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L^2 duality for stable reductive dual pairs

Let F be a local field not of characteristic 2. Let W be a vector space over F equipped with a symplectic form $\langle\ ,\ \rangle$. Let $Sp(W, \langle\ ,\ \rangle) = Sp$ be the isometry group of $\langle\ ,\ \rangle$. Let (G,G') be an irreducible reductive dual pair in Sp. (See [H1] for terminology.) From the classification [H1] of such pairs, we know there is a division algebra D over F (not necessarily central), with an involution $\natural$ (which reduces to the identity in the case of a type II pair), and vector spaces U and U' over D, with $\natural$-Hermitian or $\natural$-anti Hermitian forms $(,)$ and $(,)'$ (which are non-degenerate in the type I case and trivial in the type II case), such that G and G' are isomorphic to the isometry groups of these forms (and so are simply $GL(U)$ and $GL(U')$ respectively in the type II case). For convenience we may assume $\dim U \geq \dim U'$, and we will refer to G as the larger member of the pair, and to G' as the smaller member. We will call (G,G') stable if, in the type I case $(,)$ admits isotropic subspaces of dimension at least $\dim U'$, or in the type II case if simply $\dim_D U \geq 2 \dim_D U'$ (this condition being implicit in the type I case).

The duality phenomena exhibited by reductive dual pairs [H1] are considerably simpler for stable pairs than in general, and computations are easier to make in the range of dimensions implied by stability ([C],[G-K],[W]). The main purpose of this paper is to illustrate this assertion convincingly by giving a relatively simple proof of L^2 duality for stable pairs. Let $\widetilde{Sp}$ be the two-fold cover of Sp, and let ω be

a fixed oscillator representation of $\widetilde{Sp}$. Then ω is a unitary representation of $\widetilde{Sp}$ acting on some Hilbert space to be specified later. Let κ be the kernel of the projection from $\widetilde{Sp}$ to Sp. Then κ has order 2 and $\omega|\kappa$ is a multiple of the unique non-trivial character ε of κ. If $H \subseteq Sp$ is a subgroup, let $\widetilde{H}$ be the inverse image of H in $\widetilde{Sp}$. Then $\widetilde{H}$ projects on to H with kernel κ and any representation of $\widetilde{H}$ occurring in $\omega|\widetilde{H}$ will restrict to a multiple of ε on κ. Denote by $\widetilde{H}^\wedge(\varepsilon)$ the subset of the unitary dual of $\widetilde{H}$ whose elements restrict to multiples of ε on κ. Let $\widetilde{H}_t^\wedge(\varepsilon)$ be the support of the Plancherel measure in $\widetilde{H}^\wedge(\varepsilon)$. We will call $\widetilde{H}_t^\wedge(\varepsilon)$ the tempered dual of $\widetilde{H}$. It may also be described as the set of (irreducible unitary) representations of $\widetilde{H}$ weakly contained in $\mathrm{ind}_\kappa^{\widetilde{H}}\varepsilon$. The representation $\mathrm{ind}_\kappa^{\widetilde{H}}\varepsilon$ is roughly half of the regular representation of $\widetilde{H}$. We will refer to it as the ε-regular representation.

Our main result is

Theorem 1: Let $(G,G') \subseteq Sp$ be a stable irreducible reductive dual pair. Then the von Neumann algebras generated by $\omega(\widetilde{G})$ and by $\omega(\widetilde{G}')$ are mutual commutants. More precisely, suppose $Y \subseteq U$ is an isotropic subspace such that Y and U/Y have dimension $\geq \dim U'$. Let P be the parabolic subgroup of G preserving Y. Then $\omega(\widetilde{P})$ already generates the commutant of $\omega(\widetilde{G}')$. In particular almost all the representations of $\widetilde{G}$ occurring in $\omega|\widetilde{G}$ are already irreducible on $\widetilde{P}$.

Moreover, $\omega|\tilde{G}'$ is quasi-equivalent to the ϵ-regular representation of $\tilde{G}'$. Therefore we have the direct integral decomposition

$$\omega|\tilde{G}\cdot\tilde{G}' \simeq \int_{\tilde{G}_t'^{\wedge}(\epsilon)} \sigma \otimes \sigma' \, d\mu(\sigma') \tag{1}$$

where σ' ranges over $\tilde{G}_t'^{\wedge}(\epsilon)$ and σ varies in $\tilde{G}^{\wedge}(\epsilon)$, and $d\mu(\sigma')$ is Plancherel measure on $\tilde{G}_t'^{\wedge}(\epsilon)$. In particular, the discrete spectrum of $\omega|\tilde{G}\cdot\tilde{G}'$ is parametrized by the discrete series of $\tilde{G}'$ which restrict to ϵ on x. The correspondence

$$\sigma' \to \sigma$$

defined by (1) is an injection of $\tilde{G}_t'^{\wedge}(\epsilon)$ into $\tilde{G}(\epsilon)$ up to a set of Plancherel measure zero.

Remark: There seems a good chance that the discrete spectrum of $\omega|\tilde{G}\cdot\tilde{G}'$ would give rise to representations constructed infinitesimally by an "orbit quantization" procedure using sheaf cohomology by G. Zuckerman [Z]. If so, then (ω) would provide global realizations of these representations and establish their unitarity.

Proof: The procedure of proof is to consider a convenient particular realization of (ω) in which the operators coming from G and G' are easy to analyze. Eventually we can reduce the proof to the classical fact [D] that the right and left actions of a group on its own L^2-space are mutual commutants.

The cases when (G,G') is type II or type I with split G are easier than the general stable case. Since type I and type II must be treated separately anyway, we begin with type II for purposes of illustrating the general idea. So let (G,G') be an irreducible type II pair, and let D be the associated division algebra. It will be convenient to alter slightly the description of (G,G') given in [H1], by mixing the contragredient and co-gredient actions of G and G'. Specifically, we can find two left D vector spaces U and U' such that

$$W \simeq \mathrm{Hom}_D(U,U') \oplus \mathrm{Hom}_D(U',U) \tag{2}$$

in such fashion that $G \simeq GL(U)$ acts by premultiplication (by g^{-1}) on $\mathrm{Hom}_D(U,U')$ and by postmultiplication on $\mathrm{Hom}_D(U',U)$, and vice versa for $G' \simeq GL(U')$. The form $<\ ,\ >$ is given by

$$< (S_1,T_1),(S_2,T_2) > = \mathrm{tr}\ T_1S_2 - \mathrm{tr}\ S_1T_2, \tag{3}$$

where $S_i \in \mathrm{Hom}_D(U,U')$ and $T_i \in \mathrm{Hom}_D(U',U)$, and where tr is the trace map of $\mathrm{End}_D(U)$ or $\mathrm{End}_D(U')$ as algebra over F.

We have supposed $\dim_D U \geq 2\dim_D U'$. Choose in U two subspaces U_1 and U_2 such that $\dim U_i \geq \dim U'$ and $U = U_1 \oplus U_2$. Set

$$X = \mathrm{Hom}_D(U_1,U') \oplus \mathrm{Hom}_D(U',U_2) \tag{4}$$

$$Y = \mathrm{Hom}_D(U_2,U') \oplus \mathrm{Hom}_D(U',U_1)$$

We see that (X,Y) form a complete polarization for W, and we can form the Schrodinger model [C], [G2] of ω associated to (X,Y). The space on which $\omega(\widetilde{Sp})$ acts is then $L^2(X)$.

Since G' preserves both X and Y, the action of $\omega(\tilde{G}')$ will come from the linear action of G' on X (obtained by restricting the action of G' on W). Then [G2] for $\tilde{g} \in \tilde{G}'$, with image $g \in G'$,

$$(5) \qquad \omega(\tilde{g})(f)(x) = \sigma(\tilde{g})|\det g|^b f(g^{-1}(x)) \qquad x \in X,\ f \in L^2(X)$$

where $\sigma(\tilde{g})$ is an eighth root of unity, and $b = 2^{-1}(\dim U_2 - \dim U_1)$, and $|\det g|$ is most conveniently defined by

$$(6) \qquad |\det g| \int_{U'} f(g(u'))du' = \int_{U'} f(u')du'$$

where du' is Haar measure on U'.

The isotropy group M of X and Y in G is just the isotropy group of U_1 and U_2, and is isomorphic to $GL(U_1) \times GL(U_2)$. $\tilde{M}$ will also act on $L^2(X)$ essentially through the action of M on X, by a formula analogous to (5). The isotropy group in $\mathcal{G}$ of Y alone is a parabolic subgroup P of G containing M as Levi component. Let N be the unipotent radical of P. Then $P = M \cdot N$, and $N \simeq Hom_D(U_2,U_1)$. Also N may be lifted uniquely to a subgroup of $\tilde{S}p$, so we will not speak of $\tilde{N}$, but regard ω directly as a representation of N, which we identify with $Hom_D(U_2,U_1)$. In our present realization of ω, the group $\omega(N)$ consists of multiplication operators. Explicitly,

$$(7) \qquad \omega(n)f(x) = \chi(\tfrac{1}{2}B_n(x,x))f(x) \qquad n \in N$$

where χ is some unitary character of (the additive group of) F determined by ω, and B_n is a symmetric bilinear form on X, defined by

$$(8) \qquad B_n((S_1,T_1),(S_2,T_2)) = tr(S_1 n T_2 + S_2 n T_1)$$

where $S_i \in Hom_D(U_1,U')$ and $T_i \in Hom_D(U',U_2)$, and tr is, as above in (3), the trace on $End_D(U')$ as algebra over F.

Having described the actions of $\omega(\tilde{G}')$ and of $\omega(\tilde{P})$ on $L^2(X)$, we next investigate how they are related. The action of G' on X partitions X into G'-orbits and it is clear from (5) that $\omega|\tilde{G}'$ decomposes into a direct integral of what are essentially permutation representations on the various G'-orbits. These orbits may be described as follows. Given (S,T) in X, with $S \in Hom_D(U_1,U')$ and $T \in Hom_D(U',U_2)$, we see that the subspace $\ker S$ of U_1 and the subspace $\operatorname{im} T \subseteq U_2$ are invariant under G', and so is the map

$$TS: U_1/\ker S \to \operatorname{im} T.$$

Conversely, Witt's Theorem (see [J] and [H2]) says these 3 invariants: $\ker S$, $\operatorname{im} T$, and TS, characterize the G' orbit of (S,T) in X. In particular, we get a mapping

$$(9) \qquad \tau: X/G' \to Hom_D(U_1,U_2)$$
$$\tau(S,T) = TS$$

from G' orbits to $Hom_D(U_1,U_2)$. We will call τ the <u>orbit parameter map</u>.

Since $\tau(S,T)$ factors through U', it can have rank at most $\dim U'$. If $\tau(S,T)$ has rank exactly $\dim U'$, we will call (S,T) <u>generic</u>. The G' orbit of a generic point will be called a <u>generic G'-orbit</u>. Evidently (S,T) will be generic if and only if rank ST = rank S = rank T = $\dim U'$ so that S is surjective and T is injective. It follows that

$\ker S = \ker TS$ and $\operatorname{im} T = \operatorname{im} TS$, so that the orbit parameter map is a bijection from the generic G'-orbits to the subvariety of $\operatorname{Hom}_D(U_1,U_2)$ of maps of rank $\dim U'$. Also, if we fix a surjection $S_0: U_1 \to U'$, then the points (S_0,T) form a cross-section to the generic G'-orbits whose orbit parameter has fixed kernel equal to $\ker S_0$ as T ranges over the injections from U' to U_2. Also, the generic points are open and form a set of full measure in X, i.e., its complement is of measure zero and closed. Finally, observe that G' acts freely on each generic orbit. Hence $\omega|\tilde{G}'$ is seen to be the direct integral over the generic G'-orbits of copies of the ε-regular representation of $\tilde{G}'$. In particular, $\omega|\tilde{G}'$ is quasi-equivalent to the ε-regular representation of $\tilde{G}'$.

Next consider $\omega(\tilde{P})$. The space $\operatorname{Hom}_D(U_1,U_2)$, the range of the orbit parameter maps, is canonically the dual vector space to $N \simeq \operatorname{Hom}_D(U_2,U_1)$. By a standard construction, given a unitary character χ of F, one may identify N to the Pontrjagin dual of $\operatorname{Hom}_D(U_1,U_2)$, and in particular, points of N to characters of $\operatorname{Hom}_D(U_1,U_2)$, by the recipe

$$\psi_n(x) = \chi(\operatorname{tr}(xn)) \qquad n \in N,\ x \in \operatorname{Hom}_D(U_1,U_2) \tag{10}$$

Let χ be as in (7). Then the formulas (7), (8), and (9) say that the multiplication operator $\omega(n)$ is simply multiplication $\psi_n \circ \tau$. In particular, since τ is injective on the generic G'-orbits, the multiplication operators of $\omega(N)$ separate these orbits.

The effect of M acting on X is to permute the G'-orbits among themselves. Since $M \simeq GL(U_1) \times GL(U_2)$, it acts in an obvious way on

$\operatorname{Hom}_D(U_1,U_2)$. It is simple to check that the orbit parameter map τ of (9) is equivariant for the actions of M on domain and range. We may also observe that M acts transitively on the set of generic points, from which we conclude that:

i) M acts transitively on the set of generic G'-orbits, and

ii) the isotropy group in M of a given generic G'-orbit acts transitively on the orbit.

We now have the following data. The representation $\omega|\tilde{G}'$ is the direct integral of copies of the right ε-regular representation over a certain parameter space. The group $\omega(\tilde{P})$ provides multiplication operators separating points in the parameter space and permutation operators permuting transitively the points of the parameter space, and such that the isotropy group at any point generates the left ε-regular representation which, as is well-known [D], generates the full commutant of the right regular representation. It is not hard to convince oneself (it is a straightforward exercise of moderate length to deduce) from this data that $\omega(\tilde{P})$ indeed generates the full commutant of $\omega(\tilde{G}')$. One first shows, by considering sums of the permutation operators truncated by characteristic functions, that the full algebra generated by multiplications on the parameter space and the left ε-regular representation at each point is in the algebra generated by $\omega(\tilde{P})$. Then the full commutant of $\omega(\tilde{G}')$ is generated by this algebra and the permutations in $\omega(\tilde{P})$, since these act transitively on the parameter space. We omit the details. This concludes the proof of Theorem 1 in the type II stable case.

We proceed to consider the type I case. If the form $(,)$ on U is completely split, i.e., is a sum of hyperbolic planes, the proof proceeds in almost complete analogy with the proof for type II pairs, but in general it is slightly more involved. The computations take place not in a Schrödinger model, but in a mixed Fock-Schrödinger model. These mixed models have received less attention than pure Fock or Schrodinger models, so for reference we will write down some of the basic formulas for realizing ρ by these mixed models.

Recall that W is our basic symplectic vector space. Let $Y \subseteq W$ be an isotropic subspace, not necessarily maximal. Let $Y^{\perp}$ be the annihilator in W with respect to $<\,,\,>$. Let X be an isotropic complement to $Y^{\perp}$ in W, and let W_0 be the orthogonal complement of $X \oplus Y$. We then have the direct sum decomposition

(11) $$W = X \oplus W_0 \oplus Y.$$

Let P be the parabolic subgroup of $Sp = Sp(W,<\,,\,>)$ whose elements preserve Y. Let $M \subseteq P$ be the subgroup whose elements also preserve X, ence W_0. Then M is a Levi component for P, and $P = MN$ where N is the unipotent radical of P. We may also describe N as the subgroup of P which acts trivially on Y, on $Y^{\perp}/Y$, and on $W/Y^{\perp}$. By contrast, the restriction of M to $X \oplus W_0$ is faithful, and induces an isomorphism

(12) $$\alpha: M \xrightarrow{\sim} GL(X) \times Sp(W_0)$$

where $Sp(W_0)$ is, as you would expect, the isometry group of the restriction of $<\,,\,>$ to W_0. (This restriction is non-degenerate.)

The group N is two-step nilpotent. The center $Z(N)$ of N is the subgroup of N that acts trivially on $Y^{\perp}$, or equivalently on W/Y. It is canonically isomorphic to $B^2(X)$, the space of symmetric bilinear forms via the mapping

(13) $$\beta: z \to \beta_z \qquad z \in Z(N)$$

$$\beta_z(x_1,x_2) = < zx_2, x_1 > = < x_1, (1-z)x_2 >$$

The quotient $N/Z(N)$ is canonically isomorphic to $\mathrm{Hom}(X,W_0)$, by the recipe

(14) $$\gamma: n \to \gamma_n \qquad n \in N$$

$$\gamma_n(x) = nx - x \quad \text{modulo } Y$$

There is also a convenient cross section to γ, which by abuse of notation we will denote γ^{-1}. Write a general element in W as $w = (x,w_0,y)$ according to the decomposition (11). Via $<\,,\,>$, the dual of X is identified to Y, and W_0 is self-dual. Thus given a map $T \in \mathrm{Hom}(X,W_0)$, we may consider the adjoint T^* to be a map from W_0 to Y satisfying

$$< Tx, w_0 > + < x, T^*w_0 > = 0$$

With this convention, define for such T

(16) $$\gamma^{-1}(T)(x,w_0,y) = (x, w_0 + T(x), y + T^*(w_0) + (\tfrac{1}{2})T^*T(x)).$$

Then $\gamma^{-1}(T)$ is in N and $\gamma(\gamma^{-1}(T)) = T$, as one easily verifies.

By virtue of the decomposition we have the product $H = Sp(W_0) \times Sp(X \oplus Y)$ embedded as a subgroup of $Sp(W)$. The restriction of the oscillator representation ω to $\mathbb{H}$ and pulled back to $\widetilde{Sp}(W_0) \times \widetilde{Sp}(X \oplus Y)$ is just the outer tensor product of oscillator representations ω_1 of $\widetilde{Sp}(W_0)$ and ω_2 of $Sp(X \oplus Y)$. The mixed model of ω we will be working with has as space the tensor product of the space of ω_1 realized by an anisotropic or Fock model and the space of the Schrödinger model for ω_2 corresponding to (X,Y). The space of ω_1 is an appropriate subspace F of $L^2(W_0)$. (See [H3].) Its precise form will not be of great concern to us. The space of ω will thus be $L^2(X) \otimes F \simeq L^2(X,F)$, the space of F-valued functions on X with square integrable norm. The subgroups $Sp(W_0)$, $GL(X)$ and $Z(N)$ are all subgroups of $Sp(W_0) \times Sp(X \oplus Y)$, so they act according to their known action on one or the other factors of ω. We record the formulas.

(17) a) $\omega(\tilde{g})(\varphi)(x) = \omega_1(\tilde{g})(\varphi(x))$ $\qquad \tilde{g} \in \widetilde{Sp}(W_0),\ x \in X,\ \varphi \in L^2(X,F)$

b) $\omega(z)(\varphi)(x) = \chi(\tfrac{1}{2}B_z(x,x))\varphi(x)$ $\qquad z \in Z(N)$

c) $\omega(\tilde{m})(\varphi)(x) = \sigma(m)|\det m|^{-1/2}\varphi(m^{-1}(x))$ $\qquad \tilde{m} \in \widetilde{GL}(X)$

where σ, $|\det m|$ and χ have essentially the same meanings as in (5), (6) and (7).

The new and interesting feature of this mixed model is a formula for the action of a general element of N. By our discussion above, we see it will be enough to write this formula for the elements $\gamma^{-1}(T)$, where $T \in Hom(X,W_0)$. There is a map ρ of W_0 into the group of unitary operators on F satisfying

(18) $\rho(w_1)\rho(w_2) = \rho(w_1+w_2)\chi(\tfrac{1}{2}<w_1,w_2>)$ $\qquad w_i \in W_0$

$\omega_1(\tilde{g})\rho(w)\omega_1(\tilde{g})^{-1} = \rho(g(w))$ $\qquad \tilde{g} \in \widetilde{Sp}(W_0),\ w \in W_0$

It will be recognized that ρ is the restriction to W_0 of the representation with central character χ of the Heisenberg group attached to W_0. See for example [H1]. A computation now shows that for $T \in Hom(X,W_0)$

(19) $\omega(\gamma^{-1}(T))\varphi(x) = \rho(-T(x))(\varphi(x))$

With these formulas established, we return to our stable irreducible type I reductive dual pair (G,G'). Let D be the division algebra associated to (G,G') with involution $\natural$. Let U and U' be the basic D-modules associated to G and G' respectively, and let $(,)$ and $(,)'$ be the $\natural$-Hermitian or anti-Hermitian (one of each) forms on U and U' of which G and G' are the isometry groups. Then we may take

(20) $W \simeq Hom_D(U,U')$

Moreover there is a natural identification

$$^*: Hom_D(U,U') \to Hom_D(U',U)$$
$$T \to T^*$$

defined by the identity

(21) a) $(u,T^*u') = (Tu,u')'$ $\qquad u \in U,\ u' \in U',\ T \in Hom_D(U,U')$.

There is an analogous map

$$^{*'}: Hom_D(U',U) \to Hom_D(U,U')$$
$$S \to S^{*'}$$

defined by the analogous identity

(21) b) $\qquad (u',S^{*'}u)' = (Su',u).$

Using the fact that one of $(,)$ and $(,)'$ is $\natural$-Hermitian and the other $\natural$-anti-Hermitian we see that

(22) $\qquad T^{**'} = -T \qquad T \in \mathrm{Hom}_D(U,U')$

$\qquad S^{*'*} = -S \qquad S \in \mathrm{Hom}_D(U',U)$

In these coordinates on W, we may express $<\,,\,>$ by the formula

(23) $\qquad < S,T > = \mathrm{tr}(S^*T) = \mathrm{tr}(TS^*) \quad S,T \in \mathrm{Hom}_D(U,U')$

Here tr is the trace on $\mathrm{End}_D(U)$ or on $\mathrm{End}_D(U')$ as algebra over F.

By stability of (G,G'), we can find an isotropic subspace $U_1 \subseteq U$ such that $\dim U_1 \geq \dim U'$. Let U_2 be an isotropic complement to $U_1^{\perp}$ in U, and set $U_0 = (U_1 + U_2)^{\perp}$, so that $U = U_2 \oplus U_0 \oplus U_1$. Set

(24) $\qquad Y = \mathrm{Hom}_D(U_1,U')$

$\qquad W_0 = \mathrm{Hom}_D(U_0,U')$

$\qquad X = \mathrm{Hom}_D(U_2,U')$

Then Y is an isotropic subspace of W, and X is an isotropic complement to $Y^{\perp}$, and $W_0 = (X + Y)^{\perp}$, so $W = X \oplus W_0 \oplus Y$. Thus we may realize the oscillator representation of $\widetilde{Sp}$ by means of a mixed polarization on a space $L^2(X,F)$ where F is a suitable anisotropic model for a suitable oscillator representation of $\widetilde{Sp}(W_0)$. We thus have the formulas (17), (18), and (19) at our disposal.

We observe that G' is contained in $H = Sp(W_0) \times Sp(X \oplus Y)$. Thus $\omega|\widetilde{G}'$ is essentially a tensor product. That is, we have a canonical surjection

$$\check{H} = \widetilde{Sp}(W_0) \times \widetilde{Sp}(X \oplus Y) \to \widetilde{H}$$

and, as noted above, the pullback of $\omega|\widetilde{H}$ to $\check{H}$ is an outer tensor product $\omega_1 \otimes \omega_2$ of oscillator representations of the factors of $\check{H}$. Let κ_1 and κ_2 be the kernels of the projections from $\widetilde{Sp}(W_0)$ to $Sp(W_0)$ and from $\widetilde{Sp}(X \oplus Y)$ to $Sp(X \oplus Y)$ respectively and let ϵ_1 be the character by which κ_1 acts under ω_1. Let $\check{G}'$ be the inverse image of G' in $\check{H}$. Then $\check{G}'$ is an extension of G' by $\kappa_1 \times \kappa_2$. Moreover $\omega_1 \otimes \omega_2|\check{G}'$ factors to $\check{G}'/\ker(\epsilon_1 \otimes \epsilon_2) \simeq \widetilde{G}'$, and the pushdown to $\widetilde{G}'$ of $\omega_1 \otimes \omega_2|\check{G}'$ equals $\omega|\widetilde{G}'$. Hereafter it will be convenient to consider $\omega|\widetilde{G}'$ as a representation of $\check{G}'$ so that we can decompose it into its tensor factors and consider them separately. To begin, let us note the formula for $\omega|\check{G}'$. Combining (17) a) and c) we find for $\check{g}$ in $\check{G}'$ with image g in G',

(25) $$\omega(\check{g})(\varphi)(x) = \sigma(\check{g})\omega_1(\check{g})(\varphi(g^{-1}(x)))$$

From (25) we see that as with type II pairs, $\omega|\check{G}'$ will decompose into a direct integral of representations associated to the orbits of G' in X. Consider the structure of these orbits. If $T \in \mathrm{Hom}_D(U_2,U') = X$, then $G'(T) = \{gT : g \in G'\}$ is the G' orbit of T. Evidently $\ker T$ and the form $(,)' \circ T$ on $U_2/\ker T$ are constant on the orbit $G'(T)$. Conversely, Witt's Theorem ([J] and [H2]) tells us that the data $\ker T$ and

$(,)' \circ T$ determine $G'(T)$. Let $B'(U_2)$ denote the F-vector space of forms on U_2 of the same sort as $(,)'$. Then we have the mapping

$$\text{(26)} \qquad \tau : X/G' \to B'(U_2)$$
$$\tau(T) = (,)' \circ T$$

We will, as previously, call τ the orbit parameter map.

Since $\dim U_2 \geq \dim U'$, the set of T which are surjective onto U' will be an open set in X with complement of measure zero. We will call surjective T generic and their G'-orbits generic also. If T is generic, then $\ker T$ coincides with the radical of the form $\tau(T)$ and is therefore determined by $\tau(T)$. Hence the orbit parameter map separates generic G'-orbits.

It is obvious that G' acts freely on each generic G'-orbit. Hence $\omega_2|\check{G}'$ is the integral over the set of generic G'-orbits of the $(1 \otimes \varepsilon_2)$-regular representation. Whence $\omega|\check{G}'$ is the integral over the set of generic G'-orbits of ω_1 tensored with the $(1 \otimes \varepsilon_2)$-regular representation of $\check{G}'$. Since ω_1 behaves like $\varepsilon_1 \otimes 1$ on $x_1 \times x_2$, the tensor product of ω_1 with the $(1 \otimes \varepsilon_2)$-regular representation is isomorphic to $\dim \omega_1$ copies of the $(\varepsilon_1 \otimes \varepsilon_2)$-regular representation. This will be seen from the following discussion which is quite general, and which will be useful for computing of the commutant of $\omega(\check{G}')$.

Let L be a locally compact group with compact central subgroup C. Let ψ_1 and ψ_2 be characters of C. Let $\nu(\psi_1)$ be the ψ_1-regular representation of L and let μ be some unitary representation of L which is

a multiple of ψ_2 on C. Then there is a natural isomorphism between $\mu \otimes \nu(\psi_1)$ and $(\dim \mu)1 \otimes \nu(\psi_1\psi_2)$, where 1 here denotes the trivial representation. To construct the isomorphism, let H be the Hilbert space on which μ acts, and let $L^2(L,H,\psi_1)$ be the space of square integrable functions f from L to H such that $f(c\ell) = \psi_1(c)f(\ell)$ for $c \in C$ and $\ell \in L$. Then $\mu \otimes \nu(\psi_1)$ is the action of L on $L^2(L,H,\psi_1)$ defined by

$$\mu \otimes \nu(\psi_1)(\ell')f(\ell) = \mu(\ell')(f(\ell\ell')).$$

For $f \in L^2(L,H,\psi_1)$, define $A(f)$ by

$$A(f)(\ell) = \mu(\ell)(f(\ell)).$$

Then it is easy to see that A is an isomorphism from $L^2(L,H,\psi_1)$ to $L^2(L,H,\psi_1\psi_2)$. We further compute

$$\begin{aligned} A(\mu \otimes \nu(\psi_1)(\ell')(f))(\ell) &= \mu(\ell)(\mu \otimes \nu(\psi_1)(\ell')(f)(\ell)) \\ &= \mu(\ell)\mu(\ell')(f(\ell\ell')) \\ &= \mu(\ell\ell')(f(\ell\ell')) \\ &= A(f)(\ell\ell') \quad . \end{aligned}$$

Thus A is seen to be an intertwining operator between $\mu \otimes \nu(\psi_1)$ and the action $(\dim H)1 \otimes \nu(\psi_1\psi_2)$ on $L^2(L,H,\psi_1\psi_2)$. Hence A is the isomorphism we wanted.

The isomorphism A allows us to compute the intertwining operators for $\mu \otimes \nu(\psi_1)$. For it is well-known, and a trivial consequence of the case $H = \mathbb{C}$, that the commutant of L acting by $(\dim H)1 \otimes \nu(\psi_1,\psi_2)$ on

$L^2(L,H,\psi_1\psi_2)$ is spanned in the weak topology by operators

$$\lambda(\ell') \otimes V: f(\ell) \to V(f(\ell'^{-1}\ell))$$

where V is any bounded operator on H. Conjugating by A, we find the commutant of L acting by $\mu \otimes \nu(\psi_1)$ on $L^2(L,H,\psi_1)$ is spanned by operators of the form

(27) $$A^{-1}(\lambda(\ell') \otimes V)A: f(\ell) \to \mu(\ell)^{-1}V\mu(\ell)(f(\ell'^{-1}\ell))$$

Consider now the action of $\omega(\widetilde{G})$. More precisely, let P be the parabolic subgroup of Sp preserving Y, and let $P_1 = P \cap G$, and consider $\omega(\widetilde{P}_1)$. We must first study the structure of P_1. Observe that P_1 is composed of parts similar to the parts of P discussed above. For example, if M is the Levi component of P described above, then $M_1 = M \cap G = M \cap P_1$ is a Levi component for P_1 and $M_1 \simeq GL(U_2) \times G_0$, where G_0 is the isometry group of $(,)$ restricted to U_0. And if $N_1 = N \cap P_1$, then N_1 is the unipotent radical of P_1. N_1 is two-step nilpotent. The center of N_1, denoted $Z(N_1)$, is isomorphic to a certain space $B'(U_2^*)$ of forms on U_2^*, and $N_1/Z(N_1)$ is isomorphic to $Hom_D(U_0,U_2)$. We will describe explicitly the appropriate isomorphisms. Consider first $N_1/Z(N_1)$. If $n \in N_1$, the image of n in $N_1/Z(N_1)$ defines by (13) a map $\gamma(n)$ from $Hom_D(U_2,U')$ to $Hom_D(U_0,U')$. Since n is in G, we see $\gamma(n)$ must commute with G' acting on these spaces by postmultiplication. It follows there is a map $\gamma_1(n)$ in $Hom_D(U_0,U_2)$ such that

(28) $$\gamma(n)(T) = T\gamma_1(n) \qquad T \in Hom_D(U_2,U')$$

The map

$$\gamma_1: N_1/Z(N_1) \qquad n \to \gamma_1(n)$$

is the desired isomorphism. We may also define γ_1^{-1} from $Hom(U_0,U_2)$ to N_1 in analogy with (14).

Consider $Z(N_1)$. Since U_2 is left D vector space, U_2^* is naturally considered a right D vector space with scalar multiplication given by

$$u^*d(u) = u^*(u)d \qquad d \in D,\ u \in U,\ u^* \in U^*$$

Let $B'(U_2^*)$ be the space of forms β on U_2^* such that

$$\beta(vd,\widetilde{v}) = \beta(v,\widetilde{v})d \qquad d \in D;\ v,\widetilde{v} \in U_2^*$$

and having the same symmetry as $(,)'$ under interchange of v and $\widetilde{v}$. Take $z \in Z(N_1)$. By formula (13) we may attach to z a symmetric bilinear form β_z on $X = Hom_D(U_2,U') = U_2^* \otimes_D U'$. Since z commutes with G', the form β_z is invariant under the action of G' on X. This allows one to show there is a well defined form $\beta_z^1 \in B'(U_2^*)$ such that for $v,\widetilde{v} \in U_2^*$, and $u',\widetilde{u}' \in U'$

$$\beta_z(v \otimes u', \widetilde{v} \otimes \widetilde{u}') = tr(\beta_z^1(v,\widetilde{v})(u',\widetilde{u}')')$$

Then the map

(29) $$\beta^1: Z(N_1) \to B'(U_2^*) \qquad z \to \beta_z^1$$

is the isomorphism we want.

The space $B'(U_2^*)$ is naturally dual to $B'(U_2)$, the range of the orbit parameter map for G' orbits. Thus $B'(U_2^*)$ may be also identified to the Pontrjagin dual of $B'(U_2)$. To accomplish this explicitly, let $\{e_i\}_{i=1}^k$ be a basis for U_2 and let $\{f_i\}_{i=1}^k$ be the dual basis for U_2^*. Given β in $B'(U_2^*)$ and τ in $B'(U_2)$, put

$$\psi_\beta(\tau) = \chi(\tfrac{1}{2}\operatorname{tr}(\sum_{i,j} \tau(e_i,e_j)\beta(f_i,f_j))) \tag{30}$$

Then ψ_β is a unitary character of $B'(U_2)$.

Now we may describe the action of $\omega(P_1')$. First look at $P_1 \cap Sp(X \oplus Y) = GL_D(U_2)\cdot Z(N_1)$. From (17), (26), (29) and (30) we see that for z in $Z(N_1)$, the operator $\omega(z)$ is multiplication by $\psi_z \circ \tau$, where τ is the orbit parameter map. In particular $\omega(Z(N_1))$ acts by a scalar on each generic G'-orbit and separates these orbits. From (17) we see $GL_D(U_2)$ acts essentially to permute the G'-orbits. It is clear that $GL_D(U_2)$ will act transitively on the generic points of X. Hence $GL_D(U_2)$ permutes the generic G'-orbits transitively, and the isotropy group in $GL_D(U_2)$ of each generic G' orbit acts transitively on that orbit. Therefore, if U_0 were trivial we would be in precisely the same situation as we were when dealing with type II pairs, and would therefore be done. Here we must also take into account the action of $\omega(N_1)$.

Fix a generic G'-orbit $\mathcal{O}$ in X, and a point T in $\mathcal{O}$. Given φ in $C_c^\infty(X,F)$ (smooth functions of compact support with values in F), define $\tilde{\varphi}$ in $C_c^\infty(\check{G}',F,\varepsilon_2)$ by

$$\tilde{\varphi}(\check{r}) = \sigma(\check{r})\varphi(g^{-1}(T))$$

Then formula (25) says for $\check{r} \in \check{G}'$

$$(\omega(\check{r})(\varphi))^\sim(\check{r}_0) = \omega_1(\check{r})(\tilde{\varphi}(\check{r}_0\check{r}))$$

so that the map $\varphi \to \tilde{\varphi}$ intertwines ω with the action $\omega_1 \otimes \nu(\varepsilon_2)$ of $\check{G}$ on $C_c^\infty(\check{G}',F,\varepsilon_2)$, where $\nu(\varepsilon_2)$ is the $(1 \otimes \varepsilon_2)$-regular representation of $\check{g}$. We also see from (19) that if $S \in Hom_D(U_0,U_2)$, then

$$\omega(\gamma_1^{-1}(S)\varphi)^\sim(\check{r}) = \rho(-r^{-1}TS)(\tilde{\varphi}(\check{r})) \tag{31}$$

$$= \omega_1(\check{r})^{-1}\rho(-TS)\omega_1(\check{r})(\tilde{\varphi}(\check{r}))$$

Formula (31) shows that via $\varphi \to \tilde{\varphi}$, an operator $\omega(\gamma_1^{-1}(S))$ goes to an operator of the form $A^{-1}(\lambda(1) \otimes \rho(-TS))A$ in (27). Since T is generic, the map $S \to TS$ from $Hom_D(U_0,U_2)$ to $Hom_D(U_0,U')$ is surjective. Further it is well known (see for example [C]) that the operators $\rho(w)$ given by (18) for $w \in V_0' = Hom_D(U_0,U')$ act irreducibly on F. Hence the weak closure of the span of the $\rho(w)$ is the space of all bound operators on F.

Combining this fact with our previous observations we find ourselves in the following situation:

i) $\omega|\check{G}'$ is a direct integral over the generic G'-orbits of copies of the representation $\omega_1 \otimes \nu(\varepsilon_2)$, where $\nu(\varepsilon_2)$ is the right $(1 \otimes \varepsilon_2)$-regular representation.

ii) $\omega(Z(N_1))$ provides multiplication operators acting as scalars on each generic orbit and separating the generic orbits.

iii) $\omega(GL_D(U_2))$ permutes the orbits transitively.

iv) Additionally, the isotropy group of a given orbit, acts on the restrictions of functions to that orbit by $(\dim \omega)1 \otimes \lambda$, where λ is the left ε_2-regular representation.

v) $\omega(N_1)$ acts on the restrictions of functions to a given orbit by operators of the form (27) with $\ell' = 1$, and weakly spans all such operators.

From (iv) and (v), we see by our general discussion before (27) that $\omega(GL_D(U_2)\cdot N_1$, acting on the restriction of functions to any generic G'-orbit, generates the full commutant of $\omega_1 \otimes \nu(\epsilon_2)$. Hence using (ii) and (iii) an argument similar to the one sketched in the type II case establishes the result here also. This concludes Theorem 1.

On reading [A], I find the above proof is very similar to Asmuth's argument for the pair (Sp_{2n}, O_m), where C_m is anisotropic p-adic of dimension 2 or 4 and $n \geq m$ (so the pair is stable).

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