MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Lecture 14: The Diagram Lemma

Tin Lok Wong

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The ultimate aim of these three lectures is to show Presburger's theorem mentioned in the previous lecture, that $\mathrm{Th}(\mathbb{N},+)$ is recursive.

Plan of the proof. Find a recursive complete $T \subseteq \mathrm{Th}(\mathbb{N},+)$. Then a straightforward adaptation of Proposition 13.4 implies that T is *decidable*, meaning

$$\{\sigma : \sigma \text{ is a sentence in the language of } (\mathbb{N},+) \text{ and } T \vdash \sigma\}$$

is recursive. This set must be $\mathrm{Th}(\mathbb{N},+)$.

To show the completeness of T, we employ an intermediate notion called *model completeness*, which we will discuss in more detail in the next lecture. In this lecture, we develop some basic notions in model theory in terms of which model completeness can be formulated.

14.1 General languages

So far in the module, we restricted ourselves to the language $\mathcal{L}_{\mathrm{A}}(\exp)$. Most definitions and theorems we saw actually make sense for other languages too, especially those about first-order logic. Most of this generalization is straightforward. We go through them quickly here.

Definition. A *language* $\mathcal{L}$ consists of

- a set of constant symbols $\mathrm{Const}(\mathcal{L})$;
- a set of function symbols $\mathrm{Fn}(\mathcal{L})$;
- a set of relation symbols $\mathrm{Rel}(\mathcal{L})$; and
- an arity function $\mathrm{Arity}(\mathcal{L})\colon \mathrm{Fn}(\mathcal{L}) \cup \mathrm{Rel}(\mathcal{L}) \to \mathbb{N}$

such that $\mathrm{Const}(\mathcal{L})$, $\mathrm{Fn}(\mathcal{L})$, and $\mathrm{Rel}(\mathcal{L})$ are mutually disjoint. A language is *countable* if $\mathrm{Const}(\mathcal{L}) \cup \mathrm{Fn}(\mathcal{L}) \cup \mathrm{Rel}(\mathcal{L})$ is countable.

Example 14.1. $\mathcal{L}_{\mathrm{A}}(\exp)$ is a language with

$$\mathrm{Const}(\mathcal{L}_{\mathrm{A}}(\exp)) = \{0,1\}, \quad \mathrm{Fn}(\mathcal{L}_{\mathrm{A}}(\exp)) = \{+,\times,\exp\}, \quad \mathrm{Rel}(\mathcal{L}_{\mathrm{A}}(\exp)) = \{<\},$$

and $\mathrm{Arity}(\mathcal{L}_{\mathrm{A}}(\exp))$ maps $+,\times,\exp,<$ to $2,2,1,2$ respectively.

Definition. Let $\mathcal{L}$ be a language. The notion of an $\mathcal{L}$ *term* is defined by recursion as follows.

- Every variable is an $\mathcal{L}$ term.
- All $c \in \mathrm{Const}(\mathcal{L})$ are $\mathcal{L}$ terms.
- If $F \in \mathrm{Fn}(\mathcal{L})$ and $t_1, t_2, \dots, t_{\mathrm{Arity}(\mathcal{L})(F)}$ are $\mathcal{L}$ terms, then $F(t_1, t_2, \dots, t_{\mathrm{Arity}(\mathcal{L})(F)})$ is an $\mathcal{L}$ term.
- Every $\mathcal{L}$ term is obtained by applying the rules above finitely many times.

All our languages has equality hardwired into it.

Definition. We define the notion of $\mathcal{L}$ *formulas* by recursion as follows.

(i) $\top$ is an $\mathcal{L}$ formula.

(ii) If t, s are $\mathcal{L}$ terms, then $t = s$ is an $\mathcal{L}$ formula.

(iii) If $R \in \operatorname{Rel}(\mathcal{L})$ and $t_1, t_2, \dots, t_{\operatorname{Arity}(\mathcal{L})(R)}$ are $\mathcal{L}$ terms, then $R(t_1, t_2, \dots, t_{\operatorname{Arity}(\mathcal{L})(R)})$ is an $\mathcal{L}$ formula.

(iv) If θ, η are $\mathcal{L}$ formulas, then $\neg\theta$ and $\theta \vee \eta$ are $\mathcal{L}$ formulas.

(v) If y is a variable and θ is an $\mathcal{L}$ formula which does not contain $\exists y$ as a substring, then $\exists y\ \theta$ is an $\mathcal{L}$ formula.

(vi) Every $\mathcal{L}$ formula is obtained by applying the construction rules above finitely many times.

The formulas in (i)–(iii) are called *atomic* $\mathcal{L}$ formulas.

Definition. Let $\mathcal{L}$ be a language. An $\mathcal{L}$ *structure* $\mathfrak{M}$ consists of

- a nonempty set M called the *universe* of $\mathfrak{M}$;
- an element $c^{\mathfrak{M}} \in M$ for each $c \in \operatorname{Const}(\mathcal{L})$;
- a function $F^{\mathfrak{M}} \colon M^{\operatorname{Arity}(\mathcal{L})(F)} \to M$ for each $F \in \operatorname{Fn}(\mathcal{L})$;
- a relation $R^{\mathfrak{M}} \subseteq M^{\operatorname{Arity}(\mathcal{L})(R)}$ for each $R \in \operatorname{Rel}(\mathcal{L})$.

When there is no risk of ambiguity, we often identify $\mathfrak{M}$ with M.

Definition. We define the evaluation of $\mathcal{L}$ terms by recursion on the term as follows. Let $t(x_1, x_2, \dots, x_k)$ be a term in a language $\mathcal{L}$, and $a_1, a_2, \dots, a_k$ be elements of an $\mathcal{L}$ structure M.

- If $t(\bar{x}) = x_i$, then $t^M(\bar{a}) = a_i$.
- If $t(\bar{x}) = c \in \operatorname{Const}(\mathcal{L})$, then $t^M(\bar{a}) = c^M$.
- If $t(\bar{x}) = F(t_1(\bar{x}), t_2(\bar{x}), \dots, t_k(\bar{x}))$ where $F \in \operatorname{Fn}(\mathcal{L})$, then

$$t^M(\bar{a}) = F^M(t_1^M(\bar{a}), t_2^M(\bar{a}), \dots, t_k^M(\bar{a})).$$

Definition (Tarski)**.** Let M be a structure for a language $\mathcal{L}$. We define $M \models \theta(\bar{a})$, where $\bar{a} \in M$, by recursion on the $\mathcal{L}$ formula $\theta(\bar{x})$ as follows.

- $M \models \top(\bar{a})$.
- For all $\mathcal{L}$ terms $t(\bar{x}), s(\bar{x})$,

$$M \models (t = s)(\bar{a}) \quad \Leftrightarrow \quad t^M(\bar{a}) = s^M(\bar{a}).$$

- For all $R \in \operatorname{Rel}(\mathcal{L})$ and all $\mathcal{L}$ terms $t_1(\bar{x}), t_2(\bar{x}), \dots, t_{\operatorname{Arity}(\mathcal{L})(R)}(\bar{x})$,

$$M \models R(t_1, t_2, \dots, t_{\operatorname{Arity}(\mathcal{L})(R)})(\bar{a}) \quad \Leftrightarrow \quad (t_1^M(\bar{a}), t_2^M(\bar{a}), \dots, t^M_{\operatorname{Arity}(\mathcal{L})(R)}(\bar{a})) \in R^M.$$

- For all $\mathcal{L}$ formulas $\theta(\bar{x})$,

$$M \models (\neg\theta)(\bar{a}) \quad \Leftrightarrow \quad M \not\models \theta(\bar{a}).$$

- For all $\mathcal{L}$ formulas $\theta(\bar{x}), \eta(\bar{x})$,

$$M \models (\theta \vee \eta)(\bar{a}) \quad \Leftrightarrow \quad M \models \theta(\bar{a}) \text{ or } M \models \eta(\bar{a}).$$

- For all $\mathcal{L}$ formulas $\theta(\bar{x}, y)$ which does not contain $\exists y$ as a substring,

$$M \models (\exists y\ \theta)(\bar{a}) \quad \Leftrightarrow \quad M \models \theta(\bar{a}, b) \text{ for some } b \in M.$$

Definition. Fix a language $\mathcal{L}$. Let Φ be a set of $\mathcal{L}$ formulas and $\theta(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_k)$ be an $\mathcal{L}$ formula.

- $\Phi \models \theta$ means whenever M is an $\mathcal{L}$ structure and $a_0, a_1, a_2, \ldots \in M$, if $M \models \varphi(a_0, a_1, \ldots, a_\ell)$ for all $\varphi(v_0, v_1, \ldots, v_\ell) \in \Phi$, then $M \models \theta(a_0, a_1, \ldots, a_k)$.
- $\Phi \vdash \theta$ means the sequent $\Phi \vdash \theta$ can be obtained by applying the deduction rules in Figure 6.1 finitely many times.

Since the deduction rules stay the same, proofs remain finitary.

Compactness Lemma. Fix a language $\mathcal{L}$. Let Φ be a set of $\mathcal{L}$ formulas and θ be an $\mathcal{L}$ formula. Then $\Phi \vdash \theta$ if and only if $\Phi_0 \vdash \theta$ for some finite $\Phi_0 \subseteq \Phi$. □

Our proof of the Completeness Theorem (from Lecture 7) directly generalizes to arbitrary *countable* languages. For larger languages, one needs additional constant symbols to witness existential statements; we used variables to do this, and there are only countably many variables.

Soundness and Completeness (Gödel)**.** Let $\mathcal{L}$ be a language. For all sets of $\mathcal{L}$ formulas Φ and all $\mathcal{L}$ formulas θ,

$$\Phi \vdash \theta \quad \Leftrightarrow \quad \Phi \models \theta.$$ □

14.2 Normal form theorems

One can transform a given formula using the laws of logic to an equivalent one in 'normal form' with which one is easier to work. Fix a language $\mathcal{L}$ throughout this section.

Laws of logic. Let $\theta, \eta, \varphi, \psi$ be $\mathcal{L}$ formulas. Then the following are provable (from the empty theory $\varnothing$):

$$\begin{array}{ccc l}
\theta \vee (\varphi \vee \psi) & \leftrightarrow & (\theta \vee \varphi) \vee \psi & \multirow{2}{*}{\text{associativity}} \\
\theta \wedge (\varphi \wedge \psi) & \leftrightarrow & (\theta \wedge \varphi) \wedge \psi & \\
\theta \vee \eta & \leftrightarrow & \eta \vee \theta & \text{commutativity} \\
\theta \wedge \eta & \leftrightarrow & \eta \wedge \theta & \\
\neg\neg\theta & \leftrightarrow & \theta & \text{double negation} \\
\theta \wedge (\varphi \vee \psi) & \leftrightarrow & (\theta \wedge \varphi) \vee (\theta \wedge \psi) & \text{distributivity} \\
\theta \vee (\varphi \wedge \psi) & \leftrightarrow & (\theta \vee \varphi) \wedge (\theta \vee \psi) & \\
\neg(\theta \vee \eta) & \leftrightarrow & \neg\theta \wedge \neg\eta & \text{De Morgan Laws} \\
\neg(\theta \wedge \eta) & \leftrightarrow & \neg\theta \vee \neg\eta & \\
\neg\exists y\ \theta & \leftrightarrow & \forall y\ \neg\theta & \\
\neg\forall y\ \theta & \leftrightarrow & \exists y\ \neg\theta &
\end{array}$$

If the variable x does not appear in $\exists y\ \eta$ and the variable y does not appear in $\forall x\ \theta$, then the following are also provable:

$$\left.\begin{array}{ccc}
\exists x\ \theta \vee \exists y\ \eta & \leftrightarrow & \exists x\ \exists y\ (\theta \vee \eta) \\
\forall x\ \theta \vee \forall y\ \eta & \leftrightarrow & \forall x\ \forall y\ (\theta \vee \eta) \\
\forall x\ \theta \vee \exists y\ \eta & \leftrightarrow & \forall x\ \exists y\ (\theta \vee \eta)
\end{array}\right\}\ \text{restricted distributivity}$$

Definition. A formula is in *disjunctive normal form (DNF)* if it is of the form

$$\bigvee_{i<k} \bigwedge_{j<\ell} \alpha_{i,j},$$

where each $\alpha_{i,j}$ is either atomic or negated atomic.

Proposition 14.2. Every quantifier-free formula is equivalent to one in DNF.

Proof. It suffices to show that the set of formulas in DNF contains all the atomic formulas and is closed under negation and disjunction, because the set of quantifier-free formulas is the smallest set with these properties.

Atomic formulas are already in DNF (with exactly one disjunct and one conjunct). The disjunction of two formulas in DNF is a bigger disjunction in which every disjunct is a conjunction of atomic or negated atomic formulas. Thus formulas in DNF are closed under disjunction. The only remaining case is the one for negation.

Consider the formula $\bigvee_{i<k} \bigwedge_{j<\ell} \alpha_{i,j}$, where each $\alpha_{i,j}$ is either atomic or negated atomic. By the De Morgan Laws and the Law of Double Negation, the negation of this formula is equivalent to $\bigwedge_{i<k} \bigvee_{j<\ell} \alpha'_{i,j}$, where

$$\alpha'_{i,j} = \begin{cases} \neg\alpha_{i,j} & \text{if } \alpha_{i,j} \text{ is atomic;} \\ \beta_{i,j} & \text{if } \alpha_{i,j} = \neg\beta_{i,j}. \end{cases}$$

The formula $\bigwedge_{i<k} \bigvee_{j<\ell} \alpha'_{i,j}$, with the $\bigwedge$ and $\bigvee$ notation unravelled, is

$$(\alpha'_{0,0} \vee \alpha'_{0,1} \vee \cdots \vee \alpha'_{0,\ell-1}) \wedge (\alpha'_{1,0} \vee \alpha'_{1,1} \vee \cdots \vee \alpha'_{1,\ell-1}) \wedge \cdots \wedge (\alpha'_{k-1,0} \vee \alpha'_{k-1,1} \vee \cdots \vee \alpha'_{k-1,\ell-1}).$$

Conjuncting through using the Distributive Law gives an equivalent formula that is a disjunction of conjunctions of the $\alpha'_{i,j}$'s, and is hence in DNF. More precisely, this equivalent formula is

$$\bigvee_{f\colon \{0,1,\ldots,k-1\}\to\{0,1,\ldots,\ell-1\}} \bigwedge_{i<k} \alpha'_{i,f(i)}. \qquad \square$$

Definition. A formula is in *prenex normal form (PNF)* if it is of the form

$$\mathrm{Q}_1 y_1\ \mathrm{Q}_2 y_2\ \cdots\ \mathrm{Q}_n y_n\ \alpha(\bar{x}, \bar{y}),$$

where $\mathrm{Q}_1, \mathrm{Q}_2, \ldots, \mathrm{Q}_n \in \{\exists, \forall\}$ and α is quantifier-free.

Proposition 14.3. Every formula is equivalent to one in PNF.

Proof. It suffices to show that the set of formulas in PNF contains all the atomic formulas and is closed under negation, disjunction and existential quantification, because the set of all formulas is the smallest set with these properties.

Atomic formulas are already in PNF (with zero quantifier). Formulas in PNF are clearly closed under existential quantification. So the only non-trivial cases are the ones for negation and for disjunction. Consider the formulas $\mathrm{Q}_1 y_1\ \mathrm{Q}_2 y_2\ \cdots\ \mathrm{Q}_k y_k\ \alpha(\bar{x}, \bar{y})$ and $\mathrm{Q}'_1 z_1\ \mathrm{Q}'_2 z_2\ \cdots\ \mathrm{Q}'_k z_\ell\ \beta(\bar{x}, \bar{z})$, where $\mathrm{Q}_1, \mathrm{Q}_2, \ldots, \mathrm{Q}_n, \mathrm{Q}'_1, \mathrm{Q}'_2, \ldots, \mathrm{Q}'_n \in \{\exists, \forall\}$ and α, β are quantifier-free.

$\boxed{\neg}$ In view of the De Morgan Laws,

$$\vdash \neg \mathrm{Q}_1 y_1\ \mathrm{Q}_2 y_2\ \cdots\ \mathrm{Q}_k y_k\ \alpha(\bar{x}, \bar{y}) \leftrightarrow \overline{\mathrm{Q}}_1 y_1\ \overline{\mathrm{Q}}_2 y_2\ \cdots\ \overline{\mathrm{Q}}_k y_k\ \neg\alpha(\bar{x}, \bar{y}),$$

where $\overline{\exists} = \forall$ and $\overline{\forall} = \exists$.

$\boxed{\vee}$ Applying a change of variables if needed, we may assume $\bar{y}$ do not appear in $\exists\bar{z}\ \beta$ and $\bar{z}$ do not appear in $\forall\bar{y}\ \alpha$. Then by restricted distributivity,

$$\begin{aligned} \vdash\ & \mathrm{Q}_1 y_1\ \mathrm{Q}_2 y_2\ \cdots\ \mathrm{Q}_k y_k\ \alpha(\bar{x}, \bar{y}) \vee \mathrm{Q}'_1 z_1\ \mathrm{Q}'_2 z_2\ \cdots\ \mathrm{Q}'_k z_\ell\ \beta(\bar{x}, \bar{z}) \\ & \leftrightarrow\ \mathrm{Q}_1 y_1\ \mathrm{Q}_2 y_2\ \cdots\ \mathrm{Q}_k y_k\ \mathrm{Q}'_1 z_1\ \mathrm{Q}'_2 z_2\ \cdots\ \mathrm{Q}'_k z_\ell\ \big(\alpha(\bar{x}, \bar{y}) \vee \beta(\bar{x}, \bar{z})\big). \end{aligned} \qquad \square$$

Since every formula is equivalent to one in PNF, one can classify formulas in terms of the (minimum) number of quantifier blocks in equivalent PNF formulas, as in arithmetic.

Definition. Let $n \in \mathbb{N}$.

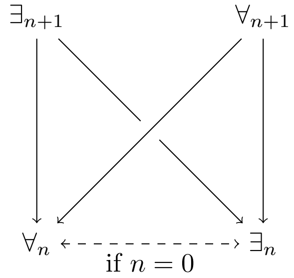

Figure 14.1: Inclusions between the $\exists_n$'s and the $\forall_n$'s

- $\exists_n(\mathcal{L})$ denotes the class of all $\mathcal{L}$ formulas of the form

$$\exists \bar{y}_1 \, \forall \bar{y}_2 \, \cdots \, \mathrm{Q} \bar{y}_n \; \alpha(\bar{x}, \bar{y}),$$

  where $\mathrm{Q} \in \{\exists, \forall\}$ and α is quantifier-free.

- $\forall_n(\mathcal{L})$ denotes the class of all $\mathcal{L}$ formulas of the form

$$\forall \bar{y}_1 \, \exists \bar{y}_2 \, \cdots \, \mathrm{Q}' \bar{y}_n \; \alpha(\bar{x}, \bar{y}),$$

  where $\mathrm{Q}' \in \{\exists, \forall\}$ and α is quantifier-free.

When there is no risk of ambiguity, we omit the reference to $\mathcal{L}$.

Note that both $\exists_0$ and $\forall_0$ refer to the set of quantifier-free formulas. Note also that a tuple $\bar{y}_i$ in the definition of $\forall_n$ and $\exists_n$ can be empty. By appending an empty quantifier block on the left/right of the quantifier blocks of an $\exists_n/\forall_n$ formula, one sees that $\exists_n \cup \forall_n \subseteq \forall_{n+1} \cap \exists_{n+1}$ for every $n \in \mathbb{N}$; cf. Figure 14.1. In particular, we know $\bigcup_{n \in \mathbb{N}} \exists_n = \bigcup_{n \in \mathbb{N}} \forall_n$, and every formula is equivalent to one in this big union.

14.3 Embeddings and diagrams

One of the most basic relations between structures are *embeddings*. Roughly speaking, they are functions that preserve the interpretations of all the symbols in the language, including equality.

Definition. Let K, M be structures in a language $\mathcal{L}$. A function $j \colon K \to M$ is an $\mathcal{L}$ *embedding* if

- for all $a, b \in K$, $\quad j(a) = j(b) \quad \Leftrightarrow \quad a = b$;
- for all $c \in \mathrm{Const}(\mathcal{L})$, $\quad j(c^K) = c^M$;
- for all $F \in \mathrm{Fn}(\mathcal{L})$ and all $a_1, a_2, \ldots, a_{\mathrm{Arity}(\mathcal{L})(F)} \in K$,

$$j\big(F^K(a_1, a_2, \ldots, a_{\mathrm{Arity}(\mathcal{L})(F)})\big) = F^M\big(j(a_1), j(a_2), \ldots, j(a_{\mathrm{Arity}(\mathcal{L})(F)})\big);$$

- for all $R \in \mathrm{Rel}(\mathcal{L})$ and all $a_1, a_2, \ldots, a_{\mathrm{Arity}(\mathcal{L})(R)} \in K$,

$$(a_1, a_2, \ldots, a_{\mathrm{Arity}(\mathcal{L})(R)}) \in R^K \quad \Leftrightarrow \quad \big(j(a_1), j(a_2), \ldots, j(a_{\mathrm{Arity}(\mathcal{L})(R)})\big) \in R^M.$$

In this case, we often identify K with its image under j, and call K a *substructure* of M, or M an *extension* of K. In symbols, we write $K \subseteq M$.

Example 14.4. Recall from Convention 3.5 that $\mathbb{N}$ is a substructure of every model of $\mathrm{R}(\exp)$.

One of the most basic piece of information about a structure is what atomic and negated atomic formulas are true on which parameters. For instance, since every quantifier-free formula is equivalent to one in DNF, this information determines completely what quantifier-free formulas are true on which parameters in the structure. The *diagram* of a structure records such information.

Definition. Let M be a structure in a language $\mathcal{L}$.

- Let A be a set. Denote by $\mathcal{L}(A)$ the language obtained from $\mathcal{L}$ by adding a new constant symbol a for every $a \in A$.
- If $A \subseteq M$, then one can view M as an $\mathcal{L}(A)$ structure by setting $a^M = a$ for every $a \in A$.
- The *diagram* of M is defined by

$$\mathrm{Diag}(M) = \{\sigma : \sigma \text{ is a quantifier-free } \mathcal{L}(M) \text{ sentence and } M \models \sigma\}.$$

Some authors define the diagram of a structure M in a language $\mathcal{L}$ to be the set of all atomic and negated atomic $\mathcal{L}(M)$ sentences true in M. As we saw, their definition is equivalent to ours.

Example 14.5. The $\mathcal{L}_{\mathrm{A}}(\exp)$ theory $\mathrm{R}(\exp)$ is defined to make $\mathrm{R}(\exp) \vdash \mathrm{Diag}(\mathbb{N})$; see Observation 3.4.

The coupling of Example 14.4 and Example 14.5 is a particular case of a general phenomenon known as the *Diagram Lemma*. It makes various logical tools, such as the Compactness Lemma, applicable to the study of extensions. The proof is simply a matter of chasing definitions: since an embedding $K \to M$ preserves the interpretations of all symbols in the language, the structure M satisfies all the atomic and negated atomics formulas true in K via the embedding; conversely, if a structure M can be expanded to satisfy the diagram of another structure K, then this expansion identifies a copy of K inside M in which the interpretations of all symbols agree with those in M.

Diagram Lemma. The following are equivalent for all structures K, M in a language $\mathcal{L}$.

(i) There is an $\mathcal{L}$ embedding $j : K \to M$.

(ii) There are elements $(c_a)_{a \in K}$ of M such that

$$M^* = (M, c_a)_{a \in K} \models \mathrm{Diag}(K),$$

where M^* is the $\mathcal{L}(K)$ structure

- whose universe and whose interpretations of the symbols in $\mathcal{L}$ are the same as those in M; and
- such that $a^{M^*} = c_a$ for all $a \in K$.

Proof. For (i) $\Rightarrow$ (ii), let $c_a = j(a)$ for every $a \in K$. We leave the routine verification that $(M, c_a)_{a \in K} \models \mathrm{Diag}(K)$ to the reader.

For (ii) $\Rightarrow$ (i), we show that $j : a \mapsto c_a$ is an $\mathcal{L}$ embedding $K \to M$. Equality can be dealt with in the same way as any other relation in the language, and constants can be treated as functions of arity 0. Thus we only need to look at functions and relations.

Let $F \in \mathrm{Fn}(\mathcal{L})$ **and** $k = \mathrm{Arity}(\mathcal{L})(F)$**.** Whenever $a_1, a_2, \ldots, a_k \in K$ and $b = F^K(\bar{a})$,

	$K \models b = F(\bar{a})$	by the truth definition;
$\therefore$	$(b = F(\bar{a})) \in \mathrm{Diag}(K)$	by the definition of Diag;
$\therefore$	$M^* \models b = F(\bar{a})$	as $M^* \models \mathrm{Diag}(K)$;
$\therefore$	$c_b = F^{M^*}(c_{a_1}, c_{a_2}, \ldots, c_{a_k}) = F^M(c_{a_1}, c_{a_2}, \ldots, c_{a_k})$	by the definition of M^*;
$\therefore$	$j(F^K(\bar{a})) = j(b) = F^M(j(a_1), j(a_2), \ldots, j(a_k))$	by the definitions of b and j.

Let $R \in \mathrm{Rel}(\mathcal{L})$ and $\ell = \mathrm{Arity}(\mathcal{L})(R)$. If $(a_1, a_2, \dots, a_\ell) \in R^K$, then

	$K \models R(\bar{a})$	by the truth definition;
$\therefore$	$R(\bar{a}) \in \mathrm{Diag}(K)$	by the definition of Diag;
$\therefore$	$M^* \models R(\bar{a})$	as $M^* \models \mathrm{Diag}(K)$;
$\therefore$	$\big(j(a_1), j(a_2), \dots, j(a_\ell)\big) = (c_{a_1}, c_{a_2}, \dots, c_{a_\ell}) \in R^{M^*} = R^M$	by the definitions of j and M^*.

The converse is similar: if $(a_1, a_2, \dots, a_\ell) \in M^\ell \setminus R^K$, then

	$K \models \neg R(\bar{a})$	by the truth definition;
$\therefore$	$(\neg R(\bar{a})) \in \mathrm{Diag}(K)$	by the definition of Diag;
$\therefore$	$M^* \models \neg R(\bar{a})$	as $M^* \models \mathrm{Diag}(K)$;
$\therefore$	$\big(j(a_1), j(a_2), \dots, j(a_\ell)\big) = (c_{a_1}, c_{a_2}, \dots, c_{a_\ell}) \notin R^{M^*} = R^M$	by the definitions of j and M^*.

□

We also have an analogue of Proposition 3.6, which says that, in an extension, quantifier-free formulas are absolute, and $\exists_1$ formulas are preserved upwards.

Lemma 14.6. Let K, M be structures in a language $\mathcal{L}$ such that $K \subseteq M$.

(1) For all $\theta(\bar{x}) \in \forall_0$ and $\bar{a} \in K$,

$$K \models \theta(\bar{a}) \quad \Leftrightarrow \quad M \models \theta(\bar{a}).$$

(2) For all $\theta(\bar{x}) \in \exists_1$ and $\bar{a} \in K$,

$$K \models \theta(\bar{a}) \quad \Rightarrow \quad M \models \theta(\bar{a}).$$

Proof of (2). Suppose $K \models \exists \bar{y}\ \eta(\bar{a}, \bar{y})$, where $\eta \in \forall_0$. Find $\bar{b} \in K \models \eta(\bar{a}, \bar{b})$ using the truth definition. Then

	$M \models \eta(\bar{a}, \bar{b})$	by (1);
$\therefore$	$M \models \exists \bar{y}\ \eta(\bar{a}, \bar{y})$	by the truth definition. □

Assignment 14.7. Prove Lemma 14.6(1). (Hint: Both Proposition 14.2 and the Diagram Lemma can be helpful.) [5 points]

In the next lecture, we will see how the Diagram Lemma, combined with some ideas from algebra, helps in establishing the completeness of theories.