MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Lecture 15: Model completeness

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The aim of this lecture is to introduce the notion of model completeness. This notion will be key in our proof that $\mathrm{Th}(\mathbb{N}, +)$ is recursive next time. The results presented below were developed by Abraham Robinson. Fix a language $\mathcal{L}$ throughout this lecture.

In view of the Diagram Lemma, to show that a structure K has an extension satisfying a theory T, it suffices to establish the consistency of $T + \mathrm{Diag}(K)$. This consistency is often established via the Compactness Lemma. Since $\mathrm{Diag}(K)$ is closed under conjunction, showing the consistency of T with any finite subset of $\mathrm{Diag}(K)$ is equivalent to showing the consistency of T with any singleton subset of $\mathrm{Diag}(K)$. This observation will simplify the presentation of our proofs slightly.

Remark 15.1. Let Φ, Ψ be sets of $\mathcal{L}$ formulas such that Ψ is closed under conjunction up to equivalence. If $\Phi + \psi \nvdash \bot$ for each $\psi \in \Psi$, then $\Phi + \Psi \nvdash \bot$.

Proof. In view of the Compactness Lemma, it suffices to show the consistency of every finite subset of $\Phi + \Psi$. Take $\psi_1, \psi_2, \ldots, \psi_k \in \Psi$. As Ψ is closed under conjunction, we get $\psi \in \Psi$ such that $\vdash \psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_k \leftrightarrow \psi$. By hypothesis, we know $\Phi + \psi \nvdash \bot$. So the equivalence implies $\Phi + \{\psi_1, \psi_2, \ldots, \psi_k\} \nvdash \bot$. □

We saw in Lemma 14.6(2) that the truth of all $\exists_1$ formulas are preserved upwards in extensions. Contrapositively, the truth of all $\forall_1$ formulas are preserved downwards in substructures. In fact, the $\forall_1$ formulas are the only formulas whose truth are always preserved downwards in substructures.

Preservation Theorem for $\forall_1$ formulas. For every $\mathcal{L}$ theory T and every $\mathcal{L}$ formula $\varphi(\bar{x})$, the following are equivalent.

(i) For all $K, M \models T$ with $K \subseteq M$ and all $\bar{a} \in K$,

$$M \models \varphi(\bar{a}) \quad \Rightarrow \quad K \models \varphi(\bar{a}).$$

(ii) There is $\psi(\bar{x}) \in \forall_1$ such that $T \vdash \forall \bar{x}\ \big(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})\big)$.

Proof. The implication (ii) ⇒ (i) follows directly from (the contrapositive of) Lemma 14.6(2). So let us concentrate on (i) ⇒ (ii). Suppose (ii) fails. The main idea is to consider the set of all $\forall_1$ consequences of $\varphi(\bar{x})$ over T:

$$\Psi(\bar{x}) = \big\{\psi(\bar{x}) \in \forall_1 : T \vdash \forall \bar{x}\ \big(\varphi(\bar{x}) \rightarrow \psi(\bar{x})\big)\big\}.$$

Since $\varphi(\bar{x})$ is not equivalent to any $\forall_1$ formula over T by the failure of (ii), we know that $\Psi(\bar{x})$ is strictly weaker than $\varphi(\bar{x})$ over T. So there is a model of T in which $\Psi(\bar{x})$ holds but $\varphi(\bar{x})$ does not. This is our structure K. Since K satisfies all the $\forall_1$ consequences of $\varphi(\bar{x})$ over T, it can be extended to $M \models T + \varphi(\bar{x})$. The pair of structures (M, K) thus witnesses the failure of (i).

To execute the proof, first note that $T + \Psi(\bar{x}) + \neg\varphi(\bar{x}) \nvdash \bot$ by Remark 15.1, because $\Psi(\bar{x})$ is closed under conjunction up to equivalence, and for all $\psi(\bar{x}) \in \Psi(\bar{x})$,

$$\begin{array}{lll} & T \nvdash \forall \bar{x}\ \big(\psi(\bar{x}) \rightarrow \varphi(\bar{x})\big) & \text{as (ii) fails;} \\ \therefore & T + \psi(\bar{x}) + \neg\varphi(\bar{x}) \nvdash \bot & \text{by logic.} \end{array}$$

Use the Completeness Theorem to find $\bar{a} \in K \models T + \Psi(\bar{a}) + \neg\varphi(\bar{a})$.

To get the structure M we want, it suffices to show

$$T + \varphi(\bar{a}) + \mathrm{Diag}(K) \nvdash \bot$$

in view of the Diagram Lemma. Note that $\mathrm{Diag}(K)$ is closed under conjunction. So we can apply Remark 15.1 again. Take $\alpha(\bar{x}, \bar{y}) \in \forall_0(\mathcal{L})$ and $\bar{b} \in K \setminus \{\bar{a}\}$ such that $\alpha(\bar{a}, \bar{b}) \in \mathrm{Diag}(K)$. Then

	$K \models \alpha(\bar{a}, \bar{b})$	by the definition of $\mathrm{Diag}(K)$;
$\therefore$	$K \models \exists \bar{y}\ \alpha(\bar{a}, \bar{y})$	by the truth definition;
$\therefore$	$\big(\forall \bar{y}\ \neg\alpha(\bar{a}, \bar{y})\big) \notin \Psi(\bar{a})$	as $K \models \Psi(\bar{a})$;
$\therefore$	$T \nvdash \forall \bar{x}\ \big(\varphi(\bar{x}) \to \forall \bar{y}\ \neg\alpha(\bar{x}, \bar{y})\big)$	by the definition of $\Psi(\bar{x})$, as $\forall \bar{y}\ \neg\alpha(\bar{x}, \bar{y}) \in \forall_1$;
$\therefore$	$T + \varphi(\bar{x}) + \exists \bar{y}\ \alpha(\bar{x}, \bar{y}) \nvdash \bot$	by logic;
$\therefore$	$T + \varphi(\bar{a}) + \alpha(\bar{a}, \bar{b}) \nvdash \bot$	

as T does not mention $\bar{a}, \bar{b}$, and $\bar{x}, \bar{y}$ do not appear free in T, and $\bar{y} \notin \{\bar{x}\}$ and $\bar{b} \notin \{\bar{a}\}$. □

In the context of arithmetic, there is a similar preservation theorem for Π_1 formulas in terms of initial segments or end extensions. There one needs to assume that the theory involved contains some combinatorial axioms necessary for the existence of end extensions.

The second half of our proof of the Preservation Theorem actually shows a more general fact: if a structure satisfies all the $\forall_1$ consequences of a theory, then it must have an extension satisfying this theory. Sometimes $\forall_1\text{-Th}(T)$ below is denoted $T_\forall$ in the literature.

Assignment 15.2. Show that if T is an $\mathcal{L}$ theory and $K \models \forall_1\text{-Th}(T)$, where

$$\forall_1\text{-Th}(T) = \{\sigma : \sigma \text{ is an } \forall_1 \text{ sentence and } T \vdash \sigma\},$$

then K has an extension $M \models T$. [4 points]

Note that the Preservation Theorem for $\forall_1$ formulas does *not* give formulas that are not preserved downwards in substructures: although there are many formulas which are (syntactically) not $\forall_1$, perhaps all of them are equivalent to some $\forall_1$ formula over some special theories. Such special theories satisfy the conditions listed in the theorem below.

Theorem 15.3. The following are equivalent for any $\mathcal{L}$ theory T.

(i) Whenever $K, M \models T$ with $K \subseteq M$, we have $K \preccurlyeq M$, i.e., for all $\mathcal{L}$ formulas $\varphi(\bar{x})$ and $\bar{a} \in K$,

$$M \models \varphi(\bar{a}) \quad \Rightarrow \quad K \models \varphi(\bar{a}).$$

(ii) Whenever $K, M \models T$ with $K \subseteq M$, we have $K \preccurlyeq_1 M$, i.e., for all $\varphi(\bar{x}) \in \exists_1$ and $\bar{a} \in K$,

$$M \models \varphi(\bar{a}) \quad \Rightarrow \quad K \models \varphi(\bar{a}).$$

(iii) For every $\varphi(\bar{x}) \in \exists_1$, there is $\psi(\bar{x}) \in \forall_1$ such that

$$T \vdash \forall \bar{x}\ \big(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})\big).$$

(iv) For every $\mathcal{L}$ formula $\varphi(\bar{x})$, there is $\psi(\bar{x}) \in \forall_1$ such that

$$T \vdash \forall \bar{x}\ \big(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})\big).$$

Proof. The implication (i) ⇒ (ii) is obvious because all $\exists_1$ formulas are $\mathcal{L}$ formulas. For (ii) ⇒ (iii) and (iv) ⇒ (i), simply apply the Preservation Theorem for $\forall_1$ formulas. So it remains to show (iii) ⇒ (iv). The idea is to use (iii) to turn every quantifier block in the given formula into a universal one inductively.

To see how this is done in more detail, assume (iii) holds. In view of the PNF Theorem (i.e., Proposition 14.3), it suffices to show that for every $n \in \mathbb{N}$,

$$\forall \varphi(\bar{x}) {\in} \forall_n \quad \exists \psi(\bar{x}) {\in} \forall_1 \quad T \vdash \forall \bar{x}\ \big(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})\big).$$

We proceed by induction on n. The $n = 0$ case is true because $\forall_0 \subseteq \forall_1$. For the induction step, let $n \in \mathbb{N}$ such that the displayed line above holds. Pick $\varphi(\bar{x}) \in \forall_{n+1}$. Suppose $\varphi(\bar{x}) = \forall \bar{y}\ \varphi_0(\bar{x}, \bar{y})$ where $\varphi_0 \in \exists_n$. Apply the induction hypothesis to find $\varphi_0'(\bar{x}, \bar{y}) \in \forall_1$ such that

$$T \vdash \forall \bar{x}, \bar{y}\ \big(\neg\varphi_0(\bar{x}, \bar{y}) \leftrightarrow \varphi_0'(\bar{x}, \bar{y})\big).$$

Using (iii), we obtain $\psi_0(\bar{x}, \bar{y}) \in \forall_1$ which satisfies

$$T \vdash \forall \bar{x}, \bar{y}\ \big(\neg\varphi_0'(\bar{x}, \bar{y}) \leftrightarrow \psi_0(\bar{x}, \bar{y})\big).$$

Then $\big(\forall \bar{y}\ \psi_0(\bar{x}, \bar{y})\big) \in \forall_1$ and $T \vdash \forall \bar{x}\ \big(\varphi(\bar{x}) \leftrightarrow \forall \bar{y}\ \psi_0(\bar{x}, \bar{y})\big)$, as the reader can readily verify. □

Definition. An $\mathcal{L}$ theory T is *model complete* if it satisfies one (or equivalently all) of the conditions in Theorem 15.3.

In condition 15.3(i), the converse to the $\Rightarrow$ arrow actually also holds because $\mathcal{L}$ formulas are closed under negation. The converse to the $\Rightarrow$ arrow in condition 15.3(ii) follows from Lemma 14.6(2). In model-theoretic terminology, condition 15.3(ii) says that every model $M \models T$ is *existentially closed*, i.e., if some quantifier-free formula (possibly with parameters from M) is satisfiable in some model of T extending M, then it is already satisfiable in M. This is reminiscent of, and in fact closely related to, the notion of algebraic closedness in the context of fields.

Example 15.4. The theory ACF of algebraically closed fields is formulated in $\mathcal{L}_\mathrm{R} = \{0, 1, +, \times\}$. It consists of the field axioms and a scheme asserting 'every non-constant polynomial with coefficients in the field has a zero'.

- ACF is model complete. The definition of algebraic closedness concerns only solutions to polynomial equations, while $\exists_1$ formulas can assert the existence of solutions to a finite system of equations and inequations. The equivalence of the two in this particular example follows from a theorem of Hilbert's often referred to as the *Nullstellensatz*.
- ACF is not complete because there are algebraically closed fields of different characteristics. The specification of a characteristic makes ACF complete. In particular,

  ACF + "the field has characteristic 0"

  axiomatizes $\mathrm{Th}(\mathbb{C}, 0, 1, +, \times)$.

One can describe the theory of real numbers in a similar way. One cannot use the Completeness Axiom here because it is second-order: it quantifies over arbitrary subsets of real numbers. In a first-order axiomatization, one is only able to quantify over numbers. Amongst the many options, we choose the Intermediate Value Theorem.

Example 15.5. The theory RCOF of real-closed ordered fields is formulated in $\mathcal{L}_\mathrm{OR} = \{0, 1, +, \times, <\}$. It consists of the ordered field axioms and a scheme asserting the following intermediate value property for every polynomial $p(X)$ with coefficients in the field:

$$\forall a, b\ \big(a < b \wedge p(a) < 0 \wedge 0 < p(b) \rightarrow \exists c\ (a < c \wedge c < b \wedge p(c) = 0)\big).$$

RCOF is model complete and complete. Thus it axiomatizes $\mathrm{Th}(\mathbb{R}, 0, 1, +, \times, <)$.

Condition 15.3(iv) says that the whole $\forall_n$ hierarchy collapses to the $\forall_1$ level over a model complete theory. In arithmetic, such a collapse usually does not happen. The properness of the formula hierarchy distinguishes arithmetic from algebraic theories.

Non-example 15.6. No consistent $\mathcal{L}_\mathrm{A}(\exp)$ theory $T \supseteq \mathrm{I}\Delta_0(\exp)$ is model complete.

Proof sketch. By negating a Σ_1 formula which corresponds to a suitable $\mathcal{L}_\mathrm{A}(\exp)$ interpreter, one can construct a Π_1 formula $\psi(y)$ such that for all Π_1 sentences σ,

$$\mathrm{I}\Delta_0(\exp) \vdash \sigma \leftrightarrow \psi(\underline{\sigma}).$$

This $\psi(y)$ is not equivalent to any Σ_1 formula over T by Assignment 5.3. So $\neg\psi(y)$ is not equivalent to any Π_1 (and a fortiori $\forall_1$) formula over T. □

In the next lecture, we will present a recursive $T \subseteq \mathrm{Th}(\mathbb{N}, +)$ whose completeness will be established via model completeness.