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CONTENTS

Oscillator Representation: Algebraic Preliminaries

1. Heisenberg groups.

Let A be a commutative ring with unit. By $H_n(A)$, the <u>standard Heisenberg group of degree n over A</u>, we mean the group of matrices of the form

$$(1.1) \qquad \begin{bmatrix} 1 & x_1 & \cdots & x_n & z \\ 0 & 1 & & & y_1 \\ \vdots & & \ddots & & \vdots \\ \vdots & 0 & & 1 & y_n \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \qquad x_i,\ y_i,\ z \in A.$$

Notation: Put $(x_1, \ldots, x_n) = x$ and $(y_1, \ldots, y_n) = y$. The typical element (1.1) of $H_n(A)$ will be denoted $h(x,y,z)$. When no confusion will arise, we will abbreviate $H_n(A) = H$.

For reference, we present some easily checked formulas expressing the group law of H in the coordinates of (1.1).

(1.2) i) $h(x,y,z)h(x',y',z') = h(x+x', y+y', z+z' + x\cdot y')$, where

ii) $x\cdot y = \sum_{i=1}^{n} x_i y_i$ as usual.

iii) $h(x,y,z) = h(0,y,0)h(x,0,0)h(0,0,z)$

$= h(x,0,0)h(0,y,0)h(0,0,z-x\cdot y)$

iv) $h(x,y,z)^{-1} = h(-x, -y, -z + x\cdot y)$

Abbreviating $h = h(x,y,z)$ and $h' = h(x',y',z')$, we have

v) $hh'h^{-1}h'^{-1} = h(0,0, x\cdot y' - x'\cdot y)$.

From the formulas (1.2) certain facts about the structure of H may be read off. We will state them explicitly. Set

$$(1.3) \qquad X_n(A) = X = \{h(x,0,0): x \in A^n\}, \; Y_n(A) = Y = \{h(0,y,0): y \in A^n\}$$
$$Z_n(A) = Z = \{h(0,0,z): z \in A\}$$

Proposition 1.1.1: a) H is a two-step nilpotent group. The subgroup Z is simultaneously the center and commutator subgroup of H.

(b) The map $z \to h(0,0,z)$ is an isomorphism of abelian groups from A to Z. We have $X \simeq A^n \simeq Y$ in similar fashion.

(c) $X \oplus Z$ and $Y \oplus Z$ are maximal abelian subgroups of H. We have the semidirect product decomposition
$H \simeq X \overset{x}{s} (Y \oplus Z) \simeq Y \overset{x}{s} (X \oplus Z)$

(d) Put $W_{2n}(A) = W = H/Z$. Then W is an abelian group and $W \simeq X \oplus Y \simeq A^{2n}$.

Consider two elements $h, h' \in H$. According to formula (1.2)v), the commutator of h and h' is in Z and depends only on the images of h and h' in W. Denote the resulting function on W by $< , >$. That is, if h and h' have images w and w' in W, then

$$(1.4) \qquad < w,w'> = hh'h^{-1}h'^{-1}$$

By abuse of notation, we may sometimes write $< h,h'>$ instead of $< w,w'>$. Also, we identify Z with A as in porposition 1.3 b) so that we regard $< , >$ as taking values in A. The following statements are clear from (1.1)iv).

Proposition 1.2: Give W the structure of A-module as in proposition 1.1 d). Then $< , > : W \times W \to A$ is an A-bilinear form which

is skew-symmetric and non-degenerate in the strong sense that the map $\alpha: W \to Hom_A(W,A)$ defined by

$$(1.5) \qquad \alpha(W)(W') = < W',W >$$

is an isomorphism of A-modules.

Remark: Via $< , >$, Y is identified with $Hom_A(X,A)$ and vice-versa.

We call a form with properties like those of $< , >$ listed in proposition 1.2 a <u>symplectic form</u>. Given an A-module V and a symplectic form $< , >'$ on V, and a subset $U \subseteq V$, define

$$(1.6) \qquad U^{\perp} = \{v \in V: < u,v >' = 0\}.$$

Then $U^{\perp}$ is the orthogonal complement or annihilator of U with respect to $< , >'$. Clearly $U^{\perp}$ is an A-submodule of V. Also $U \subseteq U^{\perp\perp}$ and $U^{\perp\perp\perp} = U^{\perp}$. When A is a field, we have $U^{\perp\perp} = U$ if U is an A-submodule (subspace), but this important property of $\perp$ fails for general rings A. Thus we may assert that $(U_1 + U_2)^{\perp} = U_1^{\perp} \cap U_2^{\perp}$, but only that $(U_1 \cap U_2)^{\perp} \supseteq U_1^{\perp} + U_2^{\perp}$. If $U_1 \subseteq U_2$, then $U_2^{\perp} \subseteq U_1^{\perp}$. If $U_2 \subseteq U_1^{\perp}$, then also $U_1 \subseteq U_2^{\perp}$ and we say U_1 and U_2 are <u>orthogonal</u>. If U is an A-submodule of V and $U \subseteq U^{\perp}$, we call U <u>isotropic</u>. If $U = U^{\perp}$, then U is automatically an A-submodule of V and we say U is <u>maximal isotropic</u>, or a <u>polarization</u> for $< , >$.

In terms of these definitions, we may state the following reformulations of formula (1.2)iv) and definition (1.4).

Proposition 1.3: Let E be a subgroup of H and let $\overline{E}$ be its image in W. Then E is abelian if and only if $\overline{E} \subseteq \overline{E}^{\perp}$. More generally, if E_1 and E_2 are two subgroups of H, then E_2 centralizes E_1 if and only if $\bar{E}_2 \subseteq \bar{E}_1^{\perp}$. In particular the centralizer of E in H is the inverse image in H of $\bar{E}^{\perp}$. Thus E is maximal abelian in H if and only if $Z \subseteq E$ and $\bar{E}$ is maximal isotropic.

Consider now $H_m(A) = H_m$ and $H_n(A) = H_n$ for two integers m and n. Suppose $m \leq n$. We will define two homomorphisms $i^1_{mn} = i^1$ and $i^2_{mn} = i^2$ from H_m into H_n. These are given by:

(1.7)

$$i^1: \begin{bmatrix} 1 & x_1 \cdots x_m & z \\ & \ddots \quad \bigcirc & y_1 \\ \bigcirc & & \vdots \\ & & y_m \\ & & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & x_1 \cdots x_m \; 0 \cdots 0 & z \\ & \ddots \quad \bigcirc & y_1 \\ & & \vdots \\ & & y_m \\ \bigcirc & & 0 \\ & & \vdots \\ & & 1 \end{bmatrix}$$

and

$$i^2: \begin{bmatrix} 1 & x_1 \cdots x_n & z \\ & \ddots \quad \bigcirc & y_1 \\ \bigcirc & & \vdots \\ & & y_m \\ & & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \cdots 0 \; x_1 \cdots x_m & z \\ & \ddots \quad \bigcirc & 0 \\ & & \vdots \\ & & 0 \\ \bigcirc & & y_1 \\ & & \vdots \\ & & y_m \\ & & 1 \end{bmatrix}$$

The following statement is easily verified by inspection.

Proposition 1.4: Consider the homomorphisms $i^1 = i^1_{m\,m+n}$ from H_m to H_{m+n} and $i^2_{n\,m+n} = i^2$ from H_n to H_{m+n}. Then

a) Each of the homomorphisms i^1 and i^2 is injective.

b) The product homomorphism $i^1 \times i^2 : H_m \times H_n \to H_{m+n}$ is surjective.

c) The kernel of $i^1 \times i^2$ is the antidiagonal of $Z_m \times Z_n$, consisting of elements $h(0,0,z) \times h(0,0,-z)$.

d) The images $i^1(H_m)$ and $i^2(H_n)$ are mutual centralizers in H_{m+n}.

e) Let $\tau^1 : W_{2n} \to W_{2(m+n)}$ and $\tau^2 : W_{2n} \to W_{2(m+n)}$ be the maps induced by i^1 and i^2. Then $\tau^1 \times \tau^2$ is an isomorphism from $W_{2m} \oplus W_{2n}$ to $W_{2(m+n)}$ and an isometry for the associated symplectic forms, so that $W_{2(m+n)} \simeq \tau^1(W_{2m}) \oplus \tau^2(W_{2n})$ (orthogonal direct sum).

2: The Heisenberg Lie Algebra

By the <u>standard Heisenberg Lie algebra of degree n</u> over A, denoted $\mathcal{H}_n(A)$, or just $\mathcal{H}$ when n and A are understood, we mean the Lie algebra of matrices

$$(2.1)\qquad \begin{bmatrix} 0 & a_1 & \cdots & a_n & c \\ & 0 & & & b_1 \\ & & \ddots & & \vdots \\ & & & 0 & b_n \\ & & & & 0 \end{bmatrix} \qquad a_i,\ b_i,\ c \in A.$$

Notation: Put $(a_1, \ldots, a_n) = a$ and $(b_1, \ldots, b_n) = b$. We will denote the typical element (2.1) of $\mathcal{H}_n(A)$ by $h(a,b,c)$.

We record the effect of the standard operations in $\mathcal{H}$. As usual, the Lie bracket is indicated by [,].

(2.2) i) $h(a,b,c) + h(a',b',c') = h(a+a', b+b', c+c')$

ii) $[h(a,b,c), h(a',b',c')] = h(0,0, a\cdot b' - b\cdot a')$

where $a\cdot b$ is as in (2.1)ii).

Put

(2.3) $\mathcal{W}_{2n}(A) = \mathcal{W} = \{h(a,b,0)\}$ $\quad a,b \in A^n$

$\mathcal{Z}_n(A) = \mathcal{Z} = \{h(0,0,c)\}$ $\quad c \in A$

From formulas (2.2) and (2.3) the following facts may be read off.

Proposition 2.1: a) The Lie algebra $\mathcal{H} = \mathcal{H}_n(A)$ is two-step nilpotent with center and commutator ideal equal to $\mathcal{Z}$

b) The map $c \to h(0,0,c)$ defines an A-module isomorphism from A to $\mathcal{Z}$.

c) The map $(a,b) \to h(a,b,0)$ defines an A-module isomorphism from A^{2n} to $\mathcal{W}$.

d) $\mathcal{H} = \mathcal{W} \oplus \mathcal{Z}$, so that $\mathcal{W}$ is a free A-module complementary to $\mathcal{Z}$ in $\mathcal{H}$.

e) If $\mathcal{Z}$ is identified to A as in b), then the restriction of [,] to $\mathcal{W}$ defines a symplectic form on $\mathcal{W}$.

We next observe that $\mathcal{H}$, as a set of matrices, is invariant under both left and right multiplication by elements of H. In particular, we can conjugate elements of $\mathcal{H}$ by elements of H. Thus we define

$$(2.4)\qquad \mathrm{Ad}\, h(h) = h h h^{-1}$$

and Ad then is a representation of H on $\mathcal{H}$ by A-linear Lie algebra automorphisms. Explicitly, in coordinates we have the formula

$$(2.5)\qquad \mathrm{Ad}\, h(x,y,z)(h(a,b,c)) = h(a,b,c + x\cdot b - y\cdot a)$$

Proposition 2.2: a) The kernel of $\mathrm{Ad}: H \to \mathrm{Aut}_A(\mathcal{H})$ is Z, so Ad is effectively an action of W

b) Ad h acts trivially on $\mathcal{Z}$ and on $\mathcal{H}/\mathcal{Z}$.

c) Ad W acts simply transitively on all A submodules complementary to $\mathcal{Z}$ in $\mathcal{H}$. Thus, if $\mathcal{M}$ is such a complement to $\mathcal{Z}$, then there is $h \in H$, determined uniquely modulo Z, such that $\mathrm{Ad}\, h(\mathcal{W}) = \mathcal{M}$, with $\mathcal{W}$ as in (2.3).

Proof: Both a) and b) are immediate from (2.5) by inspection. For c), consider an A-module $\mathcal{M} \subseteq \mathcal{H}$ complementary to $\mathcal{Z}$. Then for any $w \in \mathcal{W}$, there is an unique $\lambda(w)$ in A such that $m = w + \lambda(w)$ is in $\mathcal{M}$. Clearly the map $\lambda: w \to \lambda(w)$ determines $\mathcal{M}$ and vice-versa. Since

$\mathcal{M}$ is an A-module, the map λ is A-linear, that is, $\lambda \in \mathrm{Hom}_A(\mathcal{W}, A)$. But it is clear from (2.5) that the map $h \to \lambda(h)$, defined by $\lambda(h)(w) = \mathrm{Ad}\, h(w) - w$ is an isomorphism from W to $\mathrm{Hom}_A(\mathcal{W}, A)$, so the proposition follows.

3: The exponential map; Heisenberg groups of symplectic modules.

In this section, we assume that 2 is a unit in A. We can then define a map

$$\exp\colon \mathcal{H} \to H$$

by the formula

(3.1) $$\exp \mathfrak{h} = I + \mathfrak{h} + (\tfrac{1}{2})\mathfrak{h}^2$$

It is easy to see that exp is a bijection from $\mathcal{H}$ to H. Indeed, we may explicitly write down the inverse mapping, which we denote by ℓg. Thus

(3.2) $$\ell g \ \exp \mathfrak{h} = \mathfrak{h}, \quad \exp \ \ell g\ h = h$$

$$\ell g\ h = (h - I)^2 - \tfrac{1}{2}(h - I)^2$$

In terms of the coordinates defined in §§1 and 2 we may write

(3.3) $$\exp \mathfrak{h}(a,b,c) = h(a,b,c + (\tfrac{1}{2})\, a\cdot b)$$

$$\ell g\ h(x,y,z) = \mathfrak{h}(x,y,z - (\tfrac{1}{2})\, x\cdot y)$$

We record some identities whose verifications are straightforward.

(3.4) i) $\exp(\mathfrak{h}_1 + \mathfrak{h}_2) = \exp \mathfrak{h}_1 \exp \mathfrak{h}_2 \exp(\tfrac{1}{2}[\mathfrak{h}_1, \mathfrak{h}_2])$

ii) $\exp \mathfrak{h}_1 \exp \mathfrak{h}_2 = \exp(\mathfrak{h}_1 + \mathfrak{h}_2 - (\tfrac{1}{2})[\mathfrak{h}_1, \mathfrak{h}_2])$

iii) $\exp(\mathrm{Ad}\, h(\mathfrak{h})) = h(\exp \mathfrak{h})h^{-1} = \mathrm{Ad}\, h(\exp \mathfrak{h})$

iv) $\exp([\mathfrak{h}_1, \mathfrak{h}_2]) = \exp \mathfrak{h}_1 \exp \mathfrak{h}_2 \exp \mathfrak{h}_1^{-1} \exp \mathfrak{h}_2^{-1}$

v) $(\exp \mathfrak{h})^{-1} = \exp(-\mathfrak{h})$

vi) $\mathrm{Ad}(\exp \mathfrak{h}_1)(\mathfrak{h}_2) = \mathfrak{h}_1 + [\mathfrak{h}_1, \mathfrak{h}_2]$.

From these formulas, the following facts may be read off.

Proposition 3.1: a) The map $\exp: \mathcal{H} \to H$ is a bijection. It defines isomorphisms of A-modules from $\mathcal{Z}$ to Z and from $\mathcal{W}$ to W.

b) If $\mathcal{U} \subseteq \mathcal{H}$ is an abelian Lie subalgebra, then $\exp \mathcal{U}$ is an abelian subgroup of H and $\exp: \mathcal{U} \to \exp \mathcal{U}$ an isomorphism of groups.

c) The set $\exp \mathcal{W} \subseteq H$ is a cross-section to Z in H. That is $H = \exp \mathcal{W} \cdot Z = Z \cdot \exp \mathcal{W}$.

d) The map $w \to \exp w \pmod Z$ (as in a)) defines an isometry between the symplectic form on $\mathcal{W}$ induced by Lie bracket and the form on W induced by commutator.

Terminology: We will call $\exp \mathcal{W}$ the <u>standard isotropic cross-section</u> to Z in H. We will denote either of the forms identified in d) above by $< , >$.

We can use the above results to give a different parametrization of H whenever 2 is invertible in A. Namely, we can define a bijection

(3.5) $$e: \mathcal{W} \oplus A \to H \qquad e(w,r) = \exp w\, h(0,0,r)$$

In these coordinates, the group law of H becomes

(3.6) $$e(w,r)e(w',r') = e(w+w', r+r' + \tfrac{1}{2} < w,w' >)$$

Formulas (3.5) and (3.6) make contact with an alternate construction of Heisenberg groups, valid when 2 is invertible in A. Let W be any symplectic module over A. That is, we suppose that there is defined on W a skew-symmetric A-bilinear form via which W is isomorphic with $\mathrm{Hom}_A(W,A)$. Then we define H(W), the <u>Heisenberg group attached to W</u> by

$$H(W) = W \oplus A$$

as set, and has group law

(3.7) $$(w,r)(w',r') = (w+w', r+r' + \tfrac{1}{2} < w,w' >)$$

What we have done is to show that when 2 is a unit, the standard Heisenberg groups are isomorphic to Heisenberg groups attached to free symplectic A-modules. The converse is also true if A is not to exotic. Probably the next proposition is much too restrictive in its hypotheses on A and on W.

Proposition 3.2: Let A be a ring such that

i) stably free A-modules are free,

ii) the rank of a free A-module is well-defined.

Then if W is a free symplectic A-module, W has a symplectic basis $\{e_i, f_i\}_{i=1}^n$ such that

(3.8) $$< e_i, f_j > = \delta_{ij}$$
$$< e_i, e_j > = 0 = < f_i, f_j >$$

Hence, if 2 is a unit in A, the Heisenberg group based on W is isomorphic to a standard Heisenberg group.

Remark: a) The hypotheses i) and ii) on A permit the mindless parroting of the usual "elementary divisors" argument. Both fields and rings of integers in local or global fields satisfy these hypotheses.

b) Let us call a symplectic module with symplectic basis <u>standard</u>. The proposition says conditions i) and ii) guarantee all free symplectic A-modules are standard. Such modules are the sum of two-dimensional standard modules, called <u>hyperbolic planes</u>.

Proof: Let $\{x_i\}$ be a basis for W and let $\{y_i\}$ be the dual basis, that is, $< x_i, y_j > = \delta_{ij}$. Let W_1 be the span of x_1 and y_1. I claim

$$W = W_1 \oplus W_1^{\perp}$$

Indeed, if $w \in W$, then $w - < w, y_1 > x_1 + < w, x_1 > y_1$ is easily checked to be in $W_1^{\perp}$, and the claim follows easily. By our assumptions on A, the module $W_1^{\perp}$ is free and of lower rank than W. Hence we may assume $W_1^{\perp}$ has a symplectic basis $\{e_i, f_i\}_{i=2}^{n}$. Then set $x_1 = e_1$, and $y_1 = f_1$ and the first statement of

If W is a symplectic A-module with a symplectic basis $\{e_i, f_i\}_{i=1}^{n}$, then define a map

$$\beta : H(W) \to H_n(A)$$

by

(3.9) $$\beta\,(\Sigma\, a_i e_i + b_i f_i,\ c) = h(a, b, c + (\tfrac{1}{2}) \sum_i a_i b_i)$$

where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. Then an easy computation based on (3.7) and (1.2) shows β is a group isomorphism.

Let W again be a symplectic A-module. We denote by $\mathcal{H}(W)$ the <u>Heisenberg Lie algebra</u> attached to W. We define $\mathcal{H}(W)$ by

(3.10) $$\mathcal{H}(W) = W \oplus A$$

as A-module, with bracket operation

(3.11) $$[(w,a),\ (w',a')] = (0,\ < w, w' >)$$

Observe that both $H(W)$ and $\mathcal{H}(W)$ have the same underlying set. The difference between them lies in the structure imposed on that set by (3.7) and (3.11) respectively. However the identity map on $W \oplus A$ may be considered to define mutually inverse bijections

(3.12) $$e: \mathcal{H}(W) \to H(W)$$
$$\ell g: H(W) \to \mathcal{H}(W).$$

These maps will satisfy the analogues of (3.4). We record the most basic one.

(3.13) $$e(\mathfrak{h})e(\mathfrak{h}') = e(\mathfrak{h} + \mathfrak{h}')e(\tfrac{1}{2}[\mathfrak{h}, \mathfrak{h}']) \quad \mathfrak{h}, \mathfrak{h}' \in \mathcal{H}$$

We will make some conventions about the relation between W, $H(W)$ and $\mathcal{H}(W)$ that will simplify notation and hopefully want to be too confusing. We will identify W with the subgroup $W \times \{0\}$ in $\mathcal{H}(W)$. Then $e(W) \subseteq H$ is the standard isotropic cross-section to Z, the center of H. Define maps

$$\eta : H \to W \qquad z : H \to Z$$

by formulas

(3.14) $$\eta(w,a) = w \qquad z(w,a) = a \qquad \text{for } (w,a) = h \in H.$$

When we identify Z with A by means of the parametrization of (1.3) or (3.7), the map z will be considered as a map from H to A. The following formulas are easily checked.

(3.15) a) $\eta\, e(w) = w,\ w \in W$ b) $h = e\,\eta(h) z(h),\ h \in H$
c) $\eta(hh') = \eta(h) + \eta(h')$ d) $z(e(w)e(w')) = \frac{1}{2} < w, w' >$

Using the group law (3.7) we have the following restatement of part of proposition 1.3.

Proposition 3.3: If $X \subseteq W$ is an isotropic A submodule, then $e(X) \subseteq H$ is an abelian subgroup of H, and with η as in (3.14), $\eta^{-1}(X) = e(X)\cdot Z$, so that $e(X)$ is a complement to Z in $\eta^{-1}(X)$. Thus $X \to e(X)\cdot Z$ sets up a bijection between maximal isotropic submodules of W and maximal abelian subgroups of $H(W)$.

The constructions of $H(W)$ and $\mathcal{H}(W)$ from W have certain fairly obvious functorial properties. We will state them explicitly for $H(W)$. Consider two symplectic A-m odules W_1 and W_2, with forms $<\ ,\ >_i$. We can define the orthogonal direct sum $W_1 \oplus W_2$ of the W_i by letting $W_1 \oplus W_2 = W_3$ be the usual direct sum of spaces, and defining a form $<\ ,\ >_3$ on W_3 by

(3.16) $$< (W_1,W_2), (W_1',W_2') >_3 = < W_1,W_1' >_1 + < W_2, W_2' >_2$$

Let

(3.17) $$i_1 : W_1 \to W_1 \oplus W_2 \qquad i_2 : W_2 \to W_1 \oplus W_2$$

denote the obvious inclusions. The following proposition describes the relation between the $H(W_i)$ for $i = 1,2,3$. Its proof is obvious.

Proposition 3.4: There are unique embeddings

(3.18) $$i_j : H(W_j) \to H(W_1 \oplus W_2) \qquad j = 1,2$$

such that the diagrams

(3.19) $$\begin{array}{ccc} W_j & \xrightarrow{i_j} & W_1 \oplus W_2 \\ e \downarrow & & \downarrow e \\ H(W_j) & \xrightarrow{i_j} & H(W_1 \oplus W_2) \end{array}$$

and

(3.20) $$\begin{array}{ccc} H(W_j) & \xrightarrow{i_j} & H(W_1 \oplus W_2) \\ \eta \downarrow & & \downarrow \eta \\ W_j & \xrightarrow{i_j} & W_1 \oplus W_2 \end{array}$$

commute. One also has

(3.21) $$z \circ i_j = i_j \circ z$$

with z as in (3.10). The images $i_j(H(W_j))$ are mutual centralizers in $H(W_1 \oplus W_2)$. Furthermore, the map

(3.22) $$i_1 \times i_2 : H(W_1) \times H(W_2) \to H(W_1 \oplus W_2)$$

is a surjective homomorphism of groups, whose kernel is the kernel of the homomorphism

$$z_1 \times (-z_2) : Z_1 \times Z_2 \to F$$

where $Z_j = Z(H(W_j))$, and z_j is the map attached to $H(W_j)$ by (3.10) (and (1.3.1)).

Because of proposition 3.4, we will say that $H(W_1 \oplus W_2)$ is the <u>central direct sum</u> of $H(W_1)$ and $H(W_2)$. Thus if W is a symplectic A-module and is the direct sum of two orthogonal submodules W_1 and W_2, then $H(W)$ is the central direct sum of $H(W_1)$ and $H(W_2)$. In particular, if W is a standard symplectic module, then $H(W)$ is the central direct sum of Heisenberg groups attached to hyperbolic planes, i.e., of copies of the standard Heisenberg group $H_1(A)$.

In a related vein, let W_1 and W_2 be two symplectic A-modules, and let $T:W_1 \to W_2$ be an isometric embedding. That is,

$$< , >_1 = < , >_2 \circ T$$

Then evidently,

$$T:(w,a) \to (T(w), a)$$

extends T to an embedding T of $H(W_1)$ in $H(W_2)$. Moreover, this extension has the property that the diagram

$$(3.23) \qquad \begin{array}{ccccc} H(W_1) & & \xrightarrow{\quad T \quad} & & H(W_2) \\ {\scriptstyle z_1}\downarrow & & & & \downarrow{\scriptstyle z_2} \\ Z_1 & \simeq & F & \simeq & Z_2 \end{array}$$

commutes, extending (3.16). In particular, if W_1 and W_2 are isometric, then $H(W_1)$ and $H(W_2)$ are isomorphic by a map T fitting in (3.18). Thus the isometries of W give rise to automorphisms of $H(W)$. This situation is studied in more detail in §4.

As a special case, consider a symplectic A-module W with form $< , >$. Let b be a unit in A. Let W^b be the symplectic A-module obtained by replacing $< , >$ by $b < , >$, that is, by multiplying the values of $< , >$ by b. Suppose

$$g:W \to W$$

is an A-linear automorphism of W such that

$$(3.24) \qquad < , > \cdot g = b < , > .$$

Then g induces an automorphism

$$g:H(W) \xrightarrow{\sim} H(W^b)$$

again satisfying (3.18). If W is a standard symplectic module with symplectic basis $\{e_i, f_i\}$, then g defined by

$$g(e_i) = e_i \qquad g(f_i) = bf_i$$

will satisfy (3.19). If b is a square, say $b = c^2$, then scalar multiplication by c will satisfy (3.19).

4: Automorphisms

We will give a fairly precise description of the automorphism group of $H_n(A)$ (as abstract group). Note to begin that if θ is an automorphism of A, then

$$\tilde{\theta}(h(x,y,z)) = h(\theta x, \theta y, \theta z) \tag{4.1}$$

defines an automorphism of $H = H_n(A)$, so that we have a natural injection

$$\sim : \mathrm{Aut}(A) \to \mathrm{Aut}(H)$$

Second, let $A^\times$ denote the units of A. We may embed $A^\times$ in $\mathrm{Aut}(H)$ via a map

$$\gamma : A^\times \to \mathrm{Aut}(H) \tag{4.2}$$

$$\gamma(r)(h(x,y,z)) = h(x,ry,rz), \quad r \in A^\times .$$

Let $\mathrm{Aut}^o(H)$ be the subgroup of $\mathrm{Aut}(H_n)$ consisting of the automorphisms leaving Z pointwise fixed.

Proposition 4.1: $\mathrm{Aut}^o(H)$ is a normal subgroup of $\mathrm{Aut}(H)$ containing the inner automorphism of H. Moreover,

$$\mathrm{Aut}(H) = (\mathrm{Aut}(A))^\sim \cdot \gamma(A^\times)\cdot \mathrm{Aut}^o(A). \tag{4.3}$$

That is, given ϕ in $\mathrm{Aut}(H)$, there is θ in $\mathrm{Aut}\, A$ and $r \in A^\times$, both determined by ϕ, such that $\gamma(r)^{-1} \tilde{\theta}^{-1} \phi$ belongs to $\mathrm{Aut}^o(H)$.

Proof: Since Z is characteristic in H, it must be preserved by $\mathrm{Aut}(H)$, so there is a natural restriction map from $\mathrm{Aut}(H)$ to $\mathrm{Aut}(Z)$. Clearly $\mathrm{Aut}^o(H)$ is the kernel of the restriction map and so is normal in $\mathrm{Aut}(H)$. Inner automorphisms are in $\mathrm{Aut}^o(H)$ because Z is central in H.

Consider a general ϕ in $\mathrm{Aut}(H)$. Then ϕ induces automorphisms (of abelian groups) ϕ_1 of Z and ϕ_2 of W. Moreover, ϕ_1 and ϕ_2 are compatible with $< , >$. That is,

$$< \phi_2(w), \phi_2(w')> \; = \; \phi_1(< w,w'>) \tag{4.4}$$

(Here $< , >$ is only considered as a Z-valued biadditive form; no A-linearity is assumed). Suppose there are elements w_1 and w_2 of W such that $\phi_2(Aw_1) \subseteq A\, \phi_2(w_1)$. Then for each b in A, there is $\lambda_1(b)$ in A such that

$$\phi_2(bw_1) = \lambda_1(b)\, \phi_2(w_1).$$

Put $< w_1,w_2 > = z$. From the relation

$$< b\, c\, w_1, w_2> \; = \; < b\, w_1, c\, w_2 > \; = \; < w_1, b\, c\, w_2 > \; = \; b\, c < w_1,w_2>$$

we conclude

$$\phi_1(bc\, z) = \lambda_1(bc)\, \phi_1(z) = \lambda_1(b)\, \lambda_2(c)\, \phi_1(z) = \lambda_2(bc)\, \phi_1(z) \tag{4.5}$$

If we assume $\phi_1(z)$ is not a zero-divisor, then we can conclude

$$\lambda_1(bc) = \lambda_1(b)\, \lambda_2(c) = \lambda_2(bc)$$

Putting $b = 1$, and letting c vary and then vice-versa, we conclude $\lambda_1 = \lambda_2$ and that their common value λ satisfies $\lambda(bc) = \lambda(b)\,\lambda(c)$ and therefore defines a ring endomorphism of A. If further $\phi_1(z) = 1$, the identity of A, then $1 = \phi_1(z) = \phi_1(1\cdot z) = \lambda(1)\, \phi_1(1) = \lambda(1)$, so λ is an automorphism of A. Then we see from (4.5)

$$\phi_1(bz) = \lambda(b)$$

and since $\phi_1(Az) = Z$, we must have $Az = Z$ since ϕ_1 is injective. Hence z is a unit in A. Thus we can write, putting $c = 1$ and $b' = bz$ in (4.5)

$$\phi_1(b') = \lambda(b' z^{-1})$$

Therefore $\gamma(z) \circ (\tilde{\lambda})^{-1} \circ \phi_1$ belongs to $Aut^o(H)$ as desired.

It remains to produce w_1 and w_2 as specified. Let X and Y be as in (1.3). Let $\{e_i\}$, $1 \le i \le n$, be a basis for X and let $\{f_i\}$ be the dual basis for Y. (We can find $\{f_i\}$ according to the remark following proposition 1.2). Then, taking images in W, we have

$$A\, e_1 = \{e_1, \ldots, e_n, f_2, \ldots, f_n\}^{\perp}$$

Thus if $w_1 = \phi^{-1}(e_1)$, clearly

$$Aw_1 \subseteq \{\phi^{-1}(e_1), \ldots, \phi^{-1}(e_n), \phi^{-1}(f_2), \ldots, \phi^{-1}(f_n)\}^{\perp}$$

Hence $\phi(Aw_1) \subseteq A\,\phi(w_1)$ as desired. If $w_2 = \phi^{-1}(f_1)$ then $\phi(Aw_2) \subseteq A\,\phi(w_2)$ similarly. Furthermore, $\phi_1(< w_1,w_2 >) = \phi(< \phi^{-1}e_1, \phi^{-1}(f_1) >) = < e_1, f_1 > = 1$, so w_1 and w_2 satisfy all assumptions needed above and the proposition is proved.

Remark: There is something in the above proof reminiscent of the fundamental theorem of projective geometry.

Now consider the structure of $Aut^o(H)$. We note that $\phi \in Aut^o(H)$ factors to an automorphism of W. Let the group of A-linear isometries of $< , >$ on W be denoted by $Sp(W, < , >) = Sp(W) = Sp$.

Proposition 4.2: a) The push down ϕ_2 of any $\phi \in Aut^o(H)$ to W is A-linear and preserves $< , >$. Thus there is a natural homomorphism

(4.6) $$\pi : Aut^o(H) \to Sp(W, < , >), \quad \pi(\phi) = \phi_2$$

b) Let $Aut^{oo}(H)$ be the kernel of π. Then

(4.7) $$Aut^{oo}(H) \simeq Hom_{\mathbb{Z}}(W,A)$$

in such fashion that $Ad(H)$, the group of inner automorphisms is identified to $Hom_A(W,A)$.

Proof: If $\phi \in Aut^o(H)$, then ϕ_1 in (4.4) is the identity so that ϕ_2 will indeed preserve $<, >$ and it remains only to show ϕ_2 is A-linear. But under the relevant assumptions, (4.5) holds with $\lambda_1(b) = \lambda_2(b) = b$, so that certainly $\phi_2(bw_1) = b\,\phi_2(w_1)$ for w_1 in the discussion of (4.5). But the latter part of the proof of proposition 4.1 shows that the w_1 which may be used in (4.5) span W as A-module. Since ϕ_2 is certainly additive, it is therefore A-linear.

If ϕ_2 is trivial as well as ϕ_1, then we may certainly write

(4.8) $$\phi(h(x,y,z)) = h(x,y,z + m(x,y))$$

for some $m(x,y)$ in A. From the group law (1.2)i) it follows immediately that m must be a homomorphism of abelian groups from W to A. Conversely, if $m \in Hom_{\mathbb{Z}}(W,A)$ and ϕ is defined by (4.8), then one checks by (1.2)i) that ϕ is an automorphism of H; and evidently $\phi \in Aut^{oo}(H)$ also. Thus $\phi \longleftrightarrow m$ is the isomorphism of (4.7). From (1.2)v) or from the proof of proposition 2.2 the identification of $Ad(H)$ with $Hom_A(W,A)$ is clear.

We now assume that 2 is invertible in A. Let $Aut_A(\mathcal{H})$ denote the group of automorphisms of $\mathcal{H}$ as Lie algebra over A. It is clear that if $\phi \in Aut_A(\mathcal{H})$, then $\exp \circ \phi \circ \ell g = \exp^* \phi$, is in $Aut(H)$. Let $Aut^o_A(\mathcal{H})$ be the subgroup of $Aut_A(\mathcal{H})$ whose elements act trivially on $\mathcal{Z}$. The following result is evident from formulas (3.4) through (3.7).

Proposition 4.3: a) The subgroup of $Aut^o_A(\mathcal{H})$ leaving $\mathcal{W}$ invariant is isomorphic, by restriction to $\mathcal{W}$, to $Sp(\mathcal{W})$, and forms a complement to $Ad(H)$ in $Aut^o_A(\mathcal{H})$. Thus we have a semi-direct product decomposition

$$Aut^o_A(\mathcal{H}) \simeq Sp(\mathcal{W}) \underset{s}{\times} Ad\, H \tag{4.9}$$

b) When 2 is invertible in A transfer $exp^*(Sp(\mathcal{W}))$ to H maps via π surjectively onto $Sp(W)$, and forms a complement to $Aut^{oo}(H)$ in $Aut^o(H)$, so that one has a split exact sequence

$$1 \to Aut^{oo}(H) \to Aut^o(H) \overset{\pi}{\rightleftarrows} Sp(W) \to 1 \tag{4.10}$$

The cross-section is defined as the subgroup of $Aut^o(H)$ preserving the standard isotropic cross-section $exp\ \mathcal{W}$.

Remarks: a) We will use (4.10) to identify $Sp(W)$ with the subgroup of $Aut^o(H)$ preserving $exp\ \mathcal{W}$.

b) If 2 is not invertible in A, then π of (4.6) may not be surjective, and $Aut^{oo}(H)$ may not be complemented in $Aut^o(H)$. See [ ] for more light on this matter.

5: Structure of $Sp(W)$; free polarizations.

We saw that in §4 that the main part of $Aut(H)$ was identifiable with $Sp(W)$ if 2 was invertible in A. In this section, we describe some basic features of $Sp(W)$. Recall from § 1 that if W is a symplectic A-module with form $<\ ,\ >$, and $U \subseteq W$ is a submodule then we say U is a polarization for $<\ ,\ >$ if $U = U^\perp$. If U is a free A-module and is complemented in W, we will call U a <u>free polarization</u> of W. If U_1 and U_2 are two <u>free polarizations</u> such that $W = U_1 \oplus U_2$, then we call (U_1,U_2) a <u>complete polarization</u> of W.

Proposition 5.1: a) If U_1 is a free polarization of rank m, and U_2 is complementary to U_1 in W, then $U_2 \simeq Hom_A(U_1,A)$, so that U_2 is also free of rank m.

b) U_2 may be taken to be isotropic, so that (U_1,U_2) form a complete polarization. Thus any free polarization can be embedded in a complete polarization.

Proof: Via the homomorphism α of (1.5) we have $W \simeq Hom_A(W,A)$. Since U_1 is complemented in W, the restriction map $r:Hom_A(W,A) \to Hom_A(U_1,A)$ is surjective. By definition of polarization $U_1 = ker(r \circ \alpha)$, so that $r \circ \alpha : W/U_1 \xrightarrow{\sim} Hom_A(U_1,A)$ is an isomorphism. Since by definition $U_2 \simeq W/U_1$, part a) is proven.

For part b), choose a basis $\{e_i\}_{i=1}^m$ for U_1. We may find f_1 in U_2 such that $< e_1,f_1 > = 1$, and $< e_i,f_1 > = 0$ for $i > 1$. Let V be the span of e_1 and f_1. Then $W = V \oplus V^\perp$. Because, if $w \in W$, then $W + < w,e_1 > f_1 + < w,f_1 > e_1$ is in $V^\perp$. Now $U_1 \cap V^\perp$ is just the span of $e_2,e_3, \ldots, e_n$, and is clearly maximal isotropic in $V^\perp$. Moreover $(V + U_2) \cap V^\perp$ is easily seen to be a complement to $U_1 \cap V^\perp$ in $V^\perp$. By an obvious induction on the number of elements in a basis for U_1, the proposition follows.

Corollary: Given any basis $\{e_i\}_{i=1}^m$ of a free polarization, there is a basis $\{e_i,f_j\}_{i,j=1}^m$ of W such that

$$(5.1) \qquad < e_i,f_j > = \delta_{ij} \qquad < e_i,e_j > = 0 = < f_i,f_j >$$

A basis of the type in (5.1) will be called a <u>symplectic basis of rank m</u> for W. For reasonable rings A, such as we shall encounter, the rank of W, hence of any symplectic basis for W, will be uniquely defined. It is patently clear that Sp(W) acts simply transitively on the symplectic basis of rank m for W. We, record some consequences of this fact and proposition 5.1 for the structure of Sp.

Proposition 5.2: a) Sp acts transitively on the set of free polarizations of rank m.

b) Let $U \subseteq W$ be a free polarization, and let $P(U,W) = P(U) = P$ be the subgroup of Sp leaving U invariant. Then the restriction map

$$(5.2) \qquad r:P \to GL(U)$$

is surjective.

c) Let $N(U,W) = N(U) = N$ be the kernel of the map r of (5.2). Then N acts simply transitively on the set of complete polarizations with U as first member. Equivalently N acts simply transitively on the set of free polarizations of W complementary to U.

d) Fix a free polarization U' complementary to U. Then there is an isomorphism

$$(5.3) \qquad \beta:N \xrightarrow{\sim} S^{2*}(U')$$

where $S^{2*}(U')$ denotes the space of symmetric bilinear forms on U'.

e) Let $M = P(U) \cap P(U')$. Then M is a complement to N in P, so that P is a semidirect product.

$$(5.4) \qquad P \simeq M \ltimes N$$

Proof: If U and $\tilde{U}$ are free polarizations of rank m, then we can choose bases $\{e_i\}_{i=1}^m$ and $\{\tilde{e}_i\}_{i=1}^m$ for U and $\tilde{U}$ respectively. By the corollary to 5.1, these can be embedded in symplectic bases, which then can be transformed one into the other by Sp. Looking only at the e's, we find an element, $g \in Sp$ such that $g(e_i) = \tilde{e}_i$ <u>A fortiori</u> $g(U) = \tilde{U}$. This proves a). By choosing 2 different bases of the same cardinality for U, we can use the same argument to give b).

Consider two free polarizations U' and U" complementary to U. Fix a basis $\{e_i\}_{i=1}^m$ for U. Let $\{f_i'\}$ and $\{f_i''\}$ be the dual bases for U' and U' respectively. Then $\{e_i,f_i'\}$ and $\{e_i,f_i''\}$ are two symplectic bases for W of rank m. Clearly the element of Sp which moves one basis to the other belongs to N. This proves transitivity of N on the free polarizations complementary to U. That the isotropy group of a given complement is trivial is very easy to see. This finishes c).

Now fix U' complementary to U. Given $n \in N$, define

$$(5.5) \qquad \beta_n(u_1,u_2) = < u_1, n(u_2)>$$

Since n acts trivially on U, we see that u_2 and $n(u_2)$ represent the same element of $Hom_A(U)$. Hence $(n-1)(u_2) \in U$. Hence also $(n-1)^2(u_2) = 0$. Therefore $(n-1)^2 = 0$, whence $2-n = n^{-1}$. Now we compute

$$\beta_n(u_1,u_2) = < u_1, n(u_2) > = < n^{-1}(u_1),u_2 > = < (2-n)(u_1),u_2 >$$

(by isotropy of U') $= - < n(u_1),u_2 > = < u_2, n(u_1) > = \beta_n(u_2,u_1)$.
Hence β_n belongs to $S^{2*}(U')$. Conversely, given $B \in S^{2*}(U')$, we see there is an unique element γ_B in $Hom_A(U',U)$ such that

$$< u_1, \gamma_B(u_2) > = B(u_1,u_2)$$

Then define n_B by

$$n_B(u) = \begin{cases} u & \text{for } u \in U \\ u + \gamma_B(u) & \text{for } u \in U' \end{cases}$$

It is clear n_B will preserve $< u_1,u_2 >$, if at least one u_i is in U. So take u_1 and u_2 in U' and compute

$$< n_B(u_1), n_B(u_2) > = < u_1 + \gamma_B(u_1), u_2 + \gamma_B(u_2) > =$$

(by isotropy of U) $\quad = < u_1,u_2 > + < \gamma_B(u_1),u_2 > + < u_1, \gamma_B(u_2) >$

$\quad = < u_1,u_2 > - B(u_2,u_1) + B(u_1,u_2)$

(by symmetry of B) $\quad = < u_1,u_2 >$.

Thus $n_B \in N$. The maps $n \to \beta_n$ and $B \to n_B$ are easily seen to be inverse, so d) is proved.

Finally, we see that if $\{e_i\}$ is a basis for U, and $\{f_i\}$ the dual basis for U', and if $g \in GL(U)$, then the transformation of $W = U \oplus U'$ which is g on U, and $(g^t)^{-1}$ (which takes the $\{f_i\}$ to the dual basis of $\{g(e_i)\}$) on U' belongs to $P(U) \cap P(U') = M$, which therefore maps surjectively to $GL(U)$ by r of (5.2). But

$M \cap N = \{1\}$ by c), so the proposition is complete.

Remark: According to parts c) and d) of the proposition, the polarizations of W complementary to U are in bijection with the symmetric bilinear forms on U'.

We denote the set of free polarizations of W by Ω ($= \Omega(W) = \Omega(W, < , >)$. We will give a parametrization of Ω in the case when A is a field. Then all $U \in \Omega$ have the same rank or dimension-half the dimension of W. Also any maximal isotropic subspace is a free polarization.

Recall that when A is a field, we have the important relation $U^{\perp\perp} = U$ whenever U is a subspace of W. (When A is a field, we will refer to the A-submodules of W as subspaces.) Fix a complete polarization (U_1,U_2) and let U be any polarization. Set $W_o = U \cap U_1$ and $W_1 = U + U_1$. Evidently $W_o = W_1^{\perp}$, so also $W_1 = W_o^{\perp}$, and $< , >$ induces a symplectic form on W_1/W_o. Furthermore the pair $(U_1/W_o, U_2 \cap W_1)$ derine a complete polarization in W_1/W_o, and U/W_o is also complementary to U_1/W_o in W_1/W_o. It is clear that U is determined by $U_2 \cap W_1$ and the symmetric bilinear form on $U_2 \cap W$ corresponding to U/W_o by the remark following proposition 5.2. Thus we have established part a) of the following statement. Part b) is easily checked.

Proposition 5.3: a) Suppose A is a field. Let (U_1,U_2) be a complete polarization of W. Let $P \subseteq Sp$ be the isotropy group of U_1. Then $\Omega \simeq Sp/P$ can be parametrized by pairs (E,B) where $E \subseteq U_2$ is a subspace and B is a symmetric bilinear form on E. If $U \in \Omega$ and $(E(U),B(U))$ is the corresponding pair, then $E = (U_1 + U) \cap U_2$, and $B(e_1,e_2) = < e_1, e_2 + \gamma_B(e_2) >$, where $\gamma_B(e_2) \in U_1$ is such that $e_2 + \gamma_B(e_2)$ is in U.

b) Under this identification the N orbits are those (E,B) with fixed E. The P orbits are those pairs with $\dim E$ fixed. If $U \in \Omega$, then $\dim(E(U)) = (\frac{1}{2}) \dim W - \dim(U \cap U_1)$.

6: Reductive dual pairs, definition and classification.

In this section and henceforth, we take A to be a field, not of characteristic 2. To emphasize this restriction we use F instead of A to denote our base field.

Definition: Let Γ be a group and let (G,G') be a pair of subgroups of Γ. We will say (G,G') form a dual pair of subgroups of Γ if G is the centralizer in Γ of G' and vice-versa.

It is not hard to find dual pairs of subgroups in a group. Start with any subgroup $G \subseteq \Gamma$. Let G' be the centralizer of G in Γ and let G'' be the centralizer of G'. Then (G'',G') is a dual pair in Γ.

When we take $\Gamma = Sp(W)$ for some symplectic vector space, we can refine the concept slightly. We say $(G,G') \subseteq Sp$ form a reductive dual pair if first (G,G') is a dual pair in Sp, and moreover G and G' are redu tive in the sense that they act (absolutely) reductively on W.

The goal of this section is to describe reductive dual pairs in Sp. Let (G,G') be such a pair. Suppose $W = W_1 \oplus W_2$ is an orthogonal direct sum, and that each W_i is invariant by G and by G'. Let $G_i = G|W_i$ be the group of transformations of W_i obtained by restricting elements of G. Define G_i similarly. Let $r_i:G \to G_i$, and r_i' be the obvious maps. Then half a moment's thought convinces one that $r_1 \times r_2:G \to G_1 \times G_2$ is an isomorphism, and similarly for $r_1' \times r_2'$. Moreover, (G_i,G_i') will be a reductive dual pair in $Sp(W_i)$. To describe this situation we will say that (G,G') is the direct sum of the (G_i,G_i'). If (G,G') has no non-trivial direct sum decomposition, then we will call (G,G') irreducible.

Proposition 6.1: a) Every reductive dual pair is the direct sum of irreducible subpairs in an essentially unique way (i.e., up to numbering of the pairs)

b) If (G,G') is irreducible, then either:

i) $G\cdot G'$ acts irreducibly on W, and W consists of a single isotypic component (which is self-dual) for G or for G'; or

ii) $W = U_1 \oplus U_2$, where each U_i is invariant and irreducible for $G\cdot G'$, and is maximal isotropic in W. The rewtriction maps then take (G,G') to a dual pair in $GL(U_i)$.

Proof: Consider first the action of G alone on W. Let V_1 and V_2 be irreducible G-subspaces of W. If the pairing between V_1 and V_2 induced by $<\ ,\ >$ is non-trivial, then it must be non-degenerate, by irreducibility of the V_i, and thus will induce a G-equivariant isomorphism between V_2 and $Hom(V_1,F) = V_1^*$. (We use the conventional $*$ to denote dual space now that we are working over a field.) Thus V_1 must be orthogonal to any G-submodule of W except one isomorphic to V_1^*; and since W is the direct sum of irreducible G-modules, since G is assumed to act reductively on W, there will be a G-submodule of W isomorphic to V_1^* paired non-degenerately with V_1.

Consider the decomposition $W = \oplus_i U_i$ of W into isotypic components for G. That is, each U_i is the sum of all G-submodules of some given isomorphism type. By the preceding paragraph, we see that either the isomorphism type defining U_i is self-dual and that $<\ ,\ >|\,U_i$ will be non-degenerate; or there is another isotypic component U_j containing the dual isomorphism type to that of U_i, and that each of U_i and U_j are isotropic, but $<\ ,\ >$ is non-degenerate on $U_i \oplus U_j$.

Thus we may write $W = \oplus_j U_j'$ where each U_j' is either self-dual

isotypic for G, or the sum of two mutually contragredient isotypic components. This sum is orthogonal. The U_j' are clearly determined uniquely up to order by G and are invariant by G'. Therefore they give a direct sum decomposition of (G,G'). To prove the proposition, therefore, it will suffice to show that each G-isotypic component is irreducible under $G\cdot G'$.

Consider a subspace $U \subseteq W$ which is irreducible for $G\cdot G'$. Then U must consist of a single isotypic component for either G or G'. The form $<\ ,\ >$ is either trivial or non-degenerate on U. Suppose first $<\ ,\ >$ is non-degenerate. Then $W = U \oplus U^{\perp}$ is a decomposition of (G,G'). Since the operator which is the identity on U and minus the identity on $U^{\perp}$ commutes with both G and G', it must belong to both. Therefore U and $U^{\perp}$ contain no isomorphic G-submodules, so U is a full isotypic component in W for G and G'. This is case b)i).

Secondly, suppose $<\ ,\ >$ is trivial on U. Then there is another $G\cdot G'$-irreducible subspace V which is paired non-trivially, hence non-degenerately, with U. I claim V must also be isotropic. Otherwise, we could write $W = V \oplus V^{\perp}$ and reason as just above. But then U would be the graph of a non-trivial $G\cdot G'$ intertwining morphism from $V^{\perp}$ to V, which, we saw above, does not exist. Thus U and V are both isotropic. Write $W = (U \oplus V) \oplus (U \oplus V)^{\perp}$. Observe that multiplication by a scalar t on U, by t^{-1} on V and by 1 on $(U \oplus V)^{\perp}$ defines an element of Sp commuting with both G and G', so belonging to both. Thus again U must be a full G-isotypic component in W, and the proposition is proved.

Let (G,G') be a reductive dual pair in Sp. We will say (G,G') is of <u>type I</u> or <u>type II</u> according as possibility i) or possibility ii) of proposition 6.1 b) obtains. We proceed to describe more precisely the pairs of the two types. We begin with type II pairs, as these are somewhat simpler than type I pairs.

Proposition 6.2: Let $(G,G') \subseteq Sp$ be an irreducible type II reductive dual pair. Let (U_1,U_2) be the complete polarization of W invariant by $G\cdot G'$, so that $W = U_1 \oplus U_2$ and each of U_1 and U_2 are isotropic and irreducible under $G\cdot G'$. Then restriction to U_1 embeds (GG') as a reductive dual pair in $GL(U_1)$. Thus there is

i) a division algebra D, and

ii) a right vector space V and a left vector space V' over D,

such that U_1 is isomorphic to $V \underset{D}{\otimes} V'$ in such fashion that G is identified to $GL_D(V) \otimes I_{V'}$ and G' is identified to $I_V \otimes GL_D(V')$ where I_V and $I_{V'}$ are the identity operators on V and V' respectively.

Proof: Both G and G' are in the subgroup M preserving U_1 and U_2, as described in proposition 5.2 e). By that result M is identified by restriction with $GL(U_1)$. Clearly, if G' is the centralizer of G in Sp, it is the centralizer of G in M also. We now may apply the classical description of reductive dual pairs in the general linear group, essentially amounting to the Double Commutant Theorem of linear algebra, to obtain the existence of D and the decomposition of U_1 as a tensor product.

Remark: Since U_2 may be identified to U_1^* as in (1.5), we may write $W \simeq V \underset{D}{\otimes} V' \oplus V'^* \underset{D}{\otimes} V^*$ in such a way that (G,G) is identified to $(GL_D(V), GL_D(V'))$ acting in the obvious way. Sometimes a slight variant of this construction is useful. Namely, we can take two right D vector spaces V and V' and identify U_1 with $\mathrm{Hom}_D(V,V')$. Then G acts by right multiplication by inverses and G' acts by left multiplication. Then

(6.1) $$W \simeq \mathrm{Hom}_D(V,V') \oplus \mathrm{Hom}_D(V',V)$$

and if $S \in \mathrm{Hom}_D(V,V')$ and $T \in \mathrm{Hom}_D(V',V)$, and $g \in G$; $g' \in G'$, then

(6.2) $$g\cdot g'(X,T) = (g'S\,g^{-1},\ g\,T\,g'^{-1})$$

This formulation has the advantage that there is a simple formula for $<\ ,\ >$. Assuming the identification (6.1) is normalized properly, we have

(6.3) $$<(S_1,T_1),(S_2,T_2)> = \mathrm{tr}(S_1T_2 - S_2T_1)$$

where notation is parallel to (6.2), and tr here is reduced trace over F on $\mathrm{End}_D(V')$. Of course $\mathrm{tr}(T_2S_1 - T_1S_2)$ gives the same answer.

We turn to type I pairs, which present more variety.

Proposition 6.3: Let (G,G') be an irreducible type I reductive dual pair. Then there exist

i) a division algebra D

ii) with involution (i.e., involutory antiautomorphism) $\natural$,

iii) vector spaces V and V' over D

iv) with forms $(\ ,\ )$ and $(\ ,\ )'$, which are D-linear in the first variable and either $\natural$-hermitian or $\natural$-skew hermitian (one of each type)

such that $W \simeq V \underset{D}{\otimes} V'$ in such fashion that G is identified to the isometry group of $(\ ,\ )$ and G' to the isometry group of $(\ ,\ )'$.

Remarks: a) Just as in the type II case, we can be slightly less symmetric and write

(6.4) $$W \simeq \mathrm{Hom}_D(V, V')$$

Then the action of G is by premultiplication (by inverses) and that of G' is by postmultiplication. Of course we could also write $W \simeq \mathrm{Hom}_D(V',V)$. Note that there are nice maps mediating between these two alternatives. Namely, define

$$*: \mathrm{Hom}_D(V,V') \to \mathrm{Hom}_D(V',V), \quad \text{and}$$
$$*': \mathrm{Hom}_D(V',V) \to \mathrm{Hom}_D(V,V') \quad \text{by}$$

(6.5) $$(Tv, v')' = (v, T^{*}v') \qquad \text{for } T \in \mathrm{Hom}_D(V,V')$$
$$(Sv',v) = (v', S^{*'}v)' \qquad \text{for } S \in \mathrm{Hom}_D(V',V)$$

Since (,) and (,)' have opposite parties, we see that

(6.6) $$T^{**'} = T \qquad S^{*'*} = -S$$

Again, this slightly assymmetric approach allows one to give a convenient formula for < , > .

(6.7) $$T_1, T_2 = tr(T_2^{*} T_1) \qquad \text{for } T_i \in \mathrm{Hom}_D(V,V')$$

where T again is reduced trace on $\mathrm{End}_D(V)$. Implicit in (6.7) is the fact that $tr(T^{*}T) = 0$. This is indeed the case. For the moment we merely record (6.7). We will discuss it more fully in §7.

b) The classification of irreducible reductive dual pairs given by this proposition and the previous one is a reworking from a new viewpoint of classical results, going back through Weil [ ] and Siegel to Albert. These results can also be dug out of Satake [ ], and proposition 6.4 following is stated in Shimure [ ]. What seems to be new here is the insistence on the equality of status of G and G', and of the mutuality of their relation to one another.

Proof: We must begin with some generalities on D-semi-linear forms where D is a division algebra with involution $\natural$ over some field F. Consider D as a left vector space over itself, and define

(6.8) $$(x,y)_o = x\, y^{\natural}$$

for x,y in D. Then $(,)_o$ satisfies

(6.9) i) $(ax, y)_o = a(x,y)_o \qquad (x,ay)_o = (x,y)_o\, a^{\natural} \qquad$ for $a,x,y \in D$

ii) $(y,x)_o = (x,y)_o^{\natural}$.

If E is any left D vector space, call a form (,) satisfying (6.9) i) $\natural$ -sesquilinear. If it also satisfies (6.9) ii) call it $\natural$ -hermitian. Evidently, if we choose a basis for E and so identify E with D^k, then the k-fold direct sum of $(,)_o$ as in (6.8) will define a non-degenerate $\natural$ -hermitian form on V.

Let tr denote the reduced trace map from D to F. We recall

(6.10) $$tr(xy) = tr(yx) \qquad tr(x^{\natural}) = tr(x) \qquad \text{for } x,y \in D.$$

The D vector space E may be regarded as an F vector space, denoted E_F, by restriction of scalars. If (,) is a $\natural$ -hermitian form on E, then

$$tr(,): x,y \to tr((x,y)) \qquad x,y \in E = E_F$$

defines a symmetric bilinear form on E_F. Moreover (6.9)i) and (6.10) imply

(6.11) $$tr(ax,y) = tr(x, a^{\natural} y) \qquad \text{for } a \in D,\ x,y \in E_F$$

Let $(,)_1$ be a fixed non-degenerate $\natural$ -hermitian form on E.

Let $\{\ ,\ \}$ be an F-bilinear form on E_F satisfying (6.11). Since $tr(\ ,\ )_1$ is non-degenerate we can write

(6.12) $$\{x,y\} = tr(Tx,y)_1 \qquad x,y \in E_F$$

for some $T \in Hom_F(E_F)$. We compute

$$tr(Tax,y)_1 = \{ax,y\} = \{x, a^{\natural} y\} = tr(Tx, a^{\natural} y)_1 = tr(aTx, y)_1$$

Therefore actually T is in $Hom_D(E)$, so that if

(6.13) $$(x,y)_2 = (tx, y)_1$$

then $(\ ,\ )_2$ is a $\natural$-sesquilinear form on E such that

(6.14) $$\{\ ,\ \} = tr(\ ,\ )_2$$

Thus we have shown

Proposition 6.4: If D is a division algebra over a field F, with involution $\natural$, and E is a (left) vector space over D, then the map

$$(\ ,\ ) \to tr(\ ,\ )$$

establishes an isomorphism between $\natural$-sesquilinear forms on E and bilinear forms on E_F satisfying (6.11). Under this map $\natural$-(anti) hermitian forms correspond to (anti) symmetric forms.

Remark: This shows in particular we may always lift forms satisfying (6.11) to bilinear forms over the $\natural$-fixed subfield of the center of D.

Return to our reductive dual pair (G,G') acting irreducibly on W. From the standard double commutant theory [ ] we know we can find a division algebra D and left D-vector spaces V and V' such that $W \simeq Hom_D(V,V')$ in such a manner that the action of G is identified to right multiplication by D linear mappings of V and the action of G' is given by left multiplication by D linear maps of V'. To prove proposition 6.3 we have to find an involution $\natural$ of D and $\natural$-sesquilinear forms $(\ ,\ )$ and $(\ ,\ )'$ on V and V' respectively, invariant by G and G' respectively, one $\natural$-hermitian and the other $\natural$-antihermitian.

Suppose we find two involtuions $\natural$ and $\#$ for D which have the same restriction to the center of D. Then $\natural\#$ is an automorphism of D over its center, so by Skolem-Noether [ ]

(6.15) $$\natural\#(d) = \delta\, d\, \delta^{-1} \qquad \text{for } d \in D$$

for some $\delta \in D$. Thus involutions on D having a given restriction to the center differ by inner automorphisms of D.

Consider a vector space U over F, and a non-degenrate bilinear form $(\ ,\ )$ on F, either symmetric or antisymmetric. Then $(\ ,\ )$ induces an $End(U)$ an involution $\#$ defined by

(6.16) $$(Tu_1, u_2) = (u_1, T^{\#}u_2) \qquad u_1 \in U,\ T \in End(U)$$

and satisfying the familiar rules:

(6.17)
i) $(ST)^{\#} = T^{\#} S^{\#}$ $\qquad (S + T)^{\#} = S^{\#} + T^{\#}$
ii) $T^{\#\#} = T$
iii) The isometries of $(\ ,\ )$ are $\{g \in GL(U): g^{\#} = g^{-1}\}$.

Suppose G is a group acting irreducibly on U. Let L be the span of G in $\mathrm{End}(U)$ and let D be the commuting algebra. Then L is a simple algebra, D is a division algebra and $D \cap L$ is their common center. Suppose G preserves the form $(\, , \,)$ of the last paragraph. Then clearly, from (6.17)iii) we see $\#$ preserves G, hence L hence D. Thus D is a division algebra with involution. If $(\, , \,)_1$ is another G-invariant form inducing the involution $\#_1$, then again by (6.17)iii), both $\#$ and $\#_1$ agree on L, hence they agree on the center of D. We can say more. Since G acts irreducibly, the form $(\, , \,)$ must be non-degenerate. Hence we may write

$$(u_1,u_2)_1 = (Su_1, u_2)$$

for some S in $\mathrm{End}(U)$. Since $(\, , \,)_1$ is also G-invariant, we must have $S \in D$. The relation between $\#$ and $\#_1$ is easily seen to be

$$S^{\#_1} = (T^{\#})^{-1} S^{\#} T^{\#}$$

Hence we can demonstrate directly the conjugacy between $\#$ and $\#_1$ in this situation.

Now take G to be the first member of our pair (G,G') and take $U \subseteq W$ an irreducible subspace for G. Then the D of the above paragraph is isomorphic to the D in the isomorphism $W \simeq \mathrm{Hom}_D(V,V')$. Further U is isomorphic to V as a joint G and D module. On U there are many bilinear forms invariant by G, which we may transfer to V. Namely, for g_1, g_2 in G', consider

$$B_{g_1,g_2}(u_1,u_2) = \langle g_1'(u_1), g_2'(u_2) \rangle \qquad (6.18)$$

Some of these forms will be non-degenerate, and so will their symmetric or antisymmetric parts. Thus we may find at least one form B, either symmetric or antisymmetric on V invariant by G. If $\natural$ is the involution of D induced by B, then according to proposition 6.4, there is a $\natural$-hermitian or $\natural$-antihermitian form $(\, , \,)$ on V invariant by G.

By similar reasoning, and by adjusting our involutions as explained above, we can also find a form $(\, , \,)'$ on V' either $\natural$-hermitian or $\natural$-antihermitian, and invariant by G'. Thus we have forms on V and V'. It remains only to show we can take exactly one to be $\natural$-hermitian and the other to be $\natural$-antihermitian. To do this, it is more convenient to use the involution on D to regard V' as a right vector space and write $W \simeq V \otimes_D V'$. Take $v_i \in V$ and $v_i' \in V'$, and consider the quanity

$$\{v_1 \otimes v_1', v_2 \otimes v_2'\} = \mathrm{tr}((v_1,v_2)(v_1', v_2')') \qquad (6.19)$$

If $d \in D$, we compute

$$\begin{aligned}\{(dv_1) \otimes v_1', v_2 \otimes v_2'\} &= \mathrm{tr}((dv_1,v_2)(v_2',v_1')') \\ &= \mathrm{tr}(d(v_1,v_2)(v_2',v_1')') = \mathrm{tr}((v_1,v_2)(v_2',v_1')d) \\ &= \mathrm{tr}((v_1,v_2)(v_2', d^{\natural} v_1')) = \{v_1 \otimes d^{\natural} v_1', v_2 \otimes v_2'\}\end{aligned}$$

A similar relation holds in the second variable. It follows that $\{ \cdot , \cdot \}$ factors to define an F-bilinear form on $V \otimes_D V'$. Moreover $\{ \cdot , \cdot \}$ will be symmetric if $(\, , \,)$ and $(\, , \,)'$ have the same parity, and will be antisymmetric if they have opposite parties. Since $\{ \cdot , \cdot \}$ is obviously $G \cdot G'$ invariant, we have $\{w_1,w_2\} = \langle c\, w_1, w_2 \rangle$ where c commutes with $G \cdot G'$.

That is, c is in the center of D. If we replace $(\, , \,)$ with the form

$$v_1, v_2 \to (c^{-1} v_1, v_2)$$

before we perform the construction (6.19), then we will have $c = 1$, or $\{ \, , \, \} = \langle \, , \, \rangle$. Since $\langle \, , \, \rangle$ is antisymmetric, the symmetry properties of $(\, , \,)$ and $(\, , \,)'$ are as desired.

Finally, we see that the full isometry group of $(\, , \,)$ preserves $\langle \, , \, \rangle$, commutes with G' and contains G, so it is equal to G. Similarly, G' is the full isometry group of $(\, , \,)'$. This concludes proposition 6.3.

To round out this section, we will review the rudiments of the classification theory for division algebras with involution. See [ ] for a more complete discussion. Given a field F, the set of isomorphism classes of simple algebras central over F forms a semigroup under tensor product and if one factors out by the matrix algebras over F, one obtains a torsion group called the Brauer group of F. Given a simple algebra M over F, let $\{M\}$ denote its class in the Brauer group. Then $\{M\}^{-1}$ is represented by the opposed algebra $\tilde{M}$ of M - the same space with reversed order of multiplication.

Let $\overline{F}$ be a separable algebraic closure of F, with multiplicative group $\overline{F}^{\times}$. Let Γ be the Galois group of $\overline{F}$ over F. A basic result [ ] is that the Brauer group of F has an alternative description as the Galois cohomology group $H^2(\Gamma; \overline{F}^{\times})$. Suppose $F' \subseteq \overline{F}$ is a finite Galois extension of F. Then $Gal(F'/F)$ acts in a natural way (via its action on cocycles) on the Brauer group $H^2(\Gamma'; \overline{F}^{\times})$, where Γ' is the Galois group of $\overline{F}$ over F'. Call this action α. Let M_1 and M_2 be simple algebras central over F' and let $\phi : M_1 \to M_2$ be an F-linear isomorphism. If $\phi|F' = h \in Gal(F'/F)$, then $\{M_2\} = \alpha(h) \{M_1\}$.

Suppose now M is a simple algebra over F' with involution $\natural$, and let $F \subseteq F'$ be the fixed field of $\square$. Then $F' = F$ or F' is quadratic over F. In any case, the restriction $\natural|F'$ is a generator σ of $Gal(F'/F)$. Combining $\natural$ with the standard antiautomorphism of L with $\tilde{L}$, the opposite algebra, we conclude that $\alpha(\sigma)\{L\} = \{\tilde{L}\} = \{L\}^{-1}$, or in other words,

$$(\alpha(\sigma) + 1)\{L\} = 0 \qquad (6.20)$$

If $\alpha(\sigma)$ reduces to the identity, then (6. 20) just says $2\{L\} = 0$, or $\{L\}$ is of order 2 in the Brauer group. More detailed statements depend on the finer structure of F'.

Let us now take F' to be a local field. The Brauer groups of local fields are known [ ]. If F' is non-Archimedean , then its Brauer group is $\mathbb{Q}/\mathbb{Z}$. The Brauer group of $\mathbb{R}$ is $\mathbb{Z}/2$ and that of $\mathbb{C}$ is trivial. Shafarevich's Theorem [ ] shows the Galois action is trivial. Therefore in these cases, the only possibilities for division algebras over F such that F is the fixed field of the center are a) F itself; b) a quadratic extension F' over F; c) the unique quaternion algebra over A; and d) the quaternion algebra over a quadratic extension of A. However, as it turns out, possibility d) is also impossible. Hence there are in fact only 3 possibilities for D.

The situation for global fields is more complicated because the Galois action on the Brauer group can be non-trivial. See [ ] for some discussion of this.

7: Lie algebras of the classical groups; Cayley transform

The groups which emerged from the discussion of §6, that is, the general linear group of a vector space over a division algebra, and the isometry group of a hermitian or antihermitian sesquilinear form over a division algebra with involution, are often referred to as classical groups. Sometimes other groups closely related to these, e.g., special linear groups, etc., are also called classical groups. However, in the present paper, a classical group will be precisely one of the groups specified in propositions 6.3 and 6.4, in other words, one member of a reductive dual pair in Sp.

In many parts of the development of the theory of reductive dual pairs, the type I and type II groups, e.g., isometry groups and GL, must be discussed separately, a circumstance which predictably leads at times to considerable tedium. Sometimes explicit separate discussion of GL can be avoided if we make the convention that GL is the isometry group of the zero form. That is, results stated for isometry groups are formally correct for GL under this interpretation. Thus below and elsewhere where there is no explicit treatment of GL separate from isometry groups, the results are to be interpreted as holding for GL as isometries of the trivial form (and ignoring the fact that D should have an involution), unless the results are stated explicitly for type I classical groups.

Let G be a classical group, the isometry group of the hermitian or antihermitian form (,) on the vector space V over the division algebra D with involution $\natural$. We will call $(V, D, \natural, (,))$ the basic data of G and will always consider G coming with this data attached.

Thus G is not simply an isomorphism class of groups, but is acting on a particular space in a particular way. Regarding the form (,), it will be said to be of type $(D, \natural, +)$ or type $(D, \natural, -)$ depending on whether it is $\natural$-hermitian or $\natural$-antihermitian. The type $(D, \natural, -)$ will be said to be of dual type to the type $(D, \natural, +)$, and vice versa.

The Lie algebra $\mathfrak{g}$ of G is defined heuristically as the collection of $T \in End_D(V)$ such that, if ϵ is an infinitesimal - a non-zero quantity so small its square is zero, then $I + \epsilon T$ is an isometry of (,). Formally this amounts to the identity

$$(7.1) \qquad (Tv, v') + (v, Tv') = 0 \qquad T \in \mathfrak{g}, \ v \in V$$

It is easy to check that $\mathfrak{g}$ as defined by (7.1) is indeed a Lie algebra, i.e., is closed under taking commutators. Also if $T \in \mathfrak{g}$ and $g \in G$, we define

$$(7.2) \qquad Ad\, g(T) = g T g^{-1}.$$

It is easy to compute that Ad G preserves $\mathfrak{g}$, so Ad defines an action of G on $\mathfrak{g}$.

Let $\tilde{B}(V)$ denote the space of forms on V of the type dual to (,). There is a natural action σ of $GL_D(V)$ on $\tilde{B}(V)$ defined by

$$(7.3) \qquad \sigma(A)\,\beta(v,v') = \beta(A^{-1}v, A^{-1}v') \quad \text{for } A \in GL_D(V), \quad \beta \in \tilde{B}(V)$$

From (7.1) it is straightforward to verify the following fact.

Proposition 7.1: Define $\beta: \mathfrak{g} \to \tilde{B}(V)$ by

(7.4) $$\beta_T(v,v') = (Tv, v')$$

Then β is an isomorphism from $\mathfrak{g}$ to $\tilde{B}(V)$ and is equivariant for the actions Ad and σ of G.

For $G = GL_D(V)$, we ignore proposition (7.1) and simply note that $\mathfrak{g}$ is all of $End_D(V)$.

Let G and G' be classical groups with basic data $(V, D, \natural, (\ ,\ ))$ and $(V', D, \natural, (\ ,\ )')$ respectively. Take $(\ ,\ )$ and $(\ ,\ )'$ to be of dual type, so that (G,G') acting on $Hom_D(V,V')$ by right and left multiplication form an irreducible type I reductive dual pair. We have defined the isomorphisms $*: Hom_D(V,V') \to Hom_D(V',V)$ and $*'$ in the reverse direction in (6.5).

Proposition 7.2: The map

(7.5) $$\tilde{\tau} : T \to T^*T \qquad T \in Hom_D(V,V')$$

has image in $\mathfrak{g}$. Similarly the map

$$\tilde{\tau}' : S \to S^{*'}S \qquad S \in Hom_D(V',V)$$

has image in $\mathfrak{g}'$. Moreover

(7.5) a) $$\tilde{\tau}(g'\,T\,g^{-1}) = Ad\ g(\tau(T))$$

$$\tilde{\tau}'(g\,S\,g'^{-1}) = Ad\ g'(\tau'(S))$$

Hence the image under $\tilde{\tau}$ of a $G\cdot G'$ orbit in $Hom_D(V,V')$ is an Ad G orbit in $\mathfrak{g}$, and the image of a G' orbit is a single point; and similarly for $\tilde{\tau}'$.

Remark: By this proposition we see that to each $G\cdot G'$ orbit $\mathcal{O}$ in

$W = Hom_D(V,V') \simeq Hom_D(V',V)$ we can attach a pair of orbits $(\tilde{\tau}(\mathcal{O}), \tilde{\tau}'(\mathcal{O}))$ in $\mathfrak{g}$ and $\mathfrak{g}'$. Thus we obtain a correspondence $\tilde{\tau}(\mathcal{O}) \leftrightarrow \tilde{\tau}'(\mathcal{O})$ between certain Ad G orbits in $\mathfrak{g}$ and certain Ad G' orbits in G'. We will see in § 8 that this correspondence is "generically", i.e., for $\mathcal{O}$ in some Zariski open set, bijective. This phenomenon was to my knowledge first made explicit in [ ].

Proof: We recall from (6.5) the definition of $*$

$$(Tv, v')' = (v, T^{*'}v') \qquad T \in Hom_D(V,V')$$

Therefore

$$(T^*Tv_1, v_2) = \pm(v_2, T^*Tv_1)^\natural = \pm(Tv_2, Tv_1)'^\natural$$
$$= -(Tv_1, Tv_2)' = -(v_1, T^*Tv_2).$$

Comparing with (7.1) we find that T^*T satisfies the condition to belong to G. We also compute

$$(g'\,T\,g^{-1}v, v')' = (T\,g^{-1}v, g'^{-1}v')' = (g^{-1}v, T^*g'^{-1}v')$$
$$= (v, g\,T\,g'^{-1}v'),$$

whence

(7.6) $$(g'\,T\,g^{-1})^* = g\,T\,g'^{-1}$$

Equation (7.5) a) is immediate from (7.6).

Remark: Let F be a subfield of the center of D, in the fixed field of $\natural$, such that D is finite dimensional over F and its center is separable over F. Then $tr = tr(D/F)$, the reduced trace of D over F is defined on D and also on $\mathrm{Hom}_D(V,V)$. Since $\mathfrak{g}$ is the Lie algebra of the isometries of some non-degenerate form, tr vanishes on $\mathfrak{g}$. Hence $\mathrm{tr}(T^*T) = 0$, for $T \in \mathrm{Hom}_D(V,V')$, which shows that (6.7) does define an alternating form on $W = \mathrm{Hom}_D(V,V')$. Formula (7.5)a) shows (6.7) is invariant by G and by G'. Hence altering the identification of $\mathrm{Hom}_D(V,V')$ with W by an element of the center of D if necessary, we can indeed arrange that the form $\langle , \rangle$ on W be given by (6.7).

Let F continue as in the above remark. Although $\mathfrak{g} \subseteq \ker \mathrm{tr}$, the bilinear form

(7.7) $$T, S \to \mathrm{Tr}\ TS \qquad T, S \in \mathfrak{g}$$

is a non-degenerate symmetric, Ad-invariant bilinear form on $\mathfrak{g}$. Indeed symmetry and Ad-invariance are standard facts. As to non-degeneracy, recall that the form (,) an involution $\#$ on $\mathrm{End}_D(V)$ according to the recipe (6.16). In terms of $\#$, formula (7.1) can be written

(7.8) $$\mathfrak{g} = \{T \in \mathrm{End}_D(V) : T^{\#} = -T\}$$

Therefore the map $T \to (\frac{1}{2})(T - T^{\#})$ projects $\mathrm{End}_D(V)$ onto $\mathfrak{g}$, and we have

(7.9) $$\mathrm{End}_D(V) = \mathfrak{g} \oplus \mathfrak{s}$$

where

(7.10) $$\mathfrak{s} = \{T \in \mathrm{End}_D(V) : T = T^{\#}\}$$

Since $\mathrm{tr}\ T^{\#} = \mathrm{tr}\ T$, the decomposition (7.9) is orthogonal for the pairing (7.7). Since tr TS is non-degenerate on $\mathrm{End}_D(V)$, it is also non-degenerate on $\mathfrak{g}$.

For a type II pair, the analogue of proposition 7.2 comes by considering $(G,G') \simeq (GL_D(V), GL_D(V'))$ and writing $W = \mathrm{Hom}_D(V,V') \oplus \mathrm{Hom}_D(V',V)$. Then if $T \in \mathrm{Hom}_D(V,V')$ and $S \in \mathrm{Hom}_D(V',V)$, the pair (T,S) is a typical point of W. We may consider the maps

(7.11) $$\tilde{\tau}(T,S) = ST \qquad \tilde{\tau}'(T,S) = TS$$

Then clearly $\tilde{\tau}$ maps W to $\mathrm{End}_D(V)$ which is the Lie algebra of GL_D, and $\tilde{\tau}'$ maps W to $\mathrm{End}_D(V')$. It is obvious that formulas (7.5), and the consequent remarks, hold in the type II case also.

For $T \in \mathrm{End}_D(V)$, such that $I + T$ is invertible (where as usual I is the identity map of V) define the <u>Cayley transform</u>, $c(T)$, by

(7.12) $$c(T) = \frac{I-T}{I+T}$$

The following formulas are immediate from the definition.

(7.13) i) $c(c(T)) = T$ ii) $c(0) = I$; $c(I) = 0$

iii) $c(-T) = c(T)^{-1}$ iv) $c(T^{-1}) = -c(T)$

v) $c(\mathrm{Ad}\ S(T)) = \mathrm{Ad}\ S(c(T)')$

Proposition 7.3: Let G be a classical group with basic data $(V, D, \natural, (,))$. Let $\mathfrak{g}$ be the Lie algebra of G. Then

(7.14) $$c(\mathfrak{g}) \subseteq G \quad \text{and} \quad c(G) \subseteq \mathfrak{g}$$

Proof: Taking $T \in \mathfrak{g}$, and with $\#$ as in (6.16), we use (7.8) and (7.13) to compute

$$c(T)^{\#} = c(T^{\#}) = c(-T) = c(T)^{-1}$$

so that $c(T) \in G$ by (6.17). The other inclusion is similar.

Remark: a) The Cayley transform is a rational map, so proposition (7.3) can be sharpened by using terminology from algebraic geometry. Precisely, c gives a birational equivalence between $\mathfrak{g}$ and the (Zariski) connected component of I in G.

b) It is curious that the set

$$\mathfrak{g} \cap G = \{T : T^{\#} = -T \text{ and } T^{\#} = T^{-1}\}$$

is just the set of complex structures in $\mathfrak{g}$ or G. A <u>complex structure</u> is simply a $J \in End_D(V)$ such that $J^2 = -I$. But if $T \in \mathfrak{g} \cap G$, then $-T = T^{\#} = T^{-1}$, or $-T^2 = I$, so T is a complex structure. By the proposition 7.3, the Cayley transform will preserve $\mathfrak{g} \cap G$. Indeed, we can also characterize the complex structures as maps J such that $c(J) = -J$, for $\frac{I-J}{I+J} = -J$ implies $I-J = -J-J^2$, or $-J^2 = I$.

If we regard T as an indeterminate, we may expand $c(T)$ in a formal power series.

(7.15) $$c(T) = I + 2 \sum_{n=1}^{\infty} (-1)^n T^n = I - 2T + 2T^2 - 2T^3 + \ldots$$

Comparing this with the exponential series

(7.16) $$\exp T = \sum_{n=1}^{\infty} \frac{T^n}{n!} = I + T + \frac{T^2}{2} + \frac{T^3}{6} + \ldots$$

we find

(7.17) $$c(T) \equiv \exp(-2T) (\mathrm{mod}\ T^3).$$

Thus, the following truncated version of the Campbell-Hausdorff formula holds for $c(T)$.

(7.18) $$c(T)\, c(S) \equiv c(T + S - [T,S]) \quad \text{modulo 3rd order terms}$$

Some examples of the foregoing topics will figure in later discussion. We give a compendium of some formulas that will be pertinent. The simplest operators $End_D(V)$ are the dyads E_{xy}, defined by

(7.19) $$E_{xy}(z) = (z,x)y \qquad x,y,z \in V$$

Here $(\, , \,)$ is our usual form, part of the basic data of the type I classical group G. Of course, the E_{xy} span $End_D(V)$. We can easily compute

(7.20)

i) $E_{xy} E_{zu} = E_{z(u,x)y}$

ii) $E_{xy}^{\#} = E_{yx}$ or $-E_{yx}$ according as $(\, , \,)$ is $\natural$-hermitian or $\natural$-antihermitian.

iii) $E_{TxSy} = S E_{xy} T^{\#}$ for $T,S \in End_D(V)$

Thus we get by (7.8) a spanning set for $\mathfrak{g}$ made of the elements

(7.21) $$\hat{E}_{xy} = E_{xy} \mp E_{yx}$$

where we take the $-$ sign if $(\, , \,)$ is $\natural$-hermitian and the $+$ sign if $(\, , \,)$ is $\natural$-antihermitian. If T is a general element of G, then we can compute

(7.22)
$$\begin{aligned} tr(T \hat{E}_{xy}) &= tr(T(E_{xy} - E_{xy}^{\#})) \\ &= tr(T E_{xy} + (E_{xy} T)^{\#}) \\ &= 2\, tr(T E_{xy}) = 2(Ty\ x) \end{aligned}$$

We can specialize this to our original symplectic group Sp, whose defining data is $(W, F, 1, < , >)$ where F is our base field and $< , >$ is the symplectic form on W. In this case for any $w \in W$ the dyad E_{ww} itself belongs to the Lie algebra of Sp, which we shall denote $\mathfrak{sp}$. We note that since $< w,w > = 0$, we have $E_{ww}^2 = 0$. Hence the power series (7.15) for the Cayley transform degenerates and we find

$$c(a\,E_{ww}) = I - 2a\,E_{ww}. \tag{7.23}$$

Symplectic maps of the form (7.23) are called <u>transvections</u>. The transvections

$$t_w = I + E_{ww} = c(-\tfrac{1}{2}\,E_{ww}) \tag{7.24}$$

will be called <u>unit transvections</u>. Clearly the map $w \to t_w$ is a quadratic map from W to Sp, equivariant with respect to the appropriate actions of Sp. Since Sp acts transitively on W, the unit transvections form a single conjugacy class in Sp. Somewhat more generally, it may be computed that $I + a\,E_{ww}$ and $I + b\,E_{w'w'}$ are conjugate in Sp if and only if ab is a square in F. However, any two transvections are conjugate via outer automorphisms of Sp.

We conclude with a computation. If (G,G') is an irreducible type I reductive dual pair in Sp, then $\mathfrak{g} \subseteq \mathfrak{sp} \supseteq \mathfrak{g}'$. Write $W \simeq \mathrm{Hom}_D(V,V')$ where V and V' are the spaces in the basic data for G and G', and D is the division algebra. If $T \in \mathrm{Hom}_D(V,V')$, then $E_{TT} \in \mathfrak{sp}$. We wish to compute $\mathrm{tr}(M\,E_{TT})$, for $M \in \mathfrak{g}$. From (7.22) we see

$$\mathrm{tr}(M\,E_{TT}) = < M(T), T > .$$

But from (6.7), noting $M(T) = -TM$, we find

$$\mathrm{tr}(M\,E_{TT}) = -\mathrm{tr}(T^*TM). \tag{7.24}$$

Here the tr on the left hand side is in $\mathrm{Hom}_F(W)$ and the tr on the right hand side is in $\mathrm{Hom}_D(V)$.

8: Witt's Theorem and orbits

Let G be a classical group with basic data $(V, D, \square, (,))$. We want to know the orbits for G acting on severa- copies of V. In case $G = GL_D(V)$, we want to consider G acting on copies of V and of V^*. First take G of type I. The basic result, of which our various results will be adaptations is Witt's Theorem, as stated for example in [ ]. We record it here for convenience.

Witt's Theorem: Let G be a type I classical group with basic data $(V, D, \square, (,))$ (with D not of characteristic 2). If U_1 and U_2 are two subspaces of V, and $T:U_1 \to U_2$ is isometric with respect to the restrictions of $(,)$ to the U_i, then there is $g \in G$ such that $g|U_1 = T$.

Now consider the action of G on k-tuples $(v_1, v_2, \ldots, v_k)$ from V. Let U be an auxiliary vector space of dimension k over D, and fix a base $\{e_i\}_{i=1}^{k}$ for U. Then we may regard the k-tuple $(v_1, \ldots, v_k)$ as defining a D-linear map from U to V by specifying that e_i is mapped to v_i. Thus the action of G on k-tuples from V is identified to the action of G on $Hom_D(U,V)$ by postmultiplication.

Proposition 8.1: Let G be a type I classical group with basic data $(V, D, \square, (,))$. Let U be a vector space over D, and let $S, T \in Hom_D(U,V)$. Then there is $g \in G$ such that $gS = T$ if and only if

i) $\ker S = \ker T$, and

ii) the forms $(,) \circ S$ and $(,) \circ T$ on $U/\ker S$ are equal.

Thus the G orbits in $Hom_D(U,V)$ may be parametrized by pairs (U_0,B_0) where $U_0 \subseteq U$ is a subspace and B_0 is form on U/U_0 of the same type as

$(,)$. A given pair (U_0,B_0) will actually correspond to a G orbit if and only if

i) $\dim U - \dim U_0 \leq \dim V$ and

ii) there is a subspace V_1 of V of dimension $\dim U - \dim U_0$ such that $(,)|V_1$ is isomorphic to B_0.

Remark: We will call the pair (U_0,B_0) corresponding to an orbit $\mathcal{O}$ the <u>fine orbit parameters</u> of $\mathcal{O}$. We observe that $GL_D(U)$ will act on $Hom_D(U,V)$ by right multiplication and that this action will permute the G-orbits. If (U_0,B_0) are the parameters of an orbit $\mathcal{O}$, then it is easy to compute that the parameters of $A(\mathcal{O})$ are $(A(U_0), B_0 \circ A^{-1})$ for $A \in GL_D(U)$.

Proof: The conditions that S and T be in the same G-orbit are clearly necessary. On the other hand, if $\ker S = \ker T$, then TS^{-1} is a well defined map from $\text{im}\, S$ to $\text{im}\, T$ and the second condition says TS^{-1} is an isometry. Thus Witt's Theorem in its standard form says there is $g \in G$ such that $g|\text{im}\, S = TS^{-1}$. In other words $T = gS$. The criteria that a given pair (U_0,B_0) actually come from an orbit are self-evident.

Given $T \in Hom_D(U,V)$, we can if we wish, ignore the fact that $(,) \circ T$ is defined modulo U_0 and simply consider it as a form on all of U. We call the form $(,) \circ T$ the <u>crude orbit parameter</u> of $\mathcal{O}$. Considering only $(,) \circ T$ allows us to form an <u>orbit parameter map</u>

$$\tau_{UV} = \tau : Hom_D(U,V) \to \tilde{\mathcal{B}}(U) \qquad (8.1)$$
$$T \to (,) \circ T$$

where $\tilde{\mathcal{B}}(U)$ denotes the space of forms on U of the same type as $(,)$.

Remark: The orbit parameter map (8.1) is closely related to the map $\tilde{\tau}$ of proposition (7.2). Indeed suppose $U = V'$ has a form $(\, , \,)'$ of the type dual to $(\, , \,)$, and consider $S \in Hom(V', V)$. Then as in the proof of proposition 7.2, we may compute

$$(S^{*'} S v_1', v_2')' = -(Sv_1', Sv_2'),$$

whence, comparing (7.5) and (7.4) with (8.1) we find

$$(8.2) \qquad \tau_{V'V} = -\beta' \tilde{\tau}'$$

Here β' is the map of (7.4) with V' replacing V. A similar equation holds with V and V' reversed, a situation which would have been more consistent with previous practice. In view of (8.2), the generic one-to-one-ness of the correspondence between conjugacy classes in $\mathfrak{Y}$ and $\mathfrak{Y}'$, discussed in the remark after proposition 7.2, will be seen to follow from the next result, specifying some of the algebra-geometrical properties of the orbit parameter map τ.

Clearly, the orbit parameter map τ of (8.1) has image in the subvariety $\mathfrak{B}(U)_{(k)}$ of forms of rank at most k where $k = \min(\dim U, \dim V)$. Let $\mathfrak{B}(U)^{(k)}$ be the variety of forms of degree exactly k, and let $Hom_D(U,V)^{(k)}$ be the subvariety of maps of degree exactly k. Then $\mathfrak{B}(U)_{(k)} - \mathfrak{B}(U)^{(k)}$ is a proper closed subvariety of $\mathfrak{B}(U)_{(k)}$, and $Hom_D(U,V) - Hom_D(U,V)^{(k)}$ is a proper closed subvariety of $Hom_D(U,V)$. If $\dim U \leq \dim V$, then $\mathfrak{B}(U)_{(k)} = \mathfrak{B}(U)$ and $\mathfrak{B}(U)^{(k)}$ consists of the non-singular forms. Let $\Gamma(U,k)$ denote the set (Grassman variety) of subspaces of U of codimension k. Of course if $\dim U \leq \dim V$, $\Gamma(U,k)$ reduces to a point. We have obviously a fibration

$$(8.3) \qquad \gamma : Hom_D(U,V)^{(k)} \to \Gamma(U,k)$$
$$\gamma(T) = \ker T.$$

Evidently the fibers of γ are the maps with a given kernel of codimension k. They are also just the $GL_D(V)$ orbits in $Hom_D(U,V)^{(k)}$. We may define a vector bundle $\tilde{\mathfrak{B}}(\Gamma(U,k))$ over $\Gamma(U,k)$ which assigns to U_0 in $\Gamma(U,k)$ the space $\tilde{\mathfrak{B}}(U/U_0)$. Evidently the fine orbit parameters may be considered to define a map

$$(8.4) \qquad \tau_k : Hom_D(U,V)^k \to \tilde{\mathfrak{B}}(\Gamma(U,k))$$

Proposition 8.2: a) On $\tau^{-1}(\tilde{\mathfrak{B}}(U)^{(k)}) \subseteq Hom_D(U,V)^{(k)}$, the orbit parameter map separates G-orbits.

b) The map τ_k of (8.4) is submersive, i.e., has differential everywhere of maximal rank.

Proof: If T is in $\tau^{-1}(\tilde{\mathfrak{B}}(U)^{(k)})$ then $(\, , \,) \circ T$ has rank k, the maximal rank it possibly could have under the circumstances. Hence $\ker T$ must be precisely the radical of $\tau(T)$, so the fine orbit parameters are redundant in this situation, and $\tau(T)$ alone specifies the orbit of T.

Since $GL(U)$ acting by premultiplications preserves $Hom(U,V)^{(k)}$ and acts transitively on $\Gamma(U,k)$, it will be enough to show τ_k is submersive from the fibers of γ to the fibers of $\tilde{\mathfrak{B}}(\Gamma(U,k))$. In other words, it will suffice to assume $k = \dim U$, so $\Gamma(U,k)$ reduces to a point, and to consider τ on $Hom(U,V)^{(k)}$, which is now the set of injective maps from U to V. We compute the differential of τ at $T \in Hom(U,V)^{(k)}$. We have

$$\frac{d}{dr}\left(\tau(T + rS)(u_1,u_2)\right)\Big|_{r=0} = (Su_1, Tu_2) + (Tu_1, Su_2)$$

Since T is injective and $(\ ,\ )$ is non-degenerate, the form (Su_1, Tu_2) is an arbitrary sesquilinear form on U. Hence $(Su_1, Tu_2) + (Tu_1, Su_2)$ is an arbitrary member of $\tilde{\ }(U)$. This proves b).

We now repeat the same considerations somewhat more summarily for $GL_D(V)$. In discussing orbits for GL, it is appropriate to consider mixed tuples of vectors and covectors. Thus we take two auxiliary vector spaces U_1 and U_2, one for each type of vector. We consider

(8.5) $$Y = Hom_D(U_1, V) \oplus Hom_D(U, U_2)$$

Let (S,T), with $S \in Hom_D(U_1,V)$ and $T \in Hom_D(V,U_2)$ be a point of Y. Then if $g \in GL_D(V)$,

(8.6) $$g(S,T) = (gS, Tg^{-1})$$

Proposition 8.3: With Y as in (8.5), in order that $g(S_1,T_1) = (S_2,T_2)$ for some $g \in GL_D(V)$, it is necessary and sufficient that

i) $\ker S_1 = \ker S_2$

ii) $\operatorname{im} T_1 = \operatorname{im} T_2$

iii) $T_1S_1 = T_2S_2$

Thus a $GL_D(V)$ orbit in Y can be specified by a triple (U_{10}, U_{20}, T) where $U_{10} \subseteq U_1$ and $U_{20} \subseteq U_2$ are subspaces and $T \in Hom_D(U_1/U_{10}, U_{20})$. A given triple $(U_{10}; U_{20}, A)$ will actually parametrize an orbit in Y if and only if

i) $\dim U_1 - \dim U_{10} \leq \dim V$

ii) $\dim U_{20} \leq \dim V$

Remark: As in the type I case we call the triple corresponding to an orbit the fine parameters of that orbit. We note also that $GL_D(U_1) \times GL_D(U_2)$ will act on Y and will permute the orbits of $GL_D(V)$. If (U_{10}, U_{10}, T) are the parameters of an orbit $\mathcal{O}$, and $A_1 \in GL_D(U_1)$, then $A_1 \times A_2(\mathcal{O})$ has parameters $(A_1(U_{10}), A_2(U_{20}), A_2 T A_1^{-1})$.

Proof: The conditions given for (S_1,T_1) and (S_2,T_2) to be in the same $GL_D(V)$ orbit are clearly necessary. If $\operatorname{im} T_1 = \operatorname{im} T_2$ then we certainly write $T_2 = T_1g^{-1}$ for some $g \in GL_D(V)$. Thus let us assume $T_1 = T_2 = T$. The condition $TS_1 = TS_2$ says in particular that $S_1^{-1}(\ker T) = S_2^{-1}(\ker T)$. Call this subspace X and choose a complement X' to X in U_1. The restrictions $S_i' = S_i|X'$ define embeddings of X' into V, satisfying $TS_1' = TS_2'$. Therefore, we can find an element g of the subgroup of $GL_V(D)$ which acts trivially on $\ker T$ and on $V/\ker T$, such that $gS_1' = S_2'$. Note also $gT = T$. Thus we may also assume $S_1' = S_2' = S'$. Let us now consider the restriction $S_i'' = S_i|X$. These are elements of $Hom_D(X, \ker T)$. Our assumptions say $\ker S_1'' = \ker S_2''$, so there is an element g'' of $GL_D(\ker T)$ such that $S_2'' = g''S_1''$. We may implement g'' by an element g of $GL_D(V)$ which acts as the identity on a complement to $\ker T$ in V containing $\operatorname{im} S'$. Then $gS_1'' = S_2''$, and $gT = T$ and $gS' = S'$. Since S_1'' and S' determine S, we have shown $g(S_1,T_1) = (S_2,T_2)$ as desired.

As in the type I case we can disregard the spaces U_{10} and U_{20} attached to an orbit $\mathcal{O}$, and consider A as simply being a map from U_1 to U_2. We thus obtain a crude orbit parameter map

(8.7) $$\tau: Hom_D(U_1,V) \oplus Hom_D(V, U_2) \to Hom_D(U_1,U_2)$$
$$(S,T) \to TS$$

Evidently τ will have image in $\mathrm{Hom}_D(U_1,U_2)_{(k)}$, the maps of rank at most k, where $k = \min\{\dim U_1, \dim U_2, \dim V\}$. In general, it is false that when TS is of rank k it alone determines the orbit of (S,T). For example, if $\dim U_1 < \dim V < \dim U_2$, then S and TS may both be injective while T may or may not be injective. In fact for TS to determine $\mathrm{im}\, T$ it is necessary that rank TS = rank T. Similarly for TS to determine $\ker S$, one needs rank TS = rank S. This is seen to hold for maps of rank k if $k = \dim V$, or if $k = \dim U_1 = \dim U_2$. We state this formally.

Proposition 8.4: a) If $k = \dim V$, or $k = \dim U_1 = \dim U_2$, then on $\tau^{-1}(\mathrm{Hom}_D(U_1,U_2)^{(k)})$, the orbit parameter map separates GL_D orbits.

9: Isotropic subspaces; split forms

Let $(V, D, \natural, (\ ,\ ))$ be the basic data for a type I classical group. We will use $\perp$ to denote orthogonal complements respect to $(\ ,\ )$. That is, for any set $E \subseteq V$, we define $E^{\perp} = \{v \in V : (v,e) = 0,\ e \in E\}$. Since it would be very cumbersome to make orthogonality be explicitly with respect to a given form, the form will always have to be understood. We will endeavor to avoid the potential ambiguity of this notation.

A subspace $V_1 \subseteq V$ is called <u>isotropic</u> if $V_1 \subseteq V_1^{\perp}$ and <u>polarizing</u> if $V_1 = V_1^{\perp}$. If polarizing subspaces of V exist, we will say V, or $(\ ,\ )$, is <u>split.</u> This is not the same thing as to say the associated classical group G is split in the sense of reductive groups. To express that V is split when referring to G we will say G is <u>spatially split</u>.

If $V_1 \subseteq V$ is isotropic, let $P(V_1) = P$ denote the subgroup of G leaving V_1 stable. Let ${}^{\circ}P \subseteq P$ denote the subgroup of P acting trivially on $V_1^{\perp}/V_1$. Let $N = N(V_1)$ be the subgroup of P acting trivially on V_1 also. We want to extend proposition 8.1 to describe the orbits of P or ${}^{\circ}P$ acting on $\mathrm{Hom}_D(U,V)$, where U is an auxiliary vector space as in that proposition.

Before doing so, we will give a description of $P(V_1)$, as afforded by Witt's Theorem (See also proposition 5.2). Let V_2 be an isotropic subspace of V supplementary to $V_1^{\perp}$, so that $V = V_2 \oplus V_1^{\perp}$. Put $V_3 = (V_1 \oplus V_2)^{\perp}$. Let G_3 be the isometry group of V_3 equipped with the restriction of $(\ ,\ )$. Put

$$M = P(V_1) \cap P(V_2) \tag{9.1}$$

Then M also preserves V_3, and we see by Witt that

$$M \simeq G_3 \times GL(V_2) \tag{9.2}$$

by restriction.

Let N be the subgroup of $P(V_1)$ that acts trivially on V_1 and on $V_1^{\perp}/V_1$. Then Witt implies that N acts simply transitively on all possible V_2 so that

$$P(V_1) \simeq M \cdot N \quad \text{(semidirect product)} \tag{9.3}$$

Also we note that

$$^{\circ}P(V_1) = GL(V_2) \cdot N \tag{9.4}$$

We now describe N more precisely. Again by Witt we see that the restriction of N to $V_1^{\perp}$ yields a surjective homomorphism

$$\eta : N \to \mathrm{Hom}(V_3, V_1) \tag{9.5}$$

The kernel of η must preserve $V_1 \oplus V_2$. Hence from proposition 5.2 we see that N sits in an exact sequence

$$1 \to S^{2*}(V_2) \to N \overset{\eta}{\to} \mathrm{Hom}(V_3, V_1) \to 1 \tag{9.6}$$

We can construct a set theoretical cross section, which we will denote by e, to η. Let us note we may define an adjoint

$$*: \mathrm{Hom}(V_3, V_1) \to \mathrm{Hom}(V_2, V_3)$$

by the rule

$$(T^*(v_2), v_3) = (v_2, Tv_3) \qquad T \in \mathrm{Hom}(V_2, V_3) \tag{9.7}$$

Put

$$e(T)(v_1, v_2, v_3) = (v_1 + T(v_2) - \tfrac{1}{2} TT^*(v_3),\ v_2 - T^*(v_3),\ v_3). \tag{9.8}$$

In terms of the decomposition $V = V_1 \oplus V_2 \oplus V_3$, we may write $e(T)$ as the matrix

$$e(T) = \begin{bmatrix} I & T & -\frac{1}{2}T\,T^* \\ 0 & I & -T^* \\ 0 & 0 & I \end{bmatrix} \tag{9.9}$$

A straightforward computation shows that

$$e(T_1)e(T_2) = e(T_1 + T_2)(I + \tfrac{1}{2}(TS^* - ST^*)) \tag{9.10}$$

The analogy with (3.13) is evident. In particular, formula (9.10) shows that N is two step nilpotent, and $S^*(V_2)$ is the center and commutator subgroup of N.

We now proceed to the description of orbits of P and $^{\circ}P$.

Proposition 9.1: a) Given $S, T \in \mathrm{Hom}_D(U,V)$, then $T = pS$ for some p in $P = P(V_1)$ if and only if

i) $\ker S = \ker T$

ii) The forms $(\ ,\ ) \circ S$ and $(\ ,\ ) \circ T$ on $U/\ker S$ are equal

iii) $S^{-1}(V_1) = T^{-1}(V_1)$, and $S^{-1}(V_1^{\perp}) = T^{-1}(V_2^{\perp})$

b) One further has $T = pS$ for p in $^{\circ}P$ if and only if additionally

iv) The maps from $S^{-1}(V_1^{\perp})$ to $V_1^{\perp}/V_1$ induced by S and T are equal.

Remark: Since $\ker S = S^{-1}(\{0\})$, condition iii) is directly parallel to condition i).

Proof: Again the necessity of the conditions is self-evident. For sufficiency, observe first that by Witt's Theorem, the group ${}^{\circ}P$ is mapped by restriction to V_1 onto $GL_D(V_1)$. Also, via $(\, , \,)$, the quotient $V/V_1^{\perp}$ is identified ($\natural$-semilinearly) with $Hom_D(V_1, D)$, and vice versa. Put $U_1 = S^{-1}(V_1) = T^{-1}(V_1)$. Put $S_1 = S|U_1$ so $S_1 \in Hom_D(U_1, V_1)$. Define T_1 similarly. Further, define S_2 in $Hom_D(V_1, Hom_D^{\natural}(U/S^{-1}(V_1^{\perp}), D))$, where $Hom_D^{\natural}$ denotes $\natural$-semilinear maps, by $S_2(y)(u) = (y, S(u))$. Define T_2 similarly. The conditions i), ii), and iii) of the present proposition guarantee that (S_1, S_2) and (T_1, T_2) of the present proposition satisfy the conditions of proposition 8.3. We conclude we can find p in ${}^{\circ}P$ such that pS and T agree on U_1 and induce the same map to $V/V_1^{\perp}$.

Replace S by pS and begin again, assuming S and T agree on U_1 and induce the same map to $V/V_1^{\perp}$. Let X be a subspace of V_1 complementary to $S(U_1) = T(U_1)$. Define

$$\tilde{S}: U \oplus X \to V \qquad \text{by} \qquad \tilde{S}(u,x) = S(u) + x$$

Define $\tilde{T}$ similarly. Since S and T are compatible as specified just above, it is easy to check that $\tilde{S}$ and $\tilde{T}$ again satisfy the conditions of the proposition. In particular, there is certainly $g \in G$ such that $g\tilde{S} = \tilde{T}$. In particular, $gS = T$. But since $\tilde{S}^{-1}(V_1) = \tilde{T}^{-1}(V_1)$ and $\operatorname{im} \tilde{S} \supseteq V_1 \subseteq \operatorname{im} \tilde{T}$, we see that g must in fact belong to P. so part a) of the proposition is proved.

To prove b) we must be slightly more careful. Let $\tilde{S}$ and $\tilde{T}$ be as just above. We know $\tilde{S}$ and $\tilde{T}$ already agree on $\tilde{S}^{-1}(V_1)$. Assumption iv) implies $\tilde{S}$ and $\tilde{T}$ induce the same map from $\tilde{S}^{-1}(V_1^{\perp}) = U_2$ to $V_1^{\perp}/V_1$.

Let Y be a complement to $\tilde{S}(U_2)$ in $V_1^{\perp}$. Then $(\tilde{S}(u), y) = (\tilde{T}u, y)$ for $u \in U_2$ and $y \in Y$. Thus, for each $y \in Y$, the function $u \to ((\tilde{S}-\tilde{T})u, y)$ on $\tilde{U} = U \oplus X$ factors to $\tilde{U}/U_2$. Thus, for each $y \in Y$, there is $t(y) \in V_1$ such that

$$((\tilde{S}-\tilde{T})u, y) = (\tilde{S}(u), t(y)) = (\tilde{T}(u), t(y))$$

for $u \in U$. Clearly $t(y)$ may be made to depend linearly on y. Thus consider the maps $\check{S}$ and $\check{T}$ defined by

$$\check{S}: \tilde{U} \oplus Y \to V \qquad \check{S}(u,y) = \tilde{S}(u) + y$$

$$\check{T}: \tilde{U} \oplus Y \to V \qquad \check{T}(u,y) = \tilde{T}(u) + y + t(y).$$

Then $\check{S}$ and $\check{T}$ again satisfy conditions i) through iv). In particular, there is $g \in G$ such that $g\check{S} = \check{T}$. But since $\check{S}$ and $\check{T}$ agree on $\check{S}^{-1}(V_1)$, and agree on $\check{S}^{-1}(V_1^{\perp})$ modulo V_1, and since $\operatorname{im} \check{S} \supseteq V_1^{\perp} \subseteq \operatorname{im} \check{T}$, we see necessarily $g \in N(V_1)$. This proves the proposition.

There is a parallel result for GL_D. We omit the proof. If $V_1 \subseteq V$ is any subspace, let $P(V_1) = P$ be the subgroup of $GL_D(V)$ stabilizing V_1. Let ${}^{\circ}P$ be the subgroup acting trivially on V/V_1, and N be the subgroup acting trivially on V/V_1.

Proposition 9.2: a) If (S_1, T_1) and (S_2, T_2) are in $Y = Hom_D(U_1, V) \oplus Hom_D(V, U_2)$, and $V_1 \subseteq V$, then $(S_2, T_2) = p(S_1, T_1)$ if and only if

i) $\ker S_1 = \ker S_2$

ii) $\operatorname{im} T_1 = \operatorname{im} T_2$

iii) $T_1 S_1 = T_2 S_2$

iv) $S_1^{-1}(V_1) = S_2^{-1}(V_1)$

v) $T_1(V_1) = T_2(V_1)$

b) We have $(S_2,T_2) = p(S_1,T_1)$ if and only if, in addition,

vi) S_1 and S_2 induce the same map from U to V/V_1 and T_1 and T_2 induce the same map from V to $U_2/T_1(V_1)$.

We omit the proof.

We now wish to study polarizing subspaces. If V is split, let $\Omega(V) = \Omega$ be the set of all polarizing subspaces of V. Then there is a descriotion of Ω in direct analogy with proposition 5.3. We record it. As in the symplectic case, if V_1 and V_2 are polarizing subspaces of V and $V = V_1 \oplus V_2$ we call (V_1,V_2) a <u>complete polarization</u> of V. Corresponding to a complete polarization, we have a decomposition $P(V_1) = M\cdot N(V_1)$ where $M = P(V_1) \cap P(V_2) \simeq GL_D(V_1)$.

Proposition 9.3: a) Let (V_1,V_2) be a complete polarization of V. Then $\Omega(V) \simeq G/P(V)$ can be parametrized by pairs (E,B) where $E \subseteq V_2$ is a subspace and B is a form on E of the type dual to $(\ ,\ )$. If $X \in \Omega$, and $(E(X), B(X))$ is the corresponding pair, then $E = (V_1 + X) \cap V_2$, and $B(e_1,e_2) = \langle e_1, \gamma(e_2)\rangle$ where $\gamma(e_2) \in V_1$ is such that $e_2 + \gamma(e_2)$ is in U. Arbitrary (E,B) arise in this way.

b) Under this identification the $N(V_1)$ orbits are those pairs with fixed E, and the $P(V_1)$ orbits are pairs with dim E fixed. If $X \in \Omega$, then $\dim E(X) = (\frac{1}{2}) \dim V - \dim (X \cap V_1)$.

Let now W be a symplectic space, with form $\langle\ ,\ \rangle$ over a field F. Then $Sp(W)$ acts on $\Omega(W) = \Omega$ transitively so that Ω is a homogeneous space for $\Omega(W)$. Let (G,G') be an irreducible reductive dual pair in Sp. Then G and G' act on Ω. Let Ω^G be the subset of Ω of points fixed by G. Then clearly G' acts on Ω^G.

Proposition 9.4: a) Suppose (G,G') is of type I, so that if $(V, D, \natural, (\ ,\ ))$ and $(V', D, \natural, (\ ,\ )')$ are the basic data for G and G', then $W = \mathrm{Hom}_D(V,V')$. Then every G-invariant subspace Y of W has the form

$$Y = \mathrm{Hom}_D(V, Y_0) \tag{9.11}$$

where $Y_0 \subseteq V'$ is any subspace. Further Y will be isotropic if and only if Y_0 is isotropic. Hence Ω^G will be non-empty if and only if V' is split in which case the correspondence $Y \leftrightarrow Y_0$ induces a bijection

$$\Omega(W)^G \simeq \Omega(V') \tag{9.12}$$

In particular, $\Omega(W)^G$ consists of a single G' orbit.

b) Suppose (G,G') is of type II, and G and G' have basic data (V,D) and (V',D), so that $W \simeq \mathrm{Hom}_D(V,V') \oplus \mathrm{Hom}_D(V',V)$. Th n every G-invariant subspace Y of W has the form

$$Y = \mathrm{Hom}_D(V, Y_1) \oplus \mathrm{Hom}_D(V'/Y_2, V) \tag{9.13}$$

where Y_1, Y_2 are arbitrary subspaces of V'. Further Y will be isotropic if and only if $Y_1 \subseteq Y_2$, and maximal isotropic if and only if $Y_1 = Y_2$. Thus via the correspondence $Y \leftrightarrow (Y_1,Y_2)$, we see that $\Omega(W)^G$ corresponds to the union of Grassmann varieties in V':

$$\Omega(W)^G \simeq \bigcup_{k=0}^{\dim V} \Gamma(V',k) \tag{9.14}$$

where $\Gamma(V',k)$ denotes the set of subspaces of V of codimension k.

Proof: The statement of the result is virtually its own proof. The equation (9.11) follows because G acts irreducibly on V, so generates

$End_D(V)$ as algebra, and it is well-known that $End_D(V)$-invariant subspaces of $Hom_D(V,V')$ have the form of (9.11). If Y is given by (9.11), then the form $<\ ,\ >$ on Y is still given by (6.7). As T varies in Y, we can compute from (6.5) that T^* varies arbitrarily in $Hom_D(V'/V_0^{\perp}, V)$. Therefore $T_1^*T_2$, for $T_i \in Y$, can be an arbitrary endomorphism of V' of rank up to $\dim(Y_0/Y_0^{\perp} \cap Y)$. Thus for Y to be isotropic all $T_1^*T_2$ must vanish, that is $Y_0 \subseteq Y_0^{\perp}$, or Y_0 must be isotropic. The remaining assertions are even more obvious, and the proof of b) is similar.

Choose an isotropic subspace $V_1 \subseteq V$ in the type I case, or any subspace in the type II case. Let $P = P(V_1)$. We shall also describe $\Omega(W)^P$. Notations will be as in the previous proposition.

Proposition 9.5: a) If (G,G') is type I, then, providing that the commuting algebra of P acting on $V_1^{\perp}/V_1$ is again D, any P-invariant subspace Y of $Hom_D(V,V') = W$ has the form

$$Y = Hom_D(V,Y_1) + Hom_D(V/V_1, Y_2) + Hom\ (V/V_1^{\perp}, Y_3). \tag{9.15}$$

where $Y_1 \subseteq Y_2 \subseteq Y_3$ are any nested triple of subspaces of V'. In order that Y be isotropic, it is necessary and sufficient that Y_1 and Y_2 be isotropic and that Y_3 be contained in $Y_1^{\perp}$. Thus, Y will be a polarization for W if and only if $Y_3 = Y_1^{\perp}$, and Y_2 is a polarization for V' if $V_1 \neq V_1^{\perp}$. Hence $\Omega(W)^P$ is a finite union of flag manifolds for G'.

b) If (G,G') is of type II, then any P-invariant subspace Y of $W = Hom_D(V,V') \oplus Hom_D(V',V)$ has the form

$$Y = (Hom_D(V,Y_1) + Hom_D(V/V_1,Y_2)) \oplus (Hom_D(V'/Y_3,V) + Hom_D(V'/Y_4,V_1)) \tag{9.16}$$

where $Y_1 \subseteq Y_2$ and $Y_4 \subseteq Y_3$ are two nested pairs of subspaces of V'. In order that Y be isotropic, it is necessary and sufficient that $Y_2 \subseteq Y_3$ and $Y_1 \subseteq Y_4$. Thus Y will be a polarization for W if and only if $Y_1 = Y_4$ and $Y_2 = Y_3$. Hence $\Omega(W)^P$ is a finite union of flag manifolds for G'.

Remark: Evidently there is a systematic generalization of propositions 9.4 and 9.5 to describe $\Omega(W)^Q$ for any parabolic subgroup Q of G.

Proof: We will treat the type I case. Type II is similar. A subspace Y of W invariant by P will be invariant also by the algebra A spanned by P, and A, by Witt's Theorem and our non-degeneracy assumption, is all D-linear endomorphisms of V which preserve V_1 and $V_1^{\perp}$. We may regard $X = Hom(V_1,V')$ as a quotient of W. If Y is in $\Omega(W)^P$, then the image of Y in X will be invariant by $End_D(V_1)$ and hence will be of the form $X_1 = Hom_D(V_1,Y_1)$ for some $Y_1 \subseteq V'$. Let $T \in Hom_D(V_1,Y)$ have rank one. Since A contains projections onto A, we may suppose T is the restriction of a rank one element of Y, also denoted T. We may write $T(v) = (v,t)y$ for all $v \in V$, and suitable $t \in V$, $y \in Y_1$. Since T is non-trivial on V_1, the vector t cannot be in $V_1^{\perp}$. Hence the P orbit of t spans V. This means that the P-orbit of T spans $Hom_D(V, Dy)$. Letting Y range over a basis for Y_1, we find $Hom_D(V,Y_1) \subseteq Y$. Looking now at the subspace of transformations in Y which are zero in V_1, and considering their restrictions to $V_1^{\perp}$, repeating the same reasoning and continuing, we arrive at the form (9.15) for Y. The conditions for isotropy and maximal isotropy are easily derived, as in proposition 9.4.

10: Doubling

Let $(V, D, \natural, (\ ,\ ))$ be the basic data for a type I classical group G. Write $V = V^+$, and let V^- denote the same space as V endowed with the form $-(\ ,\ )$. Then $(V^-, D, \natural, (\ ,\ ))$ define the same classical group G. Set

(10.1) $$\tilde{V} = V^+ \oplus V^-$$

(orthogonal direct sum). Let $\widetilde{(\ ,\ )}$ on $\tilde{V}$ be the direct sum of the forms $(\ ,\ )$ and $-(\ ,\ )$. Then $\tilde{V}$ with the form $\widetilde{(\ ,\ )}$ will be called the <u>double</u> of V. The quadruple $(\tilde{V}, D, \natural, \widetilde{(\ ,\ )})$ are basic data for a type I classical group $\tilde{G}$. We also call $\tilde{G}$ the double of G.

There are two obvious embeddings of G into $\tilde{G}$. Let G^+ denote the subspace of $\tilde{G}$ which acts as the identity on V^-, and let G^- be the subgroup of G which acts as the identity on V^+. Then we have canonical identifications

(10.2) $$i^+ : G \tilde{\rightarrow} G^+ \subseteq \tilde{G}$$
$$i^- : G \tilde{\rightarrow} G^- \subseteq \tilde{G}$$

Of course, there are also canonical embeddings

(10.3) $$i^+ : V \tilde{\rightarrow} V^+ \subseteq \tilde{V}$$
$$i^- : V \tilde{\rightarrow} V^- \subseteq \tilde{V}$$

defined by change of notation. Obviously

(10.4) $$i^+(g)(i^+(v)) = i^+(g(v)) \qquad i^+(g)(i^-(v)) = i^-(v)$$

and similarly with $+$ and $-$ reversed. Evidently we can represent a general element of $\tilde{V}$ as $i^+(v_1) + i^-(v_2)$ with $v_i \in V$, but this is rather tedious, and we will generally write

(10.5) $$i^+(v_1) + i^-(v_2) =)\, v_1, v_2\, (.$$

(We turn the parentheses outward to avoid confusion with the value of the form $(\ ,\ )$ on $v_1 \times v_2$).

We observe that $\tilde{V}$ is split. Indeed, set

(10.6) $$\Delta^+(V) = \Delta^+ = \{)\, v,v\, (: v \in V\}$$
$$\Delta^-(V) = \Delta^- = \{)\, v,-v\, (: v \in V\}$$

Then (Δ^+, Δ^-) form a complete polarization for $\tilde{V}$. If we decompose an element $)\, v_1, v_2\, ($ of $\tilde{V}$ into its component along Δ^+ and Δ^-,

$$)\, v_1, v_2\, (\;=\;)\, v^+, v^+\, (\;+\;)\, v^-, -v^-\, ($$

then we easily find

(10.7) i) $v_1 = v^+ + v^-$ $\qquad v_2 = v^+ - v^-$

ii) $v^+ = (\tfrac{1}{2})(v_1 + v_2)$ $\qquad v^- = (\tfrac{1}{2})(v_1 - v_2)$

Write $\Omega(\tilde{V}) = \tilde{\Omega}(V) = \tilde{\Omega}$ for the space of polarizing subspaces of $\tilde{V}$. Write $P(\Delta^+) = \tilde{P}^+$ and $N(\Delta^+) = \tilde{N}^+$ for the isotropy group of Δ^+ and its unipotent radical.

Proposition 10.1: There is a natural isomorphism

$$\delta : \mathfrak{g} \rightarrow \tilde{N}^+$$

from the Lie algebra of G to $\tilde{N}^+$, defined by

(10.8) $\delta(T)()v_1,v_2() =)v_1,v_2(+(\frac{1}{2}))T(v_1-v_2), T(v_1-v_2)($

Proof: If we decompose $)v_1,v_2($ according to (10.6) then we find $\delta(T)$ may also be written

$$\delta(T))v^+,v^+(=) v^+,v^+ ($$

(10.9) $$\delta(T))v^-, -v^- (=)v^-, -v^- (+)Tv^-, Tv^- ($$

Thus it is clear that $\delta(T)$ is a shear along Δ^+, as it should be. Furthermore, the form

$$)v,-v(x)v', -v'(\to ( )v,-v(,)Tv', Tv'( ) \sim$$
$$= 2(v, Tv') = -2(Tv, v')$$

that would be associated to $\delta(T)$ as a prospective member of $\tilde{N}^+$ according to proposition 9.3, is in fact by (7.1) of the type dual to (,). Hence $\delta(T) \in \tilde{N}^+$. The map δ is clearly injective, so it is an isomorphism by dimension count.

Consider the action of $G^+ \times G^-$ on $\tilde{\Omega}$. Since $i^+(g))v_1,v_2(=)gv_1,v_2($, we see that the isotropy group of Δ^+ in G^+ (or G^-) is trivial. Thus the map

(10.10) $$c^+: g \to i^+(g)(\Delta^+)$$

embeds G in $\tilde{\Omega}$. If g leaves no non-zero vector fixed, then $c^+(g)$ is complementary to Δ^+, so we may write

(10.11) $$c^+(g) = \delta(\tilde{c}^+(g))$$

for suitable $\tilde{c}^+(g)$ in $\mathfrak{h}$. We will compute $\tilde{c}^+(g)$.

We have

$$)gv,v(= (\tfrac{1}{2}))gv + v, gv + v(+ (\tfrac{1}{2}))gv - v, -(gv-v)($$

Put $v' = (\frac{1}{2})(gv-v)$. Then

$$)gv, v(=)v', -v'(+) \frac{g+1}{g-1} v', \frac{g+1}{g-1} v'($$

Comparing this with (10.8), we see that

(10.12) $$c^+(g) = \frac{g+1}{g-1} = -c(-g)$$

where c is the Cayley transform as defined in 7.12. Indeed, sometimes c^+, rather than c, is called the Cayley transform.

Since we know that the Cayley transform maps an (Zariski) open set in G to an open set in $\mathfrak{h}$, we know that $G^+(\Delta^+) = G^-(\Delta^+) = G^+ \times G^-(\Delta^+)$ is (Zariski) open in $\tilde{\Omega}$. We will describe the complete $G^+ \times G^-$ orbit structure of $\tilde{\Omega}$. Actually, we shall consider a slightly more general situation. Let $(V_1, D, \mathfrak{h}, (,)_1)$ and $(V_2, D, \mathfrak{h}, (,)_2)$ be basic data for two classical groups G_1 and G_2 of the same type, and put

(10.13) $$V_3 = V_1^+ \oplus V_2^- \qquad \text{(orthogonal direct sum)}$$

Then if G_3 is the group with basic data $(V_3, D, \mathfrak{h}, (,)_3)$, with $(,)_3$ the direct sum of $(,)_1$ and $-(,)_2$ as indicated by (10.12), we have the obvious objections

(10.14)
$$i_1: V_1 \to V_3 \qquad i_2: V_2 \to V_3$$
$$i_1: G_1 \to G_3 \qquad i_2: G_2 \to G_3$$

Suppose that V_3 is split. Put $\Omega(V_3) = \Omega_3$ and consider the action of $i_1(G_1) \times i_2(G_2)$ on Ω_3. This leads to a decomposition of Ω_3 alternative to the one given by proposition 9.3.

Proposition 10.2: a) If $Y \in \Omega_3$, then Y may be parametrized by a triple (Y_1, Y_2, s) where $Y_1 = Y \cap V_1^+$, $Y_2 = Y \cap V_2^-$, and s is an isometry from $Y_1^{\perp}/Y_1$ (here $\perp$ is taken in V_1) to $Y_2^{\perp}/Y_2$ (here $\perp$ is taken in V_2). For such a triple to exist, besides the necessity of V_3 being split, the equation

(10.15) $$\dim V_1 - 2\dim Y_1 = \dim V_2 - 2\dim Y_2$$

must be satisfied.

b) The action of $i_1(G_1) \times i_2(G_2)$ on Ω_3, may be described in terms of this parametrization by the equation

(10.16) $$i_1(g_1) \times i_2(g_2)(Y_1, Y_2, s) = (g_1(Y_1),\ g_2(Y_2),\ g_2 s\, g_1^{-1})$$

Thus the $i_1(G_1)$ orbits in Ω_3 are described by triples where Y_2 is given, the $i_2(G_2)$ orbits are described by specifying Y_1, and the $i_1(G_1) \times i_2(G_2)$ orbits are given by specifying $\dim Y_1$ (or $\dim Y_2$). The isotropy group of (Y_1, Y_2, s) contains $i_1({}^{\circ}P(Y_1)) \times i_2({}^{\circ}P(Y_2))$.

Proof: Let the Y_i be as specified in the statement of the proposition. Obviously Y_1 and Y_2 are isotropic in V_1^+ and V_2^- respectively. Let Y_1' and Y_2' be the projections of Y into V_1^+ and V_2^-. Then equally obviously $Y_i' \subseteq Y_i^{\perp}$, the $\perp$ being taken in V_1^+ or V_2^- as appropriate. From the definitions of Y_i and Y_i', it is clear that Y defines the graph of an isomorphism s from Y_1'/Y_1 to Y_2'/Y_2. It is easy to see that the condition that Y be isotropic implies s is an isometry, and conversely. A dimension count shows that $\dim Y = \dim Y_1' + \dim Y_2 = \dim Y_1 + \dim Y_2'$. For Y to polarize V_3, we need $2\dim Y = \dim V_1 + \dim V_2$. Comparing these figures, we find we must have $Y_i' = Y_i^{\perp}$. This proves a). The computations needed for b) are straightforward.

This result specializes to the description we wanted for the $G^+ \times G^-$ orbit structure of $\tilde{\Omega}$ if we take $V_1 = V_2 = V$. However, in $\tilde{\Omega}$, there is extra structure. Let $\Xi(V) = \Xi$ denote the collection of all isotropic subspaces of V. The space Ξ is acted on by G, the subspaces of each dimension up to the dimension of the maximal isotropic subspaces constituting one orbit.

Proposition 10.3: a) The points of $\tilde{\Omega}$ may be parametrized by triples (Y_1, Y_2, s) where Y_1 and Y_2 are isotropic subspaces in V of the same dimension and s is an isometry from $Y_1^{\perp}/Y_1$ to $Y_2^{\perp}/Y_2$. The action of $G^+ \times G^-$ on $\tilde{\Omega}$ is described in this parametrization by

(10.17) $$i^+(g_1)i^-(g_2)\,(Y_1, Y_2, s) = (g_1(Y_1),\ g_2(Y_2),\ g_2 s\, g_1^{-1})$$

Here, if $Y \in \tilde{\Omega}$, then the corresponding Y_i are

(10.18) $$Y_1 = Y \cap V^+ \qquad Y_2 = Y \cap V^-$$

b) There is a natural embedding

$$\delta\colon \Xi \to \tilde{\Omega} \quad .$$

In the parameters of part a), δ may be written

(10.19) $$\delta(X) = (X, X, 1) \qquad X \in \Xi$$

where 1 denotes the identity map on $X^{\perp}/X$. We have the relation between G-actions:

(10.20) $$\delta(g(X)) = i^+(g)i^-(g)\ \delta(X).$$

Thus δ establishes a bijection between G-orbits in Ξ and $G^+ \times G^-$ orbits in $\tilde{\Omega}$.

Remark: a) We have $\Delta^+ = \delta(\{0\})$.

b) The isotropy group of $\delta(X)$ is $i^+({}^\circ P(X)) \cdot i^-({}^\circ P(X)) \cdot i^+ \times i^-(P(X))$.

According to proposition 9.3, the complete polarization (Δ^+, Δ^-) of $\tilde{V}$ gives rise to a parametrization of $\tilde{\Omega}$ by pairs (E,B) where E is a subspace of Δ^- and B is a form on E of the type dual to $(\, , \,)$. Let

(10.21) $$p^+ : \tilde{V} \to V^+ \qquad p^- : \tilde{V} \to V^-$$

be the projections implied by (10.1). Then we may also parametrize $\tilde{\Omega}$ by pairs (E^+, B^+), where $E^+ = {i^+}^{-1} p^+(E)$ is a subspace of V, and $B^+ = B \circ p^+ \circ {i^+}^{-1}$ is a form on E. Let $)v_1, v_2(= \tilde{v}$ and $)v_1', v_2'(= \tilde{v}'$ be elements of $Y \in \tilde{\Omega}$. Decompose $\tilde{v}$ and $\tilde{v}'$ according to (10.6), thus obtaining vectors $v^+, v^{+'}, v^-$ and $v^{-'}$ in V. According to the specifications of (9.3) and the definitions just given, we see that

(10.22) $$E^+(Y) = \{ v^- = \tfrac{1}{2}(v_1 - v_2) :)v_1, v_2(\in Y \}$$

and that

(10.23) $$B^+(Y)(v^-, v^{-'}) = 2(v^-, v^{+'})$$

Suppose the parameters of Y according to proposition 10.3 are (Y_1, Y_2, s). Then $X = Y_1^\perp \cap Y_2^\perp$ (these $\perp$'s are in V) is supplementary to Y_1 in $Y_1^\perp$ and to Y_2 in $Y_2^\perp$, so we may imagine that s is an endomorphism of $X/Y_1 \cap Y_2$.

Proposition 10.4: With notations as above, we have

(10.24) $$E^+(Y) = Y_1 + Y_2 + \mathrm{im}(s-1) = (Y_1 \cap Y_2 + \ker(s-1))^\perp.$$

Moreover

(10.25) $$Y_1 = \{v' \in E^+(Y) : B^+(v,v') = 2(v,v'), \ \text{all} \ v' \in E^+\}$$

and $$Y_2 = \{v' \in E^+(Y) : B^+(v,v') = -2(v,v') \ \text{all} \ v' \in E^+\}.$$

Proof: Since if $)v_1, v_2(\in Y$, then $v_2 = s(v_1)$, modulo the Y_i, we have $v_1 - v_2 = (1-s)(v_1)$ modulo the Y_i, and v_1 can be an arbitrary element of X, so (10.23) follows. If $v \in Y_1$, then $)v', 0(\in Y$, so then $v^{+'} = v^{-'} = (\tfrac{1}{2})v'$. Hence from (10.22), for $v \in E^+$,

$$B^+(v, v^{-'}) = 2(v, v^{+'}) = 2(v, v^{-'})$$

as stated. If conversely $\tilde{v}' =)v_1', v_2'($ is in Y, and $v^{-'} = (\tfrac{1}{2})(v_1' - v_2')$ satisfies (10.24), then since $v^{+'} = v^{-'} + v_2'$, we see that $v_2 \in E^{+\perp}$, whence $)v_2', v_2'(\in Y_1$, and so $\tilde{v}' -)v_2', v'(=)2v^{-'}, 0(\in Y \cap V^+$, or $v^{-'} \in Y_1$ as desired. The proof for membership in Y_2 is similar.

Let (G, G') be an irreducible type I reductive dual pair. Let $(V, D, \natural, (\, , \,))$ and $(V', D, \natural, (\, , \,)')$ be the basic data for G and G', so that $W = \mathrm{Hom}_D(V, V')$ is the symplectic space on which G and G' both act. Consider $\tilde{W}$. The following result is obvious.

Proposition 10.5: There is a natural isomorphism

(10.26) $$\tilde{W} \simeq \mathrm{Hom}_D(V, \tilde{V}')$$

Hence the pair $(i^+ \times i^-(G), \tilde{G}')$ is naturally embedded as a reductive dual pair in $\widetilde{Sp} = Sp(\tilde{W})$.

The above discussion has all been for type I groups. We now indicate fairly briefly the analogous facts for type II groups. One simply takes for $\tilde{V}$ the direct sum of two copies of V, which we may label V^+ and V^- for notational consistency. Then formulas (10.2) throight (10.6) make good formal sense, if $\tilde{\Omega}$ is taken simply as the subspaces if $\tilde{V}$ of dimension equal to $\dim V$. Proposition 10.1 remains true. Formulas (10.8) through (10.13) also make sense in the type II case. Proposition 10.2 must be reformulated as follows.

Proposition 10.6: Let $Y \subseteq V_3 = V_1 \oplus V_2$ be any subspace. Then Y may be specified by a 5-tuple $(Y_1, Y_1', Y_2, Y_2', s)$ where $Y_1 \subseteq Y_1'$ are nested subspaces of V_1 and $Y_2 \subseteq Y_2'$ are nested subspaces of V_2, and

$$\dim Y = \dim Y_1 + \dim Y_2' = \dim Y_1' + \dim Y_2 \tag{10.27}$$

and s is an isomorphism from Y_1'/Y_1 to Y_2'/Y_2. We have $Y_1 = Y \cap V_1$ and $Y_1' = p_1(Y)$, where p_1 is the projection of V_3 onto V_1. The action of the group $GL_D(V_1) \times GL_D(V_2)$ on subspaces of V_3 permutes the corresponding 5-tuples as follows.

$$i_1(g_1)i_2(g_2)(Y_1, Y_1', Y_2, Y_2', s) = (g_1(Y_1), g_1(Y_1'), g_2(Y_2), g_2(Y_2'), g_1 s g_1^{-1}) \tag{10.28}$$

This result may then of course be applied to the description of the double $\tilde{V}$ of V. One must replace (10.18) by

$$\delta(X, X') = (X, X', X, X', 1) \tag{10.29}$$

wehre (X, X') is an arbitrary nested pair (two-step flag) of subspaces of V. This map δ will have the appropriate covariance property (10.19), but it will bot be surjective onto the $GL_D(V^+) \times GL_D(V^-)$ orbits, either on all

subspaces of $\tilde{V}$, or just on the subspaces of dimension equal $\dim V$. The rest of the development is not relevant to type II.

§11: Case of non-Archimedean local fields.

In this section, we specialize F to be a non-Archimedean local field, with ring of integers R. We will also assume for convenience that the residual characteristic of F is odd, although much of the discussion will be valid when F is of characteristic zero, but has residual characteristic 2. For these F, the phenomena discussed in the preceding sections acquires a richer texture because of the possibility of consider R-modules as well as F-modules (subspaces). The purpose of this section is to discuss the extra structure. We will be somewhat brief, and only discuss in detail the aspects of the theory which seem new. We let π denote a prime of R, and put $R^i = \pi^i R$, and $R^x = R^0 - R^1$. Recall that F has a natural locally compact topology with respect to which R is an open compact subring and the R^i are a neighborhood basis of 0.

Let $(V, D, \natural, (\ ,\ ))$ be the basic data for a classical group G. We will take, without essential loss of generality as our basic field F the $\natural$-fixed subfield of the center of D. Then there are only the three possibilities for D listed at the end of §6. The integers of D will be denoted S. We will redefine $\perp$ in this context. If $E \subseteq V$ is any subset, then

(11.1) $$E^{\perp} = \{v \in V\colon (e,v) \in S, \text{ all } e \in E\}.$$

With this definition, $E^{\perp}$ is no longer a subspace of V, but only an R-module. If E is a subspace, however, so is $E^{\perp}$, and is equal to $E^{\perp}$ under the previous definition in §2 or §9. If is well-known that if E is an R-module, then $E^{\perp\perp} = E$. If $E \subseteq E^{\perp}$ we will say that E is <u>S-isotropic</u>; if $E = E^{\perp}$ we will say E is an <u>S-polarization</u> of V, or simply a polarization. We will denote the set of S-polarizations of V by $\Omega_S(V) = \Omega_S$. The subset of Ω_S consisting of subspaces, i.e., polarizations in the old sense, which we will also call F-polarizations, will be written $\Omega_F(V) = \Omega_F$. We will say V is <u>S-split</u> if $\Omega_S(V) \neq \phi$.

For $v \in V$, we let Sv and D respectively denote the S-module and (D-)subspace generated by v. Let $L \subseteq V$ be an S-module. It is well-known that we can find vectors $\{\ell_i\}_{i=1}^{k}$, independent over D such that

(11.2) $$L = \sum_{i=1}^{j} Dx_i \oplus \sum_{i=j+1}^{k} Rx_i$$

We denote by L_D the largest subspace contained in L, and by DL the subspace spanned by L. Thus, for L as in (11.2) we have

(11.3) $$L_D = \sum_{i=1}^{j} Dx_i \qquad DL = \sum_{i=1}^{k} Dx_i$$

We note that L/L_D is compact, and L is open in DL. It is easy to see that $(L^{\perp})_D = DL^{\perp}$. Hence if $L = L^{\perp}$, then $DL = L_D^{\perp}$. Hence L_D is isotropic. Also L/L_D is an S-polarization for the form induced by $(\ ,\ )$ on DL/L_D. If L is open and compact in V, in other words, if $DL = V$ and $L_D = \{0\}$, then we call L a <u>lattice</u> in V. An S-polarization which is also a lattice will be called a <u>self-dual lattice.</u> If L is a lattice, then $L^{\perp}$ is also, and then the map α of (1.5) defines an isomorphism of $L^{\perp}$ with $Hom_S(L,S)$. In general, for an S-module L, the map α of (1.5) defines an isomorphism of $V/L^{\perp}$ with $Hom_S(L, D/S)$.

Suppose V is S-split and fix a self-dual lattice L in V. Let $K_L = K$ be the subgroup of G leaving L stable. Then K is an open compact subgroup of G. Let Π be a prime of S. Put $S^i = \Pi^i S$, and

$L^i = \Pi^i L = S^i L$. Let $K^i \subseteq K$ be the subgroup of K which acts trivially on L/L^i.

Consider the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{g}_L \subseteq \mathfrak{g}$ be the lattice of elements which map L into L. Let $\mathfrak{g}^i_L$ be the sublattice of $\mathfrak{g}_L$ of elements which map L into L^i. If D is abelian, then $\mathfrak{g}^i_L = \Pi^i \mathfrak{g}_L$. Elements of $\mathfrak{g}^i_L$ will also map L^j into L^{j+1}. Note that

(11.4) $$[\mathfrak{g}^i_L, \mathfrak{g}^j_L] \subseteq \mathfrak{g}^{i+j}_L$$

Since L is self-dual, the dyads E_{xy}, given by 7.19, with $x,y \in L$ span $\mathrm{Hom}_S(L,L)$. The dyads $\hat{E}_{xy}$ of (7.12) will then be in $\mathfrak{g}_L$. It follows that, if D is commutative, and unramified over F, so the different [ ] of D over F is just S, then $\mathfrak{g}_L$ is a self-dual lattice in $\mathfrak{g}$ with respect to the self-dual form $\mathrm{tr}(xy)$ of (7.7).

Consider the Cayley transform c, given by (7.12).

Lemma 11.1: The Cayley transform establishes a bijection, for every i 1.

(11.5) $$c: \mathfrak{g}^i_L \leftrightarrow K_i$$

Moreover, c induces group isomorphisms

(11.6) $$c: \mathfrak{g}^i_L / \mathfrak{g}^j_L \leftrightarrow K_i/K_j$$

for $1 \le i \le j \le 2i$.

Proof: If $T \in \mathfrak{g}^i_L$, for $i \ge 1$, then T^m tends to zero, so that the formal power series (7.15) is a valid convergent series having the value $c(T)$. By inspection of the series, the sum will preserve L act as the identity on L/L^i. Since $c(T)$ belongs to G, it belongs to K^i.

Thus $c(T)$ maps $\mathfrak{g}^i_L$ to K^i. A similar expression of $c(T)$ in powers of $I-T$, obtained by writing

$$c(T) = (I-T)(2I - (I-T))^{-1}$$

shows, since 2 is a unit in S, that $c(K^i) \subseteq \mathfrak{g}^i_L$. Since c is involutive, the first statement is proved. The second statement follows by noting K^i/K^j is abelian if $1 \le i \le j \le 2i$, and from (11.4) and (7.18).

Lemma 11.2: Let $\{x_i\}$ be a basis for the self-dual lattice L. Put $(x_i,x_j) = a_{ij}$ so that $A = \{a_{ij}\}$ is the matrix of $(\, , \,)$ with respect to the basis $\{x_i\}$. Let $B = \{b_{ij}\}$ be a matrix representing a form of the same type as $(\, , \,)$, and suppose $b_{ij} \in S^m$ for some $m \ge 1$ and all i,j. Then there are vectors y_i in L^m such that $A + B$ is the matrix of $(\, , \,)$ with respect to the basis $\{x_i + y_i\}$.

Proof: This is of course a Hensel's lemma argument. Compare [O'm], p. . Indeed, since L is self-dual we may find elements z_i in L such that $(x_i,z_j) = \delta_{ij}$. Put

$$y_i' = \pm \left(\tfrac{1}{2}\right) \sum_k b_{ik} z_k$$

We take the + or − sign in the expression for y_i according as $(\, , \,)$ is $\natural$-hermitian or $\natural$-antihermitian. We compute

$$\begin{aligned}(x_i+y_i', x_j+y_j') &= (x_i,x_j) + (y_i',y_j') \pm \left(\tfrac{1}{2}\right)\left(\sum_k b_{ik}(z_k,x_j) + b^{\natural}_{jk}(x_i,z_k)\right) \\ &= (x_i,x_j) \pm \left(\tfrac{1}{2}\right)(\pm b_{ij} \pm b_{ij}) + (y_i', y_j') \\ &= a_{ij} + b_{ij} + (y_i', y_j') = a'_{ij}\end{aligned}$$

Since $y_i' \in L^m$, we have $(y_i', y_j') \in S^{2m}$. Hence if $A' = \{a'_{ij}\}$, then $A + B = A' + B'$ where B' has entries in S^{2m}. Continuing with the procedure just indicated, we find that in the limit the lemma follows.

Remark: In the above proof, we altered all x_i simultaneously. But suppose $b_{ij} = 0$ for $1 \le i, j \le k$. Then an obvious modification of the above procedure allows us to find the desired y_i's, with $y_i = 0$ for $i \le k$.

Consider now the structure of Ω_S. According to [ ], Ω_S can be considered to be a compact Hausdorff space, on which G acts with finitely many orbits, the open orbits being the orbits of lattices. In fact, the structure is more precise. Let Ω_S^j denote the set of L in Ω_S such that $\dim L_D = j$.

Proposition 11.3: G acts transitively on Ω_S^j for each j. Thus, if ℓ is the maximal dimension of isotropic subspaces of V, and if $\Omega_S \ne \phi$; then Ω_S consists of $\ell + 1$ G-orbits. For each $j < \ell$, Ω_S^{j+1} is in the closure of Ω_S^j. Thus Ω_S^0, the orbit of self-dual lattices, is the unique open orbit, and Ω_S^ℓ is the unique closed orbit.

Proof: The remarks on orbit closure follow from [ ]. We will just prove transitively on Ω_S^j. This involves three applications of Witt's Theorem and lemma 1.2. First, we know from Witt's Theorem that if L and L' are in Ω_S^j, then modulo the action of G we may assume $L_D = L_D'$. Then L/L_D and L'/L_D are self-dual lattices in DL/L_D. A second application of Witt's Theorem tells us that $P(L_D)$ restricts to yield all isometries of the form induced by $(\, , \,)$ on DL/L_D. Thus it will suffice to prove the proposition for Ω_S^0. If $L \ne L'$ are self-dual lattices, then there is an x in $(L-L^1) \cap L'^1$. Evidently then $(x,x) \in S^1$. By lemma 1.2, we can find x_1 in L such that x_1 is isotropic. Let $y \in L$ satisfy $(x_1,y) = 1$. Put $y_1 = y - (\frac{1}{2})(y,y)x_1$. Then y_1 is also isotropic, and $(x_1,y_1) = 1$ still. By symmetry, we can find a similar pair x_1', y_1' in L'. By Witt's Theorem we can move one pair into the other by G, so assume $x_1 = x_1'$ and $y_1 = y_1'$. Let V_1 be the span of x_1 and y_1. If $z \in L$, then $z - \langle x_1,z \rangle y_1 - \langle z,y \rangle x_1$ is again in L and is orthogonal to V. Thus

$$L = (L \cap V_1) \oplus (L \cap V_1^\perp)$$

and likewise for L'. Additionally, $L \cap V_1 = L' \cap V_1 = Sx_1 \oplus Sy_1$. Thus we have reduced the problem by two dimensions, so the result follows by induction on $\dim V$.

Remark: a) It is implicit in the above argument, that if $(\, , \,)$ is anisotropic there is at most one self-dual lattice in V.

b) Actually, we can be more precise about the mutual relation of two self-dual lattices L and L'. Let j be the smallest integer such that $L^j \subseteq L'$. Choose $x_1 \in L^j$ such that $\Pi^{-1} x_1 \notin L'$. As above, we can suppose x_1 is isotropic. Choose $y_1 \in L'$ such that y_1 is isotropic and $\langle x_1,y_1 \rangle = 1$. Since $L^j \subseteq L'$, we have $L' \subseteq L^{-j}$, so that $\Pi^{-1}x_1$ and $\Pi^{\natural -j}y_1$ are a similar pair for L. Thus we may take x_1 to $\Pi^{-j}x_1$ and y_1 to $\Pi^{\natural j}y_1$ and leave the orthogonal complement of the span of x_1 and y_1 pointwise fixed. Thus we may assert: given maximal isotropic lattices L and L', there are maximal isotropic spaces X and Y such that

$$L = (L \cap X) \oplus (L \cap Y) \oplus (L \cap (X \oplus Y)^\perp)$$

and similarly for L', and furthermore

$$L \cap X \subseteq L' \cap X \qquad L \cap Y \supseteq L' \cap Y$$

This result implies the Cartan decomposition for G.

The description of Ω_F given in proposition 9.3 extends with slight modifications to Ω_S. Let (V_1,V_2) be a complete polarization of V. Choose $L \in \Omega_S$. Put $E(L) = E = (V_1 + L) \cap V_2$. If e_1,e_2 are in E, choose x_i in V_1 such that $e_i + x_i$ is in L. Define $B_L = B$ by

$$B(e_1,e_2) = (e_1,x_2) \text{ modulo } S \tag{11.7}$$

First note that B is well-defined, for if x_2' is another element of V_1 such that $e_2 + x_2' \in L$, then $x_2 - x_2' \in V_1 \cap L$. Hence

$$(e_1,x_2) - (e_1,x_2') = (e_1, x_2-x_2') = (e_1 + x_1, x_2 - x_2') \in S$$

since $L \subseteq L^{\perp}$. Next, extending our terminology from D-valued forms to D/S-valued forms, we note that B is of the type dual to $(\, , \,)$. Indeed, again, since $L \subseteq L^{\perp}$, we see

$$B(e_1,e_2) \mp B(e_2,e_1)^{\natural} = (e_1,x_2) + (x_1,e_2) = (x_1 + e_1, x_2 + e_2) = 0 \text{ modulo } S.$$

Thus we have established half of the following result.

Proposition 11.4: Given a complete polarization $(V_1,V2)$ of V, the space Ω_S may be parametrized by pairs (E,B) where $E \subseteq V_2$ is an S-module and B is an D/S-valued form on E of the type dual to $(\, , \,)$. If $L \in \Omega_S$, then $E(L) = (V_1 + L) \cap V_2$, and B_L is given by (11.7). All possible pairs (E,B) arise in this fashion.

Proof: It remains only to show that a given pair (E,B) comes from some $L \in \Omega_S$. Via the map α of (1.5), V_1 is identified $\natural$-semilinearly with the dual of V_2. Thus given $e \in E$, there is an $x \in V_1$, defined modulo the annihilator mod S of E, such that $B(e',e) = (e',x)$.

Define L to be the collection of $e + x$ that can be constructed in this fashion. It is then easy to check that $L \in \Omega_S$, and $E = E_L$ and $B = B_L$.

Again fix some self-dual lattice $L \subseteq V$. Let $V_1 \subseteq V$ be an isotropic subspace. Set

$$L_{V_1} = (L \cap V_1^{\perp}) + V_1 \tag{11.8}$$

Then $L_{V_1}^{\perp} = (L^{\perp} + V_1) \cap V_1^{\perp} = (L \cap V_1^{\perp}) + V_1 = L_{V_1}$. Thus $L_{V_1} \in \Omega_S$. For any $L' \in \Omega_S$, let $P(L') = P$ be the subgroup of G stabilizing L'. (If L' is a lattice then $P(L') = K_L$, and we will favor the latter notation for this case.) We observe ${}^{\circ}P(L'_D) \subseteq P(L') \subseteq P(L_D')$, where ${}^{\circ}P(L_D')$ and $P(L_D')$ are as in §9. Suppose $L' = L_{V_1}$, as in (11.8). Then we may write

$$P(L_{V_1}) = (K_L \cap P(V_1) \cdot {}^{\circ}P(V_1) \tag{11.9}$$

Let us note also, the <u>Iwasawa</u> <u>decomposition</u>

$$G = K_L P(V_1) \tag{11.10}$$

for any isotropic subspace V_1. Hence K acts transitively on the space of isotropic subspaces of a given dimension. The Iwasawa decomposition is well-known, but it will also follow from the result on K-orbits to be proven below.

With L' as above, we let $P(L')^{i} = P^{i}$ to be the subgroup of $P(L')$ which acts trivially on $L'/S^{i}L'$. If $L' = L_{V_1}$, then it is not hard to see that

$$P(L_{V_1})^{i} = (K_L^{i} \cap P(V_1))\, {}^{\circ}P(V_1). \tag{11.11}$$

With these notations, we may now formulate the following result on orbits. This result may be thought of as interpolating between the two parts of proposition 9.1. As usual, U here will denote an auxiliary vector space.

Proposition 11.5: a) Given $L' \in \Omega_S$, and two elements T_1 and T_2 of $Hom(U,V)$, there is $p \in P(L')$ such that $pT_1 = T_2$ if and only if

a) $\ker T_1 = \ker T_2$

b) $(\, , \,)\circ T_1 = (\, , \,)\circ T_2$

c) $T_1^{-1}(L') = T_2^{-1}(L')$

b) We may write $pT_1 = T_2$ with $p \in P(L')^m$ if and only if, in addition,

d) the maps induced by the T_i form $T_1^{-1}(L')/S^m T_1^{-1}(L')$ to $L'/S^m L'$ are equal.

Proof: The stated conditions are obviously necessary. From condition c), we may determine that

$$T_1^{-1}(L'_D) = T_2^{-1}(L'_D) = T_1^{-1}(L')_D$$

and

$$T_1^{-1}(DL') = T_2^{-1}(DL') = DT_1^{-1}(L')$$

Suppose by action of $P(L')$ or $P(L')^1$, according to cases, we could arrange that the maps induced from $T_1^{-1}(DL')/T_1^{-1}(L'_D)$ to DL'/L'_D agree. Then Proposition 9.1 b) tells us by further action of ${}^\circ P(L'_D)$ we can get T_1 and T_2 to agree. Thus it will suffice to prove the proposition in the case when L' is a lattice. Since we have condition a) (which is implied by c) when L' is a lattice) we may as well also assume T_1 and T_2 are injective. Then $T_1^{-1}(L') = \Lambda$ will be some lattice in U. Set

$$\Lambda_1 = \{\lambda \in \Lambda : (T_1(\lambda), T_1(\lambda')) \in S^1 \text{ for all } \lambda' \in \Lambda\}$$

Evidently Λ_1 is a sublattice of Λ containing Λ^1. Let $\{\lambda_i\}_{i=1}^{\ell}$ be a basis for Λ such that $\{\lambda_i\}_{i=1}^{k}$, for some $k \le \ell$, generates Λ_1 modulo Λ^1. Reducing modulo S^1, we find that $\{T_1(\lambda_i)\}_{i=1}^{k}$ generate an isotropic S/S^1 subspace of L'/L'^1, while $\{T_1(\lambda_i)\}_{i=1}^{\ell}$ generate a non-degenerate subspace of L'/L'^1. The same holds true with T_2 replacing T_1. It follows that we can find $\{v_j\}_{j=1}^{k}$ in L' such that $(T_1\lambda_i, v_j) = \delta_{ij}$ modulo S^1 for $1 \le i \le \ell$ and $1 \le j \le k$, and $(v_i,v_j) = 0 \bmod S^1$. By lemma 11.2 and the remark following it, we can modify the v_i if necessary so that these relations hold exactly instead of only modulo S^1. Define a map $\tilde{T}_1 : U \oplus D^{\ell} \to V$;
$\tilde{T}_1(u,a) = T_1(u) + \sum_{i=1}^{\ell} a_i v_i$ where $a = (a_1, \ldots, a_\ell)$. Proceed similarly to obtain $\tilde{T}_2$. Then the $\tilde{T}_i$ still satisfy conditions a), b) and c) of the proposition. If the T_i also satisfy condition d), we may arrange the $\tilde{T}_i$ do also, by the following device. Having chosen the $\{v_i\}$ for T_1, we observe that since the T_i satisfy d), we have $(T_2(\lambda_i),v_j) \in S^m$. Hence by lemma 11.2 and the remark following it, we see that too obtain the v_j for T_2 we need only modify the v_j for T_1 by elements of L'^j. Then the $\tilde{T}_i$ will satisfy d) also. But we now have

$$L = (\text{im}\,\tilde{T}_1 \cap L') \oplus (\text{im}\,\tilde{T}_1^{\,-} \cap L')$$

and likewise for $\tilde{T}_2$. Since the conditions of this proposition are stronger than those of 8.1, there is certainly $g \in G$ such that $\tilde{T}_2 = g\,\tilde{T}_1$. This g must satisfy $g(\text{im}\,\tilde{T}_1 \cap L') = \text{im}\,\tilde{T}_2 \cap L'$. Then $g(\text{im}\,\tilde{T}_1^{\,-} \cap L)$ will be some self-dual lattice in $\text{im}\,\tilde{T}_2^{\,-}$. By proposition 11.3, we may modify g by an element of G acting as the identity on $\text{im}\,\tilde{T}_2$

so that $g(\mathrm{im}\,\tilde{T}_1^{\perp} \cap L) = \mathrm{im}\,\tilde{T}_2^{\perp} \cap L$ also. Then $g \in K$, as desired. Thus a) is proved. For b), choose a basis $\{\lambda_j\}$ for $\tilde{T}_1^{-1}(L')$. Consider $\{\tilde{T}_1(\lambda_j)\}$, which will be a basis for $\mathrm{im}\,T_1 \cap L'$. Choose also a basis $\{\mu_k\}$ for $\mathrm{im}\,T_1^{\perp} \cap L$. Since $\tilde{T}_1(\lambda_j) - \tilde{T}_2(\lambda_j) \in L'^m$, the matrix of $(\ ,\ )$ with respect to the basis $\beta_2 = \{\tilde{T}_2(\lambda_j), \mu_k\}$ of L' will equal the matrix of $(\ ,\ )$ with respect to $\beta_1 = \{\tilde{T}_1(\lambda_j), \mu_k\}$ modulo S^m. Thus we may alter the μ_k by elements of L'^m to obtain elements μ_k' such that the matrix of $(\ ,\ )$ with respect to $\beta_2' = \{\tilde{T}_2(\lambda_j), \mu_k'\}$ equals the matrix with respect to β_1. But then the linear transformation of V taking β_1 to β_2' is seen to belong to $K^m_{L'}$, so the proposition is proved.

Recall, we are dealing with a classical group G with defining data $(V, D, \natural, (\ ,\ ))$, all over the base field F, the $\natural$-fixed field in the center of D. We will say G, or V, or $(\ ,\ )$, is <u>unramified</u> if $(\ ,\ )$ is S-split and D is commutative and unramified over F. Thus, either $D = F$, or D is the unique unramified quadratic extension of F.

Let V be unramified, and let $L \subseteq V$ be a self-dual lattice. Then $L/L^1 = \overline{L}$ is a vector space over $S/S^1 = \overline{S}$, which is an extension of $\overline{F} = R/R^1$ of the same degree as D over F. Moreover $(\ ,\ )$ reduced modulo S^1 gives a form $\overline{(\ ,\ )}$ on $\overline{L}$, and K/K_1, by lemma 1.2, is isomorphic to the full isometry group of $\overline{(\ ,\ )}$. It is known [O'm] that the isometry group of forms over finite fields act absolutely irreducibly on their defining modules. Thus we see that the S-module in $\mathrm{End}_D(V)$ generated by K is all of $\mathrm{End}_S(L)$.

Let again W denote a symplectic vector space over F, and let (G,G') be a reductive type I irreducible dual pair in $Sp(W)$, so that

$W \simeq \mathrm{Hom}_D(V,V')$ where as usual $(V, D, \natural, (\ ,\ ))$ and $(V, D, \natural, (\ ,\ )')$ are the defining data for G and G' respectively. Suppose both G and G' are unramified. Let L and L' be self-dual lattices in V and V'. We may regard $\mathrm{Hom}_S(L,L')$ as a lattice in W. Furthermore, as T ranges through $\mathrm{Hom}_S(L,L')$, T^*, in the sense of (6.5), will range through $\mathrm{Hom}_S(L',L)$. Hence the elements $T_2^*T_1$ will span $\mathrm{End}_S(L,L)$. Since D is unramified over F, it follows, with $\langle\ ,\ \rangle$ given by (6.7), that $\mathrm{Hom}_S(L,L')$ is a self-dual lattice.

Proposition 11.6: a) The map $L' \to \mathrm{Hom}_S(L,L')$ establishes a bijection between self-dual lattices in V' and K_L-invariant self-dual lattices in W

b) In general, if $L_1 \in \Omega_S(V)$, then the $P(L_1)$-invariant elements in $\Omega_S(W)$ have the form

$$\mathrm{Hom}_D(V,Y_1) + \mathrm{Hom}_S(V/L_D;\ D/L_D,\ L') + \mathrm{Hom}(V/DL;\ Y_1^{\perp})$$

wehre $\mathrm{Hom}_S(V/L_D;\ L/L_D;\ L')$ indicates maps of V/L_D into V' which map V into DL' and L/D_D into L', and where $Y_1 \subseteq L_D'$, and $L' \in \Omega_S(V')$.

Proof: Since K spans $\mathrm{End}_S(L,L)$, it is clear that any K-invariant lattice in W has the form $\mathrm{Hom}_S(L,L')$ where L' is some lattice in V'. Computations such as in proposition 9.5 then show that $\mathrm{Hom}_S(L,L')^{\perp} = \mathrm{Hom}_S(L,L'^{\perp})$ the first $\perp$ being in W, the second in V'. Part a) follows immediately. Part b) is a combination of part a) abd proposition 9.5. We omit the details.

To conclude, we briefly indicate the analogous considerations in the type II case. By $\Omega_S(V)$ we shall mean the collection of all S-modules in the D-vector space V. The $GL_D(V)$ orbit of $L \in \Omega_S(V)$ is specified by dim L_D^- and dim DL. As detailed in [ ], the space $\Omega_S(V)$ has the structure of compact Hausdorff space such that the lattices from the unique open GL_D orbit, and the various Grassman varieties form the closed orbits. For any $L \in \Omega_S(V)$, let $P(L)$ be the stabilizer of L in GL_D, and if L is a lattice, write $P(L) = K_L \cong GL_S(L)$. Evidently K_L spans $End_S(L)$. Let $P^{\perp}(L)$ denote the subgroup of $P(L)$ acting trivially on $L/S^{\perp}L$. Write $P^{\perp}(L) = K_L$ when L is a lattice. We note that for a lattice L, $\mathfrak{H}_L = End_S(L)$, and lemma 11.1 is true here too. We need analogues of propositions 11.5 and 11.6. We will simply state them. As usual, U is an auxiliary vector space.

Proposition 11.7: a) If $L \in \Omega_S(V)$, and $T_1, T_2 \in Hom_D(U,V)$, then $T_2 = p\, T_1$ with $p \in P(L)$ if and only if

a) $\ker T_2 = \ker T_1$
b) $T_2^{-1}(L) = T_1^{-1}(L)$

b) Moreover, $T_2 = p\, T_1$ with $p \in P^m(L)$ if and only if, in addition

c) the maps induced by T_1 and T_2 form $T_2^{-1}(L)/S^m\, T_2^{-1}(L)$ to $L/S^m\, L$ are equal.

Remark: For a lattice L, condition b) implies condition a).

Proposition 11.8: a) The map $\Lambda \to Hom_S(\Lambda, L)$ establishes a bijection between lattices in U and K_L-invariant lattices in $Hom_D(U,V)$.

b) For any $L \in \Omega_S(V)$, the elements of $\Omega_S(Hom_D(U,V))$ which are P(L)-invariant have the form $Hom_D(U/U_1, L_D) \oplus Hom_S(U/U_2, DL; \Lambda, L) \oplus Hom_D(U/U_3, V)$, where $U_1 \subseteq U_2 \subseteq U_3$ are a sequence of 3 nested subspaces, and $Hom_S(U/U_2, DL; \Lambda, L)$ means maps from U/U_2 to DL which take Λ to L, where $\Lambda \in \Omega_S(U/U_2)$.

Let $\tilde{V} = V^+ \oplus V^-$ be the double of V. We will need a description of the $i^+(G) \times i^-(G)$ orbit structure in $\Omega_S(\tilde{V})$, in analogy with proposition 10.3.

Proposition 11.9: a) A point $\tilde{L} \in \Omega_S(\tilde{V})$ is specified by a triple (L^+, L^-, s) where L^+ and L^- are S-submodules of V defined by

$$i^+(L^+) = \tilde{L} \cap V^+ \qquad i^-(L^-) = \tilde{L} \cap V^- \tag{11.12}$$

and s is an isometry of the D/S-valued forms induced by (,) on $(L^+)^{\perp}/L^+$ and on $(L^-)^{\perp}/L^-$. The action of $i^+(G) \times i^-(G)$ on $\tilde{\Omega}_S$ is given in these coordinates by

$$i^+(g_1)\, i^-(g_2)(L^+, L^-, s) = (g_1(L^+), g_2(L^-), g_2\, s\, g_1^{-1}) \tag{11.13}$$

b) Let Ξ_S be the set of all S submodules of V which are contained in their duals. There is a natural embedding

$$\delta : \Xi_S \to \tilde{\Omega}_S$$

given in the parameters of a) by

$$\delta(L) = (L, L, 1) \qquad L \in \Xi_S \tag{11.14}$$

where 1 here denotes the identity map on $L^{\perp}/L$. We have the relation between G-actions:

$$\delta(g(L)) = i^+(g)\, i^-(g)\, \delta(L) \tag{11.15}$$

If Ξ_S^0 is the subset of Ξ_S consisting of lattices and $\tilde{\Omega}_S^0$ the analogous subset of $\tilde{\Omega}_S$, then δ establishes a bijection between G orbits in Ξ_S and the $i^+(G) \times i^-(G)$ orbits in Ω_S^0.

Remark: The map $\delta(L)$ fails to be surjective to $i^+(G) \times i^-(G)$ orbits in all of $\tilde{\Omega}_S$ since in a triple (L^+, L^-, s) the spaces L_D^+ and L_D^- can have different dimensions. However, the last statement says that s is essentially the only obstruction to surjectivity of δ.

Proof: Part a) proceeds precisely as in proposition 10.3 as does part b) through equation (11.15). We will prove the last statement of part b). It is clear from (11.15) that each G-orbit in Ξ_S^0 is embedded into a single $i^+(G) \times i^-(G)$ orbit in $\tilde{\Omega}_S^0$. It is clear from (11.13) that different G-orbits in Ξ_S^0 are taken to different $i^+(G) \times i^-(G)$ orbits. Thus we need only show δ is surjective on orbit spaces. This amounts to showing that if $(L_1, L_2, s) \in \tilde{\Omega}_S^0$, then s is the restriction to $L_1^\perp$ of an isometry of V carrying L_1 to L_2. But the proof of lemma 11.9 just below implies this. For let $V = \oplus_j V_j$ be a decomposition as specified there, and let $\{d_j\}$ or $\{e_j, f_j\}$ be generators for $L \cap V_j$ according as $\dim V_j = 1$ or 2. Let $|\ |$ be the standard absolute value in D. We may assume $|(d_j,d_j)|$ or $|(e_j,f_j)|$, which ever is appropriate, is monotone decreasing in j. Observe that if $y \in D$ is such that $y = \pm y^\natural$, then the map

$$1 + x \to (1+x)y(1+x)^{-1}$$

is surjective from ΠS to the set of all elements in $y(1 + \Pi S)$ having the same symmetry as y under $\natural$. This observation, plus the proof of lemma 11.9 says we can successively choose elements $\tilde{d}_j$ or $\{e_j, f_j\}$ in $L_2^\perp/L_2$ such that

$$\tilde{d}_j + L_2 = s(d_j + L_2)$$

or similarly for $\{e_j, f_j\}$ and such that $(\tilde{d}_j, \tilde{d}_j) = (d_j, d_j)$, or similarly for the $\{e_j, f_j\}$. This will be true so long as $(d_j, d_j) \notin L_2$. But then Witt's Theorem plus the conjugacy of self-dual lattices, proposition 11.3, allows us to completely lift s to an isometry of V, as desired. This proves proposition 11.8.

A brief word about maximal compact subgroups of classical groups over a non-Archimedean fields is required. Let $(V, D, \natural, (\ ,\ ))$ be the defining data for the group G and let D be central over a non-Archimedean local field of odd residual characteristic. Let $K \subseteq G$ be a compact subgroup. Then K will preserve some lattice $L \subseteq V$. Indeed, the set of lattices preserved by K will be closed under the operations of taking sums, intersections, scalar multiples and duals. We will study the smallest such set generated by a single lattice L.

For each $x \in L$, the set (L,x) is an fractional ideal Q^j, where Q is the maximal ideal of S, the integers of D. Choose $x \in L$ such that $(L,x) = Q^j$ with minimum possible j. Either $(x,x) \in Q^{j+1}$ or $(x,x) \notin Q^{j+1}$. In the second case, let $V_1 = Dx$ be the line through x. Then $V = V_1 \oplus V_1^\perp$. I claim also $L = (L \cap V_1) \oplus (L \cap V_1^\perp)$. Indeed, if $y \in L$, then, putting $\alpha = (y,x)(x,x)^{-1}$, we may write

$$y = (y - \alpha x) + \alpha x = y_1 + \alpha x$$

Then $y_1 \in L$, since (x,x) generates (L,x) and $y_1 \in V_1^\perp$ by direct computation. Suppose on the other hand that $(x,x) \in Q^\ell$, with $\ell \geq j$. Then we can find $y \in L$, such that $(y,x) \notin Q^{j+1}$. Suppose $(x,x) \in Q^\ell$. Put

$$x' = x - (\tfrac{1}{2})\,(x,x)(x,y)^{-1}y$$

Direct computation shows

$$(x',x') = (\tfrac{1}{4})(x,x)(x,y)^{-1}(y,y)(x,y)^{-1}\,\natural(x,x)\,\natural.$$

Thus $(x',x') \in Q^{m}$ with $m = 2\ell - j$. Hence, by successive modifications we can find $\tilde{x}$ such that $(L,\tilde{x}) = Q^{j}$, and $(\tilde{x},\tilde{x}) = 0$. Then if y is again an element of L such that $(y,\tilde{x}) \not\subseteq Q^{j+1}$, and we put

$$\tilde{y} = y - (\tfrac{1}{2})(y,y)(y,\tilde{x})^{-1}\tilde{x},$$

then also $(\tilde{y},\tilde{y}) = 0$, and of course we still have $(\tilde{y},\tilde{x}) \not\subseteq Q^{j+1}$. Let V_1 be the span of $\tilde{x}$ and $\tilde{y}$. Then $V = V_1 \oplus V_1^{\perp}$, and I claim also $L = (L \cap V_1) \oplus (L \cap V_1^{\perp})$. Indeed this is easily shown, just as in proposition 3.2. We have shown

Lemma 11.10: Given a lattice $L \subseteq V$, we can find mutually perpendicular subspaces $V_j \subseteq V$ such that:

i) $\dim V_j = 1$ or 2;

ii) $V = \bigoplus_j V_j$ and $L = \bigoplus_j (L \cap V_j)$; and

iii) if $\dim V_j = 2$, then $L \cap V_j$ is spanned by two isotropic vectors.

Suppose $\{V_j\}$ is such a collection of subspaces, and put $L_j = L \cap V_j$. Suppose $(L_j, L_j) = Q^{m_j}$, then

$$L^{-} = \bigoplus_j Q^{-m_j} L_j \tag{11.16}$$

Since $(Q^{m}L)^{\perp} = Q^{-m}L^{\perp}$, we may replace L by a multiple of L if necessary, and arrange $0 \le \min m_j \le 1$. If $m_j > 1$ for some j, consider

$$L' = Q^{-1}L \cap QL^{\perp}$$

We have

$$L_j' = \begin{cases} QL_j & \text{if } m_j = 0 \\ L_j & \text{if } m_j = 1 \\ Q^{-1}L_j & \text{if } m_j \ge 2 \end{cases}$$

Hence $L_j'^{\perp} = Q^{-m_j+2}L_j'$, if $m_j \ge 2$. Regarding L' as a new L and repeating, we can obtain a lattice L'' for which $(L_j'', L_j'') = Q^{\alpha}$ with $\alpha = 0, 1,$ or 2. Now set

$$L''' = (Q^{-1}L \cap L^{\perp})^{\perp}$$

Then $(L_j''', L_j''') = Q^{\alpha}$ with $\alpha = 0$ or 1. Thus we have shown

Proposition 11.11: Let $(V, D, \natural, (\ ,\ ))$ be defining data for the classical group G. Let $K \subseteq G$ be a compact group. Then K fixes a lattice $L \subseteq V$ such that

$$L \subseteq L^{\perp} \subseteq Q^{-1}L. \tag{11.17}$$

It can be shown, in analogy with proposition 11.3 that there are only finitely many lattices in V satisfying (11.17), up to a motion in G.

12: Complex polarizations over $\mathbb{R}$

Let W be a symplectic space over the real field $\mathbb{R}$, with form $< , >$. A <u>complex structure</u> for W is an endomorphism

(12.1) $$J:W \to W \quad \text{such that} \quad J^2 = -1$$

Evidently then 1 and J generate a subfield of $End_{\mathbb{R}}(W)$ isomorphic to $\mathbb{C}$ so that W can be given the structure of complex vector space, with J defining multiplication by $\sqrt{-1}$. A complex structure J is said to be <u>compatible with</u> $< , >$ if J is an isometry of $< , >$.

Proposition 12.1: A complex structure J on W is compatible with $< , >$ if and only if the bilinear form

(12.2) $$B_J(w,w') = < Jw, w'>$$

is symmetric; equivalently if and only if the complex valued, real-bilinear form

(12.3) $$H_J(w,w') = <Jw, w' > + i <w,w'>$$

is Hermitian with respect to the complex structure on W defined by J.

Proof: It is clear that $B_J(w,w') = B_J(w',w)$ if and only if $H_J(w,w') = \overline{H_J(w',w)}$, where $\overline{\quad}$ here denotes complex conjugation in $\mathbb{C}$. We compute

$$\begin{aligned} H_J(Jw, w') &= <J^2w, w' > + i < Jw, w'> \\ &= - <w,w'> + i < Jw, w' > \\ &= i(<Jw, w' > + i < w,w'>) = i\, H_J(w,w') \end{aligned}$$

Hence H_J is automatically complex linear in the first variable, and thus will be Hermitian if and only if B_J is symmetric. If B_J is symmetric, then

$$\begin{aligned} < Jw, Jw' > &= B_J(w, Jw') = B_J(Jw', w) \\ &= < J^2w', w > = - < w',w > = < w,w'> \end{aligned}$$

so $J \in Sp$. Conversely, if J is an isometry of $< , >$, we see

$$\begin{aligned} B_J(w,w') &= < Jw,w' > = < J^2w, Jw'> = - < w, Jw'> \\ &= < Jw', w > = B_J(w',w) \end{aligned}$$

so B_J is symmetric, and the proposition is proved.

Let J be a compatible complex structure on W. We will say J is <u>positive</u> if and only if B_J, equivalently H_J, is positive definite.

Remark: Starting from the other direction, suppose we are given a symmetric bilinear form B on $< , >$. Then we know from I, proposition 7.3 that we can write

$$B(w, w') = < Tw, w'>$$

where $T \in \mathfrak{sp}$. Let us say that B is <u>compatible</u> if $T \in Sp$ also. Then as we noted in §7, (remark after proposition 7.3) B will be compatible precisely when T is a complex structure.

Let $W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of W. We extend $< , >$ complex linearly to $W_{\mathbb{C}}$. Consider a compatible complex structure J on W and extend J to a complex linear endomorphism of $W_{\mathbb{C}}$. The eigenvalues of J will be $\pm i$, where $i = \sqrt{-1}$ as usual. Let $Y_J^+ = Y^+$ be the i-eigenspace for J, and let Y^- be the (-1)-eigenspace. Let $\overline{\quad}$ now indicate complex conjugation on $W_{\mathbb{C}}$ with respect to W, as well as complex conjugation on $\mathbb{C}$. We have

$$W_{\mathbb{C}} = Y^{+} \oplus Y^{0} \qquad Y^{-} = (Y^{+})^{-} \tag{12.4}$$

We note that Y^{+} and Y^{-} are isotropic for $<\ ,\ >$. Indeed, if w and w' are in V^{+}, then

$$< w,w' > = < -i\, Jw,\ w'> \ = -i < Jw,w' > = -i < Jw',\ w > = < w',w >$$

So $<\ ,\ >$ is both symmetric and antisymmetric on V^{+}, hence zero. Let

$$p^{+}: W_{\mathbb{C}} \to V^{+} \qquad p^{-}: W_{\mathbb{C}} \to V^{-}$$

be the projections corresponding to the decomposition (.4) . We may write explicitly

$$p^{+} = \frac{1}{2}(J+i) \qquad p^{-} = \frac{1}{2}(J-i) \tag{12.5}$$

On V^{+} consider the skew-Hermitian form

$$K_{V^{+}}(v,v') = < v,\bar{v}'> \tag{12.6}$$

A direct computation shows that on W

$$2iK_{V^{+}} \circ p^{+} = H_J \tag{12.7}$$

Conversely, suppose $V \subseteq W_{\mathbb{C}}$ is a maximal isotropic (complex) subspace which is totally complex, in the sense that $V \cap W = \{0\}$. Then $\overline{V}$ is another such, and $V \cap \overline{V} = \{0\}$ since V is assumed totally complex. Hence

$$W = V \oplus \overline{V} ,$$

or $(V,\overline{V})$ is a complete polarization. The endomorphism J_V of W_C which has V and $\overline{V}$ respectively as its $+i$ and $-i$ eigenspaces will evidently be invariant under complex conjugation on W_C, and will therefore preserve W, on which it will define a complex structure. Since $i(-i) = 1$, we see that $J_V \in Sp(W_C)$, so its restriction to W will in fact be a compatible complex structure. We will call V <u>positive</u> if J_V is a positive complex structure on W.

Let J be a compatible complex structure on W. Let U_J be the subgroup of $Sp = Sp(W)$ which commutes with J. From formula (12.3) it is clear that U_J may also be identified with the unitary group of H_J. Furthermore, the discussion just above shows that U_J is also the subgroup of Sp leaving V_J^{+} invariant. If J is a positive complex structure, then U_J will be compact. I claim then U_J will in fact be a maximal compact subgroup of Sp. Indeed, let $K \subseteq Sp$ be any compact subgroup. Then K will leave invariant some positive definite inner product B. By I, proposition 7.3 we can write

$$B(w,w') = <Tw,w' >$$

for some $T \in \mathfrak{sp}$. Then we compute

$$B(T^2 w,w') = < T^3 w,w' > = - < T^2 w,\ Tw'> \ = - B(Tw,\ Tw') \tag{12.8}$$

Equation (12.8) shows that $B(T^2 w,w')$ is a negative definite inner product. Hence T^2 is self-adjoint with respect to B, and has all negative eigenvalues. Thus T itself has purely imaginary eigenvalues, which will occur in conjugate pairs $\pm i\,\lambda_1$, $\pm i\,\lambda_2$, etc. Let V_λ^{+} and V_λ^{-} be the eigenspaces for T for eigenvalues $\pm i\,\lambda$ respectively, with $\lambda > 0$. A computation just as above for $T = J$ shows each $V_\lambda^{\pm}$ is isotropic, and moreover, $V^{\pm}_{\lambda}$ is orthogonal to $V^{\pm}_{\mu}$ for $\lambda \neq \mu$. Hence if

$$V^{+} = \Sigma V^{+}_{\lambda}$$

then V^{+} is a totally complex, isotropic subspace of $W_{\mathbb{C}}$. Furthermore, we see that

(12.9) $$J_{V^{+}} = T\sqrt{-T^{2}}^{\,-1}$$

It follows from (12.8) that $J_{V^{+}}$ is a positive complex structure. Since $g \in Sp$ will preserve B if and only if it commutes with T if and only if it preserves all the V^{+}_{λ}, we see that $K \subseteq U_{J_{V^{+}}}$. Hence the U_J are maximal as claimed.

We summarize the discussion so far

Proposition 12.2: The following sets are all in natural bijection to one another.

i) The set of positive complex structures on W compatible with $< , >$.

ii) The set of positive totally complex maximal isotropic subspaces of $W_{\mathbb{C}}$.

iii) The set of maximal compact subgroups of Sp.

Given J in set i), the corresponding space V^{+}_{J} in set ii) is the i-eigenspace of J, and the corresponding group U_J in set iii) is the centralizer of J.

Let J be a positive compatible complex structure on W, and let $X \subseteq W$ be any maximal isotropic subspace. Put $Y = J(X)$. Then Y is also maximal isotropic since $J \in Sp$, and $X \cap Y = \{0\}$ since B_J is positive definite. Hence (X,Y) is a complete polarization for W such

that X and Y are permuted by J. Let $\{e_j\}$ be a basis of X which is orthonormal with respect to B_J. Put

(12.10) $$f_j = -J(e_j)$$

It follows from (12.2) and orthonormality of the e_j that $\{e_j, f_j\}$ form a symplectic basis for W, and that the f_j are an orthonormal basis for B_J on Y. Define

(12.11) $$x_j(w) = \langle w, f_j\rangle \qquad y_j(w) = \langle w, e_j\rangle \quad w \in W$$
$$z_j(w) = x_j(w) + i\, y_j(w)$$

Proposition 12.3: The mapping

(12.12) $$\Upsilon : W \to \mathbb{C}^{n} \qquad \Upsilon(w) = (z_1(w), z_2(w), \ldots, z_n(w))$$

is a complex linear isomorphism when W is given the complex structure defined by J. That is,

(12.13) $$\Upsilon(J(w)) = i\,\Upsilon(w)$$

Moreover, H_J is just the pullback by Υ of the standard Hermitian inner product on $\mathbb{C}^{n}$:

(12.14) $$H_J(w,w') = \sum_j z_j(w)\overline{z_j(w')}$$

Proof: By definition,

$$z_j(w) = \langle w, f_j + i\, e_j\rangle$$

Thus,

$$z_j(Jw) = \langle Jw,\ f_j + i\ e_j \rangle = \langle w,\ -Jf_j - i\ Je_j \rangle$$
$$= \langle w,\ -e_j + i\ f_j \rangle = i \langle w,\ f_j + i\ e_j \rangle = i\ z_j(w),$$

proving (12.13). From (12.12) and (3.8) we compute

(12.15) $$w = \Sigma\, x_j e_j - y_j f_j \qquad Jw = -\ \Sigma\, y_j e_j - x_j f_j$$

Thus

$$\langle Jw,\ w' \rangle = -\langle \Sigma y_j e_j + x_j f_j,\ \Sigma x_j' e_j - y_j' f_j \rangle$$
$$= \Sigma\, y_j y_j' + x_j x_j'$$

and

$$\langle w, w' \rangle = \langle \Sigma\, x_j e_j - y_j f_j,\ \Sigma\, x_j' e_j - y_j' f_j \rangle$$
$$= \Sigma\, y_j x_j' - x_j y_j'$$

On the other hand,

$$\Sigma\, z_j \bar{z}_j' = \Sigma\, (x_j + i\ y_j)(x_j' - i\ y_j')$$
$$= \Sigma\, x_j x_j' + y_j y_j' + i(\ \Sigma\, y_j x_j' - x_j y_j')$$

Comparing these formulas with (12.3), we see (12.14) is true.

Conversely, if $\{e_j, f_j\}$ is any symplectic basis for W, then we may put

$$Je_j = -f_j \qquad Jf_j = e_j$$

and J will define a compatible positive complex structure on W. The above discussion shows all positive compatible complex structures arise in this way. Observe that Sp acts naturally on the set of positive

compatible complex structures and on the set of maximal compact subgroups by conjugation, and on the set of positive maximal isotropic totally complex subspaces of $W_{\mathbb{C}}$ via its action on $W_{\mathbb{C}}$; and observe further that the correspondences of proposition 12.2 are equivariant for these actions. The coordinatization given above shows

Proposition 12.4: The natural actions of Sp on each of the sets in proposition 12.2 is transitive; thus these sets are each Sp-equivariantly homeomorphic to Sp/K where K is a maximal compact subgroup of Sp.

The conditions of total complexity and positivity clearly define an open subset $\Omega^+(W_{\mathbb{C}})$ of the set $\Omega(W_{\mathbb{C}})$ of maximal isotropic subspaces of $W_{\mathbb{C}}$, so that $Sp/K \simeq \Omega^+(W_{\mathbb{C}})$ can be considered a complex manifold in a natural way on which Sp acts by holomorphic transformations. We will holomorphically embed $\Omega^+(W_{\mathbb{C}})$ as an open subset of a complex vector space. Indeed, let (X',Y') be any complete polarization of $W_{\mathbb{C}}$ such that $V \cap Y = \{0\}$ for any $V \in \Omega^+(W_{\mathbb{C}})$. Then according to I, proposition 5.3, the set $\Omega^+(W_{\mathbb{C}})$ will be parametrized by a (necessarily open) subset of the space $S^{2*}(X')$ of symmetric (complex) bilinear forms on X'. For Y it is evidently sufficient (and necessary, it may be seen), that the Hermitian symmetric form $2iKy$, defined in (12.6), be negative semi-definite. If $2iKy$ is negative definite (so $\overline{Y} \in \Omega^+(W_{\mathbb{C}})$); then the image of $\Omega^+(W_{\mathbb{C}})$ is bounded. Otherwise, it is unbounded.

We will take $X' = X_{\mathbb{C}}$, and $Y' = Y_{\mathbb{C}}$ where (X,Y) is a complete polarization of W. Then $2iKy$ is trivial, hence negative semi-definite (This shows that positivity of $V \in \Omega(W_{\mathbb{C}})$ implies V must be totally complex. In general, $(Y \cap W)_{\mathbb{C}}$ is the radical of $2iKy$). We may identify a complex-bilinear form on $X_{\mathbb{C}}$ with a complex-valued real-bilinear form on X.

Dividing this into its real and imaginary parts, we obtain two real-valued forms. Concretely, given $V \in \Omega^+(W_C)$, and $x \in X$, let

$$y_1 = T_1(x) \quad \text{and} \quad y_2 = T_2(x) \tag{12.16}$$

be such that $x + y_1 + i\, y_2$ is in V. Then for $x, x' \in X$, following I, proposition 5.3, define symmetric bilinear forms B_1 and B_2 on X by

$$B_1(x,x') = \langle x, T_1(x')\rangle \qquad B_2(x,x') = \langle x, T_2(x')\rangle \tag{12.17}$$

Then

$$\beta : V \to B_1 + i\, B_2 \tag{12.18}$$

is the map in question.

Proposition 12.5: The map (12.18) embeds $\Omega^+(W_C)$ as the subset of $S^{2*}(W_C)$, complex-valued symmetric real-bilinear forms on X, with positive definite imaginary part.

Proof: Take x, x' in X, and $y = (T_1 + i\, T_2)(x)$ and $y' = (T_1 + i\, T_2)(x')$ in Y_C, so that $z = x + i\, x' + y + iy'$ is a typical element of V. The condition that V be positive is that $2i \langle z,\bar{z}\rangle > 0$. We compute

$$\begin{aligned}
2i\langle z,\bar{z}\rangle &= 2i\langle x+ix'+y+iy', x-ix'+\bar{y}-i\bar{y}\rangle \\
&= 2i(\langle x,\bar{y}\rangle - i\langle x,\bar{y}'\rangle + i\langle x',\bar{y}\rangle + \langle x',\bar{y}'\rangle \\
&\qquad + \langle y,x\rangle + i\langle y',x\rangle - i\langle y,x'\rangle + \langle y',x'\rangle \\
&= 2i(\langle x,\bar{y}-y\rangle - i\langle x,\bar{y}'+y'\rangle + i\langle x',\bar{y}+y\rangle + \langle x',\bar{y}'-y'\rangle \\
&= 4(\langle x,y_2\rangle + \langle x,y_1'\rangle - \langle x',y_1\rangle + \langle x',y_2'\rangle \\
&= 4(B_2(x,x) + B_1(x,x') - B_1(x',x) + B_2(x',x') \\
\text{(by symmetry of } B_1) \quad &= 4(B_2(x,x) + B_2(x',x'))
\end{aligned}$$

Hence we see that V is positive if and only if B_2 is positive definite, as claimed.

Remark: The formation of $W_{\mathbb{C}}$ from W over $\mathbb{R}$ is analogous to the formation of $\tilde{W}$, the double of W, over any field. In fact, if one writes

$$(W_{\mathbb{C}})_{\mathbb{R}} = W \oplus i\, W$$

as a sum of real spaces, and if one takes the as form on $(W_{\mathbb{C}})_{\mathbb{R}}$ real part of the complex bilinear extension of $\langle\ ,\ \rangle$ to $W_{\mathbb{C}}$, then $(W_{\mathbb{C}})_{\mathbb{R}}$ is isometric to $\tilde{W}$, with $i\, W$ playing the role of W^-.