MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Lecture 20: Consistency proof

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The aim of this proof is to present a consistency proof of PA based on the plan we set out at the end of Lecture 18. In particular, we will see why ε_0 is relevant.

In the aforementioned plan, there is an unexplained notion of nicety. We now define this notion properly, in terms of heights and cut-ranks. Since **Nat**-proofs can be infinite, their heights and cut-ranks may be infinite ordinals. Note that, by the Well-Ordering Principle, every collection of ordinals has a supremum.

Definition. If C is a collection of ordinals, then

$$\sup C = \begin{cases} \min\{\alpha : \alpha \text{ is an ordinal and } \beta \leqslant \alpha \text{ for all } \beta \in C\}, & \text{if this collection is nonempty;} \\ \infty, & \text{otherwise.} \end{cases}$$

Roughly speaking, the *height* of a proof measures the number of deduction rules one meets when going from the top to the bottom of the proof. In particular, if one appends one deduction rule to the lower end of a proof, then the height increases by 1. The *cut-rank* is the least *strict* upper bound of the depths of cut-formulas in a proof.

Definition. The *height* $\mathrm{hgt}(\pi)$ and the *cut-rank* $\rho(\pi)$ of a **Nat**-proof π is defined by recursion on (the number of deduction rules in) π as follows.

- If π is an axiom, then $\mathrm{hgt}(\pi) = 0 = \rho(\pi)$.
- Suppose $\pi = \dfrac{\pi_0 \qquad \pi_1 \qquad \pi_2 \qquad \cdots}{A}\circledast$, where $\circledast$ has 1, 2, or ω-many upper sequents.
  - Let $\mathrm{hgt}(\pi) = \sup\{\mathrm{hgt}(\pi_0)+1, \mathrm{hgt}(\pi_1)+1, \mathrm{hgt}(\pi_2)+1, \dots\}$.
  - If $\circledast$ is a cut rule with cut-formula θ, then

    $$\rho(\pi) = \max\{\mathrm{dp}(\theta)+1, \rho(\pi_0), \rho(\pi_1)\},$$

    else $\rho(\pi) = \sup\{\rho(\pi_0), \rho(\pi_1), \rho(\pi_2), \dots\}$.

Note 20.1. A **Nat**-proof is cut-free if and only if its cut-rank is 0.

All **Nat**-proofs have countable heights, as one can show by an induction on the construction of such proofs. The cut-rank of a **Nat**-proof is at most ω because depths of formulas are natural numbers. A finite proof (i.e., a proof in which finitely many sequents appear) clearly has a finite height and a finite cut-rank (i.e., the value is strictly less than ω). So **Nat**-proofs of infinite heights or cut-ranks must use ωL or ωR.

Example 20.2. Let $\forall w\ \theta(w)$ be an $\mathcal{L}_{\mathrm{R}}(\exp)$ sentence of depth 5219, and π be the **Nat**-proof

$$\dfrac{\omega\mathrm{R}\,\dfrac{\overset{\vdots\ \pi_0}{\vdash \Psi, \theta(\underline{0})} \qquad \overset{\vdots\ \pi_1}{\vdash \Psi, \theta(\underline{1})} \qquad \overset{\vdots\ \pi_2}{\vdash \Psi, \theta(\underline{2})} \qquad \cdots}{\vdash \Psi, \forall w\ \theta(w)} \qquad \overset{\vdots\ \nu}{\forall w\ \theta(w) \vdash}}{\vdash \Psi}\,\mathrm{cut}$$

where $\mathrm{hgt}(\nu) = 8$ and $\rho(\nu) = 13$.

- If $\mathrm{hgt}(\pi_i) = i + 4$ for all $i \in \mathbb{N}$, then

$$\mathrm{hgt}(\pi) = \sup\bigl\{\sup\{4+1, 5+1, 6+1, 7+1, \dots\} + 1, 8+1\bigr\} = \sup\{\omega + 1, 8+1\} = \omega + 1.$$

- If $\rho(\pi_{2i}) = 2$ and $\rho(\pi_{2i+1}) = 3$ for all $i \in \mathbb{N}$, then

$$\rho(\pi) = \sup\{5219 + 1, \sup\{2, 3, 2, 3, \dots\}, 13\} = \sup\{5220, 3, 13\} = 5220.$$

- If $\rho(\pi_i) = i$ for all $i \in \mathbb{N}$, then

$$\rho(\pi) = \sup\{5219 + 1, \sup\{0, 1, 2, 3, \dots\}, 13\} = \sup\{5220, \omega, 13\} = \omega.$$

What we meant by a nice **Nat**-proof at the end of Lecture 18 is one of finite cut-rank and height strictly less than ε_0. Such proofs are already very powerful. For instance, cut-free **Nat**-proofs of finite heights are sufficient for proving all truths about $\mathbb{N}$. The essential reason is that every clause in Tarski's truth definition for $\mathbb{N}$ corresponds to an axiom or a non-cut deduction rule in **Nat**.

Proposition 20.3. Let σ be an $\mathcal{L}_{\mathrm{R}}(\exp)$ sentence.

(1) If $\mathbb{N} \models \sigma$, then there is a cut-free **Nat**-proof of $\vdash \sigma$ of finite height.

(2) If $\mathbb{N} \not\models \sigma$, then there is a cut-free **Nat**-proof of $\sigma \vdash$ of finite height.

Proof sketch. Prove the following strengthened clauses by induction on $\mathrm{dp}(\sigma)$.

(1*) If $\mathbb{N} \models \sigma$, then there is a cut-free **Nat**-proof of $\vdash \sigma$ of height $\mathrm{dp}(\sigma)$.

(2*) If $\mathbb{N} \not\models \sigma$, then there is a cut-free **Nat**-proof of $\sigma \vdash$ of height $\mathrm{dp}(\sigma)$.

A simultaneous induction helps in the $\neg$ case. Here we sketch only a proof for the $\forall$ case, because mathematically we will actually not need this proposition. Let $d \in \mathbb{N}$ such that the strengthened clauses hold for all $\mathcal{L}_{\mathrm{R}}(\exp)$ sentences σ of depth d. Consider the $\mathcal{L}_{\mathrm{R}}(\exp)$ sentence $\forall w\ \theta(w)$, where $\mathrm{dp}(\theta) = d$.

Suppose $\mathbb{N} \models \forall w\ \theta(w)$. Then $\mathbb{N} \models \theta(\underline{n})$ for all $n \in \mathbb{N}$ by the truth definition. So the induction hypothesis, applied to the depth-d sentences $\theta(\underline{0}), \theta(\underline{1}), \theta(\underline{2}), \dots$, gives the following cut-free **Nat**-proof of height $d + 1 = \mathrm{dp}(\forall w\ \theta(w))$:

$$\dfrac{\begin{array}{c}\vdots\ \text{cut-free height } d\\ \vdash \theta(\underline{0})\end{array} \qquad \begin{array}{c}\vdots\ \text{cut-free height } d\\ \vdash \theta(\underline{1})\end{array} \qquad \begin{array}{c}\vdots\ \text{cut-free height } d\\ \vdash \theta(\underline{2})\end{array} \qquad \cdots}{\vdash \forall w\ \theta(w)}\ \omega\mathrm{R}$$

Suppose $\mathbb{N} \not\models \forall w\ \theta(w)$. Use the truth definition to find $n \in \mathbb{N} \not\models \theta(\underline{n})$. Then the induction hypothesis, applied to the depth-d sentence $\theta(\underline{n})$, gives the following cut-free **Nat**-proof of height $d + 1 = \mathrm{dp}(\forall w\ \theta(w))$:

$$\dfrac{\begin{array}{c}\vdots\ \text{cut-free height } d\\ \theta(\underline{n}) \vdash\end{array}}{\forall w\ \theta(w) \vdash}\ \underline{\forall\mathrm{L}}$$

□

Proposition 20.3 shows that the generalizations of the first two families of axioms in **Nat** to all $\mathcal{L}_{\mathrm{R}}(\exp)$ sentences are derivable in **Nat**. In fact, the the generalization of the third family of axioms in **Nat** to all $\mathcal{L}_{\mathrm{R}}(\exp)$ formulas is also derivable in **Nat**. There is, however, an important difference between this derivability from the previous one: it is recursive.

To define recursiveness of proofs properly, one needs to put proofs in a form that programs can read. It is straightforward to modify our arithmetization of proofs on page 28 to arithmetize **PA** proofs. One cannot do the same for all **Nat**-proofs because there are countably many natural numbers but uncountably many **Nat**-proofs, as one can directly verify. Nevertheless, one can naturally code each **Nat**-proof as a set of natural numbers (by mimicking the usual coding in set theory, for example). Via this coding, the notion of recursiveness for sets of natural numbers naturally carries over to **Nat**-proofs. Moreover, recursive **Nat**-proofs correspond to those **Nat**-proofs that can be constructed in an algorithmic manner intuitively. To avoid cluttering the presentation with technical details, we make this an informal definition in the style of the Church–Turing Thesis.

Informal definition. A construction of a **Nat**-proof $\pi_{\bar{a}}$ from $\bar{a} \in \mathbb{N}$ is *(uniformly) recursive* if there is an algorithm which, on input $\bar{a}, b \in \mathbb{N}$, returns

- `true` when b is in (the set of natural numbers coding) the **Nat**-proof $\pi_{\bar{a}}$; and
- `false` otherwise.

A transformation $\mathscr{T}$ of **Nat**-proofs is *recursive* if there is an algorithm which, when given (the characteristic function of) a **Nat**-proof π as an additional resource, and $b \in \mathbb{N}$ as input, returns

- `true` when b is in (the set of natural numbers coding) the **Nat**-proof $\mathscr{T}(\pi)$; and
- `false` otherwise.

We will not demonstrate rigorously the recursiveness of proof constructions and transformations below because our definition is merely informal. However, we will state the recursiveness condition wherever it is applicable.

Lemma 20.4 (Leibniz's Principle in **Nat**)**.** From an $\mathcal{L}_{\mathrm{R}}(\exp)$ formula $\theta(v_0, v_1, \ldots, v_{\ell-1})$ and closed $\mathcal{L}_{\mathrm{R}}(\exp)$ terms $t_0, t_1, \ldots, t_{\ell-1}, s_0, s_1, \ldots, s_{\ell-1}$, if $t_i^{\mathbb{N}} = s_i^{\mathbb{N}}$ for all $i < \ell$, then one can recursively construct a finite-height cut-free **Nat**-proof with end-sequent $\theta(\bar{t}) \vdash \theta(\bar{s})$.

Proof. Informally, starting with the Leibnizian axioms in **Nat**, we introduce logical symbols on both sides of the turnstile symmetrically following the construction of the formula θ from atomic formulas. Formally, we prove by induction on $\mathrm{dp}(\theta)$ the stronger statement

> for all $\mathcal{L}_{\mathrm{R}}(\exp)$ formulas $\theta(v_0, v_1, \ldots, v_{\ell-1})$ and all closed $\mathcal{L}_{\mathrm{R}}(\exp)$ terms $t_0, t_1, \ldots, t_{\ell-1}$, $s_0, s_1, \ldots, s_{\ell-1}$, if $t_i^{\mathbb{N}} = s_i^{\mathbb{N}}$ for all $i < \ell$, then there exists a cut-free **Nat**-proof of height at most $3 \times \mathrm{dp}(\theta)$ with end-sequent $\theta(\bar{t}) \vdash \theta(\bar{s})$

and extract from the proof an algorithm for constructing the required **Nat**-proofs.

The base case for atomic formulas is true in view of the Leibnizian axioms in **Nat**. For the induction step, let $d \in \mathbb{N}$ such that the statement displayed above is true when restricted to formulas θ of depth d. Consider an $\mathcal{L}_{\mathrm{R}}(\exp)$ formula $\theta(v_0, v_1, \ldots, v_{\ell-1})$ of depth $d + 1$. Let $t_0, t_1, \ldots, t_{\ell-1}, s_0, s_1, \ldots, s_{\ell-1}$ be closed $\mathcal{L}_{\mathrm{R}}(\exp)$ terms such that $t_i^{\mathbb{N}} = s_i^{\mathbb{N}}$ for all $i < \ell$.

Suppose $\theta = \neg\eta$. Then apply the induction hypothesis to η to obtain

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \text{cut-free} \\ \vdots\ \text{height} \leqslant 3d\end{array}}{\eta(\bar{s}) \vdash \eta(\bar{t})}}{\neg\eta(\bar{t}), \eta(\bar{s}) \vdash}\,\neg\mathrm{L}}{\dfrac{\eta(\bar{s}), \neg\eta(\bar{t}) \vdash}{\neg\eta(\bar{t}) \vdash \neg\eta(\bar{s})}\,\neg\mathrm{R}}\,\mathrm{eL}$$

Suppose $\theta = \varphi \vee \psi$. Then apply the induction hypothesis to φ and ψ to obtain

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \text{cut-free} \\ \vdots\ \text{height} \leqslant 3d \\ \varphi(\bar{t}) \vdash \varphi(\bar{s})\end{array}}{\varphi(\bar{t}) \vdash \varphi(\bar{s}) \vee \psi(\bar{s})}\,\vee\mathrm{R} \qquad \dfrac{\begin{array}{c}\vdots\ \text{cut-free} \\ \vdots\ \text{height} \leqslant 3d \\ \psi(\bar{t}) \vdash \psi(\bar{s})\end{array}}{\psi(\bar{t}) \vdash \varphi(\bar{s}) \vee \psi(\bar{s})}\,\vee\mathrm{R}}{\varphi(\bar{t}) \vee \psi(\bar{t}) \vdash \varphi(\bar{s}) \vee \psi(\bar{s})}\,\vee\mathrm{L}$$

Suppose $\theta = \exists w\ \eta(\bar{v}, w)$**.** Then apply the induction hypothesis to η to obtain

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \text{cut-free}\\ \vdots\ \text{height} \leqslant 3d\\ \eta(\bar{t},\underline{0}) \vdash \eta(\bar{s},\underline{0})\end{array}}{\eta(\bar{t},\underline{0}) \vdash \exists w\ \eta(\bar{s},w)}\ \exists\text{R} \quad \dfrac{\begin{array}{c}\vdots\ \text{cut-free}\\ \vdots\ \text{height} \leqslant 3d\\ \eta(\bar{t},\underline{1}) \vdash \eta(\bar{s},\underline{1})\end{array}}{\eta(\bar{t},\underline{1}) \vdash \exists w\ \eta(\bar{s},w)}\ \exists\text{R} \quad \dfrac{\begin{array}{c}\vdots\ \text{cut-free}\\ \vdots\ \text{height} \leqslant 3d\\ \eta(\bar{t},\underline{2}) \vdash \eta(\bar{s},\underline{2})\end{array}}{\eta(\bar{t},\underline{2}) \vdash \exists w\ \eta(\bar{s},w)}\ \exists\text{R} \quad \cdots}{\exists w\ \eta(\bar{t},w) \vdash \exists w\ \eta(\bar{s},w)}\ \omega\text{L}$$

Suppose θ **is** $\varphi \wedge \psi$ **or** $\forall w\ \eta(\bar{v}, w)$**.** These are symmetric to the $\vee$ case and the $\exists$ cases respectively; see Remark 18.3. ☐

Let us proceed to step (0) of our plan. The idea is similar to our naive proof of the consistency of PA at the beginning of Lecture 17, but instead of showing the truth of every line in a **PA**-proof in $\mathbb{N}$, we show that every line is provable in **Nat**. The two are essentially the same because **Nat** is designed to capture truth in $\mathbb{N}$. The only difficulty is technical: variables can appear free in a **PA**-proof but not in a **Nat**-proof, because we explicitly forbade them in the definition. To make the quantifier step of the induction go through, we substitute closed terms into the free variables. The definition of this substitution requires a bit of care.

Definition. Let π be a **PA**-proof. We may write π as $\pi(z_1, z_2, \ldots, z_\ell)$ if

- $z_1, z_2, \ldots, z_\ell$ is a list of variables without repetition; and
- all the variables that appear free in the end-sequent of π are in this list.

If $r_1, r_2, \ldots, r_\ell$ are $\mathcal{L}_{\mathrm{R}}(\exp)$ terms, then we denote by $\pi(r_1, r_2, \ldots, r_\ell)$ the result obtained from $\pi(z_1, z_2, \ldots, z_\ell)$ by replacing with r_j each free occurrence of z_j that is not used as an eigenvariable in a deduction rule lower in $\pi(z_1, z_2, \ldots, z_\ell)$, for every $j \in \{1, 2, \ldots, \ell\}$.

Observation 20.5. If $\pi(\bar{z})$ is a **PA**-proof and $\bar{r}$ are closed $\mathcal{L}_{\mathrm{R}}(\exp)$ terms, then $\pi(\bar{r})$ is a **PA**-proof.

Proof. Go through each axiom and each deduction rule in **PA** and verify that it remains an axiom or a deduction rule after the substitution. The only non-trivial cases are the quantifier rules. The definition of $\pi(\bar{r})$ is made to preserve $\forall$R and $\exists$L. Since $\bar{r}$ do not contain any free variable, they do not affect any existing eigenvariable condition. ☐

If one omits the recursiveness condition below, then the proposition follows trivially from a straightforward refinement of Proposition 20.3. It is, however, important to us that this proof construction is recursive, so that one can carry it out over a weak theory. The height bound is less important, as long as it is strictly less than ε_0.

Proposition 20.6. From every **PA**-proof $\pi(\bar{z})$ and every tuple of closed $\mathcal{L}_{\mathrm{R}}(\exp)$ terms $\bar{r}$, one can recursively construct a **Nat**-proof $\pi_{\bar{r}}$ with $\mathrm{hgt}(\pi_{\bar{r}}) < \omega^2$ and $\rho(\pi_{\bar{r}}) < \omega$ whose end-sequent is the same as that of $\pi(\bar{r})$.

Proof. We prove by induction on (the number of deduction rules in) the **PA**-proof π the stronger statement

> for every **PA**-proof $\pi(\bar{z})$, there are $k, m \in \mathbb{N}$ such that whenever $\bar{r}$ are closed $\mathcal{L}_{\mathrm{R}}(\exp)$ terms, one can find a **Nat**-proof $\pi_{\bar{r}}$ with $\mathrm{hgt}(\pi_{\bar{r}}) < \omega k$ and $\rho(\pi_{\bar{r}}) = m$ whose end-sequent is the same as that of $\pi(\bar{r})$

and extract from the proof an algorithm for constructing the required **Nat**-proofs. Take any **PA**-proof $\nu(\bar{z})$. Let $\bar{r}$ be closed $\mathcal{L}_{\mathrm{R}}(\exp)$ terms. Assume that the statement displayed above is true when restricted to **PA**-proofs π with strictly fewer deduction rules than ν.

The case when ν is an axiom can readily be verified using the fact that $\mathbb{N} \models \mathrm{Q}(\exp)$. For instance, let us consider the example when $\nu(\bar{z})$ is the **PA**-axiom

$$t(\bar{z}) + 1 = s(\bar{z}) + 1 \vdash t(\bar{z}) = s(\bar{z})$$

where $t(\bar{z})$ and $s(\bar{z})$ are $\mathcal{L}_{\mathrm{R}}(\exp)$ terms. Set $k = 1$ and $m = 0$. If $\mathbb{N} \models t(\bar{r}) = s(\bar{r})$, then we can let

$$\nu_{\bar{r}} \quad = \quad \cfrac{\vdash t(\bar{r}) = s(\bar{r})}{t(\bar{r}) + 1 = s(\bar{r}) + 1 \vdash t(\bar{r}) = s(\bar{r})}\ \mathrm{wL}$$

If $\mathbb{N} \not\models t(\bar{r}) = s(\bar{r})$, then $\mathbb{N} \not\models t(\bar{r}) + 1 = s(\bar{r}) + 1$ because $\mathbb{N} \models (\mathrm{QS}_1)$, and so we can let

$$\nu_{\bar{r}} \quad = \quad \cfrac{t(\bar{r}) + 1 = s(\bar{r}) + 1 \vdash}{t(\bar{r}) + 1 = s(\bar{r}) + 1 \vdash t(\bar{r}) = s(\bar{r})}\ \mathrm{wR}$$

For the case when ν is not an axiom, we split into various cases according to what the lowest deduction rule is. All the deduction rules in **PA** translate directly to the corresponding rule in **Nat** except $\forall$R, $\exists$L, and Ind. So if one of these shared rules is the lowest rule in ν, then simply apply the induction hypothesis to the subproofs above this rule, and combine the translated subproofs using the translated deduction rule to obtain the translated proof $\nu_{\bar{r}}$. By the symmetry between $\forall$R and $\exists$L, only two cases remain.

$\boxed{\forall\mathrm{R}}$ Suppose

$$\nu(\bar{z}) \quad = \quad \cfrac{\begin{array}{c}\vdots\ \pi(v, \bar{z}) \\ \Phi(\bar{z}) \vdash \Psi(\bar{z}), \theta(v, \bar{z})\end{array}}{\Phi(\bar{z}) \vdash \Psi(\bar{z}), \forall w\ \theta(w, \bar{z})}\ \forall\mathrm{R}$$

Note that Φ and Ψ cannot mention v because of the eigenvariable condition. Apply the induction hypothesis to $\pi(v, \bar{z})$ to obtain $k, m \in \mathbb{N}$ as in the statement displayed above. Then for each $n \in \mathbb{N}$, we have a **Nat**-proof $\pi_{\underline{n},\bar{r}}$ with $\mathrm{hgt}(\pi_{\underline{n},\bar{r}}) < \omega k$ and $\rho(\pi_{\underline{n},\bar{r}}) = m$ whose end-sequent is $\Phi(\bar{r}) \vdash \Psi(\bar{r}), \theta(\underline{n}, \bar{r})$. So we can let

$$\nu_{\bar{r}} \quad = \quad \cfrac{\begin{array}{c}\vdots\ \pi_{\underline{0},\bar{r}} \\ \Phi(\bar{r}) \vdash \Psi(\bar{r}), \theta(\underline{0}, \bar{r})\end{array} \qquad \begin{array}{c}\vdots\ \pi_{\underline{1},\bar{r}} \\ \Phi(\bar{r}) \vdash \Psi(\bar{r}), \theta(\underline{1}, \bar{r})\end{array} \qquad \begin{array}{c}\vdots\ \pi_{\underline{2},\bar{r}} \\ \Phi(\bar{r}) \vdash \Psi(\bar{r}), \theta(\underline{2}, \bar{r})\end{array} \qquad \cdots}{\Phi(\bar{r}) \vdash \Psi(\bar{r}), \forall w\ \theta(w, \bar{r})}\ \omega\mathrm{R}$$

Notice $\mathrm{hgt}(\nu_{\bar{r}}) = \sup\{\mathrm{hgt}(\pi_{\underline{n},\bar{r}}) + 1 : n \in \mathbb{N}\} \leqslant \omega k$ and $\rho(\nu_{\bar{r}}) = \sup\{\rho(\pi_{\underline{n},\bar{r}}) : n \in \mathbb{N}\} = m$.

$\boxed{\mathrm{Ind}}$ Suppose

$$\nu(\bar{z}) \quad = \quad \cfrac{\begin{array}{c}\vdots\ \pi(v, \bar{z}) \\ \theta(v, \bar{z}), \Phi(\bar{z}) \vdash \Psi(\bar{z}), \theta(v + 1, \bar{z})\end{array}}{\theta(0, \bar{z}), \Phi(\bar{z}) \vdash \Psi(\bar{z}), \theta(t(\bar{z}), \bar{z})}\ \mathrm{Ind}$$

Note that Φ, Ψ and t cannot mention v because of the eigenvariable condition. Apply the induction hypothesis to $\pi(v, \bar{z})$ to obtain $k, m \in \mathbb{N}$ as in the statement displayed above. Then for each $n \in \mathbb{N}$, we have a **Nat**-proof $\pi_{\underline{n},\bar{r}}$ with $\mathrm{hgt}(\pi_{\underline{n},\bar{r}}) < \omega k$ and $\rho(\pi_{\underline{n},\bar{r}}) = m$ whose end-sequent is $\theta(\underline{n}, \bar{r}), \Phi(\bar{r}) \vdash \Psi(\bar{r}), \theta(\underline{n} + 1, \bar{r})$. Now let $n = t(\bar{r})^{\mathbb{N}}$ and $\nu_{\bar{r}}$ be the **Nat**-proof

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\begin{array}{c}\vdots\ \pi_{\underline{0},\bar{r}} \\ \theta(\underline{0}, \bar{r}), \Phi(\bar{r}) \vdash \Psi(\bar{r}), \theta(\underline{1}, \bar{r})\end{array} \qquad \begin{array}{c}\vdots\ \pi_{\underline{1},\bar{r}} \\ \theta(\underline{1}, \bar{r}), \Phi(\bar{r}) \vdash \Psi(\bar{r}), \theta(\underline{2}, \bar{r})\end{array}}{\text{exchanges and contractions}}\ \mathrm{cut}}{\theta(\underline{0}, \bar{r}), \Phi \vdash \Psi, \theta(\underline{2}, \bar{r})} \qquad \begin{array}{c}\vdots\ \pi_{\underline{2},\bar{r}} \\ \theta(\underline{2}, \bar{r}), \Phi(\bar{r}) \vdash \Psi(\bar{r}), \theta(\underline{3}, \bar{r})\end{array}}{\text{exchanges and contractions}}\ \mathrm{cut}}{\theta(\underline{0}, \bar{r}), \Phi(\bar{r}) \vdash \Psi(\bar{r}), \theta(\underline{3}, \bar{r})}}{\vdots}$$

$$\mathrm{cut}\ \cfrac{\theta(\underline{0}, \bar{r}), \Phi(\bar{r}) \vdash \Psi(\bar{r}), \theta(\underline{n}, \bar{r}) \qquad \begin{array}{c}\text{Lemma 20.4}\ \vdots \\ \theta(\underline{n}, \bar{r}) \vdash \theta(t(\bar{r}), \bar{r})\end{array}}{\theta(\underline{0}, \bar{r}), \Phi(\bar{r}) \vdash \Psi(\bar{r}), \theta(t(\bar{r}), \bar{r})}$$

Notice $\mathrm{hgt}(\nu_{\bar{r}}) < \omega k + \omega = \omega(k + 1)$ and $\rho(\nu_{\bar{r}}) = \max\{\mathrm{dp}(\theta) + 1, m\}$. $\square$

The ordinal we assign to a **Nat**-proof in step (1) of our plan is simply the height of the proof. To execute (2), we proceed as in our proof of the cut-elimination theorem for **LK**. We now need to keep track of the heights of the proofs so as to make sure that the cut-elimination process does not make the height of a **Nat**-proof grow past ε_0. Since we do not want to go into the ordinal arithmetic, we will leave the calculations vague. In particular, if it is not clear what ω^α means below, then assume additionally that $\alpha < \varepsilon_0$.

Lemma 20.7. Let $d \in \mathbb{N}$ and α be an ordinal. From a **Nat**-proof π of the form

$$\dfrac{\begin{array}{c}\vdots\ \pi_0 \\ \Phi \vdash \Psi, \theta\end{array} \qquad \begin{array}{c}\vdots\ \pi_1 \\ \theta, \Phi' \vdash \Psi'\end{array}}{\Phi, \Phi' \vdash \Psi, \Psi'}\ \text{cut}$$

where $\mathrm{hgt}(\pi) \leqslant \omega^\alpha + 1$ and $\max\{\rho(\pi_0), \rho(\pi_1)\} \leqslant d = \mathrm{dp}(\theta)$, one can recursively construct a **Nat**-proof $\tilde{\pi}$ with $\mathrm{hgt}(\tilde{\pi}) < \omega^\alpha 2$ and $\rho(\tilde{\pi}) \leqslant d$ whose end-sequent is the same as that of π.

Proof. This is similar to the proof of Lemma 19.2. We assume again $\Phi' = \Phi$ and $\Psi' = \Psi$. This assumption does not affect the heights because ω^α and $\omega^\alpha 2$ are both ordinals below which there is no largest ordinal. Since **Nat** has more or less the same structural rules and propositional rules as **LK**, the only differences lie in the atomic case and the quantifier case. In each case, the required **Nat**-proof $\tilde{\pi}$ is constructed roughly by stacking π_0 and π_1 on top of each other. So $\mathrm{hgt}(\tilde{\pi}) < \omega^\alpha + \omega^\alpha = \omega^\alpha 2$ because both $\mathrm{hgt}(\pi_0)$ and $\mathrm{hgt}(\pi_1)$ are at most ω^α.

Suppose θ is atomic. Split into two symmetric cases $\mathbb{N} \models \theta$ and $\mathbb{N} \not\models \theta$; see Assignment 20.8.

Suppose $\theta = \exists w\ \eta(w)$. Unlike the proof of Lemma 19.2, we do not need to worry about variables here because there is no variable in **Nat**.

For each closed $\mathcal{L}_\mathrm{R}(\exp)$ term t, let $\pi_1^{\flat(t)}$ be the **Nat**-proof obtained from π_1 by applying the following operations.

(1) Change all the direct ancestors of the leftmost $\exists w\ \eta(w)$ in the end-sequent to $\eta(t)$.

(2) Change each sequent $\Gamma \vdash \Delta$ to $\Gamma, \Phi \vdash \Psi, \Delta$.

(3) The tree of sequents obtained may not be an **LK**-proof because perhaps

$$\omega\mathrm{L}\ \dfrac{\begin{array}{c}\vdots \\ \eta(\underline{0}), \Gamma \vdash \Delta\end{array} \quad \begin{array}{c}\vdots \\ \eta(\underline{1}), \Gamma \vdash \Delta\end{array} \quad \cdots}{\exists w\ \eta(w), \Gamma \vdash \Delta} \quad \overset{(1)}{\mapsto} \quad \overset{(2)}{\mapsto} \quad \dfrac{\begin{array}{c}\vdots\ \nu_0 \\ \eta(\underline{0}), \Gamma, \Phi \vdash \Psi, \Delta\end{array} \quad \begin{array}{c}\vdots\ \nu_1 \\ \eta(\underline{1}), \Gamma, \Phi \vdash \Psi, \Delta\end{array} \quad \cdots}{\eta(t), \Gamma, \Phi \vdash \Psi, \Delta}$$

in which case replace the result here by

$$\dfrac{\begin{array}{c}\text{Lemma 20.4}\ \vdots \\ \eta(t) \vdash \eta(\underline{n})\end{array} \qquad \begin{array}{c}\vdots\ \nu_n \\ \eta(\underline{n}), \Gamma, \Phi \vdash \Psi, \Delta\end{array}}{\eta(\underline{t}), \Gamma, \Phi \vdash \Psi, \Delta}\ \text{cut}$$

where $n = t^{\mathbb{N}}$. The other steps remain compliant with the deduction rules in **Nat** since

$$\mathrm{wL}\ \dfrac{\Gamma \vdash \Delta}{\exists w\ \eta(w), \Gamma \vdash \Delta} \quad \overset{(1)}{\mapsto} \quad \overset{(2)}{\mapsto} \quad \dfrac{\Gamma, \Phi \vdash \Psi, \Delta}{\eta(t), \Gamma, \Phi \vdash \Psi, \Delta}\ \mathrm{wL}$$

(4) Add some exchanges and contractions at the lower end to turn the end-sequent $\eta(t, \bar{z}), \Phi, \Phi \vdash \Psi, \Psi$ into $\eta(t, \bar{z}), \Phi \vdash \Psi$.

Then we obtain $\tilde{\pi}$ from π_0 in exactly the same way as we did in the proof of Lemma 19.2. □

Assignment 20.8. Let θ be an atomic $\mathcal{L}_{\mathrm{R}}(\exp)$ sentence such that $\mathbb{N} \models \theta$. Suppose some cut-free **Nat**-proof π_1 has end-sequent $\theta, \Phi \mid\!- \Psi$. Find a cut-free **Nat**-proof $\tilde{\pi}$ with end-sequent $\Phi \mid\!- \Psi$. [5 points]

As in the case of **LK**, the cut-elimination theorem for **Nat** can be proved by applying Lemma 20.7 repeatedly. Although the strategy is similar, one now needs to be more careful in choosing where to apply Lemma 20.7 first because infinitely many cut rules may appear in a **Nat**-proof. In particular, if the cut-rank of a **Nat**-proof is ω, then it is not clear where one should start. So let us restrict ourselves to **Nat**-proofs of finite cut-ranks.

Cut-elimination theorem for Nat. Every **Nat**-proof of finite cut-rank (and height strictly less than ε_0) can be turned into a cut-free **Nat**-proof (of height strictly less than ε_0) with the same end-sequent.

Proof. Let us concentrate on the ε_0 case; one can prove the general case in exactly the same way. We claim that $P_d(\delta)$ holds for every $d \in \mathbb{N}$ and every ordinal $\delta < \varepsilon_0$, where $P_d(\delta)$ stands for the following statement:

> for each **Nat**-proof π with $\rho(\pi) \leqslant d+1$ and $\mathrm{hgt}(\pi) = \delta$, there is a **Nat**-proof $\tilde{\pi}$ with $\rho(\tilde{\pi}) \leqslant d$ and $\mathrm{hgt}(\tilde{\pi}) \leqslant \omega^{\delta}$ such that π and $\tilde{\pi}$ have the same end-sequent.

This suffices because, given any **Nat**-proof π of cut-rank $d \in \mathbb{N}$ and height $\delta < \varepsilon_0$, one can successively apply the claim d-many times to obtain a **Nat**-proof $\tilde{\pi}$ with the same end-sequent as π such that

$$\mathrm{hgt}(\tilde{\pi}) < \left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^{\delta}}}}}}\right\}{\scriptstyle d\text{-many } \omega\text{'s}} < \varepsilon_0 \quad \text{and} \quad \rho(\tilde{\pi}) = 0.$$

With $d \in \mathbb{N}$ fixed, we prove the claim by transfinite induction on δ. Take any ordinal $\delta < \varepsilon_0$ such that $P_d(\alpha)$ holds for every ordinal $\alpha < \delta$. If $\delta = 0$, then there is nothing to prove because no **Nat**-proof of height 0 can use the cut rule. So suppose $\delta > 0$. Let π be the **Nat**-proof

$$\frac{\pi_0 \qquad \pi_1 \qquad \pi_2 \qquad \cdots}{A}\circledast$$

where $\circledast$ has 1, 2, or ω-many upper sequents, and $\rho(\pi) \leqslant d+1$ and $\mathrm{hgt}(\pi) = \delta$. We see from the definition of height that $\mathrm{hgt}(\pi_i) < \mathrm{hgt}(\pi) = \delta$ for all i. Apply the induction hypothesis to $\pi_0, \pi_1, \pi_2, \ldots$ to obtain $\tilde{\pi}_0, \tilde{\pi}_1, \tilde{\pi}_2, \ldots$ respectively. Note that $\rho(\tilde{\pi}_i) \leqslant d$ and $\mathrm{hgt}(\tilde{\pi}_i) \leqslant \omega^{\mathrm{hgt}(\pi_i)} < \omega^{\mathrm{hgt}(\pi)} = \omega^{\delta}$ for all i. We split into two cases.

Suppose $\circledast \neq$ cut, or $\circledast =$ cut but the cut-formula involved has depth strictly less than d. Then let $\tilde{\pi}$ be the **Nat**-proof

$$\frac{\tilde{\pi}_0 \qquad \tilde{\pi}_1 \qquad \tilde{\pi}_2 \qquad \cdots}{A}\circledast$$

By the definition of cut-rank and height, one sees that $\rho(\tilde{\pi}) \leqslant d$ and

$$\mathrm{hgt}(\tilde{\pi}) = \sup\{\mathrm{hgt}(\tilde{\pi}_0)+1, \mathrm{hgt}(\tilde{\pi}_1)+1, \mathrm{hgt}(\tilde{\pi}_2)+1, \ldots\} \leqslant \omega^{\mathrm{hgt}(\pi)} = \omega^{\delta}.$$

Suppose $\circledast =$ cut and the cut-formula involved has depth d. Then apply Lemma 20.7 to the **Nat**-proof

$$\frac{\tilde{\pi}_0 \qquad \tilde{\pi}_1}{A}\,\mathrm{cut}$$

to obtain a **Nat**-proof $\tilde{\pi}$ with end-sequent A. According to the bounds in Lemma 20.7, we have $\rho(\tilde{\pi}) \leqslant d$ and $\mathrm{hgt}(\tilde{\pi}) < \omega^{\delta-1}2 < \omega^{\delta}$ because $\mathrm{hgt}(\pi_i) \leqslant \delta - 1$ implies $\mathrm{hgt}(\tilde{\pi}_i) \leqslant \omega^{\delta-1}$ for both $i < 2$ by the induction hypothesis. □

The cut-elimination theorem is where ε_0 comes into our study of **PA**. Roughly speaking, Lemma 20.7 tells us that reducing the depth of a cut-formula involves stacking the two subproofs above the cut rule on top of each other, and so the height increases by a factor of 2 approximately. By considering a tree of sequents full of cuts, one can bound the number of cut rules in a **Nat**-proof by an exponential of its height. Therefore, reducing the cut-rank of a **Nat**-proof from a natural number d to 0 leads to an increase in height that is a d-times iteration of exponentials. This makes ε_0 an upper bound for the heights of **Nat**-proofs after cut elimination because it is the first ordinal bigger than ω^2 below which ordinals are closed under exponentiation $\alpha \mapsto \omega^\alpha$. Here ω^2 comes from the height bound in Proposition 20.6.

Finally, let us put everything together.

Consistency proof for PA**.** Let π be a **PA**-proof in which no free variable appears in the end-sequent. Apply Proposition 20.6 to find a **Nat**-proof π_∞ with $\text{hgt}(\pi_\infty) < \omega^2$ and $\rho(\pi_\infty) < \omega$ whose end-sequent is the same as that of π. Then use the cut-elimination theorem for **Nat** to turn π_∞ into a cut-free **Nat**-proof $\tilde{\pi}_\infty$ while keeping the end-sequent. Since $\tilde{\pi}_\infty$ cannot have an empty end-sequent by Observation 18.9, we deduce that π also cannot have an empty end-sequent. This shows the consistency of PA. □

As the proof above shows, although the consistency of PA cannot be proved in PA, it is not unattainable. Moreover, a consistency proof based on limited resources can give extra information about PA. With some patience, one can verify that all the recursive proof transformations and constructions in this lecture can be carried out over PA. The only non-recursive construction in the consistency proof above is the one in the cut-elimination theorem for **Nat**, where transfinite induction below ε_0 with respect to recursive predicates was used. This principle of transfinite induction can be expressed in $\mathcal{L}_\text{A}(\text{exp})$ via the representability of recursive predicates in Q(exp) and the representation of ordinals less than ε_0 introduced in Lecture 17. So the Second Incompleteness Theorem applies: since PA plus this principle proves the consistency of PA, this intuitive principle cannot be provable in PA.

As alluded to in Lecture 17, the PA-unprovability of transfinite induction up to ε_0 entails the PA-unprovability of a suitable formalization of the statement in Theorem 17.7(ii). The proof we gave was proof-theoretic. In the next four lectures, we will see how similar unprovability theorems can be established using model-theoretic methods.