MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Lecture 2: Truth and computation

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The main question in this lecture is: what kind of truth about $\mathbb{N}$ can one compute?

We restrict ourselves to the truth of first-order formulas, the exact definition of which is given below. Note the resemblance with the definition of programs in Lecture 1.

Definition. We define the notion of $\mathcal{L}_{\mathrm{A}}(\exp)$ *formulas* and *free variables* by recursion as follows.

(i) $\top$ is an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula. No variable appears free in $\top$.

(ii) Let t, s be $\mathcal{L}_{\mathrm{A}}(\exp)$ terms. Then $t = s$ and $t < s$ are $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas. A variable appears free in these formulas if and only if it appears in t or in s.

(iii) Let θ, η be $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas. Then $\neg\theta$ and $\theta \vee \eta$ are $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas. A variable appears free in $\neg\theta$ if and only if it appears free in θ. A variable appears free in $\theta \vee \eta$ if and only if it appears free in θ or in η.

(iv) Let y be a variable and θ be an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula which does not contain $\exists y$ as a substring. Then $\exists y\ \theta$ is an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula. A variable x appears free in $\exists y\ \theta$ if and only if $x \neq y$ and x appears free in θ.

(v) Every $\mathcal{L}_{\mathrm{A}}(\exp)$ formula is obtained by applying the construction rules above finitely many times.

The formulas in (i) and (ii) are called *atomic* $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas. We may write an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula θ as $\theta(x_1, x_2, \dots, x_k)$ if $x_1, x_2, \dots, x_k$ is a list of variables without repetition, and all the variables that appear free in θ are in this list. We employ the shorthand for $\mathcal{L}_{\mathrm{A}}(\exp)$ programs to define $\neq, \leqslant, \bot, \wedge, \rightarrow, \leftrightarrow$. The precedence rules for $\mathcal{L}_{\mathrm{A}}(\exp)$ programs apply to $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas too.

We do not include a clause corresponding to **par-for** loops here because they can be defined in terms of **par-while** loops.

Definition. If $t(\bar{x})$ is an $\mathcal{L}_{\mathrm{A}}(\exp)$ term and $\theta(\bar{x}, y)$ is an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula which does not contain $\exists y$ as a substring, then

$$\exists y{<}t\ \theta \quad = \quad \exists y\ (y < t \wedge \theta).$$

Such a quantifier (occurrence) is said to be *bounded*. A $\Delta_0(\exp)$ *formula* is an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula in which all quantifiers are bounded. A Σ_1 *formula* is an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula of the form $\exists \bar{y}\ \theta$, where θ is a $\Delta_0(\exp)$ formula and $\bar{y}$ is a possibly empty tuple of variables.

Example 2.1. The following is a $\Delta_0(\exp)$ formula:

$$x \in y \quad = \quad \exists z{<}y\ \exists w{<}2^x\ (y = 2^{x+1}z + 2^x + w).$$

Compare this with Example 1.4.

Formulas are merely strings of symbols. For them to have truth values, we need to specify how the symbols are to be interpreted. For notational convenience, we sometimes write $\exp(x)$ for 2^x.

Definition. An $\mathcal{L}_{\mathrm{A}}(\exp)$ *structure* $\mathfrak{M}$ consists of

- a nonempty set M called the *universe* of $\mathfrak{M}$;
- two elements $0^{\mathfrak{M}}, 1^{\mathfrak{M}} \in M$;

- functions $+^{\mathfrak{M}}, \times^{\mathfrak{M}} \colon M^2 \to M$ and $\exp^{\mathfrak{M}} \colon M \to M$; and
- a binary relation $<^{\mathfrak{M}}$ on M.

When there is no risk of ambiguity, we identify a structure with its universe, and omit the superscripts.

Example 2.2. $\mathbb{N} = (\mathbb{N}, 0, 1, +, \times, \exp, <)$ and $\mathbb{R} = (\mathbb{R}, 0, 1, +, \times, \exp, <)$ with the usual arithmetic operations are $\mathcal{L}_{\mathrm{A}}(\exp)$ structures. The former is called the *standard model of arithmetic.*

We can evaluate $\mathcal{L}_{\mathrm{A}}(\exp)$ terms in arbitrary $\mathcal{L}_{\mathrm{A}}(\exp)$ structures, in the same way as in the standard model of arithmetic.

Definition. Let M be an $\mathcal{L}_{\mathrm{A}}(\exp)$ structure. We define the evaluation $t^M(a_1, a_2, \ldots, a_k)$ of an $\mathcal{L}_{\mathrm{A}}(\exp)$ term $t(x_1, x_2, \ldots, x_k)$ in M on elements $a_1, a_2, \ldots, a_k \in M$ by recursion on (the number of steps in the construction of) the term t as follows: whenever $t(x_1, x_2, \ldots, x_k), s(x_1, x_2, \ldots, x_k)$ are $\mathcal{L}_{\mathrm{A}}(\exp)$ terms and $a_1, a_2, \ldots, a_k \in M$,

- $t^M(\bar{a}) = \begin{cases} a_i, & \text{if } t(\bar{x}) = x_i; \\ 0^M, & \text{if } t(\bar{x}) = 0; \\ 1^M, & \text{if } t(\bar{x}) = 1; \end{cases}$
- $(t+s)^M(\bar{a}) = t^M(\bar{a}) +^M s^M(\bar{a})$;
- $(t \times s)^M(\bar{a}) = t^M(\bar{a}) \times^M s^M(\bar{a})$;
- $(\exp(t))^M(\bar{a}) = \exp^M(t^M(\bar{a}))$.

When there is no risk of ambiguity, the superscript may be omitted.

Example 2.3. Let $t(w, x, z) = 2^{x+1}z + 2^x + w$ as in Example 1.3, and $\mathbb{R}$ be the $\mathcal{L}_{\mathrm{A}}(\exp)$ structure defined in Example 2.2. Then $t^{\mathbb{R}}(-2, 2, 1) = 2^{2+1} \times 1 + 2^2 + (-2) = 8 + 4 - 2 = 10$.

Informally speaking, Tarski's definition of truth consists of a set of inductive clauses which asserts that truth commutes with all logical connectives.

Definition (Tarski)**.** Let M be an $\mathcal{L}_{\mathrm{A}}(\exp)$ structure. We define what is means for an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula $\theta(x_1, x_2, \ldots, x_k)$ to be *true* in M on $a_1, a_2, \ldots, a_k \in M$, or $M \models \theta(a_1, a_2, \ldots, a_k)$ in symbols, by recursion on (the number of steps in the construction of) the formula θ as follows.

- $M \models \top(\bar{a})$ for all $\bar{a} \in M$.
- For all $\mathcal{L}_{\mathrm{A}}(\exp)$ terms $t(\bar{x}), s(\bar{x})$ and $\bar{a} \in M$,
$$M \models (t = s)(\bar{a}) \quad \Leftrightarrow \quad t^M(\bar{a}) = s^M(\bar{a}); \quad \text{and}$$
$$M \models (t < s)(\bar{a}) \quad \Leftrightarrow \quad t^M(\bar{a}) <^M s^M(\bar{a}).$$
- For all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas $\theta(\bar{x})$ and $\bar{a} \in M$,
$$M \models (\neg\theta)(\bar{a}) \quad \Leftrightarrow \quad M \not\models \theta(\bar{a}).$$
- For all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas $\theta(\bar{x}, \bar{z}), \eta(\bar{y}, \bar{z})$ with $\{\bar{x} \cap \bar{y}\} = \varnothing$ and all tuples $\bar{a}, \bar{b}, \bar{c} \in M$ of appropriate lengths,
$$M \models (\theta \vee \eta)(\bar{a}, \bar{b}, \bar{c}) \quad \Leftrightarrow \quad M \models \theta(\bar{a}, \bar{c}) \text{ or } M \models \eta(\bar{b}, \bar{c}).$$
- For all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas $\theta(\bar{x}, y)$ which does not contain $\exists y$ as a substring, and for all $\bar{a} \in M$,
$$M \models (\exists y\ \theta)(\bar{a}) \quad \Leftrightarrow \quad M \models \theta(\bar{a}, b) \text{ for some } b \in M.$$

It is customary to write

$$M \models (\neg\theta)(\bar{a}), \quad M \models (\theta \vee \eta)(\bar{a}, \bar{b}, \bar{c}), \quad M \models (\exists y\ \theta)(\bar{a})$$

here respectively as

$$M \models \neg\theta(\bar{a}), \quad M \models \theta(\bar{a},\bar{c}) \vee \eta(\bar{b},\bar{c}), \quad M \models \exists y\ \theta(\bar{a},y).$$

We may read $M \models \theta(\bar{a})$ as '$\theta(\bar{a})$ is true in M' or 'M satisfies/models $\theta(\bar{a})$'; informally we may also say 'M thinks/believes $\theta(\bar{a})$ is true'.

Example 2.4. Let $x \in y$ be as in Example 2.1. We saw in Example 1.4 that $\mathbb{N} \models \neg(2 \in 10)$. On the contrary, Example 2.3 tells us $\mathbb{R} \models 2 \in 10$.

Remark 2.5. In the $\exists$ clause of the truth definition, we implicitly wrote $\exists y\ \theta$ as $(\exists y\ \theta)(\bar{x})$, where $y \notin \{\bar{x}\}$. Our convention actually also allows us to write $\exists y\ \theta$ as $(\exists y\ \theta)(\bar{x},y)$, in which case we define, for all $\bar{a}, b \in M$,

$$M \models (\exists y\ \theta)(\bar{a},b) \quad \Leftrightarrow \quad M \models \theta(\bar{a},b') \text{ for some } b' \in M.$$

As noted in the previous lecture, without the **par-while** construction, an $\mathcal{L}_{\mathrm{A}}(\exp)$ program always returns an answer (and this answer is always correct). Therefore, both the truth and the falsehood of a $\Delta_0(\exp)$ formula can be computed by the corresponding programs. Moreover, since **par-while** preserves positive information, the truth of any Σ_1 formula can also be computed by the corresponding program. This gives an answer to our main question.

Proposition 2.6. (1) For all $\theta(x_1,x_2,\dots,x_k) \in \Delta_0(\exp)$ and all $a_1,a_2,\dots,a_k \in \mathbb{N}$,

$$\mathbb{N} \models \theta(\bar{a}) \quad \Leftrightarrow \quad [\![\theta(\bar{a})]\!] = \texttt{true} \quad \Leftrightarrow \quad [\![\theta(\bar{a})]\!] \neq \texttt{false}.$$

(2) For all $\theta(x_1,x_2,\dots,x_k) \in \Sigma_1$ and all $a_1,a_2,\dots,a_k \in \mathbb{N}$,

$$\mathbb{N} \models \theta(\bar{a}) \quad \Leftrightarrow \quad [\![\theta(\bar{a})]\!] = \texttt{true}.$$

Proof. Notice that every $\Delta_0(\exp)$ formula is constructed from the atomic $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas by finitely many applications of negation, disjunction, and bounded quantification. The proof of (1) is by induction on (the number of steps in the construction of) the $\Delta_0(\exp)$ formula θ.

Consider $\top$ and $\bar{a} \in M$. The definitions imply $\mathbb{N} \models \top(\bar{a})$ and $[\![\top(\bar{a})]\!] = \texttt{true} \neq \texttt{false}$.

Consider $t = s$ and $\bar{a} \in M$, where $t(\bar{x}), s(\bar{x})$ are $\mathcal{L}_{\mathrm{A}}(\exp)$ terms.

$$\begin{aligned} \mathbb{N} \models (t=s)(\bar{a}) \quad &\Leftrightarrow \quad t^M(\bar{a}) = s^M(\bar{a}) && \text{by the truth definition;}\\ &\Leftrightarrow \quad [\![(t=s)(\bar{a})]\!] = \texttt{true} && \text{by the definition of } [\![\dots]\!];\\ &\Leftrightarrow \quad [\![(t=s)(\bar{a})]\!] \neq \texttt{false} && \text{by the definition of } [\![\dots]\!]. \end{aligned}$$

The case when the $\Delta_0(\exp)$ formula is $t < s$ is proved similarly.

Consider $\neg\theta$ and $\bar{a} \in M$, where $\theta(\bar{x})$ is a $\Delta_0(\exp)$ formula.

$$\begin{aligned} &\mathbb{N} \models \neg\theta(\bar{a}) && && \\ \Leftrightarrow\ &\mathbb{N} \not\models \theta(\bar{a}) && && \text{by the truth definition;}\\ \Leftrightarrow\ &[\![\theta(\bar{a})]\!] \neq \texttt{true} &\Leftrightarrow\ & [\![\theta(\bar{a})]\!] = \texttt{false} && \text{by the induction hypothesis;}\\ \Leftrightarrow\ &[\![\neg\theta(\bar{a})]\!] \neq \texttt{false} &\Leftrightarrow\ & [\![\neg\theta(\bar{a})]\!] = \texttt{true} && \text{by the definition of } [\![\dots]\!]. \end{aligned}$$

Consider $\theta \vee \eta$ and $\bar{a}, \bar{b}, \bar{c} \in M$ of appropriate lengths, where $\theta(\bar{x},\bar{z}), \eta(\bar{y},\bar{z})$ are $\Delta_0(\exp)$ formulas with $\{\bar{x}\} \cap \{\bar{y}\} = \varnothing$. By the truth definition,

$$\mathbb{N} \models \theta(\bar{a},\bar{c}) \vee \eta(\bar{b},\bar{c}) \quad \Leftrightarrow \quad \mathbb{N} \models \theta(\bar{a},\bar{c}) \text{ or } \mathbb{N} \models \eta(\bar{b},\bar{c}).$$

On the one hand,

$$\begin{aligned} &\mathbb{N} \models \theta(\bar{a},\bar{c}) \text{ or } \mathbb{N} \models \eta(\bar{b},\bar{c}) && \\ \Leftrightarrow\ &[\![\theta(\bar{a},\bar{c})]\!] = \texttt{true} \text{ or } [\![\eta(\bar{b},\bar{c})]\!] = \texttt{true} && \text{by the induction hypothesis;}\\ \Leftrightarrow\ &[\![(\theta \vee \eta)(\bar{a},\bar{b},\bar{c})]\!] = \texttt{true} && \text{by the definition of } [\![\dots]\!]. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathbb{N} \models \theta(\bar{a},\bar{c}) \text{ or } \mathbb{N} \models \eta(\bar{b},\bar{c}) \\ \Leftrightarrow\ & [\![\theta(\bar{a},\bar{c})]\!] \neq \texttt{false} \text{ or } [\![\eta(\bar{b},\bar{c})]\!] \neq \texttt{false} && \text{by the induction hypothesis;} \\ \Leftrightarrow\ & [\![(\theta \vee \eta)(\bar{a},\bar{b},\bar{c})]\!] \neq \texttt{false} && \text{by the definition of } [\![\ldots]\!]. \end{aligned}$$

Consider $\exists y{<}t\ \theta$ and $\bar{a} \in M$, where $t(\bar{x})$ is an $\mathcal{L}_{\mathrm{A}}(\exp)$ term and $\theta(\bar{x}, y)$ is a $\Delta_0(\exp)$ formula. By the truth definition,

$$\mathbb{N} \models \exists y{<}t(\bar{a})\ \theta(\bar{a}, y) \quad \Leftrightarrow \quad \mathbb{N} \models \theta(\bar{a}, b) \text{ for some } b < t(\bar{a}).$$

On the one hand,

$$\begin{aligned} & \mathbb{N} \models \theta(\bar{a}, b) \text{ for some } b < t(\bar{a}) \\ \Leftrightarrow\ & [\![\theta(\bar{a}, b)]\!] = \texttt{true} \text{ for some } b < t(\bar{a}) && \text{by the induction hypothesis;} \\ \Leftrightarrow\ & [\![(\exists y{<}t\ \theta)(\bar{a})]\!] = \texttt{true} && \text{by the definition of } [\![\ldots]\!]. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathbb{N} \models \theta(\bar{a}, b) \text{ for some } b < t(\bar{a}) \\ \Leftrightarrow\ & [\![\theta(\bar{a}, b)]\!] \neq \texttt{false} \text{ for some } b < t(\bar{a}) && \text{by the induction hypothesis;} \\ \Leftrightarrow\ & [\![(\exists y{<}t\ \theta)(\bar{a})]\!] \neq \texttt{false} && \text{by the definition of } [\![\ldots]\!]. \end{aligned}$$

This completes the induction for (1). To show (2), consider the Σ_1 formula

$$\exists y_1, y_2, \ldots, y_k\ \theta(\bar{x}, y_1, y_2, \ldots, y_k),$$

where $\theta(\bar{x}, y_1, y_2, \ldots, y_k)$ is a $\Delta_0(\exp)$ formula. By k-many applications of the truth definition,

$$\begin{aligned} & \mathbb{N} \models \exists y_1, y_2, y_3, y_4, \ldots, y_k\ \theta(\bar{a}, y_1, y_2, y_3, y_4, \ldots, y_k) \\ \Leftrightarrow\ & \mathbb{N} \models \exists y_2, y_3, y_4, \ldots, y_k\ \theta(\bar{a}, b_1, y_2, y_3, y_4, \ldots, y_k) && \text{for some } b_1 \in \mathbb{N} \\ \Leftrightarrow\ & \mathbb{N} \models \exists y_3, y_4, \ldots, y_k\ \theta(\bar{a}, b_1, b_2, y_3, y_4, \ldots, y_k) && \text{for some } b_1, b_2 \in \mathbb{N} \\ \Leftrightarrow\ & \cdots \\ \Leftrightarrow\ & \mathbb{N} \models \theta(\bar{a}, b_1, b_2, b_3, b_4, \ldots, b_k) && \text{for some } b_1, b_2, \ldots, b_k \in \mathbb{N}. \end{aligned}$$

By (1), the last statement in this chain is equivalent to

$$\begin{aligned} & [\![\theta(\bar{a}, b_1, b_2, \ldots, b_{k-2}, b_{k-1}, b_k)]\!] = \texttt{true} && \text{for some } b_1, b_2, \ldots, b_{k-2}, b_{k-1}, b_k \in \mathbb{N} \\ \Leftrightarrow\ & [\![(\exists y_k\ \theta)(\bar{a}, b_1, b_2, \ldots, b_{k-2}, b_{k-1})]\!] = \texttt{true} && \text{for some } b_1, b_2, \ldots, b_{k-2}, b_{k-1} \in \mathbb{N} \\ \Leftrightarrow\ & [\![(\exists y_{k-1}, y_k\ \theta)(\bar{a}, b_1, b_2, \ldots, b_{k-2})]\!] = \texttt{true} && \text{for some } b_1, b_2, \ldots, b_{k-2} \in \mathbb{N} \\ \Leftrightarrow\ & \cdots \\ \Leftrightarrow\ & [\![(\exists y_1, y_2, \ldots, y_k\ \theta)(\bar{a})]\!] = \texttt{true} \end{aligned}$$

by k-many applications of the definition of $[\![\ldots]\!]$. □

One does not need to go far beyond Σ_1 to find a formula whose truth value in $\mathbb{N}$ is different from the output it gives when considered as a program.

Assignment 2.7. A Π_1 *formula* is an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula of the form $\forall \bar{y}\ \theta$, where θ is a $\Delta_0(\exp)$ formula and $\bar{y}$ is a possibly empty tuple of variables. Find $\theta(x_1, x_2, \ldots, x_k) \in \Pi_1$ and $a_1, a_2, \ldots, a_k \in \mathbb{N}$ such that

$$\mathbb{N} \models \theta(\bar{a}) \quad \text{and} \quad [\![\theta(\bar{a})]\!] \neq \texttt{true}.$$

Explain briefly. [5 points]

We may view Proposition 2.6 as saying: via the truth of Σ_1 formulas, the standard model of arithmetic $\mathbb{N}$ knows when programs output `true`. In the next two lectures, we will find other $\mathcal{L}_{\mathrm{A}}(\exp)$ structures which also have this property.