Part II. Analytic Preliminaries

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0: Introduction

In this chapter we will be working over a self-dual base field F. By a self-dual field, we mean a locally compact topological field whose additive group F^+ is self-dual in the sense of harmonic analysis, that is, F^+ is isomorphic to its Pontrjagin dual $\hat{F}^+$. In [ ] it is shown that a non-discrete locally compact field is a local field in the sense of algebraic number theory, that is, either: $\mathbb{R}$; $\mathbb{C}$; a finite extension of $\mathbb{Q}_p$, the p-adic numbers; or $\mathbb{F}_q((t))$, the field of formal Laurent series over a finite field. Since the Pontrjagin dual of a discrete group is compact and vice versa, if F is discrete and self-dual, it must also be compact, thus finite. Hence a self-dual field is either a local field or a finite field.

We will always assume also that the characteristic of F is not 2. Sometimes we will also need to assume the residual characteristic of F is not 2, but we will note this explicitly when the occasion arises.

Let V be a vector space over F, with vector space dual $V^* = Hom(V,F)$. Since V is locally compact, V has also a Pontrjagin dual $\hat{V}$. Following [Wi] we note that $\hat{V}$ has naturally a structure of vector space over F. Indeed, for $\psi \in \hat{V}$, $t \in F$, we may define

$$t\,\psi(v) = \psi(tv), \tag{0.1}$$

which gives a scalar multiplication of F on $\hat{V}$ making $\hat{V}$ into a vector space over F. We may further define a morphism

$$\alpha : V^* \otimes \hat{F} \to \hat{V}$$

by

$$\alpha(v^* \otimes \psi) = \psi(v^*(v)) \tag{0.2}$$

As shown in [Wi] the map α is an isomorphism. This essentially amounts to the fact that $\hat{F}$ is one-dimensional over F. Thus when F is self-dual, we may realize the isomorphism from F to $\hat{F}$ by picking any non-trivial $\chi_0 \in \hat{F}$ and sending t to $t\chi_0$. The chosen χ_0 is called a <u>basic character</u> of F. Choice of the basic character χ_0 allows one to define an isomorphism

$$\alpha_{\chi_0} = \alpha : V^* \xrightarrow{\sim} \hat{V}$$

by

$$\alpha_{\chi_0}(v^*) = \alpha(v^* \otimes \chi_0), \text{ or } \alpha_{\chi_0}(v^*)(v) = \chi_0(v^*(v)) \tag{0.3}$$

When F is a local non-Archimedean field, we will always assume that the basic character χ_0 is trivial on the integers R of F, but not on $\pi^{-1}R$, where π is a prime of F. When $F = \mathbb{R}$, we take $\chi_0(t) = e^{2\pi i t}$. When $F = \mathbb{C}$, we take $\chi_0(z) = e^{2\pi i(z+\bar{z})}$.

If W is a symplectic vector space over F, then W is isomorphic to W^* via the isomorphism ( ). Combining this with (0.3), we get an isomorphism, again denoted α_{χ_0}, from W to $\hat{W}$.

$$\alpha_{\chi_0} : W \xrightarrow{\sim} \hat{W}; \quad \alpha_{\chi_0}(w)(w') = \chi_0(\langle w', w \rangle). \tag{0.4}$$

Suppose $X \subseteq W$ is a subgroup. Set

$$X^{\perp} = \{y \in W : \langle x, y \rangle \in \ker \chi_0, \text{ all } x \in X\} \tag{0.5}$$

Then it is obvious from (0.4) that $\alpha_{\chi_0}(X^{\perp})$ is the annihilator of X in the sense of the duality theory of locally compact abelian groups. By a basic result [ ] of that theory, we may factor α_{χ_0} to obtain an isomorphism

$$\alpha_{\chi_0} : W/X^{\perp} \xrightarrow{\sim} \hat{X} \tag{0.6}$$

Suppose F is a non-Archimedean local field, with integers R and prime π. If $X \subseteq W$ is an R-module, then $\langle X, y \rangle$ is an ideal in F for any $y \in W$. Thus by our normalization of χ_0 assumed above, we see y belongs to $X^{\perp}$ as defined in (0.5) if and only if $\langle X, y \rangle \subseteq R$. Thus for R-submodules of W, the $X^{\perp}$ given by (0.5) coincides with the $X^{\perp}$ defined in , so the notation should cause no confusion.

If $F = R$ and χ_0 is as specified above, then $X^{\perp}$ may also be described as

$$X^{\perp} = \{y \in W : \langle x, y \rangle \in \mathbb{Z}, \text{ all } x \in X\}.$$

Let V be a vector space over F. As a locally compact group, V has a Haar measure, unique up to multiples. This Haar measure on V will usually be denoted by dv but may be denoted otherwise as specified if necessary. Let $L^p(V)$ denote the space of (equivalence classes of) measurable functions on V whose absolute p-th powers are integrable. Here p is a real number ≥ 1 (usually 1 or 2). If $F = \mathbb{R}$ or $\mathbb{C}$, let $C^{\infty}(V)$ be the space of all functions on V which are smooth in the usual sense. For other F, we let $C^{\infty}(V)$ be the space of locally constant functions on V. We let $S(V)$ be the Schwartz-Bruhat space on V. If $F = \mathbb{R}$ or $\mathbb{C}$, then V has the structure of real vector space, and $S(V)$ is the usual Schwartz space of rapidly decreasing functions. That is, $S(V)$

is the space of smooth functions f such that

$$\sup\{Df(v) : v \in V\} < \infty \tag{0.7}$$

for all polynomial coefficient differential operators on V (considered as real vector space). If F is not connected, then $\mathcal{S}(V)$ is just the locally constant functions of compact support. We clearly have

$$\mathcal{S}(V) \subseteq L^p(V) \quad \text{all } p \geq 1, \text{ and } \quad \mathcal{S}(V) \subseteq C^\infty(V). \tag{0.8}$$

The space $\mathcal{S}(V)$ has a natural locally convex topology. For $F = \mathbb{R}$ or $\mathbb{C}$, this is the topology given by the semi-norms (0.7), and is a Frechet topology. Otherwise, $\mathcal{S}(V)$ is the inductive limit of the finite dimensional subspaces spanned by characteristic functions of finite collections of open compact subsets of V. In all cases $\mathcal{S}(V)$ is nuclear in the sense of [ ] and reflexive. By a <u>distribution</u> on V we mean a continuous linear functional on $\mathcal{S}(V)$, that is, an element of $\mathcal{S}^*(V)$, the dual of $\mathcal{S}(V)$. For F non-Archimedean, all linear functionals $\mathcal{S}(V)$ are continuous.

By the <u>support</u> of a function or distribution, we mean the usual thing. If $f \in C^\infty(V)$, then <u>supp f</u>, the support of f, is the closure of the set where f takes on non-zero values. If $\lambda \in \mathcal{S}^*(V)$, then

$$\begin{aligned} \operatorname{supp} \lambda = \{v \in V :\ & \text{for any } f \in \mathcal{S}(V) \text{ with } f(v) \neq 0, \\ & \text{there is } g \in \mathcal{S}(V) \text{ such that } \lambda(fg) \neq 0\}. \end{aligned} \tag{0.9}$$

Let U and V be two vector spaces over F. Since $\mathcal{S}(U)$ and $\mathcal{S}(V)$ are nuclear spaces the topological tensor product $\mathcal{S}(U) \hat{\otimes} \mathcal{S}(V)$ is well defined [ ]. (For F disconnected, this coincides with the algebraic tensor product). We may define a bilinear map

$$\alpha : \mathcal{S}(U) \times \mathcal{S}(V) \to \mathcal{S}(U \oplus V)$$

by the formula

$$\alpha(f \times g)(u,v) = f(u)g(v) \tag{0.10}$$

It is well known [ ] that α then extends to a topological isomorphism

$$\hat{\alpha} : \mathcal{S}(U) \hat{\otimes} \mathcal{S}(V) \xrightarrow{\sim} \mathcal{S}(U \oplus V). \tag{0.11}$$

1: Stone-vonNeumann Theorem; oscillator representation

Let W be a symplectic vector space over F, with form $\langle\ ,\ \rangle$. We will be dealing with $H(W) = H$, the Heisenberg group of W, as defined in I §3. We will frequently use the notations and formulas of that section, in particular (1.3.7), (1.3.10), (1.3.11), and (1.3.12). The center of H is Z and is isomorphic with F by the parametrization (1.1.3).

We will be interested in the representation theory of $H(W)$. This has been studied by numerous authors and many of the basic facts are known. References particularly germane to the present discussion are [Ca], [Ho], [Ig], [Sg], and [Wi]. In the first few sections much of the exposition will be a summary of facts from these papers.

In terms of the coordinates of (1.3.7) the Haar measure dh on H is just the product of Haar measures dw on W and dz on Z:

(1.1) $$dh = dwdz, \quad \text{where} \quad w = \pi(h) \quad \text{and} \quad z = z(h).$$

We let $L^p(H)$ be the usual L^p-space with respect to Haar measure on H. According to (1.1) the coordinate mapping

$$\pi \times z : H \to W \oplus Z$$

induces an isomorphism of $L^p(H)$ with $L^p(W \oplus Z)$. We define $C^\infty(H)$ and $\mathcal{S}(H)$ also to be the pullbacks by $\pi \times z$ of the spaces $C^\infty(W \oplus Z)$ and $\mathcal{S}(W \oplus Z)$ respectively. Since $\mathcal{S}(H) \subseteq L^1(H)$, we can convolve 2 functions in $\mathcal{S}(H)$, and the result is in $\mathcal{S}(H)$ again (this is trivial for disconnected F; for $\mathbb{R}$ or $\mathbb{C}$, c.f. the formulas in [ ]), so $\mathcal{S}(H)$ is an algebra

under convolution, by the usual formula

(1.2) $$f*f'(h) = \int_H f(h_1)f'(h_1^{-1}h)dh_1 \qquad f,f' \in \mathcal{S}(H).$$

Given a non-trivial $\chi \in \hat{Z}$ and a function f on H, we will say f <u>transforms by χ</u> under Z if

(1.3) $$f(z^{-1}h) = \chi(z)f(h) \qquad z \in Z,\ h \in H.$$

For a function f on H, we define $s^*(f)$ on W by restricting f to the canonical cross-section

(1.4) $$s^*(f) = f \circ s \quad \text{or} \quad s^*(f)(w) = f(w,0)$$

Clearly if f transforms by χ under Z, then $s(f)$ determines f. We let $\mathcal{S}(H,\chi)$ be the space of functions f on H such that f transforms by χ under Z and $s^*(f) \in \mathcal{S}(W)$. We define $L^p(H,\chi)$ and $C^\infty(H,\chi)$ similarly. Each of these spaces is isomorphic via s^* to the corresponding space on W. If $f \in \mathcal{S}(H)$, then we may define

(1.5) $$\pi_\chi(f) = \int_Z f(zh)\,\chi(z)dz$$

It is easy to check that $\pi_\chi(f) \in \mathcal{S}(H,\chi)$, and furthermore that π_χ is a surjection from $\mathcal{S}(H)$ to $\mathcal{S}(H,\chi)$. We may convolve $\mathcal{S}(H)$ with $\mathcal{S}(H,\chi)$. It is easy to verify that

(1.6) $$\pi_\chi(f*f') = f * \pi_\chi(f') = \pi_\chi(f) * f'$$

Thus $\ker \pi_\chi$ is a two-sided ideal in $\mathcal{S}(H)$, so that $\mathcal{S}(H,\chi)$ is a quotient algebra of $\mathcal{S}(H)$. We may transfer the convolution structure from $\mathcal{S}(H,\chi)$ to $\mathcal{S}(W)$ by means of s^*. Thus, for $\varphi, \varphi' \in (W)$, define

$$\varphi \# \varphi' = s^*(s^{*-1}(\varphi) * s^{*-1}(\varphi')) \tag{1.7}$$

where s^* here is understood to be the isomorphism from $\mathcal{S}(H,X)$ to $\mathcal{S}(W)$ defined by (1.4). One may compute $\varphi \# \varphi'$ directly on W. The result is

$$\varphi \# \varphi'(w) = \int_W \varphi(w_1)\, \varphi'(w-w_1)\, X(\tfrac{1}{2} < w_1, w >) dw_1 \tag{1.8}$$

Similar considerations hold for $L^1(H)$ and $L^1(H,X)$ and other appropriate combinations of spaces.

Let ρ be a unitary representation of H on a Hilbert space $\mathcal{Y}$. Recall that a vector x in $\mathcal{Y}$ is called a smooth vector for ρ if the matrix coefficient

$$\varphi_{x,y}(h) = (x, \rho(h)y) \tag{1.9}$$

where $(\, , \,)$ indicates the inner product on $\mathcal{Y}$ is a smooth function on H for any y in $\mathcal{Y}$. For disconnected F (that is except for $F = \mathbb{R}$ or $\mathbb{C}$), a vector x will be smooth if and only if it is left invariant by an open compact subgroup of F. The smooth vectors form a dense, H-invariant, subspace of $\mathcal{Y}$, denoted $\mathcal{Y}^\infty$. The restriction of ρ to $\mathcal{Y}^\infty$ is written ρ^∞. A bounded operator on $\mathcal{Y}$ is called smooth if it maps $\mathcal{Y}$ into $\mathcal{Y}^\infty$.

If ρ is an irreducible representation of H, then necessarily $\rho(z) = X(z)I$, for $z \in Z$ where I is the identity operator on $\mathcal{Y}$, and X is in $\hat{Z}$. We call X the central character of ρ. Recall the usual way of "integrating" ρ to obtain a representation, also denoted ρ, of the convolution algebra $\mathcal{S}(H)$. One puts

$$\rho(f) = \int_H f(h)\, \rho(h)\, dh \qquad f \in \mathcal{S}(H). \tag{1.10}$$

The operator $\rho(f)$ is a smooth operator for $f \in \mathcal{S}(H)$. If ρ is irreducible with central character X, then it is easy to check that ρ factors through π_X, so that ρ may be regarded as a representation of $\mathcal{S}(H,X)$. We may therefore also regard ρ as a representation of $\mathcal{S}(W)$ after identifying $\mathcal{S}(W)$ with $\mathcal{S}(H,X)$ by means of s^*. The explicit formula is

$$\rho(\varphi) = \int_W \varphi(w)\, \rho((w,0)) dw \tag{1.11}$$

The formula (1.11) defines an algebra homomorphism to the space $\mathcal{L}(\mathcal{Y})$ of bounded operators on $\mathcal{Y}$ from $\mathcal{S}(W)$ with the convolution structure $\#$ defined by (1.8).

The main fact about representations of H is the Stone-vonNeumann Theorem, which asserts that an irreducible representation is determined by its central character if this is non-trivial.

Theorem 1.1: Let ρ be an irreducible unitary representation of H with central character X on a Hilbert space $\mathcal{Y}$. If X is non-trivial, then ρ is determined up to unitary equivalence by X. Moreover, the map

$$\rho : \mathcal{S}(W) \to \mathcal{L}(\mathcal{Y})$$

is an isomorphism of $\mathcal{S}(W)$ onto the algebra of smooth operators on $\mathcal{Y}$. Furthermore, if Haar measure on H is properly normalized, ρ extends to an isometric isomorphism from $L^2(W) \simeq L^2(H,X)$ to $\mathcal{H}\mathcal{S}(\mathcal{Y})$, the algebra of Hilbert-Schmidt operators on $\mathcal{Y}$ (with the Hilbert-Schmidt norm). Thus $\mathcal{S}(W)$ acts algebraically irreducibly on $\mathcal{Y}^\infty$.

Given $X \in \hat{Z}$, we will write ρ_X for the equivalence class of irreducible unitary representations with central character χ . We will let ρ_χ^∞ denote the corresponding smooth subrepresentation. Sometimes by abuse of notation, ρ_χ or ρ_χ^∞ will stand for a particular representative of the equivalence class. The space of a typical realization (as opposed to the particular explicit realizations to be discussed in §2) of ρ_χ will be written $_\chi$.

Consider the action of $Sp(W)$ on $H(W)$ by automorphisms, as given by Part I. Then Sp also acts on the representations of H by the formula

$$(1.12) \qquad Ad^* g(\rho)(h) = \rho(g^{-1}(h)) \qquad g \in Sp,\ h \in H.$$

Clearly if ρ is irreducible, the transform $Ad^* g(\rho)$ will also be irreducible. Take $\rho = \rho_\chi$ for some $\chi \in \hat{Z}$. Since Sp acts trivially on $\hat{Z}$, the representation $Ad^* g(\rho_\chi)$ will have central character χ , for any $g \in Sp$, we see by the Stone-vonNeumann Theorem that $Ad^* g(\rho_\chi)$ is unitarily equivalent to ρ_χ . Hence there is a unitary operator $\omega_\chi(g)$ on $\mathcal{Y}_\chi$ intertwining ρ_χ and $Ad^* g(\rho_\chi)$. Concretely this means

$$(1.13) \qquad \omega_\chi(g)\ \rho_\chi(h)\ \omega_\chi(g)^{-1} = \rho_\chi(g(h)) \qquad g \in Sp,\ h \in H .$$

It is clear that equation (1.13) can determine $\omega_\chi(g)$ at most up to scalar multiples. Conversely, since ρ_χ is irreducible, equation (1.13) does determine $\omega_\chi(g)$ up to multiples by Schur's lemma [ ]. Therefore one sees immediately from (1.13) that ω_χ is a homomorphism "up to multiples", that is,

$$(1.14) \qquad \omega_\chi(g_1)\ \omega_\chi(g_2) = \mu(g_1,g_2)\ \omega_\chi(g_1 g_2)$$

where $\mu(g_1,g_2)$ is some scalar of absolute value 1. Equation (1.14) is

sometimes expressed by saying ω_χ is a <u>multiplier representation</u> of Sp, with multiplier μ . Alternatively, we note that if we consider the action of operators on $\mathbb{P}\,\mathcal{Y}_\chi$, the projective space of $\mathcal{Y}_\chi$, then scalar operators act trivially, so that (1.14) says ω_χ defines an action of G on $\mathbb{P}\,\mathcal{Y}_\chi$. Therefore ω_χ is also called a <u>projective representation</u>. We note that since $\mathcal{Y}_\chi^\infty$ is defined in terms of the structure of $\mathcal{Y}_\chi$ as H-module, it will be preserved by $\omega_\chi(g)$. Thus we also have a well defined action of Sp on $\mathbb{P}\,\mathcal{Y}_\chi$. We will denote this by ω_χ^∞.

For some purposes, it is convenient to have an action of a group on $\mathcal{Y}_\chi$, not only on $\mathbb{P}\,\mathcal{Y}_\chi$. It may not be possible to get Sp (indeed, it is not in this case) to act on $\mathcal{Y}_\chi$, but a central extension of Sp will act. We recall the details of this.

Let $\mathcal{U}$ be the unitary group of $\mathcal{Y}_\chi$. In the relative strong operator topology $\mathcal{U}$ is a complete metrizable group. Recall from §3 that we call $\exp \mathcal{W} = \{(w,0)\} \subseteq H$ the canonical cross-section to Z. It follows directly from Theorem 1.1 that $\rho_\chi(\exp \mathcal{W})$ is a closed subset of $\mathcal{U}$. Let $\mathcal{B} \subseteq \mathcal{U}$ be the normalizer of $\rho_\chi(\exp \mathcal{W})$ in $\mathcal{U}$. Then clearly $\mathcal{B}$ is a closed subgroup of $\mathcal{U}$. Since ρ_χ is irreducible the centralizer of $\rho_\chi(\exp \mathcal{W})$, the subgroup of $\mathcal{B}$ which acts trivially on $\rho_\chi(\exp \mathcal{W})$ via conjugation, reduces to $\mathbb{T}$, the circle group of scalar operators in $\mathcal{U}$. Conjugating by ρ_χ exp, we get a homomorphism

$$(1.15) \qquad \beta : \mathcal{B} \to Aut(\mathcal{W})$$

where $Aut(\mathcal{W})$ is the group of automorphisms of $\mathcal{W}$ considered as an abelian topological group. Let $\mathcal{S}p \subseteq \mathcal{B}$ be the inverse image by β of

$Sp = Sp(\mathcal{W})$ (We identify $Sp(W)$ and $Sp(\mathcal{W})$ via exp.) Then Sp is a closed subgroup of $\mathcal{B}$, hence of $\mathcal{U}$. Equation (1.13) says β maps Sp onto Sp. Therefore $Sp \simeq \mathit{Sp}/\mathbb{T}$, so Sp is locally compact and we have a central extension of locally compact groups

$$(1.16) \qquad 1 \to \mathbb{T} \to \mathit{Sp} \xrightarrow{\beta} Sp \to 1.$$

By virtue of its definition, there is a unitary representation, which we shall also denote ω_χ, of Sp on $\mathcal{V}_\chi$, such that the multiplier representation ω_χ of Sp defined above in (1.13) arises by choosing a set-theor tic cross-section to β. When dealing with Sp, equation (1.13) should be amended to read

$$(1.17) \qquad \omega_\chi(g) \quad \rho_\chi(h) \quad \omega_\chi(g)^{-1} = \rho_\chi(\beta(g)(h)) \qquad g \in \mathit{Sp}, \ h \in H$$

We shall refer to ω_χ, regarded either as a representation of Sp or a projective representation of Sp, as the <u>oscillator representation.</u>

Remark: Since, except when $\dim W = 2$ and F has 3 elements, Sp is its own commutator subgroup, the commutator subgroup $\mathit{Sp}^{(2)}$ of Sp would also be mapped surjectively by β onto Sp. Shale [ ] and Weil [ ] have shown that, via β, the group $\mathit{Sp}^{(2)}$ is at most a 2-fold cover of Sp. (For finite fields and for $\mathbb{C}$, the groups $\mathit{Sp}^{(2)}$ and Sp are isomorphic.) Thus we could make Sp smaller if we wished, and this can be important, c.f. [H-P]. However, for many purposes, including the present ones, it is satisfactory to work with Sp'.

We shall study the dependence of ω_χ on χ. Consider the effect of automorphisms which preserve the canonical cross-section but instead of being trivial on Z, act as multiplication by an arbitrary scalar. If γ is such an automorphism, let $c(\gamma)$ be the scalar by which γ acts on Z.

According to I, proposition 3.1, the automorphism induced by γ on $W \simeq H/Z$ is F-linear and multiples the form $\langle\ ,\ \rangle$ by $c(\gamma)$. From Stone-vonNeumann Theorem 1.1, it is clear that

$$(1.18) \qquad Ad^* \gamma(\rho_\chi) = \rho_{c(\gamma)^{-1}\chi}$$

Since γ normalizes Sp in $Aut(H)$, it is also clear that it will also act on the projective representations ω_χ of Sp associated to the ρ_χ. From (1.13) it is immediate that

$$(1.19) \qquad Ad^* \gamma(\omega_\chi) = \omega_{c(\gamma)^{-1}\chi}$$

It should be remarked, however, that our reasoning only shows (1.19) is valid as a relation between projective representations of Sp, not between actual representations of Sp. Indeed, it is not even clear <u>a priori</u> that γ lifts to an automorphism of Sp, or what is the same, that Sp is independent of χ as a central extension of Sp. The finer analysis of Weil [ ], however, shows that these things are true.

In the case that $c(\gamma)$ is a square in $F^\times$, say $c(\gamma) = r^2$, then, as elements of $End(W)$ we can write $\gamma = rg$, with $g \in Sp$. Since we are only interested in γ modulo Sp, we may as well take γ simply to be multiplication by r. Then it is clear that if we form the semidirect product

$$\Gamma = \mathit{Sp} \times_s H$$

where Sp acts on H through the map β of (1.16), then γ will act as an automorphism of Γ acting trivially on Sp. Combining this observation with (1.18), we find that

(1.20) $$\omega_{r^2\chi} \simeq \omega_\chi$$

Equation (1.20) can be regarded as an equivalence of projective representations of Sp or of actual representations of $\mathcal{Sp}$. According to (1.20) ω_χ depends only on the $F^{\times 2}$ -orbit of χ in $\hat{Z}$. It will be seen later (see formula (3.6)) that different $F^{\times 2}$ orbits do yield different oscillator representations, so that the set of such representations has cardinality $F^\times/F^{\times 2}$. This is 1 if $F = \mathbb{C}$, 2 if $F = \mathbb{R}$ or F_q (a finite field), 4 for local fields of odd residual characteristic, and more for 2-adic fields.

If ρ is a representation on a space $\mathcal{Y}$ let ρ^* denote the contragredient representation. For unitary representations, the contragredient is isomorphic by a conjugate linear map to the original representation. The space $\mathcal{Y}^\infty$ of smooth vectors for ρ has its own natural topology and $\rho^{\infty *}$ is the contragredient action on $\mathcal{Y}^{\infty *}$. Thus $\rho^{*\infty} \neq \rho^{\infty *}$, However $\rho^{*\infty}$ does embed in a natural way in $\rho^{\infty *}$ and under favorable circumstances we can write $\rho^{*\infty} \simeq (\rho^{\infty *})^\infty$.

For unitary characters χ one has $\chi^* = \chi^{-1}$. Hence theorem 1.1 implies that for the irreducible representation ρ_χ of H, one has

(1.21) $$(\rho_\chi)^* = \rho_{\chi^*} = \rho_{\chi^{-1}}$$

From this it is immediate that

(1.22) $$(\omega_\chi)^* = \omega_{\chi^*} = \omega_{\chi^{-1}} .$$

2: Sums and tensor products

In this section, consider relations between the ρ_χ and ω_χ corresponding to the relations between the H(W) for various symplectic spaces W, as elaborated in I §3. Given a symplectic vector space W with form $<\ ,\ >$, recall that W^b denotes the same vector space equipped with the form $b<\ ,\ >$ for $0 \neq b \in F$. Let g be an isometry from W^b to W. That is, g is a similitude of $<\ ,\ >$ such that $<\ ,\ > \circ g = b<\ ,\ >$. Then g induces maps

$$g: H(W^b) \xrightarrow{\sim} H(W)$$

$$\mathrm{Ad}\ g: Sp(W^b) \xrightarrow{\sim} Sp(W) \qquad \mathrm{ad}\ g(g') = gg'g^{-1}.$$

Let Z and Z^b be the centers of H(W) and $H(W^b)$ respectively. We may identify Z and Z^b to F in the usual way. Thus if χ is a character of F, it may be considered as a character of Z and as a character of Z^b. Let ρ_χ and ρ^b_χ be the representations of H(W) and of $H(W^b)$ corresponding to χ via theorem 1.1. Let ω_χ and ω^b_χ be the corresponding oscillator representations of Sp(W) and $Sp(W^b)$. Then equations (1.18) and (1.19) may be reinterpreted in this context to say

(2.1) $$\rho^b_\chi \circ g^{-1} = \rho_{b\chi}$$
$$\omega^b_\chi \circ \mathrm{Ad}\ g^{-1} = \omega_{b\chi}$$

Thus considering ρ_χ and ω_χ for all χ and fixed symplectic space W is the same as considering ρ_χ and ω_χ for fixed χ and letting our symplectic space vary among the W^b for all possible non-zero $b \in F$.

Thus för convenience we will fix X to be the basic character χ_0, as described in §0, and we will drop it from the notation. Thus henceforth

$$\rho = \rho_{\chi_0} \qquad \omega = \omega_{\chi_0} \tag{2.2}$$

We will, however, be considering ρ and ω for different symplectic spaces, and we will then add notation to indicate to what spaces ρ and ω are attached.

Let W_1 and W_2 be two symplectic spaces, and let $W_1 \oplus W_2 = W_3$ be their direct sum. As discussed in I, §3, we have a surjective homomorphism

$$i_1 \times i_2 : H(W_1) \times H(W_2) \to H(W_3)$$

satisfying (1.3.15) and (1.3.16). Let ρ^j, $j = 1, 2, 3$, be the irreducible representation of $H(W_j)$ with central character χ_0. Then we have the relation

$$\rho^3 \circ (i_1 \times i_2) = \rho^1 \otimes \rho^2 \quad \text{(outer tensor product)} \tag{2.3}$$

Indeed, it is obvious that $\rho^3 \circ (i_1 \times i_2)$ is irreducible, so from general theory [ ] it has the form $\sigma_1 \otimes \sigma_2$ where σ_j is an irreducible representation for $H(W_j)$. Since each σ_j clearly has central character χ_0, we have $\sigma_j = \rho^j$ by theorem 1.1, and (2.3) follows.

Corresponding to the embeddings i_j of $H(W_j)$ in $H(W_3)$, we have also embeddings

$$i_j : Sp(W_j) \to Sp(W_3)$$

as in (1.10.13). In view of (2.3), it is clear that

$$\omega^3 \circ (i_1 \times i_2) = \omega^1 \otimes \omega^2 \quad \text{(outer tensor product)} \tag{2.4}$$

where ω^j is the oscillator representation of $Sp(W_j)$ coming from ρ^j.

It is plausible from theorem 1.1, that the analogue of (2.3) for smooth vectors is true. This is the case, and we record the fact here, although the most efficient proof seems to be via an explicit realization of ρ to be given in §3.

$$(\rho^3) \circ (i_1 \times i_2) = (\rho^1)^\infty \hat{\otimes} (\rho^2)^\infty \quad \text{(topological tensor product)} \tag{2.5}$$

To finish this section, we consider the inner tensor product $\omega \otimes \omega^*$. Although ω is only a projective representation of Sp, and must be "lifted" to Sp to be considered as a linear representation, the tensor product $\omega \otimes \omega^*$ will be trivial on the center of Sp, so will factor to give a bonafide representation of Sp. The representation $\omega \otimes \omega^*$ can be thought of in several ways, but one convenient way for present purposes is as the action of conjugation by $\omega(Sp)$ on the Hilbert-Schmidt operators $\mathcal{H.S.}(\mathcal{Y})$, where $\mathcal{Y}$ is the space of ρ. Explicitly

$$\omega \otimes \omega^*(g)\ (T) = \omega(g)\, T\, \omega(g)^{-1} \qquad g \in Sp; \quad T \in \mathcal{H.S.}\ (\mathcal{Y}). \tag{2.6}$$

Comparing (2.6) with (1.13) and invoking theorem 1.1, we find the following result.

Proposition 2.1: The representation $\omega \otimes \omega^*$ of Sp is equivalent to the natural action of Sp on $L^2(W)$. That is, $\omega \otimes \omega^*$ may be realized on $L^2(W)$ by the formula

$$\omega \otimes \omega^*(g)(f)(w) = f(g^{-1}(w)) \qquad g \in Sp,\ w \in W,\ f \in L^2. \tag{2.7}$$

The representation $\omega^{\infty} \otimes \omega^{*\infty}$ is given by (2.7) where f is restricted to $\mathcal{S}(W)$.

The final statement will in fact emerge from the discussion of the next session, where we will give further perspective on equation (2.7).

3: Realizations of ρ

An essential fact about the Heisenberg group is that although there is only one equivalence class ρ of irreducible unitary representations of H with central character χ_0, there are, as already remarked by Cartier, many different ways of realizing ρ concretely. These realizations are for the most part as induced representations, and they fall naturally into families, with the families fitting together into a coherent hierarchy. The structure of these families of realizations is the basic phenomenon of study for invariant theory and many other questions related to the oscillator representation. In this section we will review the various realizations of ρ and prove some basic facts about their structure.

First let us note, as have others, that if B is a subgroup of H, containing Z, and B becomes maximal abelian in $H/\ker \chi$, and if ψ is any character of B whose restriction to Z is χ, then ρ is realized as the induced representation

$$\rho \simeq \operatorname{ind}_B^H \psi \tag{3.1}$$

Indeed, it is obvious that $\operatorname{ind}_B^H \psi = \sigma$ is a representation of H with central character χ_0, so it must be a multiple of ρ, so (3.1) amounts to the assertion that σ is irreducible. To see this, consider σ more closely. The space σ is the space $L^2(H/B; \psi)$ of functions f on H such that

$$f(bh) = \psi(b)f(h) \tag{3.2}$$

so that $|f|$, the absolute value of f, is defined on H/B and satisfies

$$\int_{H/B} |f|^2 \, d\dot{h} < \infty \tag{3.3}$$

The action of H is by right translation:

$$\sigma(h')f(h) = f(hh') \tag{3.4}$$

From formulas (1.1.2) and (1.1.4), we compute

$$\sigma(b)f(h) = \psi(b) \quad \chi_0(< h,b >)f(h) = \mathrm{Ad}^* h^{-1}(\psi)(b)f(h) \tag{3.5}$$

If $\chi_0(< h,b >) = 1$ for all $b \in B$, then by formula (1.1.4) h centralizes B modulo $\ker \chi$, whence $h \in B$ by choice of B. Therefore we see that B acts on $L^2(H/B;\psi)$ by multiplication operators which separate the points of H/B. Since also H permutes transitively the points of H/B, it is not hard to verify that σ is irreducible, hence $\sigma \simeq \rho$. Or we may appeal to a general result of Mackey [ ].

The point from now on, for the most part, is to choose B to be various subgroups of particular interest and look at the particular form assumed by ρ. A partial exception is the Fock model, about which not a great deal will be said here.

A: Schrodinger or split models.

These realizations of ρ are relevant for all (self-dual) F. Let (X,Y) be a complete polarization for W. Then, with e and η as in (1.3.12) and (1.3.14) the group $\eta^{-1}(Y) = e(Y)\cdot Z$ is maximal abelian in H, and also maximal abelian modulo $\ker \chi_0$. Indeed, according to formula (3.5) and the discussion in §0, if $U \subseteq W$ is any subspace, then the centralizer, in H or modulo $\ker X$, of $s(U)\cdot Z$ is $s(U^{\perp})\cdot Z$, this $\perp$ being the usual one taken with respect to $< , >$. Thus we can take $B = s(Y)\cdot Z$ in (3.1). For ψ we can take ψ_0, the extension of χ_0 to B that is trivial on $s(Y)$. Since $s(X)$ is a complement to B in H, the restriction to $s(X)$ of functions in $L^2(H/B;\psi_0)$ identifies this space with $L^2(X)$. Formulas (3.4) and (3.5) specialize to

(3.6) a) $\rho(e(x'))f(x) = f(x + x') \qquad x,x' \in X, \quad f \in L^2(X)$

b) $\rho(e(y))f(x) = \chi_0(< x,y >)f(x) \qquad y \in Y$

In summary:

Proposition 3.1: Given any complete polarization (X,Y) in W, the representation ρ can be realized on $L^2(X)$ via the formula 3.6. The smooth vectors in this realization are precisely $\mathcal{S}(X)$.

This realization of ρ will be called the <u>Schrödinger model</u> of ρ attached to (X,Y).

Proof: Only the last statement needs further justification. For the case of connected fields, of which it suffices to discuss the case $F = \mathbb{R}$, since $F = \mathbb{C}$ may be reduced to it, this has been shown in [ ] among other places. The point here is that the under differentiated version of ρ, the image of $\mathcal{U}(\mathcal{H})$, the universal enveloping algebra of $\mathcal{H}$, consists of all polynomial coefficient differential operators. For the case of disconnected fields, one observes a function f on X will define a smooth vector for ρ if and only if it is invariant by an open subgroup in $s(X)$ and an open subgroup in $s(Y)$. From (3.6) a) one sees that f must be locally constant to be smooth. Then an elementary computation, based on the remarks on duality in §0, (see also [We], p.) shows that to be invariant under $s(L)$ for L a lattice in Y, a function f must be supported on $L^{\perp}$, the dual lattice to L in X, where X is regarded as the dual to Y by means of $< , >$. Thus f must be locally constant and of compact support to be smooth. The converse is even easier, so the equality $L^2(X)^\infty = \mathcal{S}(X)$ is seen in the disconnected case also.

From proposition 3.1 the equation (2.5) follows. For given symplectic spaces W_1 and W_2, we may choose complete polarizations (X_i, Y_i) of the W_i, and then realize ρ^3 (notation as in (2.5)) according to proposition 3.1 using the complete polarization $(X_1 \oplus X_2, Y_1 \oplus Y_2)$ of $W_3 = W_1 \oplus W_2$. The smooth vectors $(\rho^3)^\infty$ are then equal to $\mathcal{S}(X_1 \oplus X_2)$, so (2.5) is a consequence of (0.11).

Let $P(Y) = P$ be the subgroup of $Sp = Sp(W)$ preserving Y. The group P has been sdescribed in I §5. Let $M = P(X) \cap P(Y)$, and $N \subseteq P$ be the subgroup acting trivially on W/Y. We know that by restriction to X, the group M is identified to $GL(X)$. Form the semi-direct product $P \underset{s}{\times} H$. This contains the group $P \underset{s}{\times} \eta^{-1}(Y)$ as a subgroup. The action of P on $\eta^{-1}(Y)$ preserves $s(Y)$ and leaves Z fixed pointwise. Thus the character ψ_0 of $\eta^{-1}(Y)$ is invariant by Ad^*P. Thus we may extend ψ_0 to a character ψ_0' of $P \rtimes \eta^{-1}(Y)$ by letting ψ_0' be trivial on P. Consider the induced representation

$$\operatorname{ind}_{P \rtimes \eta^{-1}(Y)}^{P \rtimes H} \psi_0' = \sigma^Y = \sigma$$

Here we mean normalized induction, so σ is unitary. Since $s(X)$ is a complement to $P \rtimes \eta^{-1}(Y)$ in $P \rtimes H$, we see

$$\sigma^Y|H = \rho \tag{3.7}$$

Also the space of σ may be taken to be $L^2(X)$. Thus σ is an extension of ρ to $P \rtimes H$. Hence the restriction of σ to P must be a lifting of the restriction of the projective representation ω to P. We can easily make explicit the action of P on $L^2(X)$. We separate out the

resulting formulas.

Proposition 3.2: The projective representation ω of Sp when restricted to $P(Y)$ may be lifted to a linear representation σ of P. In the Schrodinger model of ρ attached to (X,Y), the action of P under σ is given as follows,

(3.8)
a) $\sigma(m)f(x) = |\det_X(m)|^{-\frac{1}{2}} f(m^{-1}(x)) \qquad m \in M,\ x \in X$

b) $\sigma(n)f(x) = \chi_0(-\frac{1}{2}\ \beta_n(x,x))f(x) \qquad n \in N$

Here $\det_X(m)$ is the determinant of m considered as an element of $GL(X)$, and $|\ |$ indicates the standard absolute value on F (see [We], p.). The bilinear form β_n is given by (1.5.5).

Remarks: a) One can see from formula (3.8)b) that the oscillator representations corresponding to characters of F not in the same $F^{\times 2}$ orbit are distinct, because their N-spectra are distinct.

b) One can also give a formula for ω on elements of Sp which interchange X and Y, as a sort of Fourier transform [ ], [ ]. Since such elements together with P generate Sp this permits one to compute an explicit multiplier of the type (1.15) for ω.

c) It is not the case over local fields not equal to $\mathbb{C}$ that one can lift P so as to belong to the commutator subgroup of Sp. Thus the cocycle defining ω is trivial on P if thought of as having values in $\mathbb{T}$, the circle, but not if thought of as having values in $\{\pm 1\}$. Also, over a finite field, where ω can be made an actual representation in a unique way, the representation σ given by (3.8) differs from $\omega|P$ by the unique character of P of order 2.

It is important to realize that Theorem 1.1 leads to a particularly natural Schrodinger model for the representation $\rho \otimes \rho^*$ of $H \times H$. Let Y be the space of some realization of ρ. Then $\rho \otimes \rho^*$ may be realized on $H.S.(Y)$ by the recipe

(3.9) $$\rho \otimes \rho^*(h,h')(T) = \rho(h)\, T\, \rho(h'^{-1}) \qquad h,h' \in H,\ T \in H.S.$$

Theorem 1.1 allows us to transfer the action of $\rho \otimes \rho^*$ to $L^2(W)$. We can compute the action explicitly via (1.9) and (1.3.7). We find

(3.10) $$\rho \otimes \rho^*(e(w_1),e(w_2))f(w) = \chi_0(\tfrac{1}{2}(< w,w_1+w_2 > + < w_2,w_1 >)f(w-w_1+w_2)$$

In particular,

(3.11) $$\rho \otimes \rho^*(e(w_1),e(w_1))\, f(w) = \chi_0(< w,w_1 >)f(w)$$
$$\rho \otimes \rho^*(e(-w_1),\ e(w_1))f(w) = f(w + 2w_1)$$

Recall that W^- denotes the symplectic space whose underlying vector space is W and whose symplectic form is the negative of $<,>$. Put $H^- = H(W^-)$. Both H and H^- have as underlying set the space $W \oplus F$. We may regard the set-theoretic map

$$\nu:(w,t) \to (w,\ -t) \qquad w \in W,\ t \in F$$

as defining a map from H to H^- and if we do so it is easy to verify from the group law (1.3.7) that ν is a group isomorphism. If ρ^- denotes the representation of H^- with central character χ_0, then it is clear that

(3.12) $$\rho^- \circ \nu = \rho^*$$

Recall the double $\tilde{W} = W^+ \oplus W^-$ of W, as defined in I, §10. Set $\tilde{H} = H(\tilde{W})$ and let $\tilde{\rho}$ be the representation of $\tilde{H}$ with central character χ_0. Let i^+ and i^- be the canonical embeddings of $H = H^+$ and of H^- respectively into $\tilde{H}$, defined following (1.10.3) and proposition 3.1. Combining (3.12) with (2.3) we find

(3.13) $$\tilde{\rho} \circ (i^+ \times i^-)\circ(1 \times \nu) = \rho \otimes \rho^* \quad \text{(outer tensor product)}$$

Let (Δ^+, Δ^-) be the canonical complete polarization of $\tilde{W}$ given by (1.10.5). We may realize $\tilde{\rho}$ on $L^2(\Delta^-)$ following proposition 3.1. A comparison of formulas (3.6) and (3.11) verifies the following statement.

Proposition 3.3: Define a map

$$\delta : L^2(W) \to L^2(\Delta^-)$$

by

(3.14) $$\delta(f)()w,\ -w() = f(2w)$$

Then, with suitable normalization of measures, the map δ defines a unitary intertwining between the realization of $\rho \otimes \rho^*$ on $L^2(W)$ given by formulas (3.10) and (3.11) and the realization of $\rho \otimes \rho^*$ on $L^2(\Delta^-)$ given by combining (3.13) with (3.6).

In connection with proposition 3.3, we note that if $\tilde{P}^+$ is the parabolic subgroup of $Sp(\tilde{W})$ leaving Δ^+ stable, then by means of δ, the action $\tilde{\sigma}_{\Delta^+}$ of $\tilde{P}^+$ on $L^2(\Delta^-)$ can be transferred to $L^2(W)$. Let i^+ and i^- be the embeddings of Sp in $Sp(\tilde{W})$ given by (10.2). The subgroup $i^+ \times i^-(Sp)$ of $Sp(\tilde{W})$ preserves both Δ^+ and Δ^-, so its action under $\tilde{\sigma}$ is given by (3.8)a). We see therefore that δ of (3.14) also intertwines $\tilde{\sigma}\Delta^{+} \circ (i^{+} \times i^{-})$ with the realization of $\omega \otimes \omega^{*}$ on $L^{2}(W)$ given in (2.7). We should also note the supporting fact for outer tensor products: If $\tilde{\omega}$ is the projective representation of $Sp(\tilde{W})$ arising from $\tilde{\rho}$, then we have the following equivalence between projective representation of $Sp \times Sp$

$$\tilde{\omega} \circ (i^{+} \times i^{-}) = \omega \otimes \omega^{*} \qquad \text{(outer tensor product)} \tag{3.15}$$

B: Anisotropic models: Fock model and lattice models.

To discuss other models of ρ we must distinguish among the fields. For finite fields there do not seem to be any other natural models. For the connected fields, of which we consider only $\mathbb{R}$, since the case of $\mathbb{C}$ reduces to that of $\mathbb{R}$, there are <u>Fock models</u>, called <u>complex wave models</u> by Segal [ ], attached to certain maximal isotropic subspaces of the complexification $W_{\mathbb{C}}$ of W. For p-adic fields there are <u>lattice models</u>. There is also a notion of lattice model over $\mathbb{R}$. We shall discuss it briefly for completeness' sake.

We will begin with Fock models. Let W be our symplectic space over $\mathbb{R}$, with form $\langle \, , \, \rangle$. Let $\mathcal{H}(W) = \mathcal{H}$ be the Lie algebra of $H(W)$. From (1.3.10) we have

$$\mathcal{H}(W) = W \oplus R$$

as vector space. In the usual fashion, the representation ρ of H on the space $\mathcal{Y}$ gives an embedding

$$\rho : \mathcal{H} \to \mathrm{End}(\mathcal{Y}^{\infty}) \tag{3.16}$$

which is a homomorphism of Lie algebras. Thus

$$\rho(\mathcal{H}) = \rho(W) \oplus C \tag{3.17}$$

where C in (3.17) indicates the scalar operators on $\mathcal{Y}^{\infty}$. The homomorphism ρ of (3.16) is real linear but since $\mathrm{End}(\mathcal{Y}^{\infty})$ is a complex vector space, one can extend ρ to be a complex linear embedding of $\mathcal{H}_{\mathbb{C}}$ the complexification of $\mathcal{H}$. Thus one has the subspace

$$\rho(W)_{\mathbb{C}} = \rho(W_{\mathbb{C}})$$

as a subspace of $\mathrm{End}(\mathcal{Y}^{\infty})$. As a subspace of $\rho(W_{\mathbb{C}})$, the real points $\rho(W)$ consist of the operators which are skew symmetric with respect to the natural inner product on $\mathcal{Y}_{\infty}$. Therefore complex conjugation on $W_{\mathbb{C}}$ becomes the map

$$T \to -T^{*}$$

on $\rho(W_{\mathbb{C}})$, where T^{*} is the adjoint of T with respect to the inner product on $\mathcal{Y}^{\infty}$.

Let J be a compatible complex structure on W, and let V^{+} and V^{-} be the $+i$ and $-i$ eigenspaces for J in $W_{\mathbb{C}}$. Then $\rho(V^{+})$ is a commutative subspace of $\mathrm{End}(\mathcal{Y}^{\infty})$.

Theorem 3.4: a) Suppose J is a positive complex structure on W. Then there is a unique vector y_J in $\mathcal{Y}_{\infty}$ such that

$$\rho(V^{-})y_J = 0 \tag{3.18}$$

or equivalently

$$\rho(J(w))y_J = i\rho(w)y_J \qquad w \in W. \tag{3.19}$$

Remark: Theorem 3.4 gives rise fairly immediately to a holomorphic (relative to J) embedding of W into $\mathbb{P}\,\mathcal{Y}^{\infty}$ as a complex "subvariety" in fact, an intersection of quadric hypersurfaces. Thus here begins the approach to θ functions via the Heisenberg group suggested by Cartier [ ]. See [ ] and [ ] for further development.

Proof: According to I, proposition 12.3 we can introduce complex coordinates in W so that J becomes multiplication by i and the Hermitian inner product of (1.12.3) is the standard inner product on $\mathbb{C}^n$. Then from theorem 1.7.5 of [ ] (see also [ ] and [ ]) we can realize ρ in the Fock model corresponding to these coordinates. Thus $\mathcal{Y}$ is the space of functions holomorphic on W and square-integrable with respect to the weight $e^{-\pi H_J}\,dw$, where dw is Haar measure on W. The space $\rho(W_{\mathbb{C}})$ is spanned by the operators

$$(3.20)\qquad \frac{\partial}{\partial z_j} \quad \text{and} \quad z_j$$

and adjoints are defined by

$$(3.21)\qquad \left(\frac{\partial}{\partial z_j}\right)^* = \pi\, z_j$$

By abuse of notation we will regard J as acting on $\rho(W_{\mathbb{C}})$. Then by our choice of coordinates

$$(3.22)\qquad J\, z_j = i\, z_j$$

Hence

$$(3.23)\qquad \rho(V^+) = \text{span } \{z_j\}$$

$$\text{and} \qquad \rho(V^-) = \text{span } \left\{\frac{\partial}{\partial z_j}\right\}$$

It is clear from (3.23) that $\rho(V^-)$ annihilates precisely the constant

function and no other element in $\mathcal{Y}$. Thus (3.18) is established. The equivalence of (3.18) and (3.19) is a one-line computation.

Corresponding to the Fock model adapted to J, we get a realization of ω. As in the Schrodinger model, it is easy to describe the action of the subgroup of Sp which preserves the polarization (V^+,V^-) of W_C. This group $K_J = K$ is of course just the unitary group of the Hermitian form H_J. Let $\mathcal{Y}$ now be the space of holomorphic functions on W square integrable with respect to the measure $e^{-\pi H_J}dw$. According to [ ], the action of H on $\mathcal{Y}$ is given by

$$(3.24)\qquad \rho(e(w))f(w') = e^{-\frac{\pi}{2}H_J(w,w)}\, e^{\pi H_J(w',w)}\, f(w'-w)$$

for $f \in \mathcal{Y}$. We may define an action σ of K on $\mathcal{Y}$ by

$$(3.25)\qquad \sigma(k)\, f(y) = f(k^{-1}y)$$

It is easy to check from (3.24) and (3.25) that

$$(3.26)\qquad \sigma(k)\, \rho(e(w))\, \sigma(k)^{-1} = \rho(e(k(w)))$$

Thus ρ and σ combine to define a representation of the group $K \times H$, extending ρ. Thus σ of (3.25) gives up to scalar multiples, the restriction of the oscillator representation ω to K.

Remark: The equations (3.18) or (3.19) say that $\rho(W)y_J$ is a complex subspace of $\mathcal{Y}^{\infty}$. This is the basis of the remark following Theorem 3.4. In fact, the whole orbit $\omega(Sp)(\rho(H)(Cy_J))$ is a complex subvariety of $\mathbb{P}\,\mathcal{Y}_{\infty}$. In fact, from (3.25) we see that $\omega(K_J)$ preserves Cy_J. And we know from I, §12 that Sp/K is naturally a complex manifold. It turns out that the map $g \to \omega(g)(Cy_J)$ factors to a holomorphic embedding of Sp/K_J into $P\mathcal{Y}_\infty$. From here to the realization of fibre systems of abelian varieties via θ-series is a short step. See [ ], [ ], [ ].

We now pass to lattice models over p-adic fields. Let F be a p-adic field, not of characteristic 2, as always. Let W be a symplectic vector space over F, with form $< , >$. We consider $\rho = \rho_\chi$, with χ normalized as specified in §2. Let $L \subseteq W$ be a lattice which is self-dual with respect to $< , >$. Consider the group $\eta^{-1}(L) = e(L)\cdot Z \subseteq H(W) = H$, with e and η as in I, §3. The remarks in §0 (c.f. (0.5) and (0.6)) say that $\eta^{-1}(L)$ is maximal abelian modulo ker χ, so that the considerations at the beginning of this section apply and we may realize ρ as $\mathrm{ind}^{H}_{\eta^{-1}(L)}\psi$ where ψ is an extension of χ on Z to $\eta^{-1}(L)$. We may choose ψ wisely as follows. If p is odd, define

$$\psi(e(\ell)\cdot z) = \chi(z) \qquad \ell \in L, \quad z \in Z. \tag{3.27}$$

From formula I(3.15d) we see that ψ is a character of $\eta^{-1}(L)$ since 2 is a unit in R, the integers of F. If however, F is a 2-adic field of characteristic zero formula (3.27) will not define a character of $\eta^{-1}(L)$, and we must modify it as follows. Let $\{e_i, f_i\}$ be a symplectic basis for L. Let L_1 be the R-module generated by $2L$ and the e_i, and let L_2 be the R-module generated by $2L$ and the f_i. Then formula (3.27) will define characters ψ_i on $\pi^{-1}(L_i)$ for $i = 1,2$, and ψ_1 will agree with ψ_2 on $\pi^{-1}(L_1) \cap \pi^{-1}(L_2) = \pi^{-1}(2L)$. Since $L_1 + L_2 = L$, there is a unique character ψ on $\eta^{-1}(L)$ restricting to ψ_i on $\eta^{-1}(L_i)$. We call such a ψ a <u>standard extension</u> of χ to $\eta^{-1}(L)$.

Consider $\mathrm{ind}^{H}_{\eta^{-1}(L)}\psi$ where ψ is a standard extension of χ to $\eta^{-1}(L)$. The functions in the space of this realization of ρ are determined by their restrictions to $e(W)$ and so may be pulled back to W. There results a space $\mathcal{Y} \subseteq L^2(W)$. The action of H is by a twisted translation:

$$\rho(e(w))f(w') = \chi(\tfrac{1}{2} < w', w >)f(w' + w) \tag{3.28}$$

The function $\mathcal{Y}_0$, equal to $\psi \circ e$ times the characteristic function of L is in $\mathcal{Y}$, and is clearly an eigenvector, with eigencharacter ψ, for $\eta^{-1}(L)$. If Haar measure on W is normalized so that L has measure 1, then $\mathcal{Y}_0$ is a unit vector. If w runs through a set of coset representatives for L in W then the translates

$$\mathcal{Y}_w = \rho(e(w))\mathcal{Y}_0 \tag{3.29}$$

$$\mathcal{Y}_w(\ell - w) = \psi(\ell) \quad \chi(\tfrac{1}{2} < \ell , w >)\ \phi_{L-w}$$

where ϕ_{L-w} is the characteristic function of $L-w$, run through an orthonormal basis for $\mathcal{Y}$. Moreover, the $\mathcal{Y}_w$ are eigenvectors for $\eta^{-1}(L)$ with eigencharacters

$$\lambda \to \chi(\tfrac{1}{2} < \pi(\lambda), w >) \quad \psi(\lambda) \qquad \lambda \in \pi^{-1}(L) \tag{3.30}$$

The space $\mathcal{Y}$ then is the space of linear combinations

$$\sum_w a_w \mathcal{Y}_w \tag{3.31}$$

with

$$\sum |a_w|^2 < \infty \tag{3.32}$$

The space $\mathcal{Y}^\infty$ is easily seen to be the sums of the form (3.31) with all but finitely many a_w equal to zero, i.e., of finite linear combinations of the $\mathcal{Y}_w$.

From here on we will restrict our attention to odd residual characteristic for symplicity. Let $K_L = K$ be the stabilizer of L in Sp. Then $K \simeq Sp_{2n}(R)$. The action of K on H preserves $\eta^{-1}(L)$, and preserves the cross-section $e(L)$ to Z. By formula (3.27) defining the standard extension $\dot\psi$ of X to $\eta^{-1}(L)$, we see $\dot\psi$ is invariant under K. Thus we may extend $\dot\psi$ to a character $\tilde{\dot\psi}$ of $K \ltimes \eta^{-1}(L)$ by letting $\tilde{\dot\psi}$ be trivial on K. Then we may induce $\tilde{\dot\psi}$ from $K \ltimes \eta^{-1}(L)$ up to $K \ltimes H$. The resulting representation $\tilde\rho$ extends ρ from H to $K \ltimes H$. The restriction of $\tilde\rho$ to K evidently is also the restriction of the oscillator representation ω to K, up to multiples. But, except when the residue class field of R is F_3, and $\dim W = 2$, the group K is perfect, since $Sp_{2n}(F_q)$ is perfect except when $n = 1$ and $q = 3$ [ ]. Hence, except in that case, which we will ignore henceforth, $\tilde\rho|K = \omega|K$. We have shown

Proposition 3.5: The projective representation ω of Sp when restricted to K may be lifted uniquely to an actual representation of K. Thus in the lower sequence (c.f. (1.17))

$$\begin{array}{ccccccc} 1 \to \mathbb{T} & \to & Sp & \xrightarrow{\beta} & Sp & \to & 1 \\ \uparrow & & \uparrow & & \uparrow & & \\ 1 \to \mathbb{T} & \to & \beta^{-1}(K) & \xrightarrow{\beta} & K & \to & 1 \end{array}$$

the group $\beta^{-1}(K)$ splits uniquely as a product $K \times \mathbb{T}$. This is true except when $\dim W = 2$, and $\#(R/\pi R) = 3$. (Here π is a prime of R.)

On the space $\mathcal{Y}$ described above, it is easy to compute that $\omega|K$ is just induced from the linear action of K on W. Thus for $f \in \mathcal{Y}$

$$\omega(k)\, f(w) = f(k^{-1}w) \qquad k \in K,\ w \in W \tag{3.33}$$

$$\text{or} \qquad \omega(k)\, \mathcal{Y}_w = \mathcal{Y}_{k(w)}$$

where $\mathcal{Y}_w$ is defined by (3.29).

Thus $\omega|K$ decomposes into a sum of monomial representations, one for each K-orbit in W/L. Again let π be a prime of R and put $L^m = \pi^m L$. According to I, proposition 11.5, the K orbits in W are precisely the sets $L^m - L^{m+1}$. Thus choose w in $L^0 - L^1$. Then the elements $w_m = \pi^{-m} w$ for $k \geq 0$ form a set of orbit representatives for K in W/L. Let Q be the isotropy group of w in Sp, and let Q_m be the isotropy group of $w_m + L$ in W/L. Then I, proposition 11.5 further says that

$$Q_m = (Q \cap K)\cdot K_m \tag{3.34}$$

where as before K_m is the subgroup of K acting trivially on L/L^m. Let ξ_n be the character by which $\mathcal{Y}_{w_m}$ transforms under the action of Q_m. Then we have

$$\omega|K \simeq \sum_{m=0}^{\infty} \operatorname{ind}_{Q_m}^{K} \xi_m \tag{3.35}$$

We will compute the ξ_m. From (3.33) we see that ξ_m is trivial on $Q \cap K$, so it is sufficient to compute ξ_m on K_m. Actually, for later purposes, we shall compute ξ_m on $Q_m \cap K_1$. Recall that $\mathfrak{sp}$ is the Lie algebra of Sp, and $\mathfrak{sp}_L^i$ for $i \geq 0$ is the lattice in $\mathfrak{sp}$ consisting of transformations that preserve L and moreover maps L into L^i. Let $\mathfrak{q}$ be the Lie algebra of Q. Then $\mathfrak{q}$ is the intersection of $\mathfrak{sp}$ with the ideal of operators that annihilate w. Put

$$\mathfrak{q}_m = (\mathfrak{q} \cap \mathfrak{sp}_L^i) + \mathfrak{sp}_L^m \tag{3.36}$$

Recall the Cayley transform c defined in (1.7.12).

Proposition 3.6: The Cayley transform defines a homeomorphism from $\mathfrak{p}_m$ to $Q_m \cap K_1$. Moreover

$$\xi_m \circ c(x) = \chi(- < xw_m, w_m >) \tag{3.37}$$

Proof: Take x in $\mathfrak{p}_m$ and write

$$x = y + z$$

with y in $\mathfrak{p}$ and z in $\mathfrak{sp}^m_L$. We know from I, lemma 11.1 that $c(x)$ is in K^1, and that $c(x)$ is a homeomorphism onto its image. If we can show that $c(x)$ is in Q_m, then since clearly $\#((Q_m \cap K^1)/K_m)$ and $\#(\mathfrak{p}_m / \mathfrak{sp}^m_L)$ are equal, the first statement of the proposition will be proved. Put $T = y+z$ in formula (1.7.15). One finds

$$\begin{aligned} c(y+z) &= I - 2(y+z) + 2(y^2 + yz + zy + z^2) - 2(y^3 + \dots) + \dots \\ &= c(y) - 2(1+y)^{-1} z (1+y)^{-1} + r \end{aligned} \tag{3.38}$$

where the remainder term r is a sum of monomials in y and z involving z at least to the second power. In particular

$$c(y+z)w = c(y)(w) \bmod L^m = w (\bmod L^m),$$

as asserted.

We proceed to compute $\xi_m \circ c$ using (3.38) and (3.29). We have

$$\omega(c(x))\, \mathcal{Y}_{w_m}(-w_m) = \mathcal{Y}_{c(x)w_m}(-w_m) = \chi\left(\tfrac{1}{2} < c(x)w_m - w_m, c(x)w_m > \right)$$

whence

$$\xi_m \circ c(x) = \chi\left(\tfrac{-1}{2} < w_m, c(x)\, w_m - w_m > \right). \tag{3.39}$$

We again use (3.38) to compute $c(x)$. Of course $c(y)\, w_m = w_m$, and rw_m is in L^m, so that $< w_m, rw_m > \in R \subseteq \ker \chi$. Also expanding $(1+y)^{-1}$ in powers of y we find

$$\chi\left(\tfrac{-1}{2} < w_m, c(x)\, w_m - w_m >\right) = \chi(- < (1+y)^{-1} z\, w_m, w_m >) \tag{3.40}$$

But $(1+y)^{-1} z = z - y\,(1+y)^{-1} z$, and

$$< y(1+y)^{-1} z\, w_m, w_m > = - < (1+y)^{-1} z\, w_m, y\, w_m > = 0.$$

Therefore (3.39) and (3.40) reduce to

$$\begin{aligned} \xi_m \circ c(x) &= \chi(< z\, w_m, w_m >) = \chi(< (y+z) w_m, w_m >) \\ &= \chi(< xw_m, w_m >) \end{aligned}$$

as desired.

Corollary 3.7: The rperesentations $\mathrm{ind}^K_{Q_m}\, \xi_m$ each decompose into 2 irreducible components for $m \geq 1$. The irreducible subspaces are the eigenspaces of $-1 \in K$.

Proof: The group K_m is normal in K. By the standard theory [ ] describing the relation between representations of a group and of a normal subgroup it will suffice to show that the isotropy group of $\xi_m | K_m$ under the action of K by conjugation on the characters of K_m is just $Q_m \cdot \{\pm 1\}$. If $k \notin Q_m \cdot \{\pm 1\}$ then $w' = k(w) \notin (w + L^m) \cup (-w + L^m)$. Thus, referring to (3.37), we see it will suffice to show that if $w' = \pm w \bmod L^m$, then

$$< xw, w > - < xw', w' > \neq 0 \bmod R^m$$

for some x in $\mathfrak{sp}_L$. We take $x = E_{vv}$ as defined by (1.7.19). Then one checks quickly that $< E_{vv} w, w > = < v, w >^2$. Thus

$$< E_{vv}w,w > - < E_{vv}w',w' > = < v,w >^2 - < v,w' >^2$$
$$= < v,w+w' > < v,w-w' >$$

Clearly one of $w+w'$ and $w-w'$ is in $L-L^1$. By assumption the other is not in L^m. Thus we can find v in L such that, say, $< v,w+w'>$ is in R^x and $< v,w-w' > \notin R^m$. This proves the corollary.

For completeness we should describe a slightly more general lattice model. We still require odd residual characteristic. We will not give all the details. Let $L \subseteq W$ be a lattice which, instead of being self-dual, satisfies

$$(3.41) \qquad L^{\perp} \nsubseteq L \nsubseteq \pi^{-1} L^{\perp}$$

Let J be the subgroup of $H(W)$ generated by $e(L)$, and let ${}^1J$ be the subgroup generated by $e(L^{\perp})$. Then ${}^1J$ is normal in J and $J/{}^1J$ is isomorphic to the Heisenberg group attached $L/L^{\perp}$ viewed as a vector space over the residue class field R/P of F. We note that X is trivial on ${}^1J \cap Z$, but not on $J \cap Z$. Therefore there is a unique irreducible representation ρ_L of $\eta^{-1}(L)$ such that ρ_L is trivial on ${}^1J$ and ρ_L has central character X. Consider

$$(3.42) \qquad \mathrm{ind}^H_{\eta^{-1}(L)} \rho_L$$

This induced representation is another realization of ρ. Indeed it has the proper central character, so we need only check irreducibility. If $\tilde{L}$ is a self-dual lattice containing $L^{\perp}$, then $L/\tilde{L}$ is a maximal isotropic subspace of $L/L^{\perp}$, so

$$\rho_L = \mathrm{ind}^{\eta^{-1}(L)}_{\eta^{-1}(L)} \psi$$

where ψ is the standard extension of X to $\eta^{-1}(L)$. Transitivity of induction shows that ρ_L indeed induces an irreducible representation, thus affording another realization of ρ. (See also Theorem 4.1).

We give an explicit description of this realization of ρ. Let X be the Hilbert space on which ρ_L acts. As usual, let $\mathcal{Y}$ be the space of ρ. In this realization, we have

$$(3.43) \qquad \mathcal{Y} = \{f:W \to X \text{ such that } f(w+\ell) = X(\tfrac{1}{2} < w, \ell >)\, \rho_L(e(\ell)f(w)$$
$$\text{for } w \in W,\ \ell \in L, \text{ and } \sum_{w \in W/L} \|f(\dot{w})\|^2 < \infty\}$$

The action is the standard right action:

$$(3.44) \qquad \rho(e(w))(f)(w') = X(\tfrac{1}{2} < w',w >) f(w'+w) \qquad f \in \mathcal{Y}.$$

By inspection of (3.43) and (3.44) we see that the functions supported on the coset $w+L$ are eigenfunctions for $e(L^{\perp})$ with eigencharacter $X(\frac{1}{2} < w, \ell >)$, for $\ell \in L^{\perp}$, and is irreducible for $e(L)$.

Let $K_L = K$ be the stabilizer of L in $Sp = Sp(W)$. Let K' be the subgroup of K which acts trivially on $L/L^{\perp}$. Then

$$(3.45) \qquad K/K' \simeq Sp(L/L^{\perp})$$

where $L/L^{\perp}$ is regarded as symplectic vector space over R/P. This may be seen by writing $W = W_1 \oplus W_2$, with the W_i orthogonal, such that

$$(3.46) \qquad L = (L \cap W_1) \oplus (L \cap W_2)$$
$$L^{\perp} \cap W_1 = \pi(L \cap W_1) \qquad L^{\perp} \cap W_2 = L \cap W_2$$

The existence of such a decomposition is implicit in the proof of I, lemma 11.9.

From [ ] or [ ], we see we may extend ρ_L to a representation of $K \rtimes \eta^{-1}(L)$, uniquely since K is perfect except in an exceptional case which we ignore. The restriction of this extension to K is just the oscillator representation of K/K'. Denote it by ω_L. Then the action (3.44) of H on $\mathcal{Y}$ may be extended to an action of $K \rtimes H$ on $\mathcal{Y}$ by the recipe

$$\sigma(k)f(w) = \omega_L(k)(f(k^{-1}(w))) \tag{3.47}$$

Thus ω may be lifted to an actual representation of any maximal compact subgroup of Sp, and uniquely except in the aforementioned exceptional case.

We see from (3.47) that σ will decompose into a sum over the K-orbits in W/L of certain induced representations. We will make this decomposition explicit in the case when $L^{\perp} = \pi L$. Then the K orbits in W/L are just the sets $\pi^{-j}L - \pi^{-j+1}L = L^{-j} - L^{-j+1}$, for $j \geq 0$, exactly as for the isotropy group of a self-dual lattice. Indeed, if instead of the form $< , >$ we put the form $\pi < , >$ on W, then L becomes a self-dual lattice. Thus this K is conjugate by an outer automorphism to the stabilizer of a self-dual lattice. Pick a vector w_0 in $L-L^{\perp}$, and put $w_m = \pi^{-m}w_0$. The isotropy group of $w_m + L$ is describable as before. Thus let Q be the isotropy group of w_0 in W, and let $Q_m \subseteq K$ be the isotropy group of $w_m + L$ in W/L. Let $K_m \subseteq K$ be the subgroup acting trivially on L/L^m. Then as before

$$Q_m = (Q \cap K)\cdot K_m \tag{3.48}$$

We see therefore that

$$\omega|K \simeq \omega_L \oplus \sum_{m=1}^{\infty} \mathrm{ind}_{Q_m}^{K} \overline{\omega}_m \tag{3.49}$$

where $\overline{\omega}_m$ is describable as follows.

$$\overline{\omega}_m \,|(Q \cap K) = \omega_L|\,(Q \cap K) \tag{3.50}$$

$$\overline{\omega}_m|K_m = \tau_m$$

where τ_m is the representation

$$\tau_m(k) = \rho_L(2c(k)w)\ \chi\ (< w, c(k)w >) \tag{3.51}$$

These formulas are computed from the definitions (3.47) and (3.43). We point out the affinity of (3.51) with (3.37) and (3.54)d).

To finish this subsection we very briefly describe the lattice model for $\mathbb{R}$. Let W be a symplectic vector space over $\mathbb{R}$ with form $< , >$. Let $L \subseteq W$ be a self-dual lattice, i.e.

$$L = \{w \in W: < w,\ell > \in \mathbb{Z}, \text{ all } \ell \in L\}$$

Then if $\chi = e^{2\pi i t}$ is the standard character of $\mathbb{R}$ as specified in §0, one sees that $e(L)Z = \eta^{-1}(L)$ is a maximal abelian subgroup of H modulo $\ker \chi$. To define a standard extension ψ of χ to $\eta^{-1}(L)$ we proceed in a fashion parallel to the 2-adic case. Let $\{e_i,f_i\}$ be a symplectic basis for L. Then there will be a unique character ψ of $\eta^{-1}(L)$ such that

$$\psi(e_i) = \psi(f_i) = 1$$

Using this ψ, we form $\rho = \mathrm{ind}_{\eta^{-1}(L)}^{H} \psi$. Here there is no convenient cross-section to take to $\eta^{-1}(L)$ in H, so the general description of ρ

given at the beginning of the section cannot be made substantially more precise. We note that $H/\eta^{-1}(L)$ is a 2n-dimensional torus, and that $e(L)$ acts by mutliplications by characters of the torus, as described in formula (3.5).

Let $\Gamma_L \subseteq Sp$ be the subgroup which preserves L. Then $\Gamma_L \simeq Sp_{2n}(\mathbb{Z})$. Let $\Gamma_{L,\psi}$ be the subgroup of Γ_L which leaves the character ψ of $\eta^{-1}(L)$ invariant. Since $e(2L) \subseteq \ker \psi$, the 2-congruence subgroup of Γ_L is contained in $\Gamma_{L,\psi}$. Then we can in the usual way extend ρ to a representation σ of $\Gamma_{L,\psi} \ltimes H$. The restriction of σ to $\Gamma_{L,\psi}$ must be the oscillator representation up to multiples, and on the commutator subgroup $\Gamma^{(2)}_{L,\psi}$ must be precisely the oscill tor representation. Results of Kazhdan [ ] and Bass-Milnor Serre [ ] imply that $\Gamma^{(2)}_{L,\psi}$ is a congruence subgroup of Γ_L when $\dim W \geq 4$. However, a more refined and explicit analysis yields a specific congruence level at which one obtains the oscillator representation, even for $\dim W = 2$.

C: Mixed or Partially Split Models

In addition to the realizations of ρ given in subsections A and B, it is sometimes useful, and necessary for a complete theory, to consider realizations of ρ corresponding to polarizations which are partly split and partly anisotropic. In order to avoid repeating a case by case discussion as in B, we will not be specific about the nature of the anisotropic part of the realization. Thus, let W be a symplectic vector space over the local field F, with form $< , >$, and let $Y \subseteq W$ be an isotropic subspace. Choose another isotropic space X supplementary to $Y^{\perp}$, so that X and Y are paired non-degenerately by $< , >$. Let $W_2 = (X \oplus Y)^{\perp}$. Let ρ_2 be a realization of the irreducible representation of $H(W_2)$ with standard central character χ. Let $\mathcal{Y}_2$ be the Hilbert space on which ρ_2 acts. We will be vague about the specific nature of ρ_2. Since $W = (X \oplus Y) \oplus W_2$ we may conclude the following statement from (2.4) and (2.5) together with proposition 3.1.

Proposition 3.8: The irreducible representation ρ of $H(W)$, with central character χ may be realized on $\mathcal{Y}_2 \otimes L^2(X) \simeq L^2(X; \mathcal{Y}_2)$ the space of square integrable functions on X with values in $\mathcal{Y}_2$, by the following formulas

(3.52)
a) $\rho(e(x'))(f)(x) = f(x + x') \qquad x, x' \in X,\ f \in L^2(X; \mathcal{Y}_2)$

b) $\rho(e(y))(f)(x) = \chi_0(< x, y >)f(x) \qquad y \in Y$

c) $\rho(e(w))(f)(x) = \rho_2(e(w))(f(x)) \qquad w \in W_2$

The space $L^2(X; \mathcal{Y}_2)^{\infty}$ is equal to $\mathcal{S}(X; \mathcal{Y}_2^{\infty})$, the Schwartz space of X with values in $\mathcal{Y}_2^{\infty}$. For F non-Archimedean this is the space of $\mathcal{Y}_2^{\infty}$-valued functions which are locally constant and have compact support. For $F = \mathbb{R}$ (or $\mathbb{C}$) this is the space of $\mathcal{Y}_2^{\infty}$-valued functions which are smooth and such that

$$\sup_{x \in X} \|Df(x)\| < \infty$$

where D is any polynomial-coefficient differential operator, and $\| \ \|$ is any continuous seminorm on $\mathcal{Y}_2^{\infty}$.

Let P_Y be the parabolic subgroup of $Sp(W)$ whose elements preserve Y. In the mixed realization of proposition 3.8 for ρ, one can also write convenient formulas for the operators $\omega(p)$, $p \in P_Y$. We will give these formulas. The sturcture of P_Y is described in I, § 9, from which we may glean the decomposition

(3.53) $$P_Y = (Sp(W_2) \times GL(X)) \cdot e(Hom(W_2,Y)) \cdot S^{2*}(X)$$

The action of 3 of the factors can be read off directly, 2 from (3.8). For $f \in L^2(X; \mathcal{Y}_2)$ we set

(3.54)
a) $\sigma(g)(f)(x) = \omega_2(g)(f(x)) \qquad g \in Sp(W_2),\ x \in X$

b) $\sigma(m)(f)(x) = |\det_X(m)|^{-1/2}\, f(m^{-1}(x)) \qquad m \in GL(X)$

c) $\sigma(n)(f)(x) = \chi_0(-1/2\ \beta_n(x,x))f(x), \quad n \in N^{(2)} \simeq S^{2*}(X)$.

To compute the action of $e(Hom(W_2,Y))$, the interesting new feature of this realization of ρ, we observe that it is essentially the realization

(3.55) $$\rho \simeq \mathrm{ind}_{H(W_2)\cdot e(Y)}^{H(W)}\ \tilde{\rho}_2$$

where $\tilde{\rho}_2$ is the extension to $H(W_2)\cdot e(Y)$ of ρ_2 defined by letting $e(Y)$ act trivially. Thus we may identify $L^2(X, \mathcal{Y}_2)$ with the space of functions f from $H(W)$ to $\mathcal{Y}_2$ satisfying

(3.56) $$f(h,h') = \tilde{\rho}_2(h')f(h) \qquad h \in H(W),\ h' \in H(W_2)\cdot e(Y)$$

Of course, the action σ of P_Y arises by virtue of the fact that P_Y, normalizes Y and $H(W_2)\cdot Y$. Thus (3.42)a), b) and c) are all specializations of

(3.57) $$\sigma(p)f(h) = f(p^{-1}(h)) \qquad p \in P_Y \quad h \in H(W)$$

Combining (3.57) with (1.9.9) we compute

(3.54) d) $\sigma(e(T))(f)(x) = \rho_2(e(T^*(x))(f(x)) \qquad T \in Hom(W_2,Y)$

In formula (3.54)d) the e on the left hand side is as in (1.9.8) and on the right hand side is as in I (3.12). The formula is a partial justification for the parallel notation.

4: Distributional Frobenius Reciprocity

This section is devoted to proving a result which, though technical in import and somewhat fussy in its proof because of the necessity to examine cases, is important because it expresses the rigidity of the structure of ρ in a definitive way. It does so in a fashion complementary to the Stone-vonNeumann Theorem. It may be regarded as a distributional version of the Frobenius Reciprocity Theorem for $H(W)$.

As usual, let W be our symplectic vector space over the self-dual field F. Let $Y \subseteq W$ be any R-module (over $\mathbb{R}$, a $\mathbb{Z}$-module). We will call Y a <u>subpolarization</u> if

$$(4.1) \qquad Y \subseteq Y^{\perp}$$

If Y is a subpolarization, then $\pi^{-1}(Y) \subseteq H(W)$ is commutative modulo ker χ, where χ is as before the standard character of $Z(H)$.

Let ψ be any character of $\pi^{-1}(Y)$ extending χ. Let ρ be the unique representation of H with central character χ realized on a space $\mathcal{Y}$. Let $\mathcal{Y}^{\infty} \subseteq \mathcal{Y}$ be the space of smooth vectors and $\mathcal{Y}^{\infty *}$ the dual of $\mathcal{Y}^{\infty *}$. Let ρ^* be the action of Hom $\mathcal{Y}^{\infty *}$ dual to ρ on $\mathcal{Y}^{\infty}$. Let $\mathcal{Y}^{\infty *}_{\psi}$ be the subspace of $\mathcal{Y}^{\infty *}$ consisting of functionals λ such that

$$(4.2) \qquad \rho^*(e(y)) = \psi^*(e(y)) \qquad y \in Y$$

that is $\qquad \lambda(\rho(e(y))v) = \psi(e(y))\,\lambda(y)$

One computes immediately with formula (3.4) that if $\lambda \in \mathcal{Y}^{\infty *}_{\psi}$, then

$$(4.3) \qquad \rho^*(e(y))\,\rho^*(e(w))\lambda = \psi(e(y))\,\chi(<y,w>)\rho^*(e(w))\lambda \;;\; y \in Y,\ w \in W$$

In other words

$$(4.4) \qquad \rho^*(e(w)) \quad \mathcal{Y}^{\infty *}_{\psi} = \mathcal{Y}^{\infty *}_{\mathrm{Ad}^* e(w)(\psi)}$$

Thus $\mathcal{Y}^{\infty *}_{\psi} = \mathcal{Z}$ is invariant under $\pi^{-1}(Y^{\perp})$, and so we get a representation of $\pi^{-1}(Y^{\perp})$ on $\mathcal{Y}^{\infty *}_{\psi}$ by restricting ρ. Put $\pi^{-1}(Y) = B$ and $\pi^{-1}(Y^{\perp}) = B'$ and let $S(B', X)$ be the space of functions on B' which are restrictions of functions in $S(H, X)$. Put

$$(4.5) \qquad \mathcal{Z}^{\infty} = \text{span} \quad \{\rho(f)z : f \in S(B', X)\}$$

Theorem 4.1: Notations as above. The space $\mathcal{Z}^{\infty}$ is an (algebraically) irreducible module for $S(B', X)$, and we have the equation

$$(4.6) \qquad (\dim\ \mathcal{Z}^{\infty})^2 = \#(B'/B) = \#(Y^{\perp}/Y),$$

where $\#(\ )$ indicates cardinality. The equation holds in the sense that one side is finite if and only if the other is.

Proof: As in I, §11, let Y_F stand for the largest vector subspace (F-submodule) of Y, and let FY be the (F-linear) span of Y. Of course Y_F is an isotropic subspace of W. From (4.4) and the discussion in §0, we know that if the theorem is true for one choice of ψ it is true for all. Therefore, we may choose ψ to be trivial on $e(Y_F)$. In that case $\mathcal{Z}$ is contained in the $e(Y_F)$ invariants in $\mathcal{Y}^*$, and these are a module for $\pi^{-1}(Y_F^{\perp})/e(Y_F) \simeq H(Y_F^{\perp}/Y_F)$. Of course $B' \subseteq \pi^{-1}(Y_F^{\perp})$. Hence if we can prove the theorem when for $Y = Y_F$, with the additional precision that then $\mathcal{Z}^{\infty}$ and $\mathcal{Z}$ are, respectively, the smooth vectors and the dual of the smooth vectors for the unique irreducible representation with central character χ of $H(Y_F^-/Y_F)$, then to finish the proof it will suffice to consider the case when $Y_F = \{0\}$. Thus we can essentially reduce the proof to the two special cases $Y = Y_F$ and $Y_F = \{0\}$.

Therefore, to begin, we will assume $Y = Y_F$. Choose a maximal isotropic subspace Y_1 containing Y. Choose further a complementary maximal isotropic subspace X_1 to Y_1, so that (X_1, Y_1) form a complete polarization. Then from §3A we can realize ρ on $\mathcal{Y} = L^2(X_1)$, by the formulas (3.6). Then $\mathcal{Y}^\infty = \mathcal{S}(X_1)$ and $\mathcal{Y}^{\infty *} = \mathcal{S}^*(X_1)$. Put $X = X_1 \cap Y^\perp$. Then according to formulas (3.6), any $\rho(e(Y))$ invariant distribution must be supported on X. For non-Archimedean F, this essentially proves the theorem, since a distribution supported on a closed set factors through the restriction map to that set (see, e.g. [ ]). Thus $Z \simeq \mathcal{S}^*(X)$. Then the same computation which showed $\mathcal{Y}^\infty \simeq \mathcal{S}(X_1)$ shows that $Z^\infty \simeq \mathcal{S}(X)$; and one sees that the action of $H(Y^\perp/Y)$ on Z^∞ is just the Schrodinger model for the pair $(X_1, Y_1/Y)$.

For $F = \mathbb{R}$ all the conclusions are the same but slightly more justification is required, since a distribution supported on a set may not factor through the restriction map to that set in general. We need a lemma to insure that in this case it does. The lemma is of a standard type, but we prove it for completeness' sake. We observe that if a distribution is invariant by $\rho(Y)$ then it will be killed by the linear functions $\{<\cdot, y>, y \in Y\}$ on X_1. To conclude that $Z \simeq \mathcal{S}^*(X)$ we need to know

Lemma 4.2: If $f \in \mathcal{S}(\mathbb{R}^n)$ vanishes on $\mathbb{R}^j = \{(x_1, \ldots, x_j, 0, \ldots, 0)\}$, then we can write

$$f = \sum_{k=j+1}^{n} x_k g_k \tag{4.7}$$

with $f_k \in \mathcal{S}(\mathbb{R}^n)$.

Proof: By the Fundamental Theorem of Calculus, we can write

$$f(x_1,\ldots,x_n) = f(x_1,\ldots,x_j,0,\ldots,0) + \int_0^1 \sum_{k=j+1}^{n} x_k \frac{\partial f}{\partial x_k}(x_1,\ldots,x_j,tx_{j+1},\ldots,tx_n)dt$$

$$= \sum_{k=j+1}^{n} x_k \int_0^1 \frac{\partial f}{\partial x_k}(x_1,\ldots,x_j,tx_{j+1},\ldots,tx_n)dt$$

Using the formula for differentiating an integral with respect to a parameter, one sees that the function

$$g_k' = \int_0^1 \frac{\partial f}{\partial x_k}(x_1,\ldots,x_j,tx_{j+1},\ldots,tx_n)dt$$

is smooth and vanishes rapidly at ∞ for $(x_{j+1}, \ldots, x_n)$ in any fixed compact set. Let φ be a smooth, non-negative function on $\mathbb{R}$ which vanishes near 0 and equals 1 outside a compact set. Put $\tilde{r}^2 = \sum_{k=j+1}^{n} x_k^2$. Then write

$$f = \sum_{k=j+1}^{n} x_k g_k' = \sum_{k=j+1}^{n} x_k\left(g_k'(1-\varphi(\tilde{r}^2)) + \frac{\varphi(\tilde{r}^2)}{\tilde{r}^2}(\tilde{r}^2 f)\right)$$

$$= \sum_{k=j+1}^{n} x_k \left\{g_k'(1-\varphi(\tilde{r}^2)) + \frac{\varphi(\tilde{r}^2)}{\tilde{r}^2} x_k f\right.$$

This is the desired expansion.

Thus if $\delta \in \mathcal{S}^*(X_1)$ is $\rho(e(Y))$ invariant, it vanishes on the ideal in $\mathcal{S}(X_1)$ generated by the coordinate functions $<\ , y>$ with $y \in Y$. But lemma 4.2 says this is the ideal of all functions vanishing on X, so that δ depends only on the restriction of $f \in \mathcal{S}(X_1)$ to X. Thus $Z \simeq \mathcal{S}^*(X_1)$ as claimed. The rest now follows as in the p-adic case.

We proceed to the case when $Y_F = \{0\}$. Then Y is compact if F is non-Archimedean, and discrete if $F = \mathbb{R}$. We can write $Y = Y' \oplus Y''$ where FY'' is the radical of $<\ ,\ >$ on FY, and $<,>$ is non-degenerate on FY'. The factor Y'' is unique, but Y' is not. We can write $W = FY' \oplus (FY')^{\perp} = W' \oplus W''$. Let $\mathcal{Y}'$ be the irreducible module for $H(FY')$, and similarly for $\mathcal{Y}''$. Then we know $\mathcal{Y}^{\infty *}$ is the topological tensor product of $\mathcal{Y}'^{\infty *}$ and $\mathcal{Y}''^{\infty *}$. Let Z' and Z'' be the spaces analogous to Z for the inclusions of Y' in W' or Y'' in W', and for the restrictions of ψ. We observe that Y' is of finite index in its annihilator in W'. Then the theorem asserts Z' is finite dimensional. Therefore, we have $Z \simeq Z' \otimes Z''$ (algebraic tensor product). Since also $S(Y,X)$ is the topological tensor product (see remarks at end of §0) of $S(Y', X)$ and $S(Y'',X)$, we see that we may further reduce the theorem to consideration of the two special cases i): FY is isotropic; or ii) Y is a lattice in (spans) W.

If FY is isotropic and F is non-Archimedean, then a proof very much like the one given above for the case $Y = Y_F$ suffices. We may suppose $\psi|e(Y)$ is trivial. Choose a complete polarization (X_1,Y_1) of W with $FY \subseteq Y_1$. Then formulas (3.6) show that the $\rho(e(Y))$ invariants are precisely those distributions supported on the open set $X_1 \cap Y^{\perp}$. On thissspace, $X_1 \cap Y^{\perp}$ acts transitively by translations, while Y_1 acts by multiplications by all characters of $X_1 \cap Y^{\perp}$. By a slight modification of the reasoning of §3A, the result in this case follows easily.

Next consider the case when $F = \mathbb{R}$ and $\mathbb{R}Y$ is isotropic. Again assume $\psi|e(Y)$ is trivial. Observe that $(\mathbb{R}Y)/Y$ is compact, a torus. Thus any element of Z may be expanded in a Fourier series in the characters of $(\mathbb{R}Y)/Y$. The subspace Z_φ that transforms according to a given character φ is just a subspace for $\mathbb{R}Y$ analogous to Z. Thus $S(\pi^{-1}((\mathbb{R}Y)^{\perp}),X)$ acts irreducibly on each Z_φ^{∞}; but then $e(Y^{\perp})$ will permute the Z_φ transitively. It is easy to check that if $\delta \in Z^{\infty}$, then its components in the Z_φ^{∞} must "vanish rapidly at ∞ in φ", and this observation yields the result in this case.

It remains to consider the possibility that $FY = W$. In this case as we have seen, $Y^{\perp}/Y$ is finite. Let Y_1 be a self-dual lattice containing Y, so that $Y \subseteq Y_1 \subseteq Y^{\perp}$, and $\#(Y_1/Y) = \#(Y^{\perp}/Y_1)$. We can decompose Z into eigenspaces for $e(Y_1)$, and formula (4.4) and the discussion in §0 shows these eigenspaces will be permuted transitively by $e(Y^{\perp})$. Therefore to finish the theorem, it will suffice to consider the case $Y = Y_1$, and to show then that Z is one dimensional. For F non-Archimedean, this result may be read off from the description of the lattice model in §3B. For $F = \mathbb{R}$ the lattice model in §3B plus reasoning as in the case of $Y = Y_F$ would imply the result. However, we shall take a slightly different approach, which has the virtue of connecting this result with the Poisson Summation Formula.

By the theory of elementary divisors (I, proposition 3.2) we can find mutually orthogonal 2-dimensional subspaces $W_j \subseteq W$

$$W = \bigoplus_{j=1}^{n} W_j \qquad \text{and} \qquad Y = \sum_{j=1}^{n} (Y \cap W_j)$$

Thus by the multiplicative property (2.5) of ρ it suffices to consider the case $\dim W = 2$. Choosing a symplectic basis $\{e,f\}$ for Y and realizing ρ by the Fock model in the resulting coordinates, we can finish by proving the following concrete lemma.

Lemma 4.3: Let δ be a distribution on $\mathbb{R}$ invariant under translation by $\mathbb{Z}$ and multiplication by $e^{2\pi ix}$. Then up to multiples, δ is counting measure on $\mathbb{Z}$.

Remark: Since translation by 1 and multiplication by $e^{2\pi ix}$ are interchanged under conjugation by Fourier transform, the lemma implies that counting measure on $\mathbb{Z}$, is up to multiples, invariant by Fourier transform. Thus the lemma essentially implies the Poisson Summation Formula.

Proof: If δ is invariant by multiplication by $e^{2\pi ix}$, then δ is annihilated by multiplication by $e^{2\pi ix} - 1$. The function $e^{2\pi ix} - 1$ vanishes precisely at integers, and its derivative is non-zero at each integer. A coordinate-free version of lemma 4.2 (special case when $j = 0$, $n = 1$) implies that δ must be a combination of point masses concentrated at the integers. Since δ is also to be $\mathbb{Z}$-invariant, the masses at any two integers must be equal, and the lemma follows.

5: Quantization of polarizations

The various lattices and vector spaces associated to W and used in §3 to realize the representation ρ of $H(W)$ will be referred to collectively as polarizations. To be more specific, over any self-dual field (not of characteristic 2) a maximal isotropic subspace of W will be called a split polarization of W. A self-dual lattice in W over a non-Archimedean field (not of residual characteristic 2) will be called an anisotropic polarization of W. Over $\mathbb{R}$, an anisotropic polarization will mean a negative definite subspace of $W_{\mathbb{C}}$. By a mixed polarization we mean, over a non-Archimedean field F, any self-dual module over the integers R of F, and over $\mathbb{R}$ any negative semi-definite subspace of $W_{\mathbb{C}}$. If Y is a mixed polarization of W, then we can find a (non-unique) orthogonal decomposition $W = W_1 \oplus W_2$ such that if $Y_j = Y \cap W_j$ (or $Y \cap W_{jC}$ when $F = \mathbb{R}$) then Y_1 is a split polarization of W_1 and Y_2 is an anisotropic polarization of W_2. The space Y_1 is determined by the polarization and is called the split part of the polarization.

As in I, §§9-12 we denote sets of polarizations with the letter Ω. The set of all polarizations of W will be denoted $\Omega(W)$, or just Ω if W is understood. The set of polarizations whose split part has dimension j will be denoted Ω^j. Thus

$$\Omega = \bigcup_{j=0}^{n} \Omega^j \qquad \text{where} \qquad 2n = \dim W \tag{5.1}$$

The set Ω^0 is the set of anisotropic polarizations of W. The set Ω^n, where $2n = \dim W$, is the set of split polarizations. We will also write ${}^s\Omega$ for the set of split polarizations in order to avoid having to specify that $2n = \dim W$. For finite fields or in residual characteristic 2, $\Omega = {}^s\Omega$.

The space Ω is acted on in a natural fashion by Sp, and we know from part I that the Sp orbits are precisely the Ω^j. Also, the space Ω has a natural compact Hausdorff topology. If $F = R$, then Ω is a closed subset of the Grassmannian of maximal isotropic subspaces of $W_{\mathbb{C}}$, and inherits the relative topology. For non-Archimedean F, the space Ω is a closed subset of the compact Hausdorff space of all R submodules of W. We refer to [ ] for details. In any event, the topology on Ω is such that Sp acts continuously, and each Sp orbit Ω^j is open in its closure which is

(5.2) $$(\Omega^j)^{-} = \bigcup_{k=j}^{\frac{1}{2}\dim W} \Omega^k$$

the — here denoting closure. Thus Ω^0, the set of anisotropic polarizations is the unique open orbit in Ω and is dense. And ${}^s\Omega$, the space of split polarizations, is the unique closed orbit.

Theorem 5.1: Let the representation ρ of $H(W)$ be realized on the space $\mathcal{Y}$.

a) Let $F = \mathbb{R}$. For each polarization $Y \in \Omega$, considered as a subspace of $H(W)_{\mathbb{C}}$, there is a unique (up to multiples) functional λ_Y in $\mathcal{Y}^{\infty *}$ such that

(5.3) $$\rho^*(y)(\lambda_Y) = 0 \qquad y \in Y$$

b) Let F be a non-Archimedean or finite. For each polarization $Y \in \Omega$, there is a functional λ_Y in $\mathcal{Y}^{\infty *}$, unique up to multiples, such that

(5.4) $$\rho^*(e(y))(\lambda_Y) = \lambda_Y$$

In all cases, we obtain a map

$$\mathcal{Q} : \Omega \to \mathbb{P}\,\mathcal{Y}^{\infty *}$$

defined by

(5.5) $$\mathcal{Q}(Y) = \lambda_Y$$

The map $\mathcal{Q}$ is a homeomorphism of Ω into $\mathbb{P}\,\mathcal{Y}^{\infty *}$, and is equivariant for the natural action of Sp on Ω and the action ω^* of Sp on $\mathbb{P}\,\mathcal{Y}^{\infty *}$.

Proof: This theorem is largely a consequence of theorem 4.1. The crucial points are that $\mathcal{Q}$ is well-defined one-to-one, and continuous. Then it is a fact of general topology that $\mathcal{Q}$ is a homeomorphism. And the Sp equivariance of $\mathcal{Q}$ follows from the definitions (5.3) and (5.4) and the defining equation (1.14) of the oscillator representation ω.

When F is non-Archimedean, the equation (5.4) is equivalent to the equation (4.2), where ψ is the character of $\pi^{-1}(Y)$ that extends χ and is trivial on $e(Y)$. Thus statement b) of the theorem follows from Theorem 4.1, so that $\mathcal{Q}$ is well-defined in the non-Archimedean case.

To see $\mathcal{Q}$ is one-one, consider two polarizations Y and Y' of W. Then both $\mathcal{Q}(Y)$ and $\mathcal{Q}(Y')$ lie in the space Z of $e(Y \cap Y')$ invariants in $\mathcal{Y}^{\infty *}$. According to Theorem 4.2, $S(\pi^{-1}((Y \quad Y')^{\perp}), \chi)$ acts irreducibly on its range in Z. But $\pi^{-1}(Y)$ and $\pi^{-1}(Y')$ together generate $\pi^{-1}((Y \cap Y')^{\perp})$. Thus if $\mathcal{Q}(Y) = \mathcal{Q}(Y')$, then this vector is invariant under $\pi^{-1}((Y \cap Y')^{\perp})$, so that Z must be one-dimensional. Hence $Y \cap Y'$ is self-dual, by equation (4.6). Thus $Y = Y \cap Y = Y'$, and $\mathcal{Q}$ is one-one.

To show continuity of $\mathcal{Q}$, we use the lattice model of §3B. In this realization, $\mathcal{Y}^{\infty}$ is a subspace of $S(W)$ spanned by certain functions $\mathcal{Y}_w$

as described in (3.29). The action of $\mathcal{S}(H,X)$ on $\mathcal{Y}^\infty$ is given by right convolution via the twisted convolution of formula (1.9).

For each $Y \in \Omega$, let $\delta_Y \in \mathbb{P}\,\mathcal{S}^*(W)$ be Haar measure on Y. According to [ ], the map $Y \to \delta_Y$ is a homeomorphism. (Indeed, it is via this map that the topology on Ω is defined in [ ]). Clearly δ_Y is invariant under right translation by $e(Y)$. Therefore, if $\delta_Y|\mathcal{Y}$ is non-zero, it must equal λ_Y. Clearly the restriction map to $\mathcal{Y}$ is continuous, so we need only check that λ_Y does not vanish on $\mathcal{Y}$. But, at least in odd residual characteristic, the function $\mathcal{Y}_0$ is positive, so that $\delta_Y(\mathcal{Y}_0) > 0$ for all Y. For residual characteristic 2, this argument breaks down, but then there is only one Sp orbit and continuity follows from Sp-equivariance. This proves theorem 5.1 for non-Archimedean F.

Take now $F = \mathbb{R}$. If $Y \in {}^s\Omega$, then equation (5.3) is equivalent to the invariance of λ_Y by $\rho^*(e(Y))$, so the existence and uniqueness of λ_Y is again guaranteed by Theorem 4.1. If $Y \in \Omega^0$, then existence and uniqueness of λ_Y is given by Theorem 3.4. If Y is a mixed polarization, then a mixed model (proposition 3.8) for ρ allows one to combine Theorems 3.4 and 4.1 to get the desired result. This is essentially an application of the multiplicative behavior (2.5) of ρ.

Thus $\mathcal{Q}$ exists. If $\mathcal{Q}(Y) = \mathcal{Q}(Y') = \lambda$, then we have $\rho^*(y)\lambda = 0$ for all y in $Y + Y'$, But, as before $Y + Y' = (Y \cap Y')^\perp$. Therefore, it may easily be checked, $Y + Y'$ contains a subspace W_1 of W (as opposed to W_C) on which $<\,,\,>$ is non-degenerate. We may choose a symplectic basis $\{e_i, f_i\}_{i=1}^m$ of W_1, and realize ρ in a Fock model on $L^2(\mathbb{R}^n)$ in which

$$\rho(e_j) = 2\pi i x_j \qquad j \le m$$

$$\rho(f_j) = \frac{\partial}{\partial x_j}$$

It is quite clear these operators have no common invariant distribution, for an invariant of $2\pi i x_j$ must be supported on the hyperplane where $x_j = 0$, but an invariant of $\frac{\partial}{\partial x_j}$ must be constant on lines of constant x_k, $k \ne j$. This shows that $\mathcal{Q}$ is one-one. We note that this proof could be developed into an analogue of Theorem 4.1 for complex polarizations of W over $\mathbb{R}$.

Finally, we must show that $\mathcal{Q}$ is continuous. For this we use the Cartan decomposition $Sp = K\,A\,K$, where K is a maximal compact (unitary) subgroup and A is a Cartan subgroup. If $K = K_J$ for a positive complex structure J on W, then

$$\Omega^0 = Sp(V_J^-) = K_J\, A(V_J^-)$$

By Sp-equivariance, it will be enough to show that $\mathcal{Q}$ is continuous on the closure of $A(V_J^-)$ in Ω.

Let $\{(e_j, f_j)\}$ be a symplectic basis for W such that $Je_j = -f_j$. (See I, proposition 12.4). We can realize ρ via a Schrodinger model on $L^2(\mathbb{R}^n)$ in such fashion that

$$\rho(e_j) = \frac{\partial}{\partial x_j} \qquad \rho(f_j) = 2\pi i x_j$$

The space V_J^- is spanned by the vectors $e_j - i\,f_j$. These are represented by the operators $\frac{\partial}{\partial x_j} + 2\pi x_j$. The common kernel of $\rho(V_J^-)$ is thus

(5.6) $$\mathcal{Q}(V_J^-) = e^{-\pi r^2} \qquad \text{where} \qquad r^2 = \sum_{j=1}^{n} x_i^2$$

In this realization of ρ, we may take A to act via diagonal matrices. We may parametrize A by n-tuples:

$$\alpha = (a_1, \ldots, a_n) \qquad a_i \in \mathbb{R},\ a_i > 0$$

(It suffices to take $a_i \geq 1$, since the Weyl group of A includes the mappin s $a_j \to a_j^{-1}$.) The action of α on $L^2(\mathbb{R}^n)$ is

$$\omega(\alpha)\ (f)(x_1, \ldots, x_n) = \left(\prod_{j=1}^{n} a_i\right)^{1/2} f(a_1^{-1}x_1, \ldots, a_n^{-1}x_n)$$

Thus

$$\mathcal{Q}(\alpha(V_J^-)) = \omega(\alpha)\ \mathcal{Q}(V_J^-) = C\, e^{-\pi \sum_j (\alpha_j^{-1}x_j)^2}$$

We also observe that

$$\alpha(e_j) = a_j\, e_j \qquad\qquad \alpha(f_j) = \alpha_j^{-1}\, f_j.$$

Hence $\alpha(V_J)$ is spanned by elements

$$e_j - i\, a_j^{-2}\, f_j$$

Thus, if certain a_j become infinite while others stay finite, we see that $\mathcal{Q}(\alpha(V_J^-))$ converges nicely as a distribution to the product of Lebesgue measure (in the coordinates for which $a_j \to \infty$) times a Gaussian (in the other coordinates). On the other hand, $\alpha(V_J^-)$ converges to the space spanned by the elements

$$e_j \text{ (if } a_j \to \infty) \qquad e_j - i\, a_j^{-2}\, f_j \text{ (otherwise).}$$

Thus by inspection $\mathcal{Q}(\lim\ \alpha(V_J^-)) = \lim \mathcal{Q}(\alpha(V_J^-))$. Hence $\mathcal{Q}$ is continuous in the real case also, and Theorem 5.1 is proved.

Remark: The following refinement of theorem 5.1 in the case of anisotropic polarizations is implicit from §4 and the above proof.

Supplement to Theorem 5.1: a) Let $F = \mathbb{R}$. For each anisotropic polarization $Y \in \Omega$, there is a unique (up to multiples) vector v_Y in $\mathcal{Y}^\infty$ such that

(5.3) alt $$\rho(y)(v_Y) = 0 \qquad y \in Y$$

b) Let F be a non-Archimedean. For each polarization $Y \in \Omega$, there is a vector v_Y in $\mathcal{Y}^\infty$, unique up to mutliples, such that

(5.4) alt $$\rho(e(y))(v_Y) = v_Y$$

In all cases we obtain a map

$$\mathcal{Q}^0 : \Omega^0 \to P\, \mathcal{Y}^\infty$$

defined by

(5.5) alt $$\mathcal{Q}^0(Y) = v_Y$$

Furthermore the map $\mathcal{Q}^0$ is related to the map $\mathcal{Q}$ of theorem 5.1 by the formula

(5.7) $$\mathcal{Q} = \alpha \circ \mathcal{Q}^0$$

where

$$\alpha : \mathcal{Y}^\infty \to \mathcal{Y}^{\infty *}$$

is the canonical inclusion induced by the unitary structure on $\mathcal{Y}$.

We will give a formula for the distributions $\mathcal{Q}(Y)$ in the Schrodinger model. Let (X,Y) be a complete polarization of the symplectic vector spcce W, and let us realize the irreducible representation ρ of $H(W) = H$ on $L^2(X)$ according to the prescription of §3A. We know $L^2(X)^\infty = S(X)$, so that for each $V \in \Omega(W)$, the functional $\mathcal{Q}(V)$ is a distribution (determined up to multiples) on X.

Theorem 5.2: In the Schrodinger model attached to the complete polarization (X,Y), the line $\mathcal{Q}(V)$ in $\mathbb{P}\, S^*(X)$, for $V \in \Omega$, may be represented by the distribution

$$\chi(-(\tfrac{1}{2})\, \beta_V(v,v))\overline{dv} \tag{5.8}$$

where

i) $(E_V \beta_V)$ is the pair of subspace and bilinear form associated to V' by I, proposition 5.3, 11.4, and/or 12.5.

ii) dv is Haar measure on E_V and $v \in E_V$

iii) — indicates complex conjugate

iv) χ is the standard central defining character for ρ.

Remark: Formula (5.7) makes sense in the non-Archimedean case because then $\chi(t)$ depends only on t modulo R.

Proof: We recall that $E_V = (V + Y) \cap X = (V \cap Y)^\perp \cap X$. Thus formula (3.6)b) shows that, to be invariant under $\rho(e(Y \cap V))$, the distribution $\mathcal{Q}(V)$ must be supported on E_Y, (in fact, must be a distribution on E_Y, by lemma 4.2). On the other hand, formula (3.6)a) shows $\mathcal{Q}(V)$ must be absolutely continuous with respect to Haar measure, at least in the non-Archimedean fields. Thus we need only use the formula (3.6) in a precise way to verify (5.8) directly for non-Archimedean fields.

Take $v_1 \in V$ and write $v_1 = v' + u'$ with $v' \in E_V$ and $u' \in Y$. Abbreviate $\beta_V = \beta$. We compute using formulas (3.6) and (1.11.7) and (1.3.13)

$$\begin{aligned}
&\rho^*(e(v_1))(\chi(\tfrac{1}{2}\ \beta(v,v))dv) \\
&= \chi(\tfrac{1}{2} < u',v' >)\ \rho^*(e(u'))\ \rho^*(e(v'))(\chi(\tfrac{1}{2}\ \beta(v,v))dv) \\
&= \chi(-\tfrac{1}{2}\ \beta(v',v'))\ \chi(\ <u',v>)\ (\chi(\tfrac{1}{2}\ \beta(v+v',\ v+v')dv) \\
&= \chi(-\tfrac{1}{2}\ \beta(v',v'))\ \chi(-\beta(v',v))\ \chi(\tfrac{1}{2}\ \beta(v+v',\ v+v'))dv \\
&= \chi(\tfrac{1}{2}\ \beta(v,v))dv
\end{aligned}$$

as desired. This proves the theorem for non-Archimedean F.

For $F = \mathbb{R}$, we use a different argument, relying again on the equivariance of $\mathcal{Q}$. The same argument as above proves the result for split polarizations. Let V_J^- be the anisotropic polarization used in the proof of theorem 5.1. Then computing by means of formula (1.12.17) we find that

$$\beta_{V_J^-}(x,x) = -i \sum_{j=1}^{n} x_j^2 \tag{5.9}$$

Comparing (5.9) with (5.6), and recalling that when $F = \mathbb{R}$, $\chi(t) = e^{2\pi i t}$, we see that foruula (5.8) is valid for $V = V_j^-$. By equivariance with respect to P_Y using the formulas (3.8) and the easily computed action of P_Y on the pairs (E,β), the validity of (5.8) is established for all anisotropic polarizations, since P_Y acts transitively on Ω^0. Next, suppose $Y = Y_1 \oplus Y_2$, and $X = X_1 \oplus X_2$ with $X_1 = Y_2^- \cap X$ and $X_2 = Y_1^- \cap X$, so that if $W_1 = X_1 \oplus Y_1$ and $W_2 = X_2 \oplus Y_2$, then $W = W_1 \oplus W_2$ is an orthogonal decomposition. The Schrodinger model for (X,Y) factors into a tensor product of Schrodinger models for (X_i, Y_i) (of the representations ρ_i of $H(W_i)$). Suppose $V \in \Omega(W)$ is a polarization such that if $V_i = V \cap W_{i_{\mathbb{C}}}$, then $V = V_1 \oplus V_2$ so that $V_i \in \Omega(W_i)$. Clearly we have the formula

$$(5.10) \qquad \mathcal{Q}(V) = \mathcal{Q}(V_1) \otimes \mathcal{Q}(V_2)$$

It is easy to see that formula (5.8) is also multiplicative under such a decomposition. Hence if formula (5.8) is valid for V_1 and V_2, it is valid for V. In particular, if V_1 is anisotropic and V_2 is split, then (5.8), being valid for each factor holds also for V.

Also, given V, put $Y_1 = V \cap Y$. Since V is negative, semi-definite, we must have $V \cap Y_{\mathbb{C}} = (Y_1)_{\mathbb{C}}$. Let X_1 be a complement in X to $Y_1^{\perp} \cap X$. Let $X_2 = Y_1^{\perp} \cap X$ and $Y_2 = Y \cap X_1^{\perp}$. Then $V = (W_{1\mathbb{C}} \cap V) \oplus (W_{2\mathbb{C}} \cap V)$, and $V \cap W_{1\mathbb{C}} = Y_{1\mathbb{C}}$. We know formula (5.8) is valid for $V_1 = Y_{1\mathbb{C}}$, so it is enough to prove it for V with $V \cap Y = \{0\}$.

Consider such a V. Put $V_2 = V \cap \bar{V} = (V \cap W)_{\mathbb{C}}$. That is, V_2 is the split part of V. By an element of N_Y, the unipotent radical of P_Y, we can move V to V' such that $V_2' \subseteq X_{\mathbb{C}}$. Then $(V' + X_{\mathbb{C}}) \cap Y_{\mathbb{C}} = V_2'^{\,-} \cap Y_{\mathbb{C}}$ is a real subspace, that is,

$$V_2'^{\perp} \cap Y_{\mathbb{C}} = (V_2' \cap Y)_{\mathbb{C}}$$

Let Y_2 be a complement in Y to $Y_1 = V_1' \cap Y$. Put $W_2 = (V_2' \cap X) \oplus Y_2$, and $W_1 = W_2^{\perp}$. By the reasoning of the previous paragraph

$$V' = (W_{1\mathbb{C}} \cap V') + (W_2 \cap V') = V_1' \oplus V_2'$$

But by construction V_2' is the split part of V_2, so that V_1' is anisotropic. By our discussion above, formula (5.8) is valid for V_1'. But since V is obtained from V' by a transformation from N_Y, we see formula (5.8) holds for V' also, by equivariance. This concludes Theorem 5.2.

Remark: When $F = \mathbb{R}$, the space $\mathcal{Q}({}^s\Omega)$ is a smooth compact submanifold of $\mathbb{P}\,\mathcal{Y}^{\infty *}$. The hyperplane section bundle over $\mathbb{P}\,\mathcal{Y}^{\infty *}$ defines by restriction a line bundle L over $\mathcal{Q}({}^s\Omega)$. The elements of $\mathcal{Y}^{\infty}$ define smooth sections of L. The formula (5.8) defines a cross-section of L over $\mathcal{Q}(N_Y(X))$. Thus a function $f \in \mathcal{S}(X)$, as an element of $\mathcal{Y}^{\infty}$, defines a function Δ_f on $N_Y(X)$ by the formula

$$\Delta_f(n) = \int_X f(x) \ \chi(\beta_n(x,x))\, dx$$

As n goes to ∞ in N_Y, the points $n(X)$ approach points in Ω on the boundary of the cell $N_Y(X) \subseteq \Omega$. In particular, if $n \to \infty$ in directions where β_n is non-degenerate, the space $n(X)$ approaches Y. Of course $\mathcal{Q}(Y) = \delta_0$, the Dirac delta at 0. Thus it may be seen that the classical Method of Stationary Phase (c.f. [ ], page) amounts to a Taylor expansion around $\mathcal{Q}(Y)$ of cross-sections of L, but in the coordinate system of the big cell $N_Y(X)$.

6: Quantization and Doubling; the Oscillator Semigroup

This section studies the consequences of combining the quantization map $\mathcal{Q}$ of §6 with the doubling process of I, §10. We will obtain explicit formulas for the operators of the oscillator representation, modulo a computation of the character of the oscillator representation. And we will note the existence, for finite and non-Archimedean fields, of a remarkable semi-group which we label the "oscillator semi-group". For finite fields, this semi-group constitutes a sort of "compactification" of $\omega(Sp)$.

Let W be as usual our symplectic vector space, with form $\langle\ ,\ \rangle$, over the self-dual field F. Let $\widetilde{W} = W^+ \oplus W^-$ be the double of W described in I, §10. Let $\tilde{\rho}$ denote the representation of $\widetilde{H} = H(W)$ with standard character χ as central character. According to proposition 3.3, we may realize $\tilde{\rho}$ on $L^2(W)$ in what essentially amounts to the Schrodinger model for the pair (Δ^-, Δ^+), defined in (1.10.5). Accompanying this realization of $\tilde{\rho}$, there is a realization of $\tilde{\omega}$ the oscillator representation of $\widetilde{Sp} = Sp(\widetilde{W})$. The action of the parabolic subgroup $\widetilde{P}_{\Delta^+} \subseteq \widetilde{Sp}$ will be given by appropriate adaptations of formula (3.8). Also there is the quantization map $\widetilde{\mathcal{Q}}$ embedding $\widetilde{\Omega} = \Omega(\widetilde{W})$ into $P\,S^*(W)$, as described by Theorem 5.2.

On the other hand, the Stone-vonNeumann Theorem identifies $L^2(W)$ with the Hilbert-Schmidt operators on the space of ρ, the irreducible representation of $H = H(W)$ with the standard character χ as central character. Corollary identifications are of $S(W)$ with the algebra of all smooth operators on $\mathcal{Y}^\infty$, the smooth vectors of ρ, and of $S^*(W)$ with $\mathrm{Hom}(\mathcal{Y}^\infty, \mathcal{Y}^{\infty *})$. In particular the operators $\rho(e(w))$, for $w \in W$, and $\omega(g)$,

for $g \in Sp$, (the extension of Sp by T defined in §1), may be identified to particular distributions on W. One goal of this section is to compute these distributions.

Recall the embeddings i^+ and i^- of H in $\widetilde{H}$ defined in (10.3). The representation $\tilde{\rho} \circ i^+$ of H is realized on $L^2(W)$ by left multiplication, in the twisted convolution product of (1.9). Similarly $\tilde{\rho} \circ i^-$ is realized by right multiplication by inverses. Since the Dirac delta at the origin, δ_0, is the identity of the convolution product, we see

$$\rho(e(w)) = \tilde{\rho} \circ i^+(e(w))(\delta_0) = \delta_w \tag{6.1}$$

That is, the distribution which corresponds to $\rho(e(w))$ is simply the Dirac delta concentrated at w.

Recall also the corollary embeddings i^+ and i^- of Sp into $\widetilde{Sp}$, and of Sp into $\widetilde{Sp}$. Again from proposition 3.3 we know that for $g \in Sp$, we have $\tilde{\omega} \circ i^+(g)$ is left multiplication of $\omega(g)$ on $L^2(W)$. Thus the distribution on W corresponding to $\omega(g)$ is given by the analogue of (6.1). That is,

$$\omega(g) = \overline{\tilde{\omega}^*(i^+(g))(\delta_0)}, \qquad g \in Sp \tag{6.2}$$

the $\overline{}$ indicating complex conjugate. The key to turning (6.2) into an explicit formula is to observe, by theorem 5.2, that we also have (up to multiples)

$$\delta_0 = \mathcal{Q}(\Delta^+) \tag{6.3}$$

Therefore, also from Theorem 5.2, we see that, up to multiples, $\omega(g)$ is equal to $\mathcal{Q}(i^+(g)\,\Delta^+)$. A representative for this line is given in formula (5.7). That representative has value 1 at 0, at least when it is supported on all of W. It is easily seen from §3B that ω is an admissible representation, that is, has finite dimensional K-isotypic subspaces. In fact, as we will see later, ω consists of 2 irreducible components. Thus it has a distributional character, which by the work of Harish-Chandra [ ], [ ] is a locally integrable, almost everywhere analytic (or locally constant in the non-Archimedean case) function on Sp. According to [ ], Theorem 3.5.4, the proof of which is the same for all F as for R, the trace of $\rho(f)$, for $f \in S(W)$, is just $f(0)$, given proper normalization of Haar measure dw. Thus we may assert, using (6.2), (5.7), (3.14), and (1.10.11), the following formula.

Proposition 6.1: If Haar measure on W is self-dual with respect to $<\ ,\ >$ and X, we have the following formula, valid for g in a Zariski open set in Sp:

$$\omega(g) = ch_{\omega}(g)\ X(\tfrac{1}{4} < c^{+}(g)w,w >)dw \tag{6.4}$$

where ch_{ω} is the character of ω, as a function times Haar measure on Sp, and c^{+} is the modified Cayley transform of (1.10.11).

Remark: Actually formula (6.4) properly interpreted can be extended to all of Sp but we will not spend the time necessary to accomplish this.

To make (6.4) completely explicit, it remains only to determine the character ch_{ω} of ω. This computation, though interesting and suggestive, is also somewhat fussey, and since it is not directly relevant to the main concerns of this paper, we put it into an appendix.

We now specialize our field F to be non-Archimedean of odd residual characteristic, or finite (and of odd characteristic as always). Consider the set $\tilde{Q}(\tilde{\Omega}^{0})$ in $P\,S^{*}(W)$ for F non-Archimedean, and $\tilde{Q}(\tilde{\Omega})$ for F finite.

To avoid excessive repetition, we will explicitly discuss the only non-Archimedean case. The finite field case requires only minor changes in language. Also it has been discussed [ ] and is implicitly contained in the non-Archimedean case by virtue of the lattice model for a non-self-dual lattice as discussed in §3B. The group Sp acts on $\tilde{\Omega}^{0}$ via the embeddings i^{+} and i^{-} of Sp into $\tilde{S}p$. Because of the $\tilde{S}p$-equivariance of $\tilde{Q}$ and proposition 3.3 we know

$$\tilde{Q}(i^{+}(g_1)\ i^{-}(g_2)\tilde{L}) = \omega(g_1)\ \tilde{Q}(\tilde{L})\ \omega(g_2)^{-1} \qquad g_1 \in Sp,\ \tilde{L} \in \tilde{\Omega}^{0} \tag{6.5}$$

From formula (5.7) we see that the $\tilde{Q}(\tilde{L})$ are actually of compact support, and also locally constant, so that they actually belong to $S(W)$ rather than only $S^{*}(W)$. In fact, therefore, we are dealing with $\tilde{Q}^{0}$ instead of $\tilde{Q}$. Thus they correspond to smooth (which in the non-Archimedean case implies finite rank) operators on the space of ρ. We will determine the nature of this set of operators. From I, proposition 11.8 plus formula (6.5) we see that up to right and left multiplication by $\omega(Sp)$, the set $\tilde{Q}^{0}(\tilde{\Omega}^{0})$ is generated by the operators $\tilde{Q}^{0}(\delta(L))$, where L is a lattice in W such that $L \subseteq L^{\perp}$, and δ is given by (1.11.14). Thus

$$\delta(L) = \{i^{+}(\ell) + (i^{+} \oplus i^{-})(\ell') :\ \ \ell \in L,\ \ \ell' \in L^{\perp}\}. \tag{6.6}$$

From (6.6) we see that, since 2 is a unit,

$$\delta(L) = (\delta(L) \cap \Delta^{+}) \oplus (\delta(L) \cap \Delta^{-}) \tag{6.7}$$

$$\delta(L) \cap \Delta^{+} = i^{+} \oplus i^{-}(L^{\perp}) \qquad \delta(L) \cap \Delta^{-} = (i^{+} - i^{-})(L).$$

Therefore, from (5.7) we see that, with proper scaling, we may represent $\tilde{Q}^{0}\delta(L)$ by

$$\tilde{Q}^{0}(\delta(L)) = (\textstyle\int_{L} dw)^{-1}\ \varphi_{L}\, dw \tag{6.8}$$

Here φ_L indicates the characteristic function on $L \subseteq W$, so that $\varphi_L \, dw$ is the Haar measure on L induced by dw on W. Thus (6.8) is simply normalized Haar measure on L. By a straightforward computation in, say, a lattice model defined by a self-dual lattice containing L, we see

Proposition 6.2: With the normalization of (6.8), the operator $\rho(\widetilde{\mathcal{Q}}^0(\delta(L)))$ is orthogonal projection onto the invariants of $e(L)$ in the space of ρ.

The main fact about $\widetilde{\mathcal{Q}}^0(\widetilde{\Omega}^0)$ is that it is a semigroup.

Theorem 6.3: The subset $\widetilde{\mathcal{Q}}^0((\widetilde{\Omega}^0)$ of $\mathbb{P}S(W)$ is a semigroup in the convolution product.

Proof: Given $\widetilde{L}_1$ and $\widetilde{L}_2$ in $\widetilde{\Omega}^0$, we can write

$$\widetilde{L}_1 = i^+(g_{11})i^-(g_{12})\delta(L_1) \qquad \widetilde{L}_2 = i^+(g_{21})i^-(g_{22})\delta(L_2)$$

with $g_{ij} \in Sp$ and L_j lattices in W. Because of formula (1.11.15) we can also write

$$\widetilde{L}_1 = i^+(g_{11}g_{12})\,\delta(g_{12}^{-1}(L_1)) \qquad \widetilde{L}_2 = i^-(g_{21}g_{22})\delta(g_{21}(L_2)) \tag{6.9}$$

By formula (6.5), we see it is sufficient to verify that the convolution of two $\widetilde{\mathcal{Q}}^0(\delta(L))$'s again belongs to $\widetilde{\mathcal{Q}}^0(\widetilde{\Omega}^0)$. We can compute this convolution product directly from (1.9). We find

$$\widetilde{\mathcal{Q}}^0(\delta(L_1)) \# \widetilde{\mathcal{Q}}^0(\delta(L_2))(\ell_1 + \ell_2) = \mu(L_1)^{-1}\mu(L_2)^{-1}\mu(L_1 \cap L_2)\chi(\tfrac{1}{2}\langle \ell_1,\ell_2 \rangle) \tag{6.10}$$

for $\ell_1 \in L_1$. Note that $\chi(\frac{1}{2}\langle \ell_1,\ell_2 \rangle)$ is a well-defined function of $\ell_1 + \ell_2$, because if $\ell_1' + \ell_2' = \ell_1 + \ell_2$, then $\ell_1 - \ell_1' = \ell_2 - \ell_2'$ is in $L_1 \cap L_2 \subseteq L_1^\perp + L_2^\perp$. I claim also that $\chi(\frac{1}{2}\langle \ell_1,\ell_2 \rangle)$ is of the form (5.7). It will suffice to show there is a symmetric bilinear form B on W such that $B(\ell_1 + \ell_2, \ell_1 + \ell_2) = \langle \ell_1,\ell_2 \rangle$ modulo R. Let L_1', L_2' be self-dual lattices containing L_1 and L_2 respectively. Then $L_1 + L_2 \subseteq L_1' + L_2'$, and $\langle \ell_1,\ell_2 \rangle \bmod R$ is still well-defined on $L_1 + L_2$ as a function of $L_1 + L_2$. Hence it suffices to take the L_j self-dual. In that case, we know from the remarks following I, lemma 11.2, that there is a complete polarization (X,Y) of W such that

$$L_1 + L_2 = (L_1 \cap X) \oplus (L_2 \cap Y)$$

Choosing representatives for ℓ_1 and ℓ_2 according to this decomposition clearly exhibits $\langle \ell_1,\ell_2 \rangle \bmod R$ as the restriction of a bilinear form on all W. Thus (6.10) is of the form (5.7), and the Theorem follows.

To finish, let us note the following relation between $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$. Let $L \subseteq W$ be a lattice contained in its dual. Let $\Omega^0(W,L)$ denote the collection of self-dual lattices in W containing L. The set $\Omega^0(W,L)$ is finite, and may also be described as the set of self-dual lattices contained in L^-. We can define a retraction

$$p_L : \Omega^0(W) \to \Omega^0(W,L)$$

by

$$p_L(\Lambda) = (\Lambda \cap L^-) + L \qquad \Lambda \in \Omega^0(W) \tag{6.11}$$

This map p_L arises naturally in the context of the maps $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$. In fact, with proper normalizations, we have

$$\rho(\widetilde{\mathcal{Q}}(\delta(L)))(\mathcal{Q}^0(\Lambda)) = \mathcal{Q}^0(p_L(\Lambda)) \tag{6.12}$$

For the left hand side of (6.12) is invariant under $\rho(e(L))$ by proposition 6.2. Further $\mathcal{Q}^0(\Lambda)$ is in particular invariant under $\rho(e(\Lambda \cap L^-))$. But $\rho(e(\Lambda \cap L^-))$ commutes with $\rho(e(L))$, so the left hand side of (6.12) is also invariant and $\rho(e(\Lambda \cap L^-))$. Hence it is invariant under $\rho(e(p_L(\Lambda)))$, and so must equal $\mathcal{Q}^0(p_L(\Lambda))$ up to multiples. We leave the question of what multiples arise in equation (6.12) given a priori normalizations of the $\mathcal{Q}^0(\Lambda)$ to another place.