, define a form $< , >$ by

(1) $< v \otimes u, v' \otimes u' > = < v,v' >_0 (u,u')_0$ for v,v' V_0 and $u,u' \in U_0$.

Then $(V, < , >)$ is a symplectic vector space. Define $j_1: GL(V_0) \to GL(V)$ by $j_1(g)(v \otimes u) = g(v) \otimes u$. Define $j_2: GL(U_0) \to GL(V)$ similarly. Let $O(U_0,(,)_0) = O(U) = O$ be the group of isometries of $(,)_0$ on U. Then (Sp,O) or more properly $(j_1(Sp(V_0)), j_2(O(U)))$ form a reductive dual pair of type I in $Sp(V)$.

B. Let Y_1 and Y_2 be vector spaces. Put $Y = Y_1 \otimes Y_2$ and put $V = Y \oplus Y^*$. On V, define a symplectic form $< , >$ by

(2) $< (y,y^*),(z,z^*) > = z^*(y) - y^*(z)$ for $y,z \in Y$ and $y^*,z^* \in Y^*$.

Define an injection $\alpha: GL(Y) \to Sp(V,< , >)$ by $\alpha(g)(y,y^*) = (g(y),g^{*-1}(y^*))$, where g^* is the usual adjoint of g, given by the formula $g^*(y^*)(y) = y^*(g(y))$. Define an injection $j_1: GL(Y_1) \to GL(Y)$ by $i_1(g_1)(y_1 \otimes y_2) = g_1(y_1) \otimes y_2$ for $y_k \in Y_k$ and $g_1 \in GL(Y_1)$. Define $j_2: GL(Y_2) \to GL(Y)$ similarly. Then $(GL(Y_1), GL(Y_2))$, or more precisely $(\alpha(j_1(GL(Y_1))), \alpha(j_2(GL(Y_2))))$ is a reductive dual pair of type II in $Sp(V)$.

C: Let F' be a quadratic extension of F. (For our finite F, there is precisely one F'.) Let τ be the non-trivial Galois involution of F' over F. Let $(W_i,(,)_i)$ for $i=1,2$ be two Hermitian vector spaces over F'. That is $(,)_i$ is an F'-valued, F-bilinear form on W_i satisfying $(aw,w')_i = a(w,w')_i = \sigma(w',aw)_i$ for $w,w' \in U_i$ and $a \in F'$. Put $V = W_1 \otimes_{F'} W_2$. On V, define another Hermitian form $(,)$ by

(3) $(w_1 \otimes w_2, w_1' \otimes w_2') = (w_1,w_1') \ (w_2,w_2')$ for $w_i,w_i' \in W_i$.

In F', choose any element β such that $\sigma(\beta) = -\beta$. On V, define a symplectic form $< , >$ by

(4) $$< v_1,v_2 > = \beta((v_1,v_2) - (v_2,v_1))$$

Let $U_i = U(W_i,(,)_i)$ be the isometry groups of the forms $(,)_i$, and let $U = U(V,(,))$ be the isometry group of $(,)$. Then we have injections $j_i:U_i \to U$, and U is obviously a subgroup of $Sp(V)$. Thus (U_1,U_2), or $(j_1(U(W_1,(,)_1)),j_2(U(W_2,(,)_2)))$ form a reductive dual pair of type I in $Sp(V)$.

The classes of groups listed in A, B and C above have some formal and geometric properties which will be important to us. We discuss the formal properties first. Let G denote any one of these groups. We refer to G as a classical group. We will call G type I or type II according to the type of pair to which it belongs. In any case, it will be observed that G is always defined as the isometries of a certain form $(,)$ on a certain module Z. (if G is of type II, the form may be considered to be identically zero.) We call Z the standard module of G, and $(,)$ we call the defining form. We call the pair $(Z,(,))$ together a formed space. Without defining it explicitly we clearly have a notion of different kinds of forms, e.g., symmetric, symplectic, Hermitian and trivial. Moreover if (G,G') are two classical groups, and if the defining form of G is of one kind, the defining form of G' will be of a definite other kind. The two kinds of forms are also said to be dual to one another.

Membership of the classical group G in a reductive dual pair (G,G') in $Sp(V)$ implicitly defines an embedding $j:G \to Sp(V)$, and j in turn defines an element in the semigroup $RS(G)$, defined in §2. An element in $RS(G)$ arising in this way will be called linear. From our classification A,B,C, of reductive dual pairs, we see that a linear element in $RS(G)$ arises as follows. Let $(Z,(,))$ be the formed space defining G. Let $(Z',(,)')$ be a formed space of the dual kind. Then, if G is of type I, we have $(V,\langle\ ,\ \rangle) \simeq (Z \otimes Z', (,) \otimes (,)')$. (These tensor products must be appropriately understood in case C). If G is of type II, then of course $V \simeq (Z \otimes Z') \oplus (Z^* \oplus Z'^*)$. In any case, we see that if $(Z_1', (,)_1')$ and $(Z_2, (,)_2')$ are two formed spaces of the kind dual to $(Z,(,))$, we may form the orthogonal direct sum $(Z_1' \oplus Z_2', (,)_1' \oplus (,)_2')$ in the obvious fashion. This makes formed spaces of a given kind into a semigroup, which we will call the Witt semigroup of these forms. We denote a typical such semigroup by $\mathcal{W}$, or more specifically by $\mathcal{W}Sp, \mathcal{W}O$, etc. With this language, we see that the operation which associates to a formed space $(Z',(,)')$ of the kind dual to $(Z,(,))$ the corresponding linear embedding $j \in RS(G)$ defines a homomorphism

$$(5) \qquad J_G: \mathcal{W}' \to RS(G)$$

Here $\mathcal{W}'$ is the Witt semigroup of forms of the type dual to $(Z,(,))$. The image $J_G(\mathcal{W}')$ will be called the linear semigroup of $RS(G)$ and denoted $LRS(G)$.

The Weil representations of G arising from $LRS(G)$ are called linear Weil representations of G. One of our main goals is to analyze their structure. For this we will need to know a little about the structure of $LRS(G)$, hence of $\mathcal{W}'$. We recall some standard facts and establish terminology. The type II case is trivial, so we assume the type I case. For convenience, we drop primes. First it is known that $\mathcal{W}$ is generated by one-dimensional forms (except for $\mathcal{W}Sp$, which is generated by the unique form of degree 2). (This follows from Graham-Schmidt orthogon alization.) Second, if $(Z,(,))$ is a formed space, then $z \in Z$ is called isotropic if $(z,z) = 0$. If Z has no isotropic vectors, it is called anisotropic. If Z is not anisotropic, and $z_1 \in Z$ is an isotropic vector then we may find a second isotropic vector $z_2 \in Z$ such that $(z_1,z_2) = 1$. The span of such a pair $\{z_1,z_2\}$ is called a hyperbolic plane, and the pair itself is called a standard or hyperbolic basis for the plane. If $(Z,(,))$ is the sum of hyperbolic planes we say Z is split. An arbitrary formed space $(Z,(,))$ can be expressed as the direct sum of an anisotropic space and a split space, and the anisotropic space is determined up to isomorphism. (This is because of Witt's theorem, see [ ].) Half the dimension of the split part of Z, which is also the dimension of a maximal isotropic subspace of Z is called the Witt index of Z. Two formed spaces whose anisotropic parts are isomorphic are said to be of the same Witt type. The anisotropic space of a given Witt type is the anisotropic model of that type. Given a formed space $(Z,(,))$, the space $(Z,-(,))$, also denoted Z^- when forms are implicit is called the negative or Witt inverse of $(Z,(,))$. If Z_1 and Z_2 have the same Witt type and conversely then $Z_1 \oplus Z_2^-$ is split. Therefore, if $\mathcal{W}_0$ is the set of Witt types, then $\mathcal{W}_0$ may be thought of as the quotient semigroup of $\mathcal{W}$ by the subsemigroup of split spaces, and $\mathcal{W}_0$ is actually a group. The sum $\tilde{Z} = Z \oplus Z^-$ is the double, or split double of Z.

By convention, all zero dimensional forms are anisotropic. For orientation, we remark that over the finite field F, the only other anisotropic forms are:

a) the two one-dimensional quadratic forms x^2 and ax^2, $a \notin F^2$;

b) the two-dimensional quadratic form defined by the norm form of the quadratic extension F';

c) the one-dimensional Hermitian form $x\tau(y)$ on F', where τ is the Galois involution of F' over F.

We pass now to geometric topics. Let G be a classical group. In analogy with the discussion above a linear module for G will mean the direct sum of a finite number of copies of the basic module of G with a finite number of copies of the contragredient of the standard module. (Explicit inclusion here of the dual module is necessary only in the type II case, since the standard modules of type I groups are self-dual.) Thus if Z is the standard module for G, a linear module has the form $Z^k \oplus Z^{*\ell}$. It may be conveniently represented also as $\mathrm{Hom}(F^k,Z) \oplus \mathrm{Hom}(Z,F^\ell)$ (for unitary groups, replace F by its quadratic extension.) Writing things this way turns the action of g into post and pre-multiplication.

(6) $$g(S,T) = (gS,Tg^{-1}) \quad \text{for} \quad g \in G,\ S \in \mathrm{Hom}(F^k,Z), \quad \text{and} \quad T \in \mathrm{Hom}(Z,F^\ell).$$

We may of course also replace F^k and F^ℓ by any convenient auxiliary vector space of the appropriate dimensions. For type I groups, we may take $\ell = 0$ without loss of generality.

Our first geometric fact is a description of the G-orbits in a standard G-module. For type I groups, the result follows quickly from Witt's Theorem [ ]. For type II groups, it is straightforward.

Proposition 3.3: Let G be a classical group with standard module Z.

a) Let G be of type I. Let A be an auxiliary vector space, and consider the linear G-module $Hom(A,Z)$, where G acts by multiplication on the left. If $T \in HOM(A,Z)$, then the G-orbit of T consists of all $S \in Hom(A,Z)$ such that

i) $\ker S = \ker T$

ii) the pullbacks to $A/\ker T$ $(,)\circ S$ and $(,)\circ T$ of the defining form of G are equal.

In other words, G orbits in $Hom(A,Z)$ may be parametrized by pairs (L,B) where $L \subseteq A$ is a subspace, and B is a form on A/L, of the same kind as $(,)$. B may be degenerate. The pairs (L,B) which actually come from orbits satisfy the conditions: a) $\dim A/L \leq \dim Z$, and b) B must make A/L isometric to some subspace of Z. In particular, all possible pairs arise if and only if the dimension of A is no more than the Witt index of Z.

b) Let G be of type II. Let A_1 and A_2 be auxiliary vector spaces, and consider the linear G-module $Hom(A_1,Z) \oplus Hom(Z,A_2)$. Let (S,T) be a pair of homomorphisms in this module, and let (Q,R) be another pair. Then (Q,R) is in the G-orbit of (S,T) if and only if

i) $\ker Q = \ker S$ and

ii) $\operatorname{im} R = \operatorname{im} T$ and

iii) $RQ = TS$.

Thus G orbits may be parametrized by triples (K,M,D) where $K \subseteq A_1$ and $M \subseteq A_2$ are subspaces and $D \in \operatorname{Hom}(K,M)$. In order for a triple (K,M,D) to arise, it is necessary and sufficient that: a) $\dim A_1/K \leq \dim Z$; b) $\dim M \leq \dim Z$; and c) $\dim A_1/K + \dim M - \operatorname{rank} D \leq \dim Z$. In particular all possible pairs occur if and only if $\dim A_1 + \dim A_2 \leq \dim Z$.

Next we consider the action of an irreducible reductive dual pair (G,G') in $Sp(V)$ on Ω, the variety of maximal isotropic subspaces of V. In particular we are interested in Ω^G, the fixed points of G. They are best described in terms of G'.

Proposition 3.4: Let $(G,G') \subseteq Sp(V)$ be an irreducible reductive dual pair. Let Z and Z' be the standard modules for G and G' respectively. Note that Ω^G is invariant under G'.

a) If (G,G') is of type I, then Ω^G is naturally and G-equivariantly isomorphic to the variety of isotropic subspaces of Z' of dimension $\frac{1}{2}\dim Z'$. Thus Ω^G is homogeneous for G', and is non-empty if and only if G' is split.

b) If G is of type II, then Ω^G is naturally and G-equivariantly isomorphic to the union of the Grassmann varieties of subspaces of Z' of all dimensions. Thus Ω^G is a finite union of homogeneous flag manifolds for G'.

Proof: If (G,G') is of type I, then $V \simeq Z \otimes Z'$, so any G-invariant subspace of V has the form $Z \otimes E$ where E is some subspace of Z'. The formula for $< , >$ on V in terms of the defining forms of G and G', given by formulas (1) and (3) above, shows that $Z \otimes E$ is isotropic if and only if E is isotropic. Clearly also, the equality $2 \dim (Z \otimes E) = \dim V$ holds if and only if $2 \dim E = \dim Z'$. The correspondence $Z \otimes E \to E$ is thus the desired correspondence.

If (G,G') is of type II, then $V \simeq (Z \otimes Z') \oplus (Z^* \otimes Z'^*)$. Thus any G-invariant subspace of V has the form $(Z \otimes E) \oplus (Z^* \otimes D)$, where E and D are subspaces of Z' and Z'^* respectively. We see such a space will be maximal isotropic in V if and only if D is the annilator in Z'^* of E. In particular, E may be arbitrary, and D is then determined. This proves the proposition.

For our third geometric topic we consider two classical groups, G_1 and G_2, both of the same kind. In fact, we assume Z_1 and Z_2, their standard modules, have the same Witt type. Put $Z_3 = Z_1 \oplus \bar{Z}_2$, so that Z_3 is split. Put $G_3 = G(Z_3, (,)_1 \oplus -(,)_2)$. We have the standard injections $i_k: G_i \to G_3$ for $k = 1,2$, defined as in (7) of §2. Since Z_3 is split, it contains, in the type I case, isotropic subspaces of dimension $(\frac{1}{2}) \dim Z_3$. Let Γ be the variety of these spaces. In the type II case, let Γ be a Grassmann variety of subspaces of Z_3 of some dimension. We will describe the $G_1 \times G_2$ orbit structure of Γ generalizing proposition 2.8. The proof is essentially the same as for that result.

Proposition 3.5: Notations as above.

a) Let G_1 and G_2 be of type I. Then each space $Y \in \Gamma$ can be associated to a triple $t(Y) = (D_1, D_2, S)$, where $Y \cap Z_k = D_k$ is an isotropic subspace of Z_k, and $S: D_2^{\perp}/D_2 \to D_1^{\perp}/D_1$ is an isometry. Here $\perp$ indicates the annihilator with respect to the appropriate form. (Note that then necessarily $\dim D_2^{\perp}/D_2 = \dim D_1^{\perp}/D_1$, or $\dim Z_1 - 2\dim D_1 = \dim Z_2 - 2\dim D_2$. This is the only restriction on D_1 and D_2.) The action of $G_1 \times G_2$ on Γ is given by

(7) $$t(i_1(g_1) \times i_2(g_2)(Y)) = (g_1(D_1), g_2(D_2), g_1 S g_2^{-1}), \quad \text{for } g_k \in G_k.$$

In particular the $G_1 \times G_2$ orbits consist of those Y such that $\dim D_k$ is constant. Thus the number of orbits is the minimum of the Witt indices of Z_1 and Z_2, plus one. Hence if Z_2 is anisotropic, the action of G_1 is transitive.

b) Let G_1 and G_2 be of type II. Then to $Y \in \Gamma$ can be attached a quintuple $q(Y) = (D_1, E_1, D_2, E_2, T)$ where $D_k = Y \cap Z_k$, and $E_k = p_k(Y)$, where p_k is projection onto Z_k, and $T: E_2/D_2 \to E_1/D_1$ is an isomorphism. The relations $\dim D_1 + \dim E_2 = \dim D_2 + \dim E_1 = \dim Y$ hold. Otherwise the D's and E's are arbitrary. The action of $G_1 \times G_2$ on Γ is given by

(8) $$q(i_1(g_1) \times i_2(g_2)(Y)) = (g_1(D_1), g_1(E_1), g_2(D_2), g_2(E_2), g_1 T g_2^{-1})$$

Finally we should note that if $(Z, (,))$ is a split formed space, there is a "cell decomposition" of the variety Γ of maximal isotropic subspaces of Z analogous to that for a symplectic space described in the discussion preceding proposition 1.4. We state it formally.

Proposition 3.5: Let $(Z,(,))$ be a split formed space. Let (X,Y) be a pair of maximal isotropic subspaces such that $Z = X \oplus Y$. Let π be projection along Y onto X. Let W be another maximal isotropic subspace. Define $B_W = B$ on W by $B(w_1,w_2) = (\pi(w_1),w_2)$. Then B factors to $\pi(W)$ and defines there a form, still denoted B, of the kind dual to $(,)$. The map $W \to (\pi(W),B_W)$ is bijective from the variety of maximal isotropic subspaces of Z to the set of pairs (M,B) where $M \subseteq X$ is any subspace, and B is any (possibly degenerate) form on M of the kind dual to $(,)$.

§4: Invariants and duality.

The main goal of this study of the oscillator representation is to describe of its restriction to a dual reductive pair (G,G'), and in particular to establish a certain relationship, which we will refer to as duality, between the representations of G and of G' occurring in this restriction. The central result in our development, however, appears much more modest. It describes only a small G-submodule of $\omega|_G$.

As before, we let Ω be the variety of maximal isotropic subspaces of our symplectic space V. We recall the quantization $\mathcal{Q}:\Omega \to \mathbb{P}\mathcal{H}_X$. Let (G,G') be a reductive dual pair in $Sp(V)$. From theorem 1.2 and formula (1) of §2, we know that given $Y \in \Omega^G$, there is a linear character δ, on G, such that

(1) $\omega(g)(q_Y) = \delta(g)q_Y$ for $g \in G$.

Here ω is as before the oscillator representation.

A priori, δ depends on Y, but we will see that actually it does not. (From formulas (10) of §2, we see δ must have order 2, so the possibilities for δ are very limited. In fact, δ will often be trivial.) Proceeding cautiously, we select a linear character δ of G, and put

(2) $I(G,\delta) = \{x \in H_\chi ; \omega(g)(x) = \delta(g)x, \text{ for all } g \in G\}$.

We refer to $I(G,\delta)$ as the δ-eigenspace of G for ω. We also put

(3) $\Omega^{G,\delta} = \{Y \in \Omega, q_Y \in I(G,\delta)\}$.

It will turn out that $\Omega^{G,\delta}$ is non-empty for at most one δ.

Theorem 4.1: Let (G,G') be a reductive dual pair in $Sp(V)$. Let δ be a linear character of G such that $\Omega^{G,\delta}$ is non-empty. Then $\mathcal{Q}(\Omega^{G,\delta})$ spans $I(G,\delta)$.

Remark: This theorem makes no statement if $\Omega^{G,\delta}$ is empty. We will see later, that in a certain stable sense, the theorem remains true when $\Omega^{G,\delta}$ is empty. That is, if Z and Z' are the standard modules for $\mathcal{G}$ and $\mathcal{G}'$, and if $\dim Z'$ is less than $\dim Z$, then if $\Omega^{G,\delta}$ is empty, the space $I(G,\delta)$ is zero.

Proof: As a preliminary reduction, note that if (G,G') is the direct sum of (G_1,G_1') and (G_2,G_2'), then if $\delta_i = \delta|_{G_i}$, we have $\Omega^{G,\delta} = \Omega^{G_1,\delta_1} \times \Omega^{G_2,\delta_2}$ and also $I(G,\delta) = I(G_1,\delta_1) \otimes I(G_2,\delta_2)$. Thus it suffices to prove the theorem when (G,G') is irreducible, so we now consider that case.

Take $Y \in \Omega^{G,\delta}$. Then we may find $X \in \Omega^G$ complementary to Y. We realize ρ_X by the Schrödinger model attached to (X,Y). In this case, restriction of G to X induces an injection $r:G \to GL(X)$. We use r to consider G as a subgroup of $GL(X)$, and will not distinguish between g and $r(g)$ for $g \in G$. The formula (10) of §2 gives $\omega_{|G}$. As we have arranged matters, q_Y is the point mass at the origin in X, so we see $\delta = sgn_{|G}$. Moreover $\omega_{|G}$ is simply the twisting by δ of the permutation representation of G acting on X. We can see from proposition 1.4, that if $M \in \Omega^G$, then q_Y is left-invariant by the transformation any $g \in G$ induces on X. Therefore $\Omega^{G,\delta} = \Omega^G$.

Let $\mathcal{O} \subseteq X$ be a G-orbit. From the explicit form of ω, we see that there is precisely one δ-eigenvector for G supported on $\mathcal{O}$, namely the characteristic function of $\mathcal{O}$. Call this $\Delta(\mathcal{O})$. It is clear that the $\Delta(\mathcal{O})$, as $\mathcal{O}$ runs through all orbits, form a basis (even an orthogonal basis) for $I(G,\delta)$. Thus to prove the theorem, it will suffice to show that, for each $\mathcal{O}$, $\Delta(\mathcal{O})$ is a linear combination of q_Y's for $Y \in \Omega^G$. To do this amounts to comparing propositions 3.3 and 3.4, using proposition 3.6. As in those results, it is convenient to argue separately the cases of type I and type II.

Begin with type I. We know from porposition 3.4 that if $\Omega^{G,\delta} = \Omega^G$ is non-empty then G' is split. Let Z and Z' be the standard modules for G and G', so that $V \simeq Z \otimes Z'$. By proposition 3.4 we may write $Y = Z \otimes Y'$ and $X = Z \otimes X'$, where X' and Y' are maximal isotropic subspaces of Z' such that $Z' = X' \oplus Y'$. Similarly, if W is any other point in Ω^G, then $W = Z \otimes W'$ with W' a maximal isotropic subspace of Z'. By proposition 3.6, we may attach to W' a pair (M',B') where $M' \subseteq X'$ is a subspace, and B' is a form on M', of the same kind as $(,)$, the defining form of G. Also, by the discussion preceding proposition 1.4, we may attach to W a pair (M,B) where $M \subseteq X$ is a subspace, and B is a symmetric bilinear form on M. The relation between (M',B') and (M,B) is seen easily to be

(4) $$M = Z \otimes M', \text{ and } B = (,) \otimes B'.$$

(Again one must interpret these tensor products appropriately in the unitary case (case C).)

Now consider orbits. By means of the defining form $(,)'$ on Z', there is defined a faithful pairing between X' and Y'. This allows us to define an isomorphism $\alpha : X' \to Y'^*$ by the formula

(5) $$\alpha(x)(y) = (y,x) \quad \text{for } x \in X' \text{ and } y \in Y'.$$

By means of α we have $X = Z \otimes X' \simeq \mathrm{Hom}(Y',Z)$. By proposition 3.3 we may associate to a G-orbit $\mathcal{O}$ in X a pair (L',C'), where $L'^{\perp} \subseteq Y'$ is a subspace and C' is a form on Y'/L' of the same kind as $(,)$. Note that the linear span of $\mathcal{O}$ in X is $Z \otimes \alpha^{-1}(L')$, where $L'^{\perp} \subseteq Y'^*$ is the annihilator of L.

Consider simultaneously an orbit $\mathcal{O}$ with associated pair (L',C') and a point $W \in \Omega^G$, with associated pair (M',B'). Suppose that $M = Z \otimes M'$ is the span of $\mathcal{O}$, so that $\alpha(M') = L'^{\perp}$. Then q_W is non-zero on $\mathcal{O}$, and since W is G-invariant, q_W, considered as a function on X, is constant on $\mathcal{O}$. Hence we may speak of the value $q_W(\mathcal{O})$. (We take q_W as given in proposition 1.4.). Let us compute this value. Observe that Y'/L' and $M' \simeq L'^{\perp}$ are mutually dual. Therefore, the space of forms of the same kind as $(,)$ on Y'/L' and M' are also dual, and two such forms may be paired to yield an element of F. Precisely, a form C' on Y'/L' defines a map $T_{C'}: Y'/L' \to M'$ by the formula

(6) $$\alpha(T_{C'}(y_1))(y_2) = C'(y_2, y_1) \quad \text{for } y_1, y_2 \in Y'/L'.$$

Similarly a form B' on M' defines a map $T_{B'}: M' \to Y'/L'$. We put

(7) $$\{B', C'\} = \operatorname{trace}(T_{C'} T_{B'})$$

If we trace through the relationships outlined above we find the formula

(8) $$q_W(\mathcal{O}) = \chi(-\tfrac{1}{2}\{B', C'\}).$$

Now let W vary through all elements $\check{W}$ of Ω^G whose associated pairs are $(\check{M}', \check{B}')$ with $\check{M}' = M'$ and $\check{B}'$ arbitrary. Consider the sum

(9) $$\Sigma_{\mathcal{O}} = \sum_{\check{W}} \chi(\tfrac{1}{2}\{\check{B}', C'\}) q_{\check{W}} .$$

We know $\Sigma_{\mathcal{O}}$ is supported in $M \subseteq X$. Let $\check{\mathcal{O}}$ be another orbit contained in M. If $\check{\mathcal{O}}$ spans M, then $\check{\mathcal{O}}$ is associated to the pair $(L', \check{C}')$, with $\check{C}' \neq C'$. Elementary Fourier analysis therefore says $\Sigma_{\mathcal{O}}(\check{\mathcal{O}}) = 0$. On the other hand, clearly $\Sigma_{\mathcal{O}}(\mathcal{O})$ is equal to the cardinality of the space of forms on M' of the kind of $(,)$, and in particular is non-zero. Therefore the support of $\Sigma_{\mathcal{O}}$ is $\mathcal{O} \cup_{\alpha} \{\mathcal{O}_\alpha\}$, where the $\mathcal{O}_\alpha$ are G-orbits whose linear span is a proper subspace of M. By induction on the dimension of the span of $\mathcal{O}$, we may assume each $\Delta(\mathcal{O}_\alpha)$ is a sum of q_W's. By the construction of $\Sigma_{\mathcal{O}}$, therefore, we see we can continue the induction, and the result is proved.

The proof in the type II case is similar but easier. In this case we have $V \simeq (Z \otimes Z') \oplus (Z^* \otimes Z'^*)$, where Z and Z' are the basic modules for G and G' respectively. Since $\Omega^{G,\delta} = \Omega^G$, we may as well take $X = Z \otimes Z'$ and $Y = Z^* \otimes Z'^*$. By proposition 3.4, Ω^G is in bijection with subspaces M of X of the form $M = Z \otimes M'$, with M' an arbitrary subspace of Z'. Furthermore, $\mathcal{Q}(\Omega^G)$ consists simply of the characteristic functions of the spaces M. On the other hand, proposition 3.4 tells us that each G-orbit $\mathcal{O}$ in X is determined by its linear span. Thus it is again clear that an argument by induction on the dimension of the linear span of $\mathcal{O}$ will prove the result.

By proposition 3.4, we see that implicit in theorem 4.1 is a description of $I(G,\delta)$ as a G'-module. The following point of view makes this clearer. Recall from the discussion preceding proposition 2.6 that the embedding $\mathcal{Q}:\Omega \to \mathbb{P}H_X$ allows us to pull back the hyperplane section bundle to obtain a line bundle h over Ω. The bundle h is Sp(V)-homogeneous, so Sp(V) acts on $C(h(\Omega))$, the space of sections of h. Further taking inner products with the q_Y, $Y \in \Omega$ yields an intertwining map $e:\mathcal{H}_\chi \to C(h(\Omega))$ between ω and the natural action of $Sp(V)$ on $C(h(\Omega))$. Since Ω^G is a subvariety of Ω, we may restrict h to Ω^G. This will give a G'-homogeneous line bundle over Ω^G. Theorem 4.1 says that if we restrict e to $I(G,\delta)$, and restrict the resulting sections of h to Ω^G, we get an injective mapping

$$e_G:I(G,\delta) \to C(h(\Omega^G)). \tag{10}$$

Since e_G is an intertwining operator for G', we have described $I(G,\delta)$ as a G' submodule of $C(h(\Omega^G))$. In this connection, and for general reasons also, it is of interest to know the possible G' submodules of $C(h(\Omega^G))$ and which of these are in the image of e_G. This matter will be taken up more thoroughly later on. Now we wish simply to point out that the structure of the proof of 4.1 yields some preliminary information on this score. First, it reduces the question of the dimension of $I(G,\delta)$ to counting orbits. Second, we know that Ω^G has a "cell decomposition" as described in proposition 3.6. We see from the proof that the restriction of e_G to those elements Y of Ω^G such that the support of q_Y has dimension at most $(\dim Z)^2$ is already faithful. In the other direction the following fact is a consequence of proposition 3.3.

Corollary 4.11: If (G,G') is an irreducible reductive pair, with Z and Z' as standard modules, and Ω^G non-empty, then $e_G:I(G,\delta) \to C(h(\Omega^G))$ is surjective if and only if the dimension of a maximal isotropic subspace of Z' is no greater than the dimension of a maximal isotropic subspace of Z.

Now we will see how the theorem 4.1 describing the δ-isotypic component of $\omega|_G$ leads to a description of the whole representation $\omega|_{G'}$ and to a duality between G and G'. We will assume (G,G') is irreducible for this discussion. This arises from two facts. First, as is well-known, the G-intertwining operators for ω are just the G-invariants for $\omega \otimes \omega^*$. Second, if $\tilde{V}$ is the double of V, and if $\tilde{Z}'$ is the double of the basic module for G', then $\tilde{V} = Z \otimes \tilde{Z}'$. Let $\tilde{G}' = G(\tilde{Z}', (,)' \oplus -(,)')$. Recall the injections i_1 and i_2 of $Sp(V)$ into $Sp(\tilde{V})$. Then on the one hand $\omega \otimes \omega^* = \tilde{\omega} \circ (i_1 \times i_2)$ by proposition 2.3. On the other hand $((i_1 \times i_2)(G), \tilde{G}')$ are a reductive dual pair in $\tilde{Sp}$. In the language of §3, if the pair (G,G') corresponds to an imbedding $j: G \to Sp(V)$, and if $-j$ is the corresponding mapping when Z' is replaced by Z'^-, then knowledge of the intertwining operators for the linear Weil representation $\omega(G,j)$ is the same as knowledge of the G-invariants for the linear Weil representation $\omega(G, j \oplus -j)$.

In any case, as a preliminary result we may state the following corollary of theorem 4.1. As before $\tilde{\Omega}$ is the variety of maximal isotropics in $\tilde{V}$, and $\tilde{\mathcal{Q}}: \tilde{\Omega} \to L^2(H_\chi)$ is the operator-valued quantization map. We let $\tilde{\Omega}^G$ be the $i_1 \times i_2(G)$ fixed points in $\tilde{\Omega}$.

Corollary 4.1.2: Given an (irreducible) reductive dual pair $(G,G') \subseteq Sp(V)$, the algebra of intertwining operators for $\omega|_G$ is spanned by $\tilde{\mathcal{Q}}(\tilde{\Omega}^G)$. The elements of $\tilde{\mathcal{Q}}(\tilde{\Omega}^G)$ are linearly independent if $\dim Z'$ is at most the dimension of a maximal isotropic subspace of Z.

By theorem 2.9, the (projectivization of) $\tilde{\mathcal{Z}}(\tilde{\Omega}^{G})$ is a semigroup containing $\omega(G')$ as a subgroup. Similarly $\tilde{\mathcal{Z}}(\tilde{\Omega}^{G'})$ is a semigroup containing $\omega(G)$. If we were willing to consider that these semigroups were the correct objects of study, then we could rephrase corollary 4.1.2 to say that each of $\tilde{\mathcal{Z}}(\tilde{\Omega}^{G})$ and $\tilde{\mathcal{Z}}(\tilde{\Omega}^{G'})$ spanned the others commuting algebra. Thus there is a perfect duality between the two semigroups. If, however, we are more conventional - minded and insist on a duality between G and G', rather than the semigroups containing them, we must do more work. It turns out that this extra work reveals more interesting sturcture for ω.

To begin, fix a classical group G, and consider all linear Weil representations of G simultaneously. These form a semigroup under tensor product, and this semigroup is an image of $\mathcal{W}'$, the Witt group of forms of the kind dual to the defining form of G. There is a homomorphism $\deg: \mathcal{W}' \to \mathbb{Z}^{+}$ take a formed space $(Z', (,)')$ to the integer $\dim Z'$. A Weil representation of G corresponding to an element of $\mathcal{W}'$ of degree $\mathcal{W}$ will be called a <u>Weil representation of degree ℓ</u> of G. Weil representations of degree zero are the trivial representation. Also there is in $\mathcal{W}'$ a unique smallest split form. For type I forms, this is the hyperbolic plane. For the type II situation, this is just the class of one-dimensional spaces (lines) over F. The Weil representation of G corresponding to this smallest split space is, up to twisting by δ, just the permutation action on $L^2(Z)$, where Z is the basic module for G. If this smallest split space arises by doubling an anisotropic form (e.g., for $\mathcal{W}O$ and $\mathcal{W}U$) then the Weil representation is precisely the permutation representation, that is, $\delta = 1$. For the other cases (e.g., $\mathcal{W}Sp$ and $\mathcal{W}GL$), however, δ may be non-trivial. Since in these cases, the smallest split form generates $\mathcal{W}'$, we may twist all Weil representations by an appropriate power of δ, in such a way that they still form a semigroup under tensor product, and are homomorphic with $\mathcal{W}'$, but now the smallest split form in $\mathcal{W}'$ corresponds exactly to the permutation action of G on $L^2(X)$. We call the resulting representations modified (linear) Weil representations. For the rest of this section, all Weil representations will be understood to be linear and modified. Modifying the Weil representations has the effect of making theorem 4.1 into simply a computation of invariants, rather than of δ-eigenvectors.

Let ν be the modified Weil representation corresponding to the smallest split space in $\mathcal{W}'$. Let ν^{ℓ} be the ℓ-fold tensor power of ν. Let ω be any modified Weil representation corresponding to an anisotropic space. (We include the possibility that ω is of degree zero, i.e., trivial). The representations $\omega \otimes \nu^{\ell}$ are the modified Weil representations corresponding to forms of a given Witt type. We will call them a Witt series of Weil representations. If ω_1 and ω_2 are members of the same Witt series, then $\omega_1 \otimes \omega_2^* = \nu^k$ for $k = \frac{1}{2}(\deg \omega_1 + \deg \omega_2)$ for G type I, and $k = \deg \omega_1 + \deg \omega_2$ for G type II. In particular $\omega_1 \otimes \omega_2^*$ is split. Therefore theorem 4.1 describes the invariants in $\omega_1 \otimes \omega_2^*$, that is, the intertwining operators from ω_2 to ω_1. Thus we will use theorem 4.1 to study a given Witt series of G. Then we will discuss the relations between different Witt series.

We will treat types I and II separately, beginning with type I. In the representation ν of G on $L^2(Z)$, the point mass at the origin is an invariant vector for G. Furthermore, Z is implicitly identified with the first member of a complete polarization (Z,Y) of a symplectic vector space V. Thus the point mass at the origin is actually q_Y. Further, the projection onto q_Y commutes with G, and is actually an element of $\mathfrak{H}^G$, namely the element P_Y or $q_{\delta(Y)}$ in the notations of proposition 2.7 and theorem 2.9. In the ensuing discussion we will denote it by P. We will denote the point mass at 0, the image of P, by μ. Let ω be any Weil representation of G. We define an intertwining operator $R:\omega \to \omega \otimes \nu$ by the formula

(11) $$R(x) = x \otimes \mu, \text{ for any } x \text{ in the space.}$$

It is clear that, if R^* is the adjoint of R, then

(12) $$R^*R = I_\omega, \text{ the identity of } \omega, \text{ and } RR^* = I_\omega \otimes P.$$

Fix an anisotropic Weil representation ω_0 of G_1 and put $\omega_i = \omega_0 \otimes \nu^i$. The ω_i form a Witt series of Weil representations. Let $R_i:\omega_i \to \omega_{i+1}$ be the intertwining operator defined by taking $\omega_i = \omega$ in (11). For $i > j$, define S_{ij} by the rules

(13) $$S_{i+1i} = R_i, \text{ and } S_{i+1j} = R_iS_{ij}.$$

For $i \leq j$, put $S_{ij} = S^*_{ji}$, and let S_{ii} be the identity on ω_i. From (12) one sees that the S_{ij} satisfy the following rules

(14) $S_{ij}S_{jk} = S_{ik}$ if $i \leq j \leq k$ or $i \geq j \geq k$ or $j \geq i \geq k$

$S_{ij}S_{ji} = S_{ii}$ if $j \geq i$

$S_{ij}S_{ji}$ is a projection operator in ω_i, of the form P_U, in the sense of theorem 2.9, when $j < i$.

Let Z'_0 be a formed space representing the element of $\mathcal{W}'$ to which ω_0 corresponds. Let Z'_h be the hyperbolic plane - the minimal split space. Then ω_i corresponds to $Z'_i = Z'_0 \oplus Z'^{i}_h$, and $\omega_i \otimes \omega^*_j$ corresponds to $Z'_i \oplus Z'^{-}_j = Z'_{ij}$. Let $I(\omega_i,\omega_j)$ be the space of intertwining maps from ω_j to ω_i. Let Γ_{ij} be the variety of maximal isotropic subspaces of Z'_{ij}. From theorem 4.1, we know that by means of proposition 3.4 and the quantization map $\mathcal{Q} = \mathcal{Q}_{ij}$, the variety Γ_{ij} is mapped to a spanning set in $I(\omega_i,\omega_j)$. Let G'_i be the isometry group of Z'_i. Then Γ_{ij} is acted on by $G'_i \times G'_j$, and $I(\omega_i,\omega_j)$ is a $G_i \times G_j$ module, and $\mathcal{Q}_{ij}$ is equivariant for these actions. The $G_i \times G_j$ orbit structure for Γ_{ij} is described in proposition 3.9. Supposing that $i \leq j$, that proposition says there is one $G_i \times G_j$ orbit for each possible dimension ℓ, $0 \leq \ell \leq i$, of isotropic subspace of Z'_i. We will denote the orbit corresponding to the isotropic subspaces of Z'_i of dimension ℓ $\Gamma_{ij,\ell}$, and refer to it as the <u>stratum of level</u> <u>ℓ</u> in Γ_{ij}. Note that $\Gamma_{ij,0}$, the stratum of level zero, is a homogeneous space for G_j acting alone. The following result is more or less immediate from equations (11) through (14).

Theorem 4.2: Notations as above

a) For $\ell \leq i \leq j$ the operator $S_{i\ell}S_{\ell j}$ belongs to the stratum of level $i-\ell$ in $\mathcal{Q}_{ij}(\Gamma_{ij})$. Thus

(15) $$\mathcal{Q}_{ij}(\Gamma_{ij}) = \bigcup_{\ell=0}^{i} \omega_i(G_i')S_{i\ell}S_{\ell j}\omega_j(G_j')$$

and the union is disjoint.

b) For $i \leq j$, consider the map m from $\mathcal{Q}(\Gamma_{ij,0}) \times \mathcal{Q}(\Gamma_{ij,0})$ to $I(\omega_j,\omega_j)$ given by

(16) $$m(S,T) = S^*T \quad \text{for} \quad S,T \in \mathcal{Q}(\Gamma_{ij,0}).$$

Then m is equivariant for the obvious actions of $G_j \times G_j$, and the image of m is precisely $\mathcal{Q}(\Gamma_{j,j-i})$.

As we have seen, we may always embed ω_i in ω_{i+1}. Furthermore, for small i, the representation ω_i is quite small, and it grows with i until eventually every representation of G must occur in ω_i. Let ω_{ii} be the subrepresentation of ω_i consisting of all isotypic components of ω_i which do not occur in ω_j for any $j < i$. (It would suffice to take $j = i=1$). Let ω_{ij}, for $j < i$, be the sum of all isotypic components of ω_i which occur in ω_{jj} also. Then clearly

(17) $$\omega_i = \sum_{j=0}^{i} \omega_{ij} \quad \text{(orthogonal direct sum)}$$

Of course for large i, some of the summands will be zero. Of course orthogonal projection onto ω_{ij} will be in $I(\omega_i,\omega_i)$, and will be a central idempotent, generating a two-sided ideal I_{ij} in $I(\omega_i,\omega_i)$.

Theorem 4.3: a) For any $j \leq i$, the set of operators $\bigcup_{k=i-j}^{i} \mathcal{Q}(\Gamma_{ii,k})$ form a subsemigroup of $\mathcal{Q}(\Gamma_{ii})$. This subsemigroup spans the ideal $\sum_{k=0}^{j} I_{ik}$.

b) The restriction of G_i' to ω_{ii} generates the commuting algebra of G there. In particular, G_0' generates the commuting algebra of G for ω_0.

Proof: We proceed by induction on j. It is clear that the operators S^*T, with $S,T \in I(\omega_j,\omega_i)$ generate the ideal $\sum_{k=0}^{j} I_{ik}$ in $I(\omega_i,\omega_i)$. Therefore theorem 4.2 b) shows $\mathcal{Q}(\Gamma_{ii,i-j}) \subseteq \sum_{k=0}^{j} I_{ik}$. On the other hand, we may assume the result is true for $j-i$. Further, from theorem 4.2. a), we know that $\mathcal{Q}(\Gamma_{ij,\ell})$ for $\ell > 0$ actually intertwines ω_i with $\sum_{k=0}^{j-1} \omega_{jk}$. Therefore $I(\omega_j,\omega_i)$ is spanned by $\mathcal{Q}(\Gamma_{ij,0})$ together with intertwining operators with images in $\sum_{k=0}^{j-1} \omega_{jk}$.

Therefore, using 4.2 b) again, we conclude $\sum_{k=0}^{j} I_{ik}$ is spanned by $\sum_{k=0}^{j-1} I_{ik}$ together with $\mathcal{Q}(\Gamma_{ii,i-j})$, and part a) is proved. Part b) follows now from part a), using theorem 4.1 and recalling that $\Gamma_{ii,0} = \omega(G_i')$.

Turn now to the type II case. Here ν, the permutation action of G on $L^2(Z)$, generates the semigroup of linear modified Weil representations, which are all of the form ν^{ℓ}, for some positive integer ℓ. In ν there are two G-invariant vectors, the point mass at the origin, and the constant function on Z. We call them μ_1 and μ_2. If $V = Z \oplus Z^*$, with the usual symplectic form, then Z and Z^* are the two G-invariant maximal isotropic subspaces of V, and $\mu_1 = q_{Z^*}$, and $\mu_2 = q_Z$. Let P and Q respectively be projections onto μ_1 and μ_2. These are P_Z and P_{Z^*}, or $q_{\delta(Z)}$ and $q_{\delta(Z^*)}$ in the terminology of proposition 2.7 and theorem 2.9. For $1 \leq i \leq \ell$, define on ν^ℓ an operator P_i by letting P_i act as P on the i-th factor in the tensor product and as the identity on all the other factors. Define Q_i similarly. Note that the P_i's, the Q_j's and P_i and Q_j for $i \neq j$, commute. For any triple of integers i,j,k, with $1 \leq i,j \leq k \leq 1$, define $T_{ij,k}$ by

$$T_{ij,k} = (\prod_{\alpha=1}^{i} P_\alpha)(\prod_{\beta=i+1}^{k} Q_\beta)(\prod_{\gamma=1}^{j} P_\gamma)(\prod_{\delta=j+1}^{k} Q_\delta). \tag{18}$$

As in the type I case, ν^ℓ can be embedded in $\nu^{\ell+1}$. Let ν^ℓ_ℓ be the sum of all the isotypic components in ν^ℓ which do not occur in ν^m for $m < \ell$. For $m < \ell$, let ν^ℓ_m be the sum of the isotypic components of ν^ℓ which intertwine with ν^m_m. We have the orthogonal direct sum decomposition

$$\nu^\ell = \sum_{m=0}^{\ell} \nu^\ell_m \tag{19}$$

Let $I_{\ell m}$ be the ideal in $I(\nu^\ell;\nu^\ell)$ generated by orthogonal projection onto ν^ℓ_m.

Theorem 4.4: Notations as above.

a) The operators $\{T_{ij,k}\}$ defined by (18) form a set of representatives for the $G \times G$ orbits in $\mathcal{Q}(\mathcal{R}^{\nu^\ell}(G))$.

b) For any m, $0 \leq m \leq \ell$, the $G \times G$ orbits of the $T_{ij,k}$ with $k \geq m$ span $\sum_{n=m}^{\ell} I_{\ell,\ell-n}$, and constitute a subsemigroup of

$\mathcal{Q}(\Omega^{\nu^\ell(G)})$.

c) If G'_ℓ is the group dual to $\nu^\ell(G)$ in $Sp(Z^\ell \oplus Z^{*\ell})$, then G'_ℓ restricted to ν^ℓ_ℓ generates the commuting algebra of G on ν^ℓ_ℓ.

Proof: (sketch). It is evident by inspection that $T_{ij,k}$ belongs to $\sum_{n=k}^{\ell} I_{\ell,\ell-n}$. One checks a) using the definition of P and Q and proposition 3.5. From this much c) follows. The refinement b) of corollary 4.1.2 is checked by analyzing all $I(\nu^\ell,\nu^m)$ in analogy with theorems 4.2 and 4.3