MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Lecture 21: End extension

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In the last few lectures, we saw how to prove unprovability proof-theoretically. More specifically, one proves the unprovability of Goodstein's theorem in PA by proving from it the consistency of PA, and then invoke the Second Incompleteness Theorem. It would be nice if one can avoid going through consistency in a proof-theoretic unprovability proof, e.g., by devising a suitable complete proof system which by design cannot prove certain sentences. However, such cases seem to be rare, if they exist at all.

In the next three lectures, we show how to establish unprovability results using model-theoretic methods: to prove the unprovability of a sentence in PA, we produce a model of PA which does not satisfy this sentence. As it turns out, for us, this model will be an initial segment of an existing model of PA. Let us start from the other direction in this lecture, by constructing end extensions of models of PA.

Definition. Let K, M be $\mathcal{L}_\mathrm{A}(\exp)$ structures. Then M is an *end extension* of K, or K is a *cut* of M, if $K \subseteq M$ as $\mathcal{L}_\mathrm{A}(\exp)$ structures, and

$$\forall m{\in}M \setminus K \quad \forall k{\in}K \quad m \geqslant k.$$

We use a subscript e to indicate end extension, as in $M \supseteq_\mathrm{e} K$.

Recall that the *standard model of arithmetic* is $\mathbb{N} = (\mathbb{N}, 0, 1, +, \times, \exp, <)$. An $\mathcal{L}_\mathrm{A}(\exp)$ structure is *nonstandard* if and only if it is not isomorphic to the standard model. As we saw in Lecture 3, every model of R(exp) is an extension of $\mathbb{N}$. Moreover, Observation 4.3 tells us that all such extensions are end extensions. In particular, all extensions of $\mathbb{N}$ satisfying R(exp) are end extensions. Using the Compactness Lemma, one can readily show that no nonstandard model of R(exp) has this property. Nevertheless, many nice $\mathcal{L}_\mathrm{A}(\exp)$ structures have proper end extensions. The nicer ones even have proper *elementary* end extensions, i.e., those in which the truth of all formulas, with parameters from the ground model, is preserved; see Theorem 15.3(i) for the definition of elementarity. In general, there is a tight connection between

the existence of proper elementary end extensions   and   strength/induction.

This is one of the reasons why (partially) elementary end extensions are interesting.

In the mid 1960s, Keisler isolated the so-called *regularity scheme* for the construction of elementary end extensions. The term probably came from the notion of regular cardinals in set theory. In arithmetic, it is the same as the Infinite Pigeonhole Principle: if one partitions an infinite set into finitely many parts, then one part has to be infinite. As is well known, infinitude cannot be expressed in $\mathcal{L}_\mathrm{A}(\exp)$. For the natural numbers, fortunately we can replace infinitude by unboundedness/cofinality, because a set of natural numbers is infinite if and only if it is unbounded in $\mathbb{N}$. The letters cf below abbreviate the word *cofinal*.

Definition. Let $\exists^\mathrm{cf} x \ldots$ stand for $\forall y\ \exists x{\geqslant}y \ldots$, where y is a fresh variable.

Lemma 21.1. PA proves the *regularity scheme*

$$\forall \bar{z}\ \forall a\ \big(\exists^\mathrm{cf} x\ \exists u{<}a\ \varphi(u, x, \bar{z}) \rightarrow \exists u{<}a\ \exists^\mathrm{cf} x\ \varphi(u, x, \bar{z})\big),$$

where φ ranges over all $\mathcal{L}_\mathrm{A}(\exp)$ formulas.

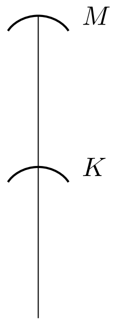

Figure 21.1: An end extension M of K, and a cut K of M

Proof. Fix $\bar{c} \in M \models \mathrm{PA}$. We show by induction on a in M that

$$M \models \forall a \left(\exists^{\mathrm{cf}} x\ \exists u{<}a\ \varphi(u, x, \bar{c}) \to \exists u{<}a\ \exists^{\mathrm{cf}} x\ \varphi(u, x, \bar{c})\right).$$

The base case when $a = 0$ holds trivially because $M \models \forall u\ (u \not< 0)$. For the induction step, let $a \in M$ such that

$$M \models \exists^{\mathrm{cf}} x\ \exists u{<}a\ \varphi(u, x, \bar{c}) \to \exists u{<}a\ \exists^{\mathrm{cf}} x\ \varphi(u, x, \bar{c}).$$

Assume $\exists^{\mathrm{cf}} x\ \exists u{<}a+1\ \varphi(u, x, \bar{c})$. If $M \models \exists^{\mathrm{cf}} x\ \varphi(a, x, \bar{c})$, then we are already done. So suppose not. Then

$$M \models \exists^{\mathrm{cf}} x\ \exists u{<}a\ \varphi(u, x, \bar{c})$$

because the union of two bounded subsets of M is bounded. Hence, by the induction hypothesis, we are also done. □

Our construction of elementary end extensions uses a bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ to let us deal with $\mathbb{N}$-many tasks $\mathbb{N}$-many times in $\mathbb{N}$-many steps. The usual such bijection works already in $\mathrm{I}\Delta_0$.

Definition (Cantor)**.** Let $\langle x, y \rangle = z$ be the Δ_0 formula

$$2z = (x + y)(x + y + 1) + 2y.$$

Let $\mathrm{left}(z) = x$ and $\mathrm{right}(z) = y$ be respectively the Δ_0 formulas

$$\exists y{\leqslant}z\ \langle x, y \rangle = z \quad \text{and} \quad \exists x{\leqslant}z\ \langle x, y \rangle = z.$$

Lemma 21.2. $\mathrm{I}\Delta_0(\exp)$ proves

(1) $\forall x, y\ \exists! z\ \langle x, y \rangle = z$;

(2) $\forall x, y, z\ \left(\langle x, y \rangle = z \to z \leqslant (x + y)(x + y + 1) + 2y \wedge x \leqslant z \wedge y \leqslant z\right)$; and

(3) $\forall z\ \left(\exists! x\ \exists y\ \langle x, y \rangle = z \wedge \exists! y\ \exists x\ \langle x, y \rangle = z\right)$.

Proof. Elementary. □

The theorem that every model of PA has a proper elementary end extension was proved by Mac Dowell and Specker in the early 1960s. It was an important theorem historically because, since the discovery of nonstandard models of PA by Gödel and Skolem in the 1930s, it had not been clear what interesting properties these models can have. The construction of a proper elementary end extensions is one of the first interesting operations one can perform on nonstandard models of PA. Similar to Skolem's construction of a nonstandard model, the original proof by Mac Dowell and Specker is essentially a definable ultrapower construction. To avoid going into the topic of ultrapowers, we unravel this construction into a Henkin construction, i.e., one that mimics our proof of the Henkinization Lemma in Lecture 7. It can be seen as a simple instance of model-theoretic forcing, using which a particular case of the Omitting-Types Theorem will be established. Here we restrict ourselves to *countable* models of PA (meaning models of PA whose universes are countable) because this is already enough for our purposes, and the restriction slightly simplifies the combinatorics involved.

The outline of the construction is as follows. Given a countable model $K \models \mathrm{PA}$, add a new element a_0 above all the elements of K. Since the extension is intended to be elementary, lots of other elements will come with a_0. We will decide the $\mathcal{L}_\mathrm{A}(\exp)$ properties of a_0 one by one so that none of these new elements sits below an old element.

Mac Dowell–Specker Theorem (countable version)**.** Every countable $K \models \mathrm{PA}$ has a proper elementary end extension M.

Proof. Let $(c_j)_{j\in\mathbb{N}}$ enumerate the elements of K. For all $j \in \mathbb{N}$, define

$$x = \mathsf{v}_0, \quad \text{and} \quad u_j = \mathsf{v}_{2j+1}, \quad \text{and} \quad z_j = \mathsf{v}_{2j+2}.$$

The x represents the new element we add to K, the $u_0, u_1, u_2, \ldots$ will be used for Henkinization, and the $z_0, z_1, z_2, \ldots$ represent the elements $c_0, c_1, c_2, \ldots$ of K respectively. Fix an enumeration

$$\big(\theta_j(u_0, u_1, \ldots, u_{2j}, x, z_0, z_1, \ldots)\big)_{j\in\mathbb{N}}$$

of all $\mathcal{L}_\mathrm{A}(\exp)$ formulas. We will construct another sequence of $\mathcal{L}_\mathrm{A}(\exp)$ formulas

$$\big(\varphi_k(u_0, u_1, \ldots, u_{k+1}, x, z_0, z_1, \ldots)\big)_{k\in\mathbb{N}}$$

with the inductive assumption that at each stage $k \in \mathbb{N}$,

$$K \models \exists^{\mathrm{cf}} x\ \exists u_0, u_1, \ldots, u_k \bigwedge\!\!\!\bigwedge \Phi_k(u_0, u_1, \ldots, u_k, x, c_0, c_1, \ldots),$$

where $\Phi_k = \{\varphi_i : i < k\}$. At the end, we will apply the Model Construction Theorem from Lecture 7 to $\Phi^* := \{\varphi_i : i \in \mathbb{N}\}$ to obtain an $\mathcal{L}_\mathrm{A}(\exp)$ structure M with an enumeration $a_0, a_1, a_2, \ldots$ of M such that

$$M \models \varphi_k(a_1, a_3, \ldots, a_{2(k+1)+1}, a_0, a_2, a_4, a_6, \ldots) \quad \text{for all } k \in \mathbb{N}.$$

The inductive assumption and Soundness ensure the consistency of every finite subset of Φ^*. So $\Phi^* \nvdash \bot$ by the Compactness Lemma.

We will use the even stages to make Φ^* Henkinized and decide all $\mathcal{L}_\mathrm{A}(\exp)$ formulas. We will use the odd stages to ensure M is an end extension of K. Assume $(\varphi_i)_{i<2j}$ is defined. Consider θ_j.

Case 1: Suppose $K \not\models \exists^{\mathrm{cf}} x\ \exists \bar{u}\ \big(\bigwedge\!\!\!\bigwedge \Phi_{2j}(\bar{u}, x, \bar{c}) \wedge \theta_j(\bar{u}, x, \bar{c})\big)$. Then

$$K \models \exists^{\mathrm{cf}} x\ \exists \bar{u}\ \Big(\bigwedge\!\!\!\bigwedge \Phi_{2j}(\bar{u}, x, \bar{c}) \wedge \neg\theta_j(\bar{u}, x, \bar{c})\Big)$$

by the inductive assumption, because the union of two bounded subsets of K is bounded. Set $\varphi_{2j} = \neg\theta_j$. Note $\Phi_{2j+1} \vdash \neg\theta_j$.

Case 2: Suppose $K \models \exists^{\mathrm{cf}} x\ \exists \bar{u}\ \big(\bigwedge\!\!\!\bigwedge \Phi_{2j}(\bar{u}, x, \bar{c}) \wedge \theta_j(\bar{u}, x, \bar{c})\big)$.

Case 2a: Suppose $\theta_j = \exists y\ \eta(u_0, u_1, \ldots, u_{2j}, y, x, \bar{z})$. Since u_{2j+1} does not appear free in Φ_{2j},

$$K \models \exists^{\mathrm{cf}} x\ \exists \bar{u}, u_{2j+1}\ \Big(\bigwedge\!\!\!\bigwedge \Phi_{2j}(\bar{u}, x, \bar{c}) \wedge \eta(\bar{u}, u_{2j+1}, x, \bar{c})\Big).$$

Set $\varphi_{2j}(u_0, u_1, \ldots, u_{2j+1}, x, \bar{z}) = \eta(\bar{u}, u_{2j+1}, x, \bar{z})$. Note $\Phi_{2j+1} \vdash \theta_j$ by $\exists$R.

Case 2b: Suppose θ_j is not of the form $\exists y\ \eta$. Then set $\varphi_{2j} = \theta_j$. Note $\Phi_{2j+1} \vdash \theta_j$.

After defining φ_{2j}, we consider u_m, where $m = \mathrm{left}(j)$. This arrangement ensures that every u_m is considered infinitely many times in the construction.

Case I: Suppose $n \in \mathbb{N}$ such that $\Phi_{2j+1} \vdash u_m < z_n$. In view of the inductive assumption, we know $\Phi_{2j+1} \nvdash \forall u\ (u < z_n)$, and so u_m must appear free in Φ_{2j+1} by $\forall$R. Thus $m \leqslant 2j+1$. Now, since $\Phi_{2j+1} \vdash u_m < z_n$, the inductive assumption implies

$$K \models \exists^{\mathrm{cf}} x\ \exists u_m{<}c_n\ \exists u_0, u_1, \ldots, u_{m-1}, u_{m+1}, u_{m+2}, \ldots, u_{2j+1} \bigwedge\!\!\!\bigwedge \Phi_{2j+1}(\bar{u}, u_{2j+1}, x, \bar{c}).$$

Apply the regularity scheme from Lemma 21.1 to find $c_\ell \in K$ such that

$$K \models \exists^{\mathrm{cf}} x\ \exists \bar{u}, u_{2j+1}\ \Big(\bigwedge\!\!\!\bigwedge \Phi_{2j+1}(\bar{u}, u_{2j+1}, x, \bar{c}) \wedge u_m = c_\ell\Big).$$

Let φ_{2j+1} be $u_m = z_\ell$.

Case II: Suppose $\Phi_{2j+1} \nvdash u_m < z_n$ for any $n \in \mathbb{N}$. Then let φ_{2j+1} be $x = x$.

By construction, the set Φ^* is consistent, Henkinized, and decides all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas. So the Model Construction Theorem is applicable. On the one hand, if $\theta(z_0, z_1, \ldots, z_n)$ is an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula such that $K \models \theta(c_0, c_1, \ldots, c_n)$, then

	$\forall k{\in}\mathbb{N}$	$\Phi_k \nvdash \neg\theta(z_0, z_1, \ldots, z_n)$	by the inductive assumption;
$\therefore$		$\Phi^* \nvdash \neg\theta(z_0, z_1, \ldots, z_n)$	by the Compactness Lemma;
$\therefore$		$\Phi^* \vdash \theta(z_0, z_1, \ldots, z_n)$	as Φ^* decides all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas;
$\therefore$		$M \models \theta(a_2, a_4, \ldots, a_{2n+2})$	by the choice of M.

Hence, identifying each $c_j \in K$ with $a_{2j+2} \in M$, we have $M \succcurlyeq K \models \mathrm{PA}$. On the other hand, note

	$\forall k, n{\in}\mathbb{N}$	$\Phi_k \nvdash x \leqslant z_n$	by the inductive assumption;
$\therefore$	$\forall n{\in}\mathbb{N}$	$\Phi^* \nvdash x \leqslant z_n$	by the Compactness Lemma;
$\therefore$	$\forall n{\in}\mathbb{N}$	$\Phi^* \vdash x \nleqslant z_n$	as Φ^* decides all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas;
$\therefore$	$\forall c{\in}K$	$M \models a_0 \nleqslant c$	by the choice of M and the identification above;
$\therefore$	$\forall c{\in}K$	$M \models a_0 > c$	as $M \models \mathrm{PA}$.

This implies $a \in M \setminus K$, and thus $M \neq K$. Finally, to verify that $M \supseteq_{\mathrm{e}} K$, we prove

$$a_i < c \in K \quad \Rightarrow \quad a_i \in K.$$

Suppose $a_i < c_n \in K$. We saw just now that $i \neq 0$ in this case. If $i = 2j+2$, then $a_i = a_{2j+2} = c_j \in K$ by the identification above. So suppose $i = 2m+1$. By the choice of M and $a_0, a_1, a_2, \ldots$, we know $\Phi^* \nvdash u_m \not< z_n$. Hence $\Phi^* \vdash u_m < z_n$ as Φ^* decides all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas. Apply the Compactness Lemma to find $k \in \mathbb{N}$ such that $\Phi_k \vdash u_m < z_n$. Without loss of generality, assume $k = 2j+1$, where $\mathrm{left}(j) = m$. Then we must be in Case I when defining φ_{2j+1}. Our construction thus gives $\ell \in \mathbb{N}$ such that φ_{2j+1} is $u_m = z_\ell$. So $a_i = a_{2m+1} = c_\ell \in K$ by the choice of M. ☐

In the proof above, we implicitly assumed that every $\mathcal{L}_{\mathrm{A}}(\exp)$ property satisfied by the new element a_0 is satisfied by arbitrarily large elements of the ground model K. This assumption is actually well justified.

Assignment 21.3. Let $K \preccurlyeq_{\mathrm{e}} M \models \mathrm{I}\Delta_0(\exp)$ and $a \in M \setminus K$. If $\theta(x)$ is an $\mathcal{L}_{\mathrm{A}}(\exp)(K)$ formula such that $M \models \theta(a)$, then $M \models \exists^{\mathrm{cf}} x\ \theta(x)$. [4 points]

As observed already in the 1960s by Keisler, the use of the regularity scheme in the construction of elementary end extensions is necessary. For the extension of this to the induction scheme, one had to wait until the 1970s. More precisely, Paris and Kirby showed that if a model of $\mathrm{I}\Delta_0(\exp)$ has a proper elementary end extension, then this model must satisfy PA. We will see in the next lecture more about what Paris and Kirby did in the late 1970s.