MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Lecture 12: Formalized Σ_1 completeness

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20 September, 2018

The main aim of this lecture is to show the Σ_1 completeness of R(exp) in $\mathrm{I}\Delta_0(\exp)$. Using this, we verify the provability of Löb's derivability conditions in $\mathrm{I}\Delta_0(\exp)$.

Global assumption. Throughout this lecture, fix a recursive $\mathcal{L}_\mathrm{A}(\exp)$ theory $T \supseteq \mathrm{R}(\exp)$. Let $\Box_T(y)$ denote a Σ_1 formula constructed in Lecture 10 which expresses 'for some $\mathcal{L}_\mathrm{A}(\exp)$ formula θ, we have $y = \ulcorner\theta\urcorner$ and $T \vdash \theta$' in $\mathbb{N}$. Recall that $\Box_T(y)$ has the form

$$\exists s, w \left(\mathrm{pf}(s) \wedge \exists a{\leqslant}s \begin{pmatrix} \mathrm{antet}((s)''_0) = a \wedge \mathrm{succt}((s)''_0) = y \\ \wedge\ \forall x{\leqslant}a\ \forall i{<}\mathrm{len}(a)\ \big((a)'_i = x \rightarrow \exists \bar{z}{\leqslant}w\ \varepsilon(x,\bar{z})\big)\end{pmatrix}\right),$$

where ε is some $\Delta_0(\exp)$ formula making

$$\{c \in \mathbb{N} : \mathbb{N} \models \exists\bar{z}\ \varepsilon(c,\bar{z})\} = \{\ulcorner\sigma\urcorner : \sigma \in T\}.$$

Notice $\Box_T(y)$ depends on the ε we choose. Since a recursive theory can be defined by many Σ_1 formulas, there are many different ways of defining $\Box_T(y)$. For our purposes, it does not matter mathematically which one we choose. However, for the sake of simplicity, let us assume the tuple $\bar{z}$ is empty, so that we can ignore the variable w as well. Hence a *witness* to our $\Box_T(y)$ is a proof s.

Proposition 12.1. For all $\mathcal{L}_\mathrm{A}(\exp)$ sentences σ, τ,

(N) if $T \vdash \sigma$, then $\mathrm{R}(\exp) \vdash \Box_T\,\underline{\sigma}$;

(IN) $\mathrm{I}\Delta_0(\exp) \vdash \Box_T\,\underline{\sigma} \rightarrow \Box_T\,\underline{\Box_T\,\underline{\sigma}}$; and

($\Box$D) $\mathrm{I}\Delta_0(\exp) \vdash \Box_T\,\underline{(\sigma\rightarrow\tau)} \rightarrow (\Box_T\,\underline{\sigma} \rightarrow \Box_T\,\underline{\tau})$.

Consequently, if $T \vdash \mathrm{I}\Delta_0(\exp)$, then $\Box_T$ satisfies Löb's derivability conditions over T.

Proof. ($\Box$D) Work in a fixed $M \models \mathrm{I}\Delta_0(\exp)$. Suppose $M \models \Box_T\,\underline{(\sigma\rightarrow\tau)} \wedge \Box_T\,\underline{\sigma}$. Let $s, t \in M$ witness these. Intuitively, such s, t are (Gödel numbers of) proofs of $T \vdash (\sigma \rightarrow \tau)$ and $T \vdash \sigma$ in M respectively, and our goal is to put them together using the tools available in M to make a proof of $T \vdash \tau$. Let a be the antecedent of the last sequent $(s)''_0$ in the proof s, and let b be the antecedent of the last sequent $(t)''_0$ in the proof t. Then

$$s \mathbin{\widehat{{}_{''}}} t \mathbin{\widehat{{}_{''}}} a \mathbin{\widehat{{}_{'}}} b \mathbin{\widehat{}} \underline{\vdash (\sigma\rightarrow\tau)} \mathbin{\widehat{{}_{''}}} a \mathbin{\widehat{{}_{'}}} b \mathbin{\widehat{}} \underline{\vdash \sigma} \mathbin{\widehat{{}_{''}}} a \mathbin{\widehat{{}_{'}}} b \mathbin{\widehat{}} \underline{\vdash \tau}$$

is a proof in the sense of M, in which the last three steps are justified by (w), (w) and (MP) respectively. It witnesses $M \models \Box_T\,\underline{\tau}$ because the elements in the comma-separated lists a, b are all in T by the choice of s and t.

(N) If $T \vdash \sigma$, then $\mathbb{N} \models \Box_T\,\underline{\sigma}$ by the choice of $\Box_T$, and so $\mathrm{R}(\exp) \vdash \Box_T\,\underline{\sigma}$ by the Σ_1 completeness of R(exp); see Corollary 4.2.

(IN) Run the proof of (N) in $\mathrm{I}\Delta_0(\exp)$. To be able to do this, we need $\mathrm{I}\Delta_0(\exp)$ to be able to prove the Σ_1 completeness of R(exp). This will be provided by Theorem 12.6(2) below.

$\Box$ modulo Theorem 12.6(2)

$\boxed{\mathrm{nmrl}(x) = y}$

Explanation: $\mathrm{nmrl}(x)$ is the Gödel number of the numeral for the number x.

Example: $\mathrm{nmrl}(2) = \ulcorner \underline{2} \urcorner = \ulcorner ((0+1)+1) \urcorner = 119\mathrm{BA2BA2}$ in hexadecimal.

Functionality: $\forall x\ \exists! y\ \mathrm{nmrl}(x) = y$.

Length bound: $\forall x, y\ \big(\mathrm{nmrl}(x) = y \to \mathrm{len}(y) = 4x + 1\big)$.

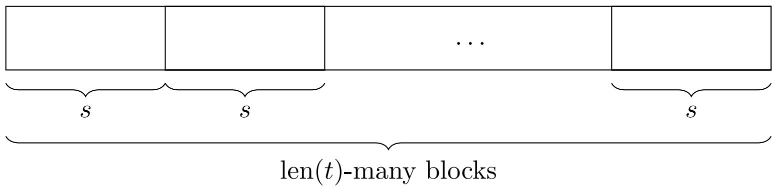
$\boxed{\mathrm{tsub}(t, x, s) = t'}$

Explanation: if t, s are Gödel numbers of the $\mathcal{L}_{\mathrm{A}}(\exp)$ terms $\hat{t}, \hat{s}$ respectively, and x is the Gödel number of a variable $\hat{x}$, then $\mathrm{tsub}(t, x, s)$ is the Gödel number of the $\mathcal{L}_{\mathrm{A}}(\exp)$ term obtained from $\hat{t}$ by replacing every occurrence of $\hat{x}$ with $\hat{s}$.

Example: $\mathrm{tsub}(\ulcorner 2^{\mathsf{v}_0+1}\mathsf{v}_1 + 2^{\mathsf{v}_0} + \mathsf{v}_2 \urcorner, \ulcorner \mathsf{v}_0 \urcorner, \mathrm{nmrl}(3)) = \ulcorner 2^{\underline{3}+1}\mathsf{v}_1 + 2^{\underline{3}} + \mathsf{v}_2 \urcorner$.

Functionality: $\forall t, x, s\ \big(\mathrm{term}(t) \wedge \mathrm{var}(x) \wedge \mathrm{term}(s) \to \exists! t'\ \mathrm{tsub}(t, x, s) = t'\big)$.

Length bound: $\forall t, x, s, t'\ \big(\mathrm{tsub}(t, x, s) = t' \to \mathrm{len}(t') \leqslant \mathrm{len}(t)\,\mathrm{len}(s)\big)$, which originates from the extreme situation where every digit in the hexadecimal representation of t is replaced by a copy of the hexadecimal representation of s:



$\boxed{\mathrm{fsub}(t, x, s) = t'}$

Explanation: if t, s are Gödel numbers of the $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas $\hat{t}, \hat{s}$ respectively, and x is the Gödel number of a variable $\hat{x}$, then $\mathrm{fsub}(t, x, s)$ is the Gödel number of the $\mathcal{L}_{\mathrm{A}}(\exp)$ formula obtained from $\hat{t}$ by replacing every free occurrence of $\hat{x}$ with $\hat{s}$.

Example: $\mathrm{fsub}(\ulcorner \mathsf{v}_0 = \mathsf{v}_0 \vee \exists \mathsf{v}_0\ (\mathsf{v}_0 = \mathsf{v}_0) \urcorner, \ulcorner \mathsf{v}_0 \urcorner, \mathrm{nmrl}(2)) = \ulcorner \underline{2} = \underline{2} \vee \exists \mathsf{v}_0\ (\mathsf{v}_0 = \mathsf{v}_0) \urcorner$.

Functionality: $\forall t, x, s\ \big(\mathrm{fma}(t) \wedge \mathrm{var}(x) \wedge \mathrm{fma}(s) \to \exists! t'\ \mathrm{fsub}(t, x, s) = t'\big)$.

Length bound: $\forall t, x, s, t'\ \big(\mathrm{fsub}(t, x, s) = t' \to \mathrm{len}(t') \leqslant \mathrm{len}(t)\,\mathrm{len}(s)\big)$, similar to our length bound for tsub.

$\boxed{\text{multi-variate versions of tsub}}$ These are defined by recursion on the number of variables:

- $\mathrm{tsub}_0(t) = t_0$ is $t = t_0$;
- for every $k \in \mathbb{N}$, let $\mathrm{tsub}_{k+1}(t, x_1, x_2, \ldots, x_{k+1}, s_1, s_2, \ldots, s_{k+1}) = t_{k+1}$ be

$$\exists t_k {<} 2^{4\,\mathrm{len}(t)\,\mathrm{len}(s_1)\,\mathrm{len}(s_2)\cdots\,\mathrm{len}(s_k)} \big(\mathrm{tsub}_k(t, x_1, x_2, \ldots, x_k, s_1, s_2, \ldots, s_k) = t_k \wedge \mathrm{tsub}(t_k, x_{k+1}, s_{k+1}) = t_{k+1}\big).$$

When there is no risk of ambiguity, we omit the subscript in the notation.

$\boxed{\text{multi-variate versions of fsub}}$ These are defined in a way analogous to that for tsub.

Figure 12.1: Numerals and substitution

We list in Figure 12.1 some $\Delta_0(\exp)$ formulas we use for stating the Σ_1 completeness of T in the language $\mathcal{L}_{\mathrm{A}}(\exp)$. We omit the exact definitions because they are cumbersome, and they can be constructed routinely by imitating the programs we wrote in Lecture 10. To avoid confusion, we need to distinguish between a syntactic object and its Gödel number clearly there.

Remark 12.2. Let $a_1, a_2, \ldots, a_\ell \in M \models \mathrm{I}\Delta_0(\exp)$ and $t(x_1, x_2, \ldots, x_\ell)$ be an $\mathcal{L}_{\mathrm{A}}(\exp)$ term. By considering the extreme situation when every symbol in t is exp, we see that $t(\bar{a}) \leqslant \exp^{(\mathrm{len}(\ulcorner t \urcorner))}\big(\sum_{i=1}^{\ell} a_i\big)$, where the empty sum is considered to be 0 here. Hence, by our length bound on nmrl,

$$\mathrm{len}(\mathrm{nmrl}(t(\bar{a}))) = 4t(\bar{a}) + 1 \leqslant 5\exp^{(\mathrm{len}(\ulcorner t \urcorner))}\Big(\sum_{i=1}^{\ell} a_i\Big), \quad \text{and}$$

$$\mathrm{len}\big(\mathrm{tsub}(\underline{\ulcorner t \urcorner}, \underline{\overline{\ulcorner x \urcorner}}, \mathrm{nmrl}(\bar{a}))\big) \leqslant \mathrm{len}\Big(\mathrm{nmrl}\Big(\sum_{i=1}^{\ell} a_i\Big)\Big) \times \mathrm{len}(\ulcorner t \urcorner) \leqslant 5\sum_{i=1}^{\ell} a_i \,\mathrm{len}(\ulcorner t \urcorner)$$
$$\leqslant 5\exp^{(\mathrm{len}(\ulcorner t \urcorner))}\Big(\sum_{i=1}^{\ell} a_i\Big).$$

Ideally, we carry out in $\mathrm{I}\Delta_0(\exp)$ the argument for Σ_1 completeness we saw in Lecture 3. The problem is that our argument involved structures, which are infinite objects, while in $\mathrm{I}\Delta_0(\exp)$ one cannot talk about infinite objects directly. One solution is to use proofs instead of structures. By the Completeness Theorem, all entailments that can be proved using structures are provable using proofs too. The theory $\mathrm{I}\Delta_0(\exp)$ can deal with proofs because of their finitary nature. In particular, it proves that all the deduction rules shown in Figure 12.2 are derivable, as one can verify. Moreover, the elimination of these additional rules only increases the lengths of proofs polynomially. If one has doubt, then one can include these rules in the definition of $\Box_T$.

We start with a syntactic analogue of Lemma 3.3, the parsing of which requires some thought. Recall from Figure 12.1 that the formula $\mathrm{nmrl}(x) = y$ expresses 'y is the Gödel number of the numeral for x'. Assuming this formula is naturally defined, it makes sense not only in $\mathbb{N}$ but also in all model of $\mathrm{I}\Delta_0(\exp)$. Therefore, as $\mathrm{I}\Delta_0(\exp) \vdash \forall x\ \exists! y\ \mathrm{nmrl}(x) = y$, from the point of view of any model $M \models \mathrm{I}\Delta_0(\exp)$, every $a \in M$ has a numeral $\mathrm{nmrl}^M(a)$, which is the unique $b \in M \models \mathrm{nmrl}(a) = b$, even when $a \notin \mathbb{N}$. For instance, the expression $\mathrm{nmrl}(t(\bar{x}))$ in the statement of Lemma 12.3 below is used in this sense: given $\bar{x} \in M \models \mathrm{I}\Delta_0(\exp)$, evaluate in M the term t on $\bar{x}$ to obtain $a \in M$, then make $\mathrm{nmrl}^M(a)$; cf. Figure 12.3.

Lemma 12.3. For all variables $\bar{x}$ and all $\mathcal{L}_{\mathrm{A}}(\exp)$ terms $t(\bar{x})$,

$$\mathrm{I}\Delta_0(\exp) \vdash \forall \bar{x}\ \Box_T\big(\mathrm{nmrl}(t(\bar{x}))\,\hat{}\!\equiv\!\hat{}\,\mathrm{tsub}(\underline{\ulcorner t \urcorner}, \underline{\overline{\ulcorner x \urcorner}}, \mathrm{nmrl}(\bar{x}))\big).$$

Proof. Fix $M \models \mathrm{I}\Delta_0(\exp)$. We proceed by induction on the $\mathcal{L}_{\mathrm{A}}(\exp)$ term t in $\mathbb{N}$. Let $\bar{a} \in M$. The blue bubbles below are meant to describe arguments from the point of view of M. For example, the numerals signaled by underlines within a bubble are equivalent to nmrl^M without any bubble.

Consider the variable x_i. The definition of term evaluation and the definition of term substitution tell us respectively that $\underline{x_i(\bar{a})} = \underline{a_i} = x_i(\underline{\bar{a}})$. So $T \vdash \underline{x_i(\bar{a})} = x_i(\underline{\bar{a}})$ by (refl).

Consider the constant symbol 0. The definition of term evaluation, the definition of $\underline{0}$, and the definition of term substitution tell us respectively that $\underline{0(\bar{a})} = \underline{0} = 0 = 0(\underline{\bar{a}})$. So $T \vdash \underline{0(\bar{a})} = 0(\underline{\bar{a}})$ by (refl).

The constant symbol 1 can be dealt with similarly, except that (R1) is used instead of (refl) and the definition of $\underline{0}$.

Consider the $\mathcal{L}_{\mathrm{A}}(\exp)$ term $t+s$. The definition of term evaluation and the definition of term substitution tell us respectively that

$$t(\bar{a}) + s(\bar{a}) = (t+s)(\bar{a}) \quad \text{and} \quad t(\underline{\bar{a}}) + s(\underline{\bar{a}}) = (t+s)(\underline{\bar{a}}).$$

$$\dfrac{\Phi \vdash \theta}{\Phi \cup \{\psi_1, \psi_2, \ldots, \psi_\ell\} \vdash \theta}\ (\mathrm{w}) \qquad \dfrac{\Phi \vdash \theta}{\Phi \vdash \neg\neg\theta}\ (\neg\mathrm{R}) \qquad \dfrac{\Phi + \theta \vdash \bot}{\Phi \vdash \neg\theta}\ (\mathrm{RAA}')$$

$$\dfrac{\Phi \vdash \neg\theta \vee \eta \qquad \Phi \vdash \theta}{\Phi \vdash \eta}\ (\mathrm{MP}) \qquad \dfrac{\Phi \vdash \neg\theta \vee \eta \qquad \Phi \vdash \neg\eta}{\Phi \vdash \neg\theta}\ (\mathrm{MT})$$

$$\dfrac{\Phi + y < s + \theta \vdash \bot}{\Phi + \exists y{<}s\ \theta \vdash \bot}\ (\exists^{<}\mathrm{L}) \qquad \dfrac{\Phi \vdash t < s \qquad \Phi \vdash \theta(\bar{x}, t)}{\Phi \vdash \exists y{<}s\ \theta}\ (\exists^{<}\mathrm{R}) \qquad \dfrac{\Phi \vdash \neg\exists y\ \neg\theta(\bar{x}, y)}{\Phi \vdash \theta(\bar{x}, t)}\ (\overline{\forall}\mathrm{R})$$

$$\dfrac{\Phi \vdash \theta(t_1, t_2, \ldots, t_k, \bar{z}) \qquad \Phi \vdash t_1 = s_1 \qquad \Phi \vdash t_2 = s_2 \qquad \cdots \qquad \Phi \vdash t_k = s_k}{\Phi \vdash \theta(s_1, s_2, \ldots, s_k, \bar{z})}\ (\mathrm{Leibniz+})$$

where

- Φ is a set of $\mathcal{L}_\mathrm{A}(\exp)$ formulas;
- $\psi_1, \psi_2, \ldots, \psi_\ell, \theta, \eta$ are $\mathcal{L}_\mathrm{A}(\exp)$ formulas;
- $t, t_1, t_2, \ldots, t_k, s, s_1, s_2, \ldots, s_k$ are $\mathcal{L}_\mathrm{A}(\exp)$ terms;
- y does not appear in s in ($\exists^{<}$L) and in ($\exists^{<}$R); and
- y does not appear free in any element of Φ in ($\exists^{<}$L).

Figure 12.2: Some derived deduction rules

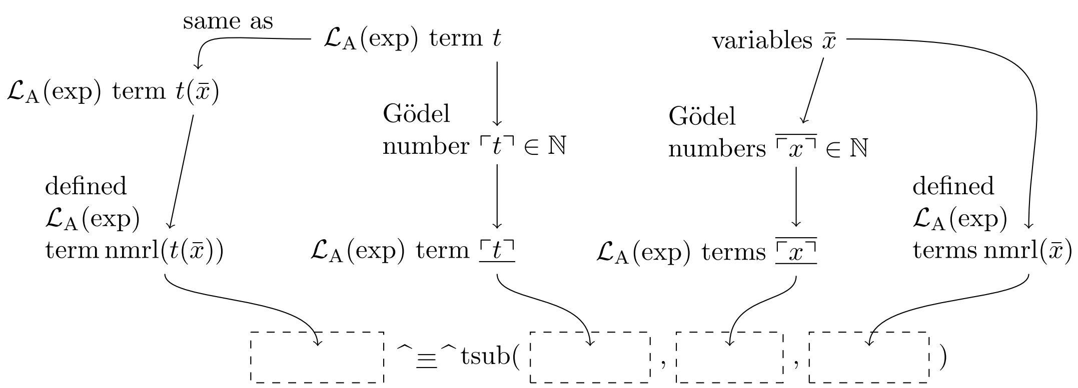


Figure 12.3: Parsing the term nmrl$(t(\bar{x}))\hat{\ }\equiv\hat{\ }$tsub$(\underline{\ulcorner t \urcorner}, \overline{\underline{\ulcorner x \urcorner}}, \mathrm{nmrl}(\bar{x}))$ in Lemma 12.3

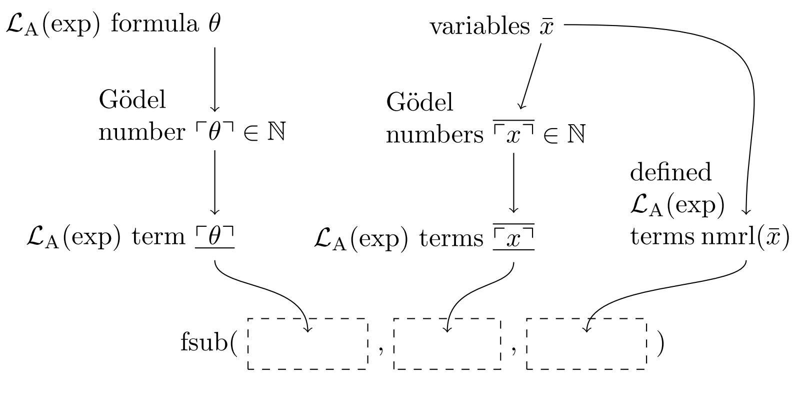


Figure 12.4: Parsing the term fsub$(\underline{\ulcorner \theta \urcorner}, \overline{\underline{\ulcorner x \urcorner}}, \mathrm{nmrl}(\bar{x}))$ in Theorem 12.6

So the following is a proof of $T \vdash \underline{(t+s)(\bar{a})} = (t+s)(\underline{\bar{a}})$:

$$
\dfrac{\dfrac{}{T \vdash \underline{t(\bar{a})} + \underline{s(\bar{a})} = \underline{t(\bar{a})} + \underline{s(\bar{a})}}\,(\text{asn}, \text{R}+) \qquad \dfrac{\vdots\ \text{induction hypothesis}}{T \vdash \underline{t(\bar{a})} = t(\underline{\bar{a}})} \qquad \dfrac{\vdots\ \text{induction hypothesis}}{T \vdash \underline{s(\bar{a})} = s(\underline{\bar{a}})}}{T \vdash \underline{t(\bar{a}) + s(\bar{a})} = t(\underline{\bar{a}}) + s(\underline{\bar{a}})}\,(\text{Leibniz})
$$

Terms of the form $t \times s$ and $\exp(t)$ are dealt with similarly. $\square$

Remark 12.4. Let us analyze our proof of Lemma 12.3 for a fixed $\mathcal{L}_{\text{A}}(\exp)$ term $t(\bar{x})$ and fixed parameters $\bar{a} \in M \models \text{I}\Delta_0(\exp)$. One has to go through at most $\text{len}(\ulcorner t \urcorner)$-many base or induction steps. One can extract from our argument above a polynomial $p(X) \in \mathbb{N}[X]$, whose choice is independent of M, t and $\bar{a}$, such that each step adds to the final proof less than $p\big(\max\{\, \text{len}(\text{nmrl}(t(\bar{a}))), \text{len}\big(\text{tsub}(\underline{\ulcorner t \urcorner}, \overline{\ulcorner x \urcorner}, \text{nmrl}(\bar{a}))\big)\,\}\big)$ symbols. Since Remark 12.2 implies

$$
\begin{aligned}
&\text{len}(\ulcorner t \urcorner) \times p\big(\max\{\, \text{len}(\text{nmrl}(t(\bar{a}))), \text{len}\big(\text{tsub}(\underline{\ulcorner t \urcorner}, \overline{\ulcorner x \urcorner}, \text{nmrl}(\bar{a}))\big)\,\}\big) \\
&\leqslant \exp^{(\text{len}(\ulcorner t \urcorner))}\Big(\sum_i a_i\Big) \times p\Big(5 \exp^{(\text{len}(\ulcorner t \urcorner))}\Big(\sum_i a_i\Big)\Big),
\end{aligned}
$$

the length of the witness to the $\square_T$ in Lemma 12.3 can be bounded above by a polynomial in $\exp^{(\text{len}(\ulcorner t \urcorner))}\big(\sum_i a_i\big)$. Moreover, this polynomial is independent of M, t and $\bar{a}$.

Remark 12.5. What we call the *length* of a proof here is essentially the number of digits in the hexadecimal representation of the Gödel number of the proof. Some authors call this the *size* of a proof, and reserve the word *length* for the number of lines in the proof.

Next, we prove the syntactic analogue of Proposition 3.6.

Theorem 12.6. Let T be a recursive consistent $\mathcal{L}_{\text{A}}(\exp)$ theory that includes $\text{R}(\exp)$.

(1) For every $\Delta_0(\exp)$ formula $\theta(\bar{x})$,

$$
\begin{aligned}
\text{I}\Delta_0(\exp) \vdash\ & \forall \bar{x}\ \big(\theta(\bar{x}) \to \square_T\, \text{fsub}(\underline{\ulcorner \theta \urcorner}, \overline{\ulcorner x \urcorner}, \text{nmrl}(\bar{x}))\big) \\
& \wedge \forall \bar{x}\ \big(\neg\theta(\bar{x}) \to \square_T\, \text{fsub}(\underline{\ulcorner \neg\theta \urcorner}, \overline{\ulcorner x \urcorner}, \text{nmrl}(\bar{x}))\big).
\end{aligned}
$$

(2) $\text{I}\Delta_0(\exp)$ proves the Σ_1 completeness of T, i.e., for every Σ_1 formula $\theta(\bar{x})$,

$$
\text{I}\Delta_0(\exp) \vdash \forall \bar{x}\ \big(\theta(\bar{x}) \to \square_T\, \text{fsub}(\underline{\ulcorner \theta \urcorner}, \overline{\ulcorner x \urcorner}, \text{nmrl}(\bar{x}))\big).
$$

Proof. Fix $M \models \text{I}\Delta_0(\exp)$.

To show (1), we proceed by induction on the $\Delta_0(\exp)$ formula θ in $\mathbb{N}$. Let $\bar{a} \in M$. Inductively, we assume the existence of a polynomial $p_\theta(X) \in \mathbb{N}[X]$, which depends only on θ and not on M or $\bar{a}$, such that the lengths of the witnesses to the $\square_T$'s can be bounded above by $p_\theta\big(\exp^{(\text{len}(\ulcorner \theta \urcorner))}\big(\sum_i a_i\big)\big)$. We reason informally in M. To improve the readability of proofs, we sometimes compress several lines into one. Such lines are indicated by double lines.

Consider $\top$. The truth definition and (top) tell us $\top$ holds and $T \vdash \top$.

Consider $t < s$, where $t(\bar{x}), s(\bar{x})$ are $\mathcal{L}_{\text{A}}(\exp)$ terms. First, suppose $(t < s)(\bar{a})$ holds. Then the truth definition implies $t(\bar{a}) < s(\bar{a})$. As $(t < s)(\underline{\bar{a}})$ is equal to $t(\underline{\bar{a}}) < s(\underline{\bar{a}})$ by the definition of formula substitution, the following is a proof of $T \vdash (t < s)(\underline{\bar{a}})$:

$$
\dfrac{\dfrac{}{T \vdash \underline{t(\bar{a})} < \underline{s(\bar{a})}}\,(\text{asn}, \text{R}<) \qquad \dfrac{\vdots\ \text{Lemma 12.3}}{T \vdash \underline{t(\bar{a})} = t(\underline{\bar{a}})} \qquad \dfrac{\vdots\ \text{Lemma 12.3}}{T \vdash \underline{s(\bar{a})} = s(\underline{\bar{a}})}}{T \vdash t(\underline{\bar{a}}) < s(\underline{\bar{a}})}\,(\text{Leibniz})
$$

Second, suppose $\neg(t < s)(\bar{a})$ holds. Then $t(\bar{a}) \not< s(\bar{a})$ by the truth definition. So $s(\bar{a})$-many applications of (asn, R$\neq$) and $(s(\bar{a})-1)$-many applications of $(\neg\vee)$ show

$$T \vdash \neg(\underline{t(\bar{a})} = \underline{0} \vee \underline{t(\bar{a})} = \underline{1} \vee \cdots \vee \underline{t(\bar{a})} = \underline{s(\bar{a})-1}).$$

This gives the following proof of $T \vdash \neg(\underline{t(\bar{a})} < \underline{s(\bar{a})})$, where we write $b = t(\bar{a})$ and $c = s(\bar{a})$:

$$\text{(MT)}\ \dfrac{\text{(}\forall\text{R)}\ \dfrac{\text{(asn, RInit)}\ \dfrac{}{T \vdash \forall x{<}\underline{c}\ (x = \underline{0} \vee \cdots \vee x = \underline{c-1})}}{T \vdash \neg(\underline{b} < \underline{c}) \vee (\underline{b} = \underline{0} \vee \cdots \vee \underline{b} = \underline{c-1})} \qquad \begin{matrix}\vdots \\ T \vdash \neg(\underline{b} = \underline{0} \vee \cdots \vee \underline{b} = \underline{c-1})\end{matrix}}{T \vdash \neg(\underline{b} < \underline{c})}$$

As in the $t < s$ case above, we can combine this with the proofs given by Lemma 12.3 using (Leibniz+) to show that $T \vdash \neg(t < s)(\underline{\bar{a}})$.

In view of Remark 12.4, the length of the proof above is bounded above by a polynomial in $\exp^{(\mathrm{len}(\ulcorner t \urcorner))}(\sum_i a_i)$ and the lengths of

$$\mathrm{nmrl}(t(\bar{a}))), \quad \mathrm{tsub}(\underline{\ulcorner t \urcorner}, \overline{\underline{\ulcorner x \urcorner}}, \mathrm{nmrl}(\bar{a})), \quad \mathrm{nmrl}(s(\bar{a})), \quad \text{and} \quad \mathrm{tsub}(\underline{\ulcorner s \urcorner}, \overline{\underline{\ulcorner x \urcorner}}, \mathrm{nmrl}(\bar{a})).$$

So the inductive bound is met by Remark 12.2. The formula $t = s$ can be dealt with similarly.

Consider $\neg\theta$, where $\theta(\bar{x})$ is a $\Delta_0(\exp)$ formula. If $\neg\theta(\bar{a})$ holds, then $T \vdash \neg\theta(\underline{\bar{a}})$ by the induction hypothesis. So suppose $\neg\neg\theta(\bar{a})$ holds. Then the truth definition tells us $\theta(\bar{a})$ is true. So $T \vdash \theta(\underline{\bar{a}})$ by the induction hypothesis. From this, we derive using $(\neg\text{R})$ that $T \vdash \neg\neg\theta(\underline{\bar{a}})$.

Consider $\theta \vee \eta$, where $\theta(\bar{x})$ and $\eta(\bar{x})$ are $\Delta_0(\exp)$ formulas. See Assignment 12.7 below.

Consider $\exists y{<}t\ \theta$, where $\theta(\bar{x}, y)$ is a $\Delta_0(\exp)$ formula and $t(\bar{x})$ is an $\mathcal{L}_{\mathrm{A}}(\exp)$ term. First, suppose $(\exists y{<}t\ \theta)(\bar{a})$ holds. Use the truth definition to find $b < t(\bar{a})$ such that $\theta(\bar{a}, b)$ is true. Then the induction hypothesis tells us $T \vdash \theta(\underline{\bar{a}}, \underline{b})$. Also $T \vdash \underline{b} < \underline{t(\bar{a})}$ and $T \vdash \underline{t(\bar{a})} = t(\underline{\bar{a}})$ by (R<) and Lemma 12.3. Hence (Leibniz) and $(\exists^{<}\text{R})$ imply $T \vdash (\exists y{<}t\ \theta)(\underline{\bar{a}})$.
Second, suppose $(\neg\exists y{<}t\ \theta)(\bar{a})$ holds. If $b < t(\bar{a})$, then $\neg\theta(\bar{a}, b)$ is true by the truth definition, and so we can form the following proof π_b of $T + \theta(\underline{\bar{a}}, y) \vdash \neg(y = \underline{b})$, where we write $\theta'(y) = \theta(\underline{\bar{a}}, y)$:

$$\dfrac{\dfrac{\text{(asn, asn, Leibniz+)}\ \dfrac{}{T + \theta'(y) + y = \underline{b} \vdash \theta'(\underline{b})} \qquad \begin{matrix}\vdots\ \text{induction} \\ \vdots\ \text{hypothesis} \\ T \vdash \neg\theta'(\underline{b})\end{matrix}}{T + \theta'(y) + y = \underline{b} \vdash \bot}\ (\text{w}, \bot)}{T + \theta'(y) \vdash \neg(y = \underline{b})}\ (\text{RAA}')$$

Putting together all these π_b's using $(t(\bar{a})-1)$-many applications of $(\neg\vee)$, we see that

$$T + \theta(\underline{\bar{a}}, y) \vdash \neg(y = \underline{0} \vee y = \underline{1} \vee \cdots \vee y = \underline{t(\bar{a})-1}).$$

This gives the following proof of $T \vdash \neg\exists y{<}\underline{t(\bar{a})}\ \theta(\underline{\bar{a}}, y)$, where we write $\theta'(y) = \theta(\underline{\bar{a}}, y)$ and $c = t(\bar{a})$:

$$\text{(RAA}')\ \dfrac{(\exists^{<}\text{L})\ \dfrac{(\text{w, asn}, \bot)\ \dfrac{\text{(MT)}\ \dfrac{\text{(}\forall\text{R)}\ \dfrac{\text{(asn, RInit)}\ \dfrac{}{T \vdash \forall x{<}\underline{c}\ (x = \underline{0} \vee \cdots \vee x = \underline{c-1})}}{T \vdash \neg(y < \underline{c}) \vee (y = \underline{0} \vee \cdots \vee y = \underline{c-1})} \qquad \begin{matrix}\vdots \\ T + \theta'(y) \vdash \neg(y = \underline{0} \vee \cdots \vee y = \underline{c-1})\end{matrix}}{T + \theta'(y) \vdash \neg(y < \underline{c})}}{T + y < \underline{c} + \theta'(y) \vdash \bot}}{T + \exists y{<}\underline{c}\ \theta' \vdash \bot}}{T \vdash \neg\exists y{<}\underline{c}\ \theta'}$$

So we know $T \vdash (\neg\exists y{<}t\ \theta)(\underline{\bar{a}})$ by Lemma 12.3 and (Leibniz+) as in the atomic case.

Apply the inductive assumption to find a polynomial $p(X) \in \mathbb{N}[X]$ such that each π_b has length at most $p\big(\exp^{(\mathrm{len}(\ulcorner\theta\urcorner))}\big(\sum_i a_i + b\big)\big)$. The proof for the $\neg\exists y{<}t\ \theta$ case is obtained by putting together $\pi_0, \pi_1, \ldots, \pi_{t(\bar{a})-1}$. So its length is bounded above by a polynomial in

$$\begin{aligned}
&\sum_{b<t(\bar{a})} p\Big(\exp^{(\mathrm{len}(\ulcorner\theta\urcorner))}\Big(\sum_i a_i + b\Big)\Big)\\
&\leqslant t(\bar{a}) \times p\Big(\exp^{(\mathrm{len}(\ulcorner\theta\urcorner))}\Big(\sum_i a_i + t(\bar{a})\Big)\Big)\\
&\leqslant \exp^{(\mathrm{len}(\ulcorner t\urcorner))}\Big(\sum_i a_i\Big) \times p\Bigg(\exp^{(\mathrm{len}(\ulcorner\theta\urcorner))}\Big(\sum_i a_i + \exp^{(\mathrm{len}(\ulcorner t\urcorner))}\Big(\sum_i a_i\Big)\Big)\Bigg) \quad \text{by Remark 12.2;}\\
&\leqslant \exp^{(\mathrm{len}(\ulcorner t\urcorner))}\Big(\sum_i a_i\Big) \times p\Bigg(\exp^{(\mathrm{len}(\ulcorner\theta\urcorner))}\Big(\exp^{(\mathrm{len}(\ulcorner t\urcorner)+1)}\Big(\sum_i a_i\Big)\Big)\Bigg) \quad \text{by (exp>) and (Qexp}_1\text{);}\\
&\leqslant \exp^{(\mathrm{len}(\ulcorner\exists y<t\ \theta\urcorner))}\Big(\sum_i a_i\Big) \times p\Big(\exp^{(\mathrm{len}(\ulcorner\exists y<t\ \theta\urcorner))}\Big(\sum_i a_i\Big)\Big).
\end{aligned}$$

All these polynomials depend only on the formula $\exists y{<}t\ \theta$, not on M or $\bar{a}$. So the inductive assumption is maintained.

This completes the induction for (1). For (2), fix $\bar{a} \in M \models \mathrm{I}\Delta_0(\exp)$ again. Suppose $M \models \exists\bar{y}\ \eta(\bar{a},\bar{y})$, where $\eta \in \Delta_0(\exp)$. Use the truth definition to find $\bar{b} \in M \models \eta(\bar{a},\bar{b})$. Reasoning informally in M, we see that $T \vdash \eta(\underline{\bar{a}},\underline{\bar{b}})$ by (1), and so repeated applications of ($\exists$R) shows $T \vdash \exists\bar{y}\ \eta(\underline{\bar{a}},\bar{y})$. ☐

Assignment 12.7. Let $M \models \mathrm{I}\Delta_0(\exp)$ and $\theta(\bar{x}), \eta(\bar{x})$ be $\Delta_0(\exp)$ formulas such that

(1) $M \models \forall\bar{x}\ \big(\theta(\bar{x}) \to \Box_T\,\mathrm{fsub}(\underline{\ulcorner\theta\urcorner}, \underline{\overline{\ulcorner x\urcorner}}, \mathrm{nmrl}(\bar{x}))\big) \wedge \forall\bar{x}\ \big(\neg\theta(\bar{x}) \to \Box_T\,\mathrm{fsub}(\underline{\ulcorner\neg\theta\urcorner}, \underline{\overline{\ulcorner x\urcorner}}, \mathrm{nmrl}(\bar{x}))\big)$;

(2) $M \models \forall\bar{x}\ \big(\eta(\bar{x}) \to \Box_T\,\mathrm{fsub}(\underline{\ulcorner\eta\urcorner}, \underline{\overline{\ulcorner x\urcorner}}, \mathrm{nmrl}(\bar{x}))\big) \wedge \forall\bar{x}\ \big(\neg\eta(\bar{x}) \to \Box_T\,\mathrm{fsub}(\underline{\ulcorner\neg\eta\urcorner}, \underline{\overline{\ulcorner x\urcorner}}, \mathrm{nmrl}(\bar{x}))\big)$.

Show that

$$\begin{aligned}
M \models\ &\forall\bar{x}\ \big((\theta\vee\eta)(\bar{x}) \to \Box_T\,\mathrm{fsub}(\underline{\ulcorner\theta\vee\eta\urcorner}, \underline{\overline{\ulcorner x\urcorner}}, \mathrm{nmrl}(\bar{x}))\big)\\
&\wedge \forall\bar{x}\ \big(\neg(\theta\vee\eta)(\bar{x}) \to \Box_T\,\mathrm{fsub}(\underline{\ulcorner\neg(\theta\vee\eta)\urcorner}, \underline{\overline{\ulcorner x\urcorner}}, \mathrm{nmrl}(\bar{x}))\big).
\end{aligned}$$

[5 points]

Remark 12.8. The bounds on the lengths of proofs we obtained are not optimal, mainly because our definition of numerals is not efficient: our numerals represent numbers in unary, and so their Gödel numbers are exponential in the numbers they represent. Binary numerals give better bounds; see the discussion in Lecture 25.

After three lectures of hard work, we finally reach the following theorem.

Second Incompleteness Theorem (Gödel)**.** If T is a recursive consistent $\mathcal{L}_\mathrm{A}(\exp)$ theory and $T \vdash \mathrm{I}\Delta_0(\exp)$, then there is a Σ_1 formula $\Box(y)$ such that $T \nvdash \neg\Box\bot$ and

$$\{a \in \mathbb{N} : \mathbb{N} \models \Box(a)\} = \{\ulcorner\theta\urcorner : \theta \text{ is an } \mathcal{L}_\mathrm{A}(\exp) \text{ formula and } T \vdash \theta\}.$$

Proof. Let $\Box = \Box_T$. By construction, it defines the set of all $\mathcal{L}_\mathrm{A}(\exp)$ formulas provable in T over $\mathbb{N}$. Since $T \vdash \mathrm{I}\Delta_0(\exp)$, this $\Box$ satisfies Löb's derivability conditions over T by Proposition 12.1. Thus Gödel's Incompleteness Theorems from Lecture 9 tell us that $T \nvdash \neg\Box\bot$. ☐

Remark 12.9. Notice $\mathrm{I}\Delta_0(\exp) \not\supseteq \mathrm{R}(\exp)$ as sets of sentences. So, strictly speaking, the Second Incompleteness Theorem as we stated it above does not apply to $\mathrm{I}\Delta_0(\exp)$ because of our global assumption on T. This problem can be overcome by extracting a polynomial $p(X) \in \mathbb{N}[X]$ from a proof of Proposition 4.5 such that whenever $\sigma \in \mathrm{R}(\exp)$, one can find a proof of $\mathrm{Q}(\exp) \vdash \sigma$ whose Gödel number has length less than $p(\ulcorner\sigma\urcorner)$.