MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Lecture 19: Cut elimination

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The aim of this lecture is to demonstrate how to eliminate the use of the cut rule from proofs. Cut elimination will be the main ingredient of our consistency proof for PA: see part (2) of our plan at the end of the previous lecture.

Instead of considering **Nat** straightaway, we first look at the analogous result for **LK**. The cut-elimination theorem for **LK** is arguably one of the most important theorems in proof theory nowadays. As discussed in the previous lecture, cut-formulas may be viewed as lemmas in proofs. Therefore, informally speaking, the cut-elimination theorem tells us that any theorem can be proved without going through any lemma. There is a price to pay though: the proofs may become much longer (more precisely, superexponentially longer) after the lemmas are eliminated. While suitable lemmas help shorten proofs, as far as the purity of methods is concerned, cut-free proofs are more preferable because cut-formulas may involve notions not present in the conclusion. Mathematically, since cut-free proofs have many nice properties, for example, the subformula property from Observation 18.5, one can derive a good number of results about proofs and provability from the cut-elimination theorem without much effort.

One way to prove the cut-elimination theorem for **LK** is to prove the completeness of **LK** for first-order logic without invoking the cut rule. We take instead the proof-theoretic approach here because it gives extra quantitative information. Let us start with some pieces of terminology from proof theory.

Definition. A variable v is *used as an eigenvariable* in an **LK**-proof π if the proof π contains a deduction rule of the form

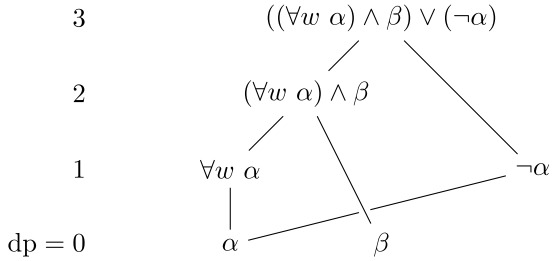
$$\exists\mathrm{L}\ \frac{\theta(v,\bar{z}),\Phi \mid\!\!- \Psi}{\exists w\ \theta(w,\bar{z}),\Phi \mid\!\!- \Psi} \quad \text{or} \quad \frac{\Phi \mid\!\!- \Psi,\theta(v,\bar{z})}{\Phi \mid\!\!- \Psi,\forall w\ \theta(w,\bar{z})}\ \forall\mathrm{R}$$

Roughly speaking, the *depth* of a formula is the number of layers of logical connectives this formula has on top the atomic subformulas. In other words, it is the height of the construction tree of the formula.

Definition. The *depth* $\mathrm{dp}(\theta)$ of a formula θ is defined by recursion on θ as follows.

- If θ is an atomic formula, then $\mathrm{dp}(\theta) = 0$.
- If θ, η are formulas, then
  - $\mathrm{dp}(\neg\theta) = \mathrm{dp}(\exists w\ \theta) = \mathrm{dp}(\forall w\ \theta) = \mathrm{dp}(\theta) + 1$; and
  - $\mathrm{dp}(\theta \vee \eta) = \mathrm{dp}(\theta \wedge \eta) = \max\{\mathrm{dp}(\theta), \mathrm{dp}(\eta)\} + 1$.

Example 19.1. From the construction tree below, one sees that $((\forall w\ \alpha) \wedge \beta) \vee (\neg\alpha)$ has depth 3 if α and β are atomic formulas.



We will eliminate the instances of the cut rule in a given **LK**-proof one by one in the order of decreasing depths of cut-formulas. The idea in each elimination step is to replace the direct ancestors of the cut-formula by its direct subformulas. By the *direct ancestors* of an occurrence of a formula in a proof, we mean here the occurrences of the same formula from which this occurrence originates. While the intuitive idea is relatively simple, the precise definition is lengthy because one has to go through all the cases.

Definition. We first define below what it means for a formula occurrence in an **LK**-proof to be an *immediate direct ancestor* of another formula occurrence in the same proof.

- In a contraction rule

  $$\text{cL}\ \frac{\theta, \theta, \Phi \models \Psi}{\theta, \Phi \models \Psi} \quad \text{or} \quad \frac{\Phi \models \Psi, \theta, \theta}{\Phi \models \Psi, \theta}\ \text{cR}$$

  the two immediate direct ancestors of the occurrence of θ in the lower sequent shown above are the two occurrences of θ in the upper sequent shown.

- In an exchange rule

  $$\text{eL}\ \frac{\Phi, \theta, \eta, \Xi \models \Psi}{\Phi, \eta, \theta, \Xi \models \Psi} \quad \text{or} \quad \frac{\Phi \models \Psi, \theta, \eta, \Xi}{\Phi \models \Psi, \eta, \theta, \Xi}\ \text{eR}$$

  – the immediate direct ancestor of a formula occurrence in the Φ, the Ψ or the Ξ in the lower sequent is the occurrence of the same formula in the same position of the upper sequent;

  – the immediate direct ancestor of the occurrence of θ in the lower sequent shown above is the occurrence of θ in the upper sequent shown; and

  – the immediate direct ancestor of the occurrence of η in the lower sequent shown above is the occurrence of η in the upper sequent shown.

- In a cut rule

  $$\frac{\Phi \models \Psi, \theta \qquad \theta, \Gamma \models \Delta}{\Phi, \Gamma \models \Psi, \Delta}\ \text{cut}$$

  – the immediate direct ancestor of a formula occurrence in the Φ or the Ψ in the lower sequent is the occurrence of the same formula in the same position of the left upper sequent; and

  – the immediate direct ancestor of a formula occurrence in the Γ or the Δ in the lower sequent is the occurrence of the same formula in the same position of the right upper sequent.

- In a deduction rule

  $$\frac{\ldots \Phi \models \Psi \ldots}{\ldots \Phi \models \Psi \ldots}$$

  the immediate direct ancestor of a formula occurrence in the Φ or the Ψ in the lower sequent is the occurrence of the same formula in the same position of the upper sequent.

- In a deduction rule

$$\frac{\ldots\Phi \vdash \Psi \ldots \qquad \ldots\Phi \vdash \Psi \ldots}{\ldots\Phi \vdash \Psi \ldots}$$

  the two immediate direct ancestors of a formula occurrence in the Φ or the Ψ in the lower sequent are the occurrences of the same formula in the same position of the upper sequents.

- No other formula occurrence has an immediate direct ancestor.

An occurrence a of a formula is a *direct ancestor* of an occurrence b of a formula in an **LK**-proof if there is a sequence $a = x_0, x_1, \ldots, x_\ell = b$ of occurrences of formulas such that x_j is an immediate direct ancestor of x_{j+1} for every $j < \ell$.

If any deduction rule is broken when replacing a direct ancestor of the cut-formula by its subformula, then one remedies the situation using the other side of the cut. This is where we exploit the symmetry of **LK**. The proof below, which comes from Buss, contains the details of this cut-elimination process. The lemma itself tells us that we can eliminate one instance of the cut rule, at the expense of introducing other instances of the cut rule with simpler cut-formulas.

Lemma 19.2. Let $d \in \mathbb{N}$ and π be an **LK**-proof of the form

$$\dfrac{\begin{matrix}\vdots\,\pi_0\\ \Phi \vdash \Psi, \theta\end{matrix} \qquad \begin{matrix}\vdots\,\pi_1\\ \theta, \Phi' \vdash \Psi'\end{matrix}}{\Phi, \Phi' \vdash \Psi, \Psi'}\ \text{cut}$$

where $\mathrm{dp}(\theta) = d$ and all the cut-formulas except the θ shown above have depths strictly less than d. Then one can find an **LK**-proof $\tilde{\pi}$ with the same end-sequent as π in which all cut-formulas have depths strictly less than d.

Proof. A proof for the special case when $\Phi' = \Phi$ and $\Psi' = \Psi$ suffices because if π is as in the statement of the lemma, then one can apply this special case to

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_0\\ \Phi \vdash \Psi, \theta\end{matrix}}{\text{weakenings and exchanges}}}{\Phi, \Phi' \vdash \Psi, \Psi', \theta} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\,\pi_1\\ \theta, \Phi \vdash \Psi\end{matrix}}{\text{weakenings and exchanges}}}{\theta, \Phi, \Phi' \vdash \Psi, \Psi'}}{\Phi, \Phi', \Phi, \Phi' \vdash \Psi, \Psi', \Psi, \Psi'}\ \text{cut}$$

and then append several exchanges and contractions to the lower end of the resulting proof to obtain the required $\tilde{\pi}$. So let us assume

$$\pi \quad = \quad \dfrac{\begin{matrix}\vdots\,\pi_0\\ \Phi \vdash \Psi, \theta\end{matrix} \qquad \begin{matrix}\vdots\,\pi_1\\ \theta, \Phi \vdash \Psi\end{matrix}}{\Phi, \Phi \vdash \Psi, \Psi}\ \text{cut}$$

Suppose θ is atomic. Apply the following operations to π_0 to obtain $\tilde{\pi}$.

(1) Remove all the direct ancestors of the rightmost θ in the end-sequent.

(2) Change each sequent $\Gamma \vdash \Delta$ to $\Gamma, \Phi \vdash \Psi, \Delta$.

(3) The tree of sequents obtained may not be an **LK**-proof because perhaps

$$\text{wR}\,\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \theta} \qquad \overset{(1)}{\mapsto} \quad \overset{(2)}{\mapsto} \qquad \dfrac{\Gamma, \Phi \vdash \Psi, \Delta}{\Gamma, \Phi \vdash \Psi, \Delta}$$

in which case remove one of the two sequents $\Gamma, \Phi \vdash \Psi, \Delta$ in the result here; and perhaps

$$\theta \vdash \theta \qquad \overset{(1)}{\mapsto} \quad \overset{(2)}{\mapsto} \qquad \theta, \Phi \vdash \Psi$$

in which case replace the result by π_1 here.

Suppose $\theta = \neg\eta$. Let $\pi_0^\sharp$ be the **LK**-proof obtained from π_0 by applying the following operations.

(1) Remove all the direct ancestors of the rightmost $\neg\eta$ in the end-sequent.

(2) Change each sequent $\Gamma \vdash \Delta$ to $\eta, \Gamma \vdash \Delta$.

(3) The tree of sequents obtained may not be an **LK**-proof because perhaps

$$\text{wR}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\eta} \qquad \overset{(1)}{\mapsto} \ \overset{(2)}{\mapsto} \qquad \frac{\eta, \Gamma \vdash \Delta}{\eta, \Gamma \vdash \Delta}$$

in which case remove one of the two sequents $\eta, \Gamma \vdash \Delta$ in the result here. The other steps remain compliant with the deduction rules in **LK** since

$$\neg\text{R}\ \frac{\eta, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\eta} \qquad \overset{(1)}{\mapsto} \ \overset{(2)}{\mapsto} \qquad \frac{\eta, \eta, \Gamma \vdash \Delta}{\eta, \Gamma \vdash \Delta}\ \text{cL}$$

Symmetrically, let $\pi_1^\flat$ be the **LK**-proof obtained from π_1 by applying the following operations.

(1) Remove all the direct ancestors of the leftmost $\neg\eta$ in the end-sequent.

(2) Change each sequent $\Gamma \vdash \Delta$ to $\Gamma \vdash \Delta, \eta$.

(3) The tree of sequents obtained may not be an **LK**-proof because perhaps

$$\text{wL}\ \frac{\Gamma \vdash \Delta}{\neg\eta, \Gamma \vdash \Delta} \qquad \overset{(1)}{\mapsto} \ \overset{(2)}{\mapsto} \qquad \frac{\Gamma \vdash \Delta, \eta}{\Gamma \vdash \Delta, \eta}$$

in which case remove one of the two sequents $\Gamma \vdash \Delta, \eta$ in the result here. The other steps remain compliant with the deduction rules in **LK** since

$$\neg\text{L}\ \frac{\Gamma \vdash \Delta, \eta}{\neg\eta, \Gamma \vdash \Delta} \qquad \overset{(1)}{\mapsto} \ \overset{(2)}{\mapsto} \qquad \frac{\Gamma \vdash \Delta, \eta, \eta}{\Gamma \vdash \Delta, \eta}\ \text{cR}$$

Notice the end-sequents of $\pi_0^\sharp$ and $\pi_1^\flat$ are respectively $\eta, \Phi \vdash \Psi$ and $\Phi \vdash \Psi, \eta$. So we can define

$$\tilde{\pi} \quad = \quad \frac{\begin{matrix} \vdots\ \pi_1^\flat \\ \Phi \vdash \Psi, \eta \end{matrix} \qquad \begin{matrix} \vdots\ \pi_0^\sharp \\ \eta, \Phi \vdash \Psi \end{matrix}}{\Phi, \Phi \vdash \Psi, \Psi}\ \text{cut}$$

Suppose $\theta = \xi \vee \zeta$. Let $\pi_0^\sharp$ be the **LK**-proof obtained from π_0 by applying the following operations.

(1) Change all the direct ancestors of the rightmost $\xi \vee \zeta$ in the end-sequent to ξ, ζ.

(2) The tree of sequents obtained may not be an **LK**-proof because perhaps

$$\text{wR}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \xi \vee \zeta} \qquad \overset{(1)}{\mapsto} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

in which case remove one of the two sequents $\Gamma \vdash \Delta$ in the result here; and perhaps

$$\vee\text{R}\ \frac{\Gamma \vdash \Delta, \zeta}{\Gamma \vdash \Delta, \xi \vee \zeta} \qquad \overset{(1)}{\mapsto} \qquad \frac{\Gamma \vdash \Delta, \zeta}{\Gamma \vdash \Delta, \xi, \zeta}$$

in which case replace the result here by

$$\cfrac{\cfrac{\Gamma \vdash \Delta, \zeta}{\Gamma \vdash \Delta, \zeta, \xi}\ \text{wR}}{\Gamma \vdash \Delta, \xi, \zeta}\ \text{eR}$$

The other steps remain compliant with the deduction rules in **LK** since

$$\vee\mathrm{R}\,\frac{\Gamma \vdash \Delta, \xi}{\Gamma \vdash \Delta, \xi \vee \zeta} \quad \overset{(1)}{\mapsto} \quad \frac{\Gamma \vdash \Delta, \xi}{\Gamma \vdash \Delta, \xi, \zeta}\,\mathrm{wR}$$

For $\eta \in \{\xi, \zeta\}$, let $\pi_1^{\flat(\eta)}$ be the **LK**-proof obtained from π_1 by applying the following operations.

(1) Change all the direct ancestors of the leftmost $\xi \vee \zeta$ in the end-sequent to η.

(2) The tree of sequents obtained may not be an **LK**-proof because perhaps

$$\vee\mathrm{L}\,\frac{\begin{array}{c}\vdots\, \nu_\xi \\ \xi, \Gamma \vdash \Delta\end{array} \qquad \begin{array}{c}\vdots\, \nu_\zeta \\ \zeta, \Gamma \vdash \Delta\end{array}}{\xi \vee \zeta, \Gamma \vdash \Delta} \quad \overset{(1)}{\mapsto} \quad \frac{\begin{array}{c}\vdots\, \nu_\xi \\ \xi, \Gamma \vdash \Delta\end{array} \qquad \begin{array}{c}\vdots\, \nu_\zeta \\ \zeta, \Gamma \vdash \Delta\end{array}}{\eta, \Gamma \vdash \Delta}$$

in which case replace the result by ν_η. The other steps remain compliant with the deduction rules in **LK** since

$$\mathrm{wL}\,\frac{\Gamma \vdash \Delta}{\xi \vee \zeta, \Gamma \vdash \Delta} \quad \overset{(1)}{\mapsto} \quad \frac{\Gamma \vdash \Delta}{\eta, \Gamma \vdash \Delta}\,\mathrm{wL}$$

Notice that, for $\eta \in \{\xi, \zeta\}$, the end-sequents of $\pi_0^\sharp$ and $\pi_1^{\flat(\eta)}$ are respectively $\Phi \vdash \Psi, \xi, \zeta$ and $\eta, \Phi \vdash \Psi$. So we can define

$$\tilde{\pi} \quad = \quad \frac{\dfrac{\dfrac{\begin{array}{c}\vdots\, \pi_0^\sharp \\ \Phi \vdash \Psi, \xi, \zeta\end{array} \qquad \begin{array}{c}\vdots\, \pi_1^{\flat(\zeta)} \\ \zeta, \Phi \vdash \Psi\end{array}}{\text{exchanges and contractions}}\,\mathrm{cut}}{\Phi \vdash \Psi, \xi} \qquad \begin{array}{c}\vdots\, \pi_1^{\flat(\xi)} \\ \xi, \Phi \vdash \Psi\end{array}}{\Phi, \Phi \vdash \Psi, \Psi}\,\mathrm{cut}$$

Suppose $\theta = \exists w\ \eta(w, \bar{z})$. Since all the axioms and deduction rules in **LK** are closed under substitution of fresh variables, by suitably applying some changes of free variables, we may assume that, in the **LK**-proof π, if a variable is used as an eigenvariable, then it appears only above the place where it is used as an eigenvariable.

For each term t in which no variable is used as an eigenvariable in π_1, let $\pi_1^{\flat(t)}$ be the **LK**-proof obtained from π_1 by applying the following operations.

(1) Change all the direct ancestors of the leftmost $\exists w\ \eta(w, \bar{z})$ in the end-sequent to $\eta(t, \bar{z})$.

(2) Change each sequent $\Gamma \vdash \Delta$ to $\Gamma, \Phi \vdash \Psi, \Delta$.

(3) The tree of sequents obtained may not be an **LK**-proof because perhaps

$$\exists\mathrm{L}\,\frac{\eta(v, \bar{z}), \Gamma \vdash \Delta}{\exists w\ \eta(w, \bar{z}), \Gamma \vdash \Delta} \quad \overset{(1)}{\mapsto}\ \overset{(2)}{\mapsto} \quad \frac{\eta(v, \bar{z}), \Gamma, \Phi \vdash \Psi, \Delta}{\eta(t, \bar{z}), \Gamma, \Phi \vdash \Psi, \Delta}$$

in which case

(a) remove the lower sequent $\eta(t, \bar{z}), \Gamma, \Phi \vdash \Psi, \Delta$ here; and
(b) replace all occurrences of v by t.

Thanks to our condition on the variables in t, the eigenvariable conditions are not destroyed by (3b) above. The other steps remain compliant with the deduction rules in **LK** since

$$\mathrm{wL}\,\frac{\Gamma \vdash \Delta}{\exists w\ \eta(w, \bar{z}), \Gamma \vdash \Delta} \quad \overset{(1)}{\mapsto}\ \overset{(2)}{\mapsto} \quad \frac{\Gamma, \Phi \vdash \Psi, \Delta}{\eta(t, \bar{z}), \Gamma, \Phi \vdash \Psi, \Delta}\,\mathrm{wL}$$

(4) Add some exchanges and contractions at the lower end to turn the end-sequent $\eta(t,\bar{z}), \Phi, \Phi \vdash \Psi, \Psi$ into $\eta(t,\bar{z}), \Phi \vdash \Psi$.

Then we obtain $\tilde{\pi}$ by applying the following operations to π_0.

(1) Remove all the direct ancestors of the rightmost $\exists w\ \eta(w,\bar{z})$ in the end-sequent.

(2) Change each sequent $\Gamma \vdash \Delta$ to $\Gamma, \Phi \vdash \Psi, \Delta$.

(3) The tree of sequents obtained may not be an **LK**-proof because perhaps

$$\text{wR}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \exists w\ \eta(w,\bar{z})} \quad \overset{(1)}{\mapsto} \overset{(2)}{\mapsto} \quad \frac{\Gamma, \Phi \vdash \Psi, \Delta}{\Gamma, \Phi \vdash \Psi, \Delta}$$

in which case remove one of the two sequents $\Gamma, \Phi \vdash \Psi, \Delta$ in the result here; and perhaps

$$\exists\text{R}\ \frac{\begin{matrix}\vdots\ \nu \\ \Gamma \vdash \Delta, \eta(t,\bar{z})\end{matrix}}{\Gamma \vdash \Delta, \exists w\ \eta(w,\bar{z})} \quad \overset{(1)}{\mapsto} \overset{(2)}{\mapsto} \quad \frac{\begin{matrix}\vdots\ \nu \text{ after (2)} \\ \Gamma, \Phi \vdash \Psi, \Delta, \eta(t,\bar{z})\end{matrix}}{\Gamma, \Phi \vdash \Psi, \Delta}$$

in which case replace the result here by

$$\frac{\dfrac{\begin{matrix}\vdots\ \nu \text{ after (2)} \\ \Gamma, \Phi \vdash \Psi, \Delta, \eta(t,\bar{z})\end{matrix} \qquad \begin{matrix}\vdots\ \pi_1^{\flat(t)} \\ \eta(t,\bar{z}), \Phi \vdash \Psi\end{matrix}}{\text{exchanges and contractions}}\ \text{cut}}{\Gamma, \Phi \vdash \Psi, \Delta}$$

Note that, by our initial assumption on the appearance of variables in π, no variable in the term t here is used as an eigenvariable in π_1, and so we know $\pi_1^{\flat(t)}$ is an **LK**-proof.

Suppose $\theta = \xi \wedge \zeta$ or $\theta = \forall w\ \eta(w,\bar{z})$. To deal with the $\wedge$ case, turn it into the $\vee$ case using the symmetry of **LK** described in Remark 18.3, reduce the depth of the cut-formula as above, then swap back using Remark 18.3. The $\forall$ case can be dealt with similarly using the $\exists$ case. □

Being the main theorem of Gentzen's original paper, the cut-elimination theorem for **LK** is often referred to as Gentzen's *Hauptsatz*.

Theorem 19.3 (Gentzen, cut-elimination theorem for **LK**)**.** Given any **LK**-proof, one can find a cut-free **LK**-proof with the same end-sequent.

Proof. Consider any **LK**-proof. Amongst the instances of the cut rule with cut-formulas of maximum depth, choose one that is closest to the axioms in the proof, and apply Lemma 19.2 to the subproof above it. Since the instance of the cut rule chosen is closest to the axioms, this subproof does not contain any cut-formula of maximum depth. So Lemma 19.2 is applicable. Repeat. Each time, one cut-formula of maximum depth is eliminated. Hence, after finitely many steps, the maximum depth of cut-formulas drops. Repeat again. Eventually, the maximum depth of cut-formulas drops beyond 0, at which point all instances of the cut-rule have been eliminated. □

Since cut-free **LK**-proofs have the subformula property (from Observation 18.5), the cut-elimination theorem allows one to carry out proof searches in a much smaller search space: one only needs to consider those proofs that are made up of subformulas of the goal we want to prove. There may be infinitely subformulas of a formula because we allow arbitrary terms to be substituted into free variables in subformulas. So the cut-elimination theorem does not give an algorithm for deciding provability; in fact, it cannot, by Theorem 9.2.

In the next lecture, we will move on to **PA** and **Nat**. Note that every closed $\mathcal{L}_{\text{R}}(\exp)$ term evaluates to a natural number in $\mathbb{N}$ because it contains no variable.

Assignment 19.4. For every closed $\mathcal{L}_{\text{R}}(\exp)$ term t, find a cut-free **Nat**-proof of

$$t \neq 0 \vdash \exists w\ (t = w + 1). \qquad \text{[3 points]}$$