Lecture 24: Indiscernibles

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The aim of this lecture is to verify that $Y_{\mathrm{PH}}(x, y) = z$ defined at the end of the previous lecture is indeed an indicator for PA over PA. The only remaining part to be checked is the following.

Theorem 24.1 (Paris–Harrington)**.** *If $a, b \in M \models \mathrm{PA}$ such that $Y^M_{\mathrm{PH}}(a, b) > \mathbb{N}$, then there is $K \subseteq_{\mathrm{e}} M$ satisfying PA with $a \in K < b$.*

Our proof relies on indiscernibles. Roughly speaking, a sequence is indiscernible if there is no formula which can tell its elements apart. Unlike the usual notion of indiscernibility, we may even allow small enough parameters in the formula. Note that what we define here are indiscernible *sequences*, not indiscernible *sets*. So the order in which the elements are taken out matters.

Definition. A $\Delta_0(\exp)$*-indiscernible sequence* over a model $M \models \mathrm{I}\Delta_0(\exp)$ is a strictly increasing sequence $(c_i)_{i\in\mathbb{N}}$ of elements of M such that for all $\Delta_0(\exp)$ formulas $\theta(\bar{u}, v_1, v_2, \dots, v_\ell)$ and all subsequences $(d_i)_{i\leqslant 2\ell}$ of (c_i),

$$M \models \forall \bar{u}{<}d_0 \ \big(\theta(\bar{u}, d_1, d_2, \dots, d_\ell) \leftrightarrow \theta(\bar{u}, d_{\ell+1}, d_{\ell+2}, \dots, d_{2\ell})\big).$$

In this case, let $c_{\mathbb{N}} = \{x \in M : x < c_i \text{ for some } i \in \mathbb{N}\}$.

Here is how indiscernibles help us prove Theorem 24.1.

Theorem 24.2. *Let $a, b \in M \models \mathrm{PA}$ with $Y^M_{\mathrm{PH}}(a, b) > \mathbb{N}$ and $a > \mathbb{N}$. Then there exists a $\Delta_0(\exp)$-indiscernible sequence over M in $[a, b]$.*

Proposition 24.3. *Let $M \models \mathrm{I}\Delta_0(\exp)$ and $(c_i)_{i\in\mathbb{N}}$ be a $\Delta_0(\exp)$-indiscernible sequence over M. Then $c_{\mathbb{N}} \models \mathrm{PA}$.*

Proof of Theorem 24.1 from Theorem 24.2 and Proposition 24.3. Let $a, b \in M \models \mathrm{PA}$ such that $Y^M_{\mathrm{PH}}(a, b) > \mathbb{N}$. Notice $b \geqslant 4Y^M_{\mathrm{PH}}(a, b) + 1 > \mathbb{N}$ by the definition of $Y_{\mathrm{PH}}(x, y) = z$. Hence if $a \in \mathbb{N}$, then we are already done because $M \supseteq_{\mathrm{e}} \mathbb{N} \models \mathrm{PA}$ and $a \in \mathbb{N} < b$. So suppose $a > \mathbb{N}$. Apply Theorem 24.2 to find a $\Delta_0(\exp)$-indiscernible sequence $(c_i)_{i\in\mathbb{N}}$ over M in $[a, b]$. Then Proposition 24.3 tells us that $c_{\mathbb{N}} \models \mathrm{PA}$. Note $a \leqslant c_0 \in c_{\mathbb{N}} < b$. □

Remark 24.4. Theorem 24.2 actually remains true even when the $a > \mathbb{N}$ condition is removed, but a small additional argument is needed to prove this.

The rest of this lecture is devoted to proving Theorem 24.2 and Proposition 24.3. We first consider Proposition 24.3, which is simpler. The following key lemma is one of the many miraculous applications of indiscernibles. To start with, one cannot have both x and $x + 1$ in the same indiscernible sequence because they must have different parity. Part (1) below says that this can be strengthened from successor to exponential. Thus elements in an indiscernible sequence must be far apart. This part also implies that $c_{\mathbb{N}}$ is an $\mathcal{L}_{\mathrm{A}}(\exp)$ substructure of M with the interpretations of symbols inherited from M if $(c_i)_{i\in\mathbb{N}}$ is a $\Delta_0(\exp)$-indiscernible sequence over M. So it makes sense to talk about what formulas are true in this cut. Part (2) below says that every $\mathcal{L}_{\mathrm{A}}(\exp)$ formula for $c_{\mathbb{N}}$ translates to a $\Delta_0(\exp)$ formula for M by bounding all the quantifiers with elements of the indiscernible sequence. Informally speaking, this is true because what happens below some c_i must happen arbitrarily high in $c_{\mathbb{N}}$ by indiscernibility.

Lemma 24.5. Let $M \models \mathrm{I}\Delta_0(\exp)$ and $(c_i)_{i\in\mathbb{N}}$ be a $\Delta_0(\exp)$-indiscernible sequence over M. Then

(1) $c_{\mathbb{N}}$ is closed under $+$, $\times$, and exp; and

(2) for all $\eta(\bar{u}, x_1, x_2, \ldots, x_\ell) \in \Delta_0(\exp)$, all subsequences $(d_i)_{i\leqslant\ell}$ of (c_i), and all $\bar{a} < d_0$,

$$c_{\mathbb{N}} \models \exists x_1\ \forall x_2\ \cdots\ \mathrm{Q}x_\ell\ \eta(\bar{a}, \bar{x}) \quad \Leftrightarrow \quad M \models \exists x_1{<}d_1\ \forall x_2{<}d_2\ \cdots\ \mathrm{Q}x_\ell{<}d_\ell\ \eta(\bar{a}, \bar{x}).$$

Proof. For (1), it suffices to show $2^{c_i} \leqslant c_{i+2}$ for all $i \in \mathbb{N}$ because doubling and squaring grow more slowly than exponential. Pick $i \in \mathbb{N}$. Indiscernibility implies

$$M \models \forall u{<}c_i\ (u \in c_{i+1} \leftrightarrow u \in c_{i+2}),$$

where $x \in y$ is the $\Delta_0(\exp)$ formula from Example 2.1. Since $c_{i+1} < c_{i+2}$, we must have $c_{i+2} \geqslant 2^{c_i}$ by extensionality.

For (2), we proceed by induction on $\ell \in \mathbb{N}$. The base case when $\ell = 0$ follows from $\Delta_0(\exp)$ absoluteness, i.e., Proposition 22.2(1). For the induction step, suppose (2) holds for $\ell \in \mathbb{N}$. Let $\eta(\bar{u}, x_1, x_2, \ldots, x_{\ell+1}) \in \Delta_0(\exp)$. Take a subsequence $(d_i)_{i\leqslant\ell+1}$ of (c_i) and $\bar{a} < d_0$.

First, suppose $M \models \exists x_1{<}d_1\ \forall x_2{<}d_2\ \cdots\ \mathrm{Q}x_{\ell+1}{<}d_{\ell+1}\ \eta(\bar{a}, x_1, x_2, \ldots, x_{\ell+1})$. Use the truth definition to find $a' < d_1$ in M such that

$$M \models \forall x_2{<}d_2\ \exists x_3{<}d_3\ \cdots\ \mathrm{Q}x_{\ell+1}{<}d_{\ell+1}\ \eta(\bar{a}, a', x_2, x_3, \ldots, x_{\ell+1}).$$

Note $\bar{a} < d_0 < d_1$. So the induction hypothesis (with both sides negated) implies

$$\begin{aligned} & c_{\mathbb{N}} \models \forall x_2\ \exists x_3\ \cdots\ \mathrm{Q}x_{\ell+1}\ \eta(\bar{a}, a', x_2, x_3, \ldots, x_{\ell+1}) \\ \therefore\quad & c_{\mathbb{N}} \models \exists x_1\ \forall x_2\ \exists x_3\ \cdots\ \mathrm{Q}x_{\ell+1}\ \eta(\bar{a}, x_1, x_2, x_3, \ldots, x_{\ell+1}) \qquad \text{by the truth definition.} \end{aligned}$$

Conversely, suppose $M \models \forall x_1{<}d_1\ \exists x_2{<}d_2\ \cdots\ \mathrm{Q}'x_{\ell+1}{<}d_{\ell+1}\ \neg\eta(\bar{a}, x_1, x_2, \ldots, x_{\ell+1})$, where

$$\mathrm{Q}' = \begin{cases} \forall, & \text{if } \mathrm{Q} = \exists; \\ \exists, & \text{if } \mathrm{Q} = \forall. \end{cases}$$

Pick any $a' \in c_{\mathbb{N}}$. Extend $(d_i)_{i\leqslant\ell}$ to a subsequence $(d_i)_{i\leqslant 2\ell+1}$ of (c_i) such that $a' < d_{\ell+1}$. Now

$$\begin{aligned} & M \models \forall x_1{<}d_{\ell+1}\ \exists x_2{<}d_{\ell+2}\ \cdots\ \mathrm{Q}'x_{\ell+1}{<}d_{2\ell+1}\ \neg\eta(\bar{a}, x_1, x_2, \ldots, x_{\ell+1}) && \text{by indiscernibility;} \\ \therefore\quad & M \models \exists x_2{<}d_{\ell+2}\ \forall x_3{<}d_{\ell+3}\ \cdots\ \mathrm{Q}'x_{\ell+1}{<}d_{2\ell+1}\ \neg\eta(\bar{a}, a', x_2, x_3, \ldots, x_{\ell+1}) && \text{as } a' < d_{\ell+1}; \\ \therefore\quad & c_{\mathbb{N}} \models \exists x_2\ \forall x_3\ \cdots\ \mathrm{Q}'x_{\ell+1}\ \neg\eta(\bar{a}, a', x_2, x_3, \ldots, x_{\ell+1}) && \text{as } \bar{a}, a' < d_{\ell+1} \end{aligned}$$

by the induction hypothesis. Since the choice of $a' \in c_{\mathbb{N}}$ was arbitrary, we conclude that

$$c_{\mathbb{N}} \models \forall x_1\ \exists x_2\ \forall x_3\ \cdots\ \mathrm{Q}'x_{\ell+1}\ \neg\eta(\bar{a}, x_1, x_2, x_3, \ldots, x_{\ell+1}). \qquad \square$$

As Lemma 24.5(2) shows, every $\mathcal{L}_{\mathrm{A}}(\exp)$ formula for the cut $c_{\mathbb{N}}$ can be translated into a $\Delta_0(\exp)$ formula for the model M with some additional parameters. Consequently, if M has $\Delta_0(\exp)$ induction, then $c_{\mathbb{N}}$ has full induction, as claimed in Proposition 24.3.

Proof of Proposition 24.3. From Lemma 24.5(1) and Proposition 22.3, we know $c_{\mathbb{N}} \models \mathrm{I}\Delta_0(\exp)$. According to Proposition 14.3, every $\mathcal{L}_{\mathrm{A}}(\exp)$ formula is equivalent to one of the form

$$\exists x_1\ \forall x_2\ \cdots\ \mathrm{Q}x_\ell\ \eta(\bar{u}, w, x_1, x_2, \ldots, x_\ell),$$

where $\eta \in \Delta_0(\exp)$. Therefore, to show induction for all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas, it suffices to deal with $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas of this form. Take $\eta(\bar{u}, w, x_1, x_2, \ldots, x_\ell) \in \Delta_0(\exp)$ and $\bar{a} \in c_{\mathbb{N}}$ such that

$$\begin{aligned} c_{\mathbb{N}} \models\ & \exists x_1\ \forall x_2\ \cdots\ \mathrm{Q}x_\ell\ \eta(\bar{a}, 0, \bar{x}) \\ & \wedge \forall w\ \big(\exists x_1\ \forall x_2\ \cdots\ \mathrm{Q}x_\ell\ \eta(\bar{a}, w, \bar{x}) \to \exists x_1\ \forall x_2\ \cdots\ \mathrm{Q}x_\ell\ \eta(\bar{a}, w+1, \bar{x})\big). \end{aligned}$$

Pick any $i \in \mathbb{N}$ with $c_i > \bar{a}$. Then Lemma 24.5(2) and some elementary logic imply

$$
\begin{aligned}
& M \models \exists x_1{<}c_{i+2}\ \forall x_2{<}c_{i+3} \cdots \mathrm{Q}x_\ell{<}c_{i+\ell+1}\ \eta(\bar{a},0,\bar{x}) \\
& \qquad \wedge \forall w{<}c_{i+1} \begin{pmatrix} \exists x_1{<}c_{i+2}\ \forall x_2{<}c_{i+3} \cdots \mathrm{Q}x_\ell{<}c_{i+\ell+1}\ \eta(\bar{a},w,\bar{x}) \\ \to \exists x_1{<}c_{i+\ell+2}\ \forall x_2{<}c_{i+\ell+3} \cdots \mathrm{Q}x_\ell{<}c_{i+2\ell+1}\ \eta(\bar{a},w+1,\bar{x}) \end{pmatrix} \\
\therefore\ & M \models \exists x_1{<}c_{i+2}\ \forall x_2{<}c_{i+3} \cdots \mathrm{Q}x_\ell{<}c_{i+\ell+1}\ \eta(\bar{a},0,\bar{x}) && \text{by indiscernibility;} \\
& \qquad \wedge \forall w{<}c_{i+1} \begin{pmatrix} \exists x_1{<}c_{i+2}\ \forall x_2{<}c_{i+3} \cdots \mathrm{Q}x_\ell{<}c_{i+\ell+1}\ \eta(\bar{a},w,\bar{x}) \\ \to \exists x_1{<}c_{i+2}\ \forall x_2{<}c_{i+3} \cdots \mathrm{Q}x_\ell{<}c_{i+\ell+1}\ \eta(\bar{a},w+1,\bar{x}) \end{pmatrix} \\
\therefore\ & M \models \forall w{<}c_{i+1}\ \exists x_1{<}c_{i+2}\ \forall x_2{<}c_{i+3} \cdots \mathrm{Q}x_\ell{<}c_{i+\ell+1}\ \eta(\bar{a},w,\bar{x}) && \text{as } M \models \mathrm{I}\Delta_0(\exp); \\
\therefore\ & c_{\mathbb{N}} \models \forall w\ \exists x_1\ \forall x_2 \cdots \mathrm{Q}x_\ell\ \eta(\bar{a},w,\bar{x}) && \text{by Lemma 24.5(2).}
\end{aligned}
$$

□

Before we move on to Theorem 24.2, let us establish a technical lemma which will help simplify our proof. This technical lemma provides a way of extracting a usual indiscernible sequence from a univariate one. We start with some observations regarding our proof of Lemma 24.5(1).

Remark 24.6. (1) Note that the proof of Lemma 24.5(1) invokes only indiscernibility for one $\Delta_0(\exp)$ formula $u \in v$ and one parameter u.

(2) In the proof of Lemma 24.5(1), one can actually show $2^{c_{i+1}} \leqslant c_{i+2}$ because by indiscernibility,

$$2^{c_{i+1}} \leqslant c_{i+3} \quad \Leftrightarrow \quad 2^{c_{i+4}} \leqslant c_{i+5} \quad \Leftrightarrow \quad 2^{c_{i+1}} \leqslant c_{i+2}.$$

Lemma 24.7. Let $(c_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence of elements of a model $M \models \mathrm{I}\Delta_0(\exp)$. Suppose that, for all $\Delta_0(\exp)$ formulas $\theta(u, v_1, v_2, \ldots, v_\ell)$ and all subsequences $(d_i)_{i\leqslant 2\ell}$ of (c_i),

$$M \models \forall u{<}d_0\ \big(\theta(u, d_1, d_2, \ldots, d_\ell) \leftrightarrow \theta(u, d_{\ell+1}, d_{\ell+2}, \ldots, d_{2\ell})\big).$$

Then $(c_{2i+1})_{i\in\mathbb{N}}$ is a $\Delta_0(\exp)$-indiscernible sequence over M.

Proof. Let $\theta(u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_\ell) \in \Delta_0(\exp)$ and $(d_i)_{i\leqslant 2\ell}$ be a subsequence of $(c_{2i+1})_{i\in\mathbb{N}}$. If $k \leqslant 1$, then there is nothing to prove. So assume $k > 1$. Suppose $d_0 = c_{2j+1}$. Take $a_1, a_2, \ldots, a_k < d_0 = c_{2j+1}$, and let $b = \langle\langle \ldots \langle\langle a_1, a_2\rangle, a_3\rangle, \ldots\rangle, a_k\rangle$. As one can verify using Remark 24.6,

$$b \leqslant \underbrace{\langle\langle \ldots \langle\langle c_{2j+1}, c_{2j+1}\rangle, c_{2j+1}\rangle, \ldots\rangle, c_{2j+1}\rangle}_{k\text{-many } c_{2j+1}\text{'s}} < 2^{c_{2j+1}} \leqslant c_{2j+2}.$$

Since $d_0 = c_{2j+1} < c_{2j+2} < c_{2j+3} \leqslant d_1 < d_2 < \cdots < d_{2\ell}$, in view of our hypothesis on (c_i),

$$
\begin{aligned}
& M \models \exists u_1, u_2, \ldots, u_k{\leqslant}b\ \big(b = \langle\langle \ldots \langle\langle u_1, u_2\rangle, u_3\rangle, \ldots\rangle, u_k\rangle \wedge \theta(\bar{u}, d_1, d_2, \ldots, d_\ell)\big) \\
& \qquad \leftrightarrow \exists u_1, u_2, \ldots, u_k{\leqslant}b\ \big(b = \langle\langle \ldots \langle\langle u_1, u_2\rangle, u_3\rangle, \ldots\rangle, u_k\rangle \wedge \theta(\bar{u}, d_{\ell+1}, d_{\ell+2}, \ldots, d_{2\ell})\big) \\
\therefore\ & M \models \theta(\bar{a}, d_1, d_2, \ldots, d_\ell) \leftrightarrow \theta(\bar{a}, d_{\ell+1}, d_{\ell+2}, \ldots, d_{2\ell})
\end{aligned}
$$

as $M \models \forall u_1, u_2, \ldots, u_k\ (b = \langle\langle \ldots \langle\langle u_1, u_2\rangle, u_3\rangle, \ldots\rangle, u_k\rangle \leftrightarrow u_1 = a_1 \wedge u_2 = a_2 \wedge \cdots \wedge u_k = a_k)$. □

Indiscernible sequences are often associated with Ramsey-style theorems. With respect to a structure, each formula can be viewed as a colouring of tuples of appropriate length in two colours: true and false. In this sense, homogeneous sets correspond naturally to indiscernible sequences/sets. Indiscernibles can thus be obtained in an elementary extension by repeated applications of the Infinite Ramsey Theorem. We cannot directly adopt this approach because (1) we are not allowed to move to an elementary extension, and (2) we need to take care of a variable number of parameters. Inside a nonstandard model of arithmetic, one can deal with (1) by applying the Finite Ramsey Theorem a nonstandard number of times, instead of applying the Infinite Ramsey Theorem an infinite number of times. To deal with (2), we invoke the relative largeness of our homogeneous sets, which tells us that, in a sense, the number of elements is large relative to the number of parameters involved.

We present below a streamlined version of the construction due to Bovykin. It avoids the iterated application of Ramsey-style theorems by employing a colouring which *compares*, as opposed to *indicates*, the truth values of formulas on tuples. The tuples in a homogeneous set for such a colouring *either* behave all in the same way with respect to the formulas considered by the colouring, in which case we obtain the indiscernibles we want, *or* behave all in different ways. Since there are a limited number of ways a tuple can behave with respect to a fixed (possibly nonstandard) number of formulas, one can eliminate the second alternative by requiring the size of the homogeneous set to be sufficiently large.

Proof of Theorem 24.2. Let $e \in M \setminus \mathbb{N}$ such that $e \leqslant Y^M_{\mathrm{PH}}(a,b)$ and $24e \leqslant a$. Fix an M-definable enumeration $(\theta_j(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_e))_{j \in M}$ of all $\Delta_0(\exp)$ formulas in M whose free variables are amongst $\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_e$ such that $(\theta_j)_{j\in\mathbb{N}}$ enumerates all standard $\Delta_0(\exp)$ formulas. For the purpose of this proof, an M-coded strictly increasing sequence $\bar{d} \in M$ of length e will be referred to as an *e-tuple*. Unless otherwise stated, all tuples $\bar{d} \in M$ are implicitly e-tuples. If $d_0, \bar{d}_1, \bar{d}_2 \in M$, then (it goes without saying that $\bar{d}_1, \bar{d}_2$ are e-tuples, and) we write $d_0 < \bar{d}_1$ and $\bar{d}_1 < \bar{d}_2$ for $d_0 < \min\{\bar{d}_1\}$ and $\max\{\bar{d}_1\} < \min\{\bar{d}_2\}$ respectively. If $d_0, \bar{d}_1, \bar{d}_2 \in M$ with $d_0 < \bar{d}_1 < \bar{d}_2$, then set

$$\mathrm{dfma}(\{d_0, \bar{d}_1, \bar{d}_2\}) = \begin{cases} (\min j < e)\big(\exists u{<}d_0\ \neg\big(\theta_j(u,\bar{d}_1) \leftrightarrow \theta_j(u,\bar{d}_2)\big)\big), & \text{if it exists;} \\ e, & \text{otherwise,} \end{cases}$$

and

$$\mathrm{dprm}(\{d_0, \bar{d}_1, \bar{d}_2\}) = \begin{cases} (\min u < d_0)\big(\neg\big(\theta_j(u,\bar{d}_1) \leftrightarrow \theta_j(u,\bar{d}_2)\big)\big), & \text{if } \mathrm{dfma}(\{d_0, \bar{d}_1, \bar{d}_2\}) = j < e; \\ d_0, & \text{otherwise.} \end{cases}$$

These definitions can be made in M using a universal $\Delta_0(\exp)$ formula, which is available when working in $\mathrm{I}\Delta_0(\exp)$, and a fortiori PA. Roughly speaking, dfma picks out the first formula on which two e-tuples differ over some small parameter, and then dprm picks out this small parameter witnessing the difference. Since our logic is two-valued, at most two e-tuples can mutually differ on the same formula over the same parameter.

Claim 24.2.1. For *no* $c, \bar{h}_1, \bar{h}_2, \bar{h}_3 \in M$ with $c < \bar{h}_1 < \bar{h}_2 < \bar{h}_3$ can we simultaneously have

- $\mathrm{dfma}(\{c, \bar{h}_1, \bar{h}_2\}) = \mathrm{dfma}(\{c, \bar{h}_2, \bar{h}_3\}) = \mathrm{dfma}(\{c, \bar{h}_1, \bar{h}_3\}) < e$; and
- $\mathrm{dprm}(\{c, \bar{h}_1, \bar{h}_2\}) = \mathrm{dprm}(\{c, \bar{h}_2, \bar{h}_3\}) = \mathrm{dprm}(\{c, \bar{h}_1, \bar{h}_3\})$.

Proof of Claim 24.2.1. Let $c, \bar{h}_1, \bar{h}_2, \bar{h}_3 \in M$ with $c < \bar{h}_1 < \bar{h}_2 < \bar{h}_3$ such that

- $j := \mathrm{dfma}(\{c, \bar{h}_1, \bar{h}_2\}) = \mathrm{dfma}(\{c, \bar{h}_2, \bar{h}_3\}) < e$; and
- $u := \mathrm{dprm}(\{c, \bar{h}_1, \bar{h}_2\}) = \mathrm{dprm}(\{c, \bar{h}_2, \bar{h}_3\})$.

Then

$$\begin{aligned} & M \models \neg\big(\theta_j(u,\bar{h}_1) \leftrightarrow \theta_j(u,\bar{h}_2)\big) & \\ & \qquad \wedge \neg\big(\theta_j(u,\bar{h}_2) \leftrightarrow \theta_j(u,\bar{h}_3)\big) & \text{by the definitions of dfma and dprm;} \\ \therefore\ & M \models \theta_j(u,\bar{h}_1) \leftrightarrow \theta_j(u,\bar{h}_3) & \text{as there are only two truth values;} \\ \therefore\ & u \neq \mathrm{dprm}(\{c, \bar{h}_1, \bar{h}_3\}) \text{ or } j \neq \mathrm{dfma}(\{c, \bar{h}_1, \bar{h}_3\}) & \text{by the definitions of dfma and dprm.} \end{aligned}$$

□ Claim 24.2.1

As alluded to before the proof, our plan is to compare the differences between pairs of e-tuples. To do this, we define in M the colouring $f\colon [[a,b]]^{4e+1} \to 3e+1$ by setting

$$f(\{d_0, \bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{d}_4\}) = \begin{cases} 3e, & \text{if } \mathrm{dprm}(\{d_0, \bar{d}_1, \bar{d}_2\}) = \mathrm{dprm}(\{d_0, \bar{d}_3, \bar{d}_4\}); \\ u, & \text{otherwise, where } u < 3e \text{ and } \mathrm{dprm}(\{d_0, \bar{d}_1, \bar{d}_2\}) \equiv u \mod 3e \end{cases}$$

for all $d_0, \bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{d}_4 \in [a,b]$ with $d_0 < \bar{d}_1 < \bar{d}_2 < \bar{d}_3 < \bar{d}_4$. Since $\mathrm{Y}^M_{\mathrm{PH}}(a,b) \geqslant e$,

$$M \models [a,b] \xrightarrow{*} (\mathrm{R}^{2e}_e(3e) + 4e + 1)^{4e+1}_{3e+1}.$$

Use this to find a relatively large f-homogeneous subset $H \subseteq [a,b]$ of cardinality at least $\mathrm{R}^{2e}_e(3e) + 4e + 1$ in M. This H is too large compared to the number of possibilities for $\mathrm{dprm}(X)$, where $X \in [H]^{4e+1}$, to allow such $\mathrm{dprm}(X)$'s to be all different. So H must have f-colour $3e$.

Claim 24.2.2. The homogeneous set H has f-colour $3e$, i.e., $f(X) = 3e$ for all/some $X \in [H]^{4e+1}$.

Proof of Claim 24.2.2. Let $c_0 = \min(H)$. Since $\mathrm{card}(H) \geqslant \mathrm{R}^{2e}_e(3e) + 4e + 1 \geqslant 4e + 1$, one can find $\bar{d}_1, \bar{d}_2, \ldots \bar{d}_{2n}, \bar{d}_{2n+1}, \bar{d}_{2n+2} \in H$ such that

$$c_0 < \bar{d}_1 < \bar{d}_2 < \cdots < \bar{d}_{2n} < \bar{d}_{2n+1} < \bar{d}_{2n+2},$$

where $n = \lfloor (\mathrm{card}(H) - (2e+1))/2e \rfloor \geqslant \lfloor (\mathrm{R}^{2e}_e(3e) + 2e)/2e \rfloor \geqslant \lfloor (3e + 2e)/2e \rfloor = 2$. The homogeneity of H for f implies, for every nonzero $k < n$,

$$f(\{c_0, \bar{d}_1, \bar{d}_2, \bar{d}_{2k+1}, \bar{d}_{2k+2}\}) = f(\{c_0, \bar{d}_1, \bar{d}_2, \bar{d}_{2n+1}, \bar{d}_{2n+2}\}) = f(\{c_0, \bar{d}_{2k+1}, \bar{d}_{2k+2}, \bar{d}_{2n+1}, \bar{d}_{2n+2}\})$$

and so, regardless of whether the f-colour of H is $3e$ or not,

$$\mathrm{dprm}(\{c_0, \bar{d}_1, \bar{d}_2\}) \equiv \mathrm{dprm}(\{c_0, \bar{d}_{2k+1}, \bar{d}_{2k+2}\}) \mod 3e.$$

Therefore, if $\mathrm{dprm}(c_0, \bar{d}_1, \bar{d}_2) \equiv u \mod 3e$, where $u < 3e$, then as such $\mathrm{dprm}(\{c_0, \bar{d}_{2k+1}, \bar{d}_{2k+2}\}) \leqslant c_0$, it must take one of the following values:

$$u, \quad u + 2(3e), \quad u + 3(3e), \quad \ldots, \quad u + \left\lceil \frac{c_0}{3e} \right\rceil (3e).$$

- The number of possible values for $\mathrm{dprm}(\{c_0, \bar{d}_{2k+1}, \bar{d}_{2k+2}\})$ where $k < n$ is $\lceil c_0/3e \rceil + 1 = \lceil \min(H)/3e \rceil + 1$.
- The number of pairs $(\bar{d}_{2k+1}, \bar{d}_{2k+2})$ where $k < n$ is $n = \lfloor (\mathrm{card}(H) - (2e+1))/2e \rfloor$.

It suffices to show $\lceil \min(H)/3e \rceil + 1 < \lfloor (\mathrm{card}(H) - (2e+1))/2e \rfloor$, because this implies, by the Finite Pigeonhole Principle, the existence of distinct $k, k' < \ell$ such that

$$\mathrm{dprm}(\{c_0, \bar{d}_{2k+1}, \bar{d}_{2k+2}\}) = \mathrm{dprm}(\{c_0, \bar{d}_{2k'+1}, \bar{d}_{2k'+2}\})$$

and hence $f(\{c_0, \bar{d}_{2k+1}, \bar{d}_{2k+2}, \bar{d}_{2k'+1}, \bar{d}_{2k'+2}\}) = 3e$. To show the required inequality, notice

$$\begin{aligned} 24e &\leqslant a && \text{by the choice of } e;\\ &\leqslant \min(H) && \text{as } H \subseteq [a,b];\\ &\leqslant \min(H) + 3\big(\mathrm{card}(H) - \min(H)\big) && \text{as } H \text{ is relatively large;}\\ &= 3\,\mathrm{card}(H) - 2\min(H). \end{aligned}$$

Rearranging gives $2(\min(H) + 6e) \leqslant 3(\mathrm{card}(H) - 4e)$. So

$$\begin{aligned} \left\lceil \frac{\min(H)}{3e} \right\rceil + 1 < \frac{\min(H) + 6e}{3e} &\leqslant \frac{\mathrm{card}(H) - 4e}{2e} && \text{by the above;}\\ &= \frac{\mathrm{card}(H) - (2e+1) - (2e-1)}{2e} \leqslant \left\lfloor \frac{\mathrm{card}(H) - (2e+1)}{2e} \right\rfloor. && \square \text{ Claim 24.2.2} \end{aligned}$$

In increasing order, enumerate the last $2e$ elements of H as $\bar{d}_{2n+1}, \bar{d}_{2n+2}$. Set $D = \{\bar{d}_{2n+1}, \bar{d}_{2n+2}\}$. This extra pair of e-tuples provides room to move the e-tuples around so that dprm gives the same value on *all* pairs of e-tuples in H below D, not only those arranged in the right order.

Claim 24.2.3. Let $c, \bar{h}_1, \bar{h}_2, \bar{h}'_1, \bar{h}'_2 \in H \setminus D$ such that $c < \bar{h}_1 < \bar{h}_2$ and $c < \bar{h}'_1 < \bar{h}'_2$. Then $\mathrm{dprm}(\{c, \bar{h}_1, \bar{h}_2\}) = \mathrm{dprm}(\{c, \bar{h}'_1, \bar{h}'_2\})$.

Proof of Claim 24.2.3. We know $f(\{c, \bar{h}_1, \bar{h}_2, \bar{d}_{2n+1}, \bar{d}_{2n+2}\}) = f(\{c, \bar{h}'_1, \bar{h}'_2, \bar{d}_{2n+1}, \bar{d}_{2n+2}\}) = 3e$ in view of Claim 24.2.2. So $\mathrm{dprm}(\{c, \bar{h}_1, \bar{h}_2\}) = \mathrm{dprm}(\{c, \bar{d}_{2n+1}, \bar{d}_{2n+2}\}) = \mathrm{dprm}(\{c, \bar{h}'_1, \bar{h}'_2\})$ by the definition of f. □ Claim 24.2.3

From dprm-homogeneity, one can derive dfma-homogeneity on a subset using a Ramsey-style argument. The size of H was chosen so that Claim 24.2.1 applies to this subset. Thus the majority of pairs of e-tuples from $H \setminus D$ must be in the 'otherwise' case in the definition of dfma and dprm homogeneously. Here we let $(c_i)_{i \leqslant 2e}$ enumerate the first $2e+1$ elements of H in increasing order, and let $C = \{c_i : i \leqslant 2e\}$.

Claim 24.2.4. Let $i \leqslant 2e$ and $H_i = \{x \in H \setminus D : x > c_i\}$. Then $\mathrm{dfma}(\{c_i\} \cup X) = e$ for all $X \in [H_i]^{2e}$.

Proof of Claim 24.2.4. In M, define $g_i \colon [H_i]^{2e} \to e+1$ by setting

$$g_i(X) = \mathrm{dfma}(\{c_i\} \cup X)$$

for all $X \in [H_i]^{2e}$. If $e \in \mathrm{Im}(g_i)$, then

$$\begin{array}{llll} & \mathrm{dfma}(\{c_i\} \cup X) = e & \text{for some } X \in [H_i]^{2e}, & \text{by the definition of } g_i; \\ \therefore & \mathrm{dprm}(\{c_i\} \cup X) = c_i & \text{for some } X \in [H_i]^{2e}, & \text{by the definition of dprm;} \\ \therefore & \mathrm{dprm}(\{c_i\} \cup X) = c_i & \text{for all } X \in [H_i]^{2e}, & \text{by Claim 24.2.3;} \\ \therefore & \mathrm{dfma}(\{c_i\} \cup X) = e & \text{for all } X \in [H_i]^{2e}, & \text{by the definition of dprm;} \\ \therefore & \mathrm{Im}(g_i) = \{e\} & & \text{by the definition of } g_i. \end{array}$$

So either $\mathrm{Im}(g_i) \subseteq e$ or $\mathrm{Im}(g_i) = \{e\}$. Since

$$\mathrm{card}(H_i) \geqslant \mathrm{card}(H \setminus (C \cup D)) \geqslant \mathrm{R}^{2e}_e(3e) + 4e + 1 - (2e + 1 + e + e) = \mathrm{R}^{2e}_e(3e),$$

the first alternative cannot hold in view of Claim 24.2.1 and Claim 24.2.3. Therefore, the second alternative must hold, as required. □ Claim 24.2.4

Claim 24.2.4 readily entails the indiscernibility of the c_i's. More specifically, if $(d_i)_{i \leqslant 2\ell}$ is a subsequence of $(c_i)_{i \in \mathbb{N}}$, then for all $\bar{h} \in H \setminus (C \cup D)$,

$$\begin{aligned} &\mathrm{dfma}(\{d_0, d_1, d_2, \ldots, d_\ell, c_{e+\ell+1}, c_{e+\ell+2}, \ldots, c_{2e}, \bar{h}\}) \\ &= \mathrm{dfma}(\{d_0, d_{\ell+1}, d_{\ell+2}, \ldots, d_{2\ell}, c_{e+\ell+1}, c_{e+\ell+2}, \ldots, c_{2e}, \bar{h}\}) = e \end{aligned}$$

by Claim 24.2.4, and so the definition of dfma implies that, whenever $j \in \mathbb{N}$,

$$\begin{aligned} M \models\ & \forall u{<}d_0\ \big(\theta_j(u, d_1, d_2, \ldots, d_\ell, c_{e+\ell+1}, c_{e+\ell+2}, \ldots, c_{2e}) \leftrightarrow \theta_j(u, \bar{h})\big) \\ & \wedge \forall u{<}d_0\ \big(\theta_j(u, d_{\ell+1}, d_{\ell+2}, \ldots, d_{2\ell}, c_{e+\ell+1}, c_{e+\ell+2}, \ldots, c_{2e}) \leftrightarrow \theta_j(u, \bar{h})\big). \end{aligned}$$

Thus $(c_{2i+1})_{i \in \mathbb{N}}$ is a $\Delta_0(\exp)$-indiscernible sequence over M by Lemma 24.7. □

This completes the proof that $Y_{\mathrm{PH}}(x, y) = z$ is an indicator for PA over PA. So Theorem 22.8 applies. From this, we deduce that the Paris–Harrington–Ramsey Theorem (PH), without the restriction to standard exponents, is unprovable in PA.

Theorem 24.8 (Paris–Harrington)**.** $\mathrm{PA} \nvdash \forall k, m, r, x{\geqslant}1\ \exists y{\geqslant}x\ [x, y] \xrightarrow{*} (m)^k_r$.

Proof. The sentence $\forall z\ \forall x\ \exists y\ Y_{\mathrm{PH}}(x, y) > z$ is not provable in PA by Theorem 22.8(2), but it is directly provable in $\mathrm{PA} + \forall k, m, r, x{\geqslant}1\ \exists y{\geqslant}x\ [x, y] \xrightarrow{*} (m)^k_r$. □

It is known that PH is equivalent over PA to a natural formalization of Goodstein's Theorem 17.7(ii). Via our consistency proof of PA in Lecture 20, this shows $\mathrm{PA} + \mathrm{PH} \vdash \mathrm{Con(PA)}$. A direct proof can be obtained by formalizing the model-theoretic construction presented in this lecture. Furthermore, there is a partial converse: PH is equivalent over PA to the so-called 1-*consistency of* PA or the *uniform* Σ_1 *reflection principle for* PA, i.e., the assertion that 'every true Π_1 formula is consistent with PA.'