MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Lecture 18: Peano Arithmetic

Tin Lok Wong

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The aim of this lecture is to introduce Peano arithmetic and explain why ordinals are relevant for proving its consistency.

Peano Arithmetic was intended to capture all truth about the natural numbers. In fact, the original second-order axioms put forward by Peano (which involve quantification over all sets of natural numbers) characterize the natural numbers completely. Nevertheless, the most commonly studied version of Peano Arithmetic nowadays is based on first-order logic.

Definition. The $\mathcal{L}_\mathrm{A}(\exp)$ theory PA consists of the elements of $\mathrm{Q}(\exp)$ and an induction axiom

$$\forall\bar{z}\ \big(\theta(0,\bar{z})\wedge\forall x\ \big(\theta(x,\bar{z})\rightarrow\theta(x+1,\bar{z})\big)\rightarrow\forall x\ \theta(x,\bar{z})\big)$$

for each $\mathcal{L}_\mathrm{A}(\exp)$ formula $\theta(x,\bar{z})$.

In our construction of $\Box(y)$ in Lecture 10, we invoked a fact about the natural numbers when extracting the bound w. This fact is called *collection* or *bounding* in the literature. As Parsons showed, collection is not provable in $\mathrm{I}\Delta_0(\exp)$, even when restricted to Δ_0 formulas. However, it is directly provable in PA.

Assignment 18.1. Show by induction on a that for every $\mathcal{L}_\mathrm{A}(\exp)$ formula $\eta(x,y)$,

$$\mathrm{PA}\vdash\forall a\ \big(\forall x{<}a\ \exists y\ \eta(x,y)\rightarrow\exists b\ \forall x{<}a\ \exists y{<}b\ \eta(x,y)\big).\qquad\text{[3 points]}$$

While restriction to first-order logic buys us a recursive finitary proof system, we lose completeness: by the First Incompleteness Theorem, PA must be incomplete. Since PA was intended to capture all truth about the natural numbers, this incompleteness is (historically) more striking than the incompleteness of $\mathrm{I}\Delta_0(\exp)$. Technically, it is not too hard to produce a true sentence that is unprovable in $\mathrm{I}\Delta_0(\exp)$, e.g., collection, without invoking the Second Incompleteness Theorem. On the contrary, doing the same for PA without invoking its consistency seems difficult.

To give a meaningful proof of the consistency of PA, we need to understand the notion of PA-provability better. This begs for a proof system that is more easily analyzable, instead of the completeness-oriented proof system we chose in Lecture 6. Some symmetry helps in this situation.

Definition (plus some explanation)**.** In the following, we introduce Gentzen's system **LK**, which stands for *logistischer klassischer Kalkül.*

- The logical symbols are

  $$(\quad)\quad,\quad\mathsf{v}_i\quad\neg\quad\wedge\quad\vee\quad\forall\quad\exists$$

  where i ranges over $\mathbb{N}$. Neither $\top$ nor $=$ is a logical symbol, but one can define them in terms of non-logical symbols (if such are present).

- To get more symmetry, we allow more than one formula to appear on either side of the turnstile in a sequent. More precisely, a *sequent* in **LK** is an expression of the form

  $$\Phi\vdash\Psi$$

  where Φ and Ψ are (possibly empty) finite sequences (not sets) of formulas. The intended meaning of the sequent $\Phi\vdash\Psi$ is '$\bigwedge\!\!\!\bigwedge\Phi$ entails $\bigvee\!\!\!\bigvee\Psi$'. Beware of the fact that the formulas on the right of the turnstile are *not put in a conjunction* but a disjunction. This ensures the two sides of a sequent are symmetric because $\Phi\vdash\Psi$ acts like $\bigvee\!\!\!\bigvee\{\neg\varphi:\varphi\in\Phi\}\cup\{\psi:\psi\in\Psi\}$.

Axioms $\theta \vdash \theta$ (identity axiom)

Structural rules

$$\text{wL}\ \frac{\Phi \vdash \Psi}{\theta, \Phi \vdash \Psi} \qquad \frac{\Phi \vdash \Psi}{\Phi \vdash \Psi, \theta}\ \text{wR} \qquad \text{(weakening)}$$

$$\text{cL}\ \frac{\theta, \theta, \Phi \vdash \Psi}{\theta, \Phi \vdash \Psi} \qquad \frac{\Phi \vdash \Psi, \theta, \theta}{\Phi \vdash \Psi, \theta}\ \text{cR} \qquad \text{(contraction)}$$

$$\text{eL}\ \frac{\Phi, \theta, \eta, \Xi \vdash \Psi}{\Phi, \eta, \theta, \Xi \vdash \Psi} \qquad \frac{\Phi \vdash \Psi, \theta, \eta, \Xi}{\Phi \vdash \Psi, \eta, \theta, \Xi}\ \text{eR} \qquad \text{(exchange)}$$

Cut rule

$$\frac{\Phi \vdash \Psi, \theta \qquad \theta, \Gamma \vdash \Delta}{\Phi, \Gamma \vdash \Psi, \Delta}\ \text{cut}$$

Propositional rules

$$\neg\text{L}\ \frac{\Phi \vdash \Psi, \theta}{\neg\theta, \Phi \vdash \Psi} \qquad \frac{\theta, \Phi \vdash \Psi}{\Phi \vdash \Psi, \neg\theta}\ \neg\text{R}$$

$$\wedge\text{L}\ \frac{\theta, \Phi \vdash \Psi}{\theta \wedge \eta, \Phi \vdash \Psi}$$

$$\wedge\text{L}\ \frac{\eta, \Phi \vdash \Psi}{\theta \wedge \eta, \Phi \vdash \Psi}$$

$$\frac{\Phi \vdash \Psi, \theta \qquad \Phi \vdash \Psi, \eta}{\Phi \vdash \Psi, \theta \wedge \eta}\ \wedge\text{R}$$

$$\vee\text{L}\ \frac{\theta, \Phi \vdash \Psi \qquad \eta, \Phi \vdash \Psi}{\theta \vee \eta, \Phi \vdash \Psi}$$

$$\frac{\Phi \vdash \Psi, \theta}{\Phi \vdash \Psi, \theta \vee \eta}\ \vee\text{R}$$

$$\frac{\Phi \vdash \Psi, \eta}{\Phi \vdash \Psi, \theta \vee \eta}\ \vee\text{R}$$

Quantifier rules

$$\forall\text{L}\ \frac{\theta(t, \bar{z}), \Phi \vdash \Psi}{\forall w\ \theta(w, \bar{z}), \Phi \vdash \Psi} \qquad \frac{\Phi \vdash \Psi, \theta(v, \bar{z})}{\Phi \vdash \Psi, \forall w\ \theta(w, \bar{z})}\ \forall\text{R}$$

$$\exists\text{L}\ \frac{\theta(v, \bar{z}), \Phi \vdash \Psi}{\exists w\ \theta(w, \bar{z}), \Phi \vdash \Psi} \qquad \frac{\Phi \vdash \Psi, \theta(t, \bar{z})}{\Phi \vdash \Psi, \exists w\ \theta(w, \bar{z})}\ \exists\text{R}$$

Eigenvariable condition. The variable v does not appear in the lower sequent of $\forall$R and $\exists$L.

- $\Phi, \Psi, \Xi, \Gamma, \Delta$ are finite sequences of formulas.
- θ, η are formulas.
- t is a term.

Figure 18.1: Gentzen's sequent calculus system **LK**

- The *antecedent* and the *succedent* of a sequent $\Phi \vdash \Psi$ are respectively Φ and Ψ.
- The notion of an **LK**-*proof* is defined by recursion as follows.
  - Every axiom in Figure 18.1 is an **LK**-proof.
  - If A is the lowest sequent in the **LK**-proof π, and $\dfrac{A}{B}$ is a rule in Figure 18.1, then
  $$\frac{\pi}{B}$$
  is an **LK**-proof.
  - If A_0, A_1 are respectively the lowest sequents in the **LK**-proofs π_0, π_1, and $\dfrac{A_0 \qquad A_1}{B}$ is a rule in Figure 18.1, then
  $$\frac{\pi_0 \qquad \pi_1}{B}$$
  is an **LK**-proof.
  - Every **LK**-proof is obtained by applying the construction rules above finitely many times.
- The lowest sequent in an **LK**-proof is sometimes referred to as the *end-sequent* of the proof. An **LK**-*proof of a sequent* A is an **LK**-proof whose end-sequent is A.

Remark 18.2. Some people consider the cut rule as a structural rule, and use the term *weak structural rules* when they specifically want to refer to a structural rule that is not the cut rule.

LK possesses all the four qualities we require of a proof system: soundness, finiteness, completeness, and recursiveness. These can be proved in a way similar to how we prove them for our proof system from Lecture 6. So we omit the proofs here.

Remark 18.3. Special care was taken when laying down the deduction rules of **LK** in Figure 18.1 to make the left–right symmetry evident: if one reflects a deduction rule of **LK** about the turnstile, and transforms the logical symbols

$$(\neg, \wedge, \vee, \forall, \exists) \quad \mapsto \quad (\neg, \vee, \wedge, \exists, \forall)$$

therein, then one obtains another deduction rule of **LK**. It follows that **LK**-proofs exhibit the same symmetry too. This symmetry becomes less evident if one defines $\wedge$ and $\forall$ in terms of the other logical symbols as we did in previous lectures.

The deduction rules of **LK** correspond naturally to the logical symbols. This makes it more easily adaptable to logics with other logical symbols.

As Gentzen remarked, one of the reasons for allowing more than one formula to appear in the succedent of an **LK**-sequent is to prove the Law of Excluded Middle, as shown in the example below. If one only allows at most one formula to appear in a succedent in **LK**, then one obtains a weaker system called **LJ**, which stands for *logistischer intuitionistischer Kalkül.* Gentzen introduced **LJ** for intuitionistic logic, where the Law of Excluded Middle does not hold in general.

Example 18.4 (Law of Excluded Middle)**.** The following is an **LK**-proof for every formula θ.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\theta \vdash \theta}{\vdash \theta, \neg\theta}\,\neg\mathrm{R}}{\vdash \theta, \theta \vee \neg\theta}\,\vee\mathrm{R}}{\vdash \theta \vee \neg\theta, \theta}\,\mathrm{eR}}{\vdash \theta \vee \neg\theta, \theta \vee \neg\theta}\,\vee\mathrm{R}}{\vdash \theta \vee \neg\theta}\,\mathrm{cR}$$

The rules $\wedge$L and $\vee$R are sometimes formulated in slightly different ways. The formulation we gave above follows the original one by Gentzen. Our deduction system from Lecture 6 also has ($\exists$L) and ($\exists$R) rules. They are essentially the same as the $\exists$L and $\exists$R rules in **LK**. Recall that the empty conjunction and the empty disjunction are by definition $\top$ and $\bot$ respectively. What we previously wrote as $\Phi \vdash \bot$ is thus written $\Phi \vdash$ in **LK**. In particular, our cut rule in Figure 6.1 can be derived from the cut rule in **LK** as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Phi, \theta \vdash}{\text{some exchanges}}}{\theta, \Phi \vdash}}{\Phi \vdash \neg\theta}\,\neg\text{R} \qquad \dfrac{\dfrac{\Phi, \neg\theta \vdash}{\text{some exchanges}}}{\neg\theta, \Phi \vdash}}{\dfrac{\dfrac{\Phi, \Phi \vdash}{\text{some exchanges and contractions}}}{\Phi \vdash}}\,\text{cut}$$

The cut rule can also be viewed as a generalization of *modus ponens.*

The cut rule is distinct from the rest of the deduction rules in **LK** in that it is the only rule in which the upper sequent may contain a formula more complicated than all the formulas in the lower sequent. For instance, all the propositional and quantifier rules introduce a new logical symbol into the lower sequent. To take into account the $\forall$L and $\exists$R rules, we need to include substitutional instances when defining subformulas.

Definition. Let θ, η be formulas. Then θ is a *subformula* of η if $\theta = \theta(\bar{t})$ for some terms $\bar{t}$ and some substring $\theta(\bar{w})$ of η.

Observation 18.5 (Subformula property of cut-free **LK**-proofs)**.** Every formula that appears in a cut-free **LK**-proof is a subformula of some formula in the end-sequent of the proof.

Proof. It can readily be seen that, in every deduction rule of **LK** except the cut rule, the upper sequent mentions only subformulas of formulas in the lower sequent. ☐

By Observation 18.5, a cut-free **LK**-proof is 'direct', in the sense that it does not involve any formula not relevant to the conclusion. The use of the cut rule thus indicates detours in proofs, which are otherwise known as *lemmas* in usual mathematical discourse. More concretely, every cut-formula represents a lemma, which is proved, used, then discarded.

Definition. In the instance of cut rule shown in Figure 18.1, the formula θ is called the *cut-formula.*

Via an argument similar to that for Observation 18.5, one sees that there can be no cut-free **LK**-proof of the sequent containing no formula.

Definition. The *empty sequent* is the sequent $\vdash$, in which both the antecedent and the succedent are empty.

Observation 18.6. No cut-free **LK**-proof has empty end-sequent.

Proof. The empty sequent is neither an identity axiom nor the lower sequent of any deduction rule in **LK** except the cut rule. ☐

Gentzen showed that the use of the cut rule in **LK** can actually be eliminated (at the cost of making the proofs longer). More precisely, every **LK**-proof can be transformed into a (longer) cut-free **LK**-proof with the same end-sequent. Hence actually there can be no **LK**-proof of the empty sequent by Observation 18.6. Recall that, in view of our convention on empty conjunction and empty disjunction, the empty sequent signifies the provability of contradiction. So the eliminability of cuts entails the consistency of the **LK** deduction system. Our plan is to prove the consistency of PA via a similar route.

To capture provability in PA, one can formulate a proof system in the style of **LK**. This system also originates from Gentzen.

Definition (plus some explanation)**.** In the following, we introduce the proof system **PA**.

- The logical symbols are the same as those in **LK**. The language is $\mathcal{L}_{\mathrm{R}}(\exp)$, which is essentially $\mathcal{L}_{\mathrm{A}}(\exp)$ without $<$. The order $<$ can be defined via (Q$<$). We need to add $=$ to $\mathcal{L}_{\mathrm{R}}(\exp)$ because **LK** does not already have $=$.

- Sequents are defined in the same way as in **LK**. The intended meaning of a sequent $\Phi \vdash \Psi$ in **PA** is 'PA + $\bigwedge\!\!\!\bigwedge \Phi$ entails $\bigvee\!\!\!\bigvee \Psi$'.

Mathematical axioms

$$\vdash 1 = 0 + 1 \tag{Q1}$$

$$t + 1 = 0 \vdash \tag{QS$_0$}$$

$$t + 1 = s + 1 \vdash t = s \tag{QS$_1$}$$

$$\vdash t + 0 = t \tag{Q+}$$

$$\vdash t + (s + 1) = (t + s) + 1 \tag{Q+$_1$}$$

$$\vdash t \times 0 = 0 \tag{Q$\times_0$}$$

$$\vdash t \times (s + 1) = (t \times s) + t \tag{Q$\times_1$}$$

$$\vdash 2^0 = 1 \tag{Qexp$_0$}$$

$$\vdash 2^{t+1} = 2^t + 2^t \tag{Qexp$_1$}$$

Equality axioms

$$\vdash t = t$$

$$t_1 = s_1, t_2 = s_2 \vdash t_1 + t_2 = s_1 + s_2$$

$$t_1 = s_1, t_2 = s_2 \vdash t_1 \times t_2 = s_1 \times s_2$$

$$t = s \vdash 2^t = 2^s$$

Induction rule

$$\frac{\theta(v, \bar{z}), \Phi \vdash \Psi, \theta(v + 1, \bar{z})}{\theta(0, \bar{z}), \Phi \vdash \Psi, \theta(t, \bar{z})} \text{ Ind}$$

Eigenvariable condition. The variable v does not appear in the lower sequent of Ind.

- Φ, Ψ are finite sequences of $\mathcal{L}_{\mathrm{R}}(\exp)$ formulas.
- $\theta(v, \bar{z})$ is an $\mathcal{L}_{\mathrm{R}}(\exp)$ formula.
- t, s, t_1, t_2, s_1, s_2 are $\mathcal{L}_{\mathrm{R}}(\exp)$ terms.

Figure 18.2: Additional axioms and deduction rules in the proof system **PA**

- **PA**-*proofs* are defined in the same way as **LK**-proofs, except that we add the axioms and the deduction rules in Figure 18.2 to those in Figure 18.1.

The additional axioms and rules in the proof system **PA** are chosen to capture exactly provability in the theory PA, except that one half of (QS_0) is missing. As the following example shows, this missing half can be derived from the other axioms. The consistency of PA thus means the impossibility for a **PA**-proof to have an empty end-sequent.

Example 18.7. The following is a **PA**-proof for every $\mathcal{L}_{\mathrm{R}}(\exp)$ term t.

$$
\begin{prooftree}
\AxiomC{$t = v+1 \vdash t = v+1$}
\LeftLabel{wR}
\UnaryInfC{$t = v+1 \vdash t = v+1, t = v$}
\LeftLabel{$\neg$L}
\UnaryInfC{$t \neq v, t = v+1 \vdash t = v+1$}
\LeftLabel{$\exists$R}
\UnaryInfC{$t \neq v, t = v+1 \vdash \exists w\ (t = w+1)$}
\LeftLabel{eL}
\UnaryInfC{$t = v+1, t \neq v \vdash \exists w\ (t = w+1)$}
\LeftLabel{$\neg$R}
\UnaryInfC{$t \neq v \vdash \exists w\ (t = w+1), t \neq v+1$}
\LeftLabel{Ind}
\UnaryInfC{$t \neq 0 \vdash \exists w\ (t = w+1), t \neq t$}
\AxiomC{$\vdash t = t$}
\RightLabel{$\neg$L}
\UnaryInfC{$t \neq t \vdash$}
\RightLabel{cut}
\BinaryInfC{$t \neq 0 \vdash \exists w\ (t = w+1)$}
\end{prooftree}
$$

The following are the key ingredients of Gentzen's consistency proof for PA.

(1) Assign to every **PA**-proof an ordinal in ε_0.

(2) Given a **PA**-proof which contains either an instance of the induction rule or a non-atomic cut (i.e., an instance of the cut rule in which the cut-formula is not atomic), construct another **PA**-proof with the same end-sequent which has a strictly smaller ordinal assigned to it.

(3) Observe that if a **PA**-proof contains neither an instance of the induction rule nor a non-atomic cut, then its end-sequent cannot be empty.

The outline of Gentzen's proof is as follows. Take any **PA**-proof π. Repeatedly apply (2) to π. By the Well-Ordering Principle for ordinals less than ε_0, this process must stop in finitely many steps, yielding a **PA**-proof π_0 which has the same end-sequent as π and contains neither an instance of the induction rule nor a non-atomic cut. Then (3) tells us that the end-sequent of π_0 is not empty. Thus the end-sequent of π is also not empty. Since the choice of the **PA**-proof was arbitrary, no **PA**-proof can have an empty end-sequent, which is to be demonstrated.

We do not follow Gentzen's proof here because it may not be clear at first sight why ε_0 should be involved in (1). Instead we follow Schütte's strategy, which goes via an infinitary proof system **Nat** in which proofs are naturally associated with ordinals in ε_0.

Definition. A term is *closed* if it does not contain any variable.

Definition (plus some explanation)**.** In the following, we introduce the proof system **Nat**.

- The logical and non-logical symbols are the same as those in **PA**.
- No free variable is allowed in sequents. In other words, all formulas in sequents are sentences. The intended meaning of the sequent $\Phi \vdash \Psi$ is $\mathbb{N} \models \bigwedge \Phi \to \bigvee \Psi$.
- The restrictions of the identity axiom, the cut rule, the structural rules and the propositional rules in **LK** to sentences, as well as the axioms and the deduction rules in Figure 18.3, are available. **Nat**-proofs are constructed from axioms and deduction rules in a way similar to that for **LK**. In particular, if $A_0, A_1, A_2, \dots$ are respectively the lowest sequents in the **Nat**-proofs $\pi_0, \pi_1, \pi_2, \dots$, and

$$\frac{A_0 \qquad A_1 \qquad A_2 \qquad \cdots}{B}$$

is a rule in Figure 18.3, then

$$\frac{\pi_0 \qquad \pi_1 \qquad \pi_2 \qquad \cdots}{B}$$

is a **Nat**-proof.

Axioms

$$\vdash \alpha \qquad \text{whenever } \mathbb{N} \models \alpha$$

$$\alpha \vdash \qquad \text{whenever } \mathbb{N} \not\models \alpha$$

$$\beta(\bar{t}) \vdash \beta(\bar{s}) \qquad \text{whenever } t_i^{\mathbb{N}} = s_i^{\mathbb{N}} \text{ for all } i < \ell$$

Quantifier rules

$$\underline{\forall\text{L}}\ \frac{\theta(t), \Phi \vdash \Psi}{\forall w\ \theta(w), \Phi \vdash \Psi} \qquad \frac{\Phi \vdash \Psi, \theta(\underline{0}) \quad \Phi \vdash \Psi, \theta(\underline{1}) \quad \Phi \vdash \Psi, \theta(\underline{2}) \quad \cdots}{\Phi \vdash \Psi, \forall w\ \theta(w)}\ \omega\text{R}$$

$$\omega\text{L}\ \frac{\theta(\underline{0}), \Phi \vdash \Psi \quad \theta(\underline{1}), \Phi \vdash \Psi \quad \theta(\underline{2}), \Phi \vdash \Psi \quad \cdots}{\exists w\ \theta(w), \Phi \vdash \Psi} \qquad \frac{\Phi \vdash \Psi, \theta(t)}{\Phi \vdash \Psi, \exists w\ \theta(w)}\ \underline{\exists\text{R}}$$

- Φ, Ψ are finite sequences of $\mathcal{L}_{\mathrm{R}}(\exp)$ sentences.
- $\theta(v)$ is an $\mathcal{L}_{\mathrm{R}}(\exp)$ formula.
- α is an atomic $\mathcal{L}_{\mathrm{R}}(\exp)$ sentence.
- $\beta(v_0, v_1, \ldots, v_{\ell-1})$ is an atomic $\mathcal{L}_{\mathrm{R}}(\exp)$ sentence.
- $t, t_0, t_1, \ldots, t_{\ell-1}, s_0, s_1, \ldots, s_{\ell-1}$ are closed $\mathcal{L}_{\mathrm{R}}(\exp)$ terms.

Figure 18.3: Additional axioms and deduction rules in the proof system **Nat**

Although the symmetry of the proof system mentioned in Remark 18.3 is broken when we go from **LK** to **PA**, a similar symmetry re-emerges in **Nat**. This suggests a cut-elimination theorem for **Nat** analogous to Gentzen's one for **LK**, and hence the following plan for a consistency proof of PA. This plan is slightly cleaner than Gentzen's one because one eliminates all the cuts here, not only the non-atomic ones.

(0) Translate every **PA**-proof in which no variable appears free in the end-sequent to a nice **Nat**-proof with the same end-sequent.

(1) Devise a natural assignment of ordinals to **Nat**-proofs in which every nice **Nat**-proof is assigned an ordinal strictly less than ε_0.

(2) Given any nice **Nat**-proof that contains an instance of the cut rule, construct another nice **Nat**-proof with the same end-sequent in which the number of logical symbols in the most complicated cut-formulas drops.

(3) Observe that if a **Nat**-proof is cut-free, then its end-sequent cannot be empty.

Step (3) can be done in a way similar to how we dealt with Observation 18.6. We collect here also the analogue for Observation 18.5.

Observation 18.8 (Subformula property of cut-free **Nat**-proofs)**.** Every formula that appears in a cut-free **Nat**-proof is a subformula of some formula in the end-sequent of the proof. □

Observation 18.9. No cut-free **Nat**-proof has empty end-sequent. □

We will deal with Steps (0), (1), and (2) in the next two lectures.