MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Lecture 3: Σ_1 completeness

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The aim of this lecture is to isolate a class of $\mathcal{L}_{\mathrm{A}}(\exp)$ structures whose idea of programs agrees reasonably well with that of $\mathbb{N}$. It will be the ability to reason about programs which enables one to formulate a sentence that is neither provable nor refutable in the Incompleteness Theorems.

As we saw in Proposition 2.6, the standard model of arithmetic $\mathbb{N}$ has a good idea of what programs output. For very good reasons, the same is not true of $\mathbb{R}$. The following is another $\mathcal{L}_{\mathrm{A}}(\exp)$ structure which has a reasonably good idea of what programs output.

Example 3.1. Let $\mathbb{N}^\infty = (\mathbb{N} \cup \{\infty\}, 0, 1, +, \times, \exp, <)$, where

- $\infty \notin \mathbb{N}$;
- $0, 1$ are the usual $0, 1 \in \mathbb{N}$;
- $+, \times, \exp, <$ are the usual $+, \times, \exp, <$ on $\mathbb{N}$;
- for all $a \in \mathbb{N} \cup \{\infty\}$,
  - $\infty + a = a + \infty = a \times \infty = \infty$;
  - $\infty \times a = \begin{cases} 0, & \text{if } a = 0; \\ \infty, & \text{otherwise;} \end{cases}$
  - $2^\infty = \infty$;
  - if $a \in \mathbb{N}$, then $a < \infty$; and
  - $\infty \not< a$.

$\mathbb{N}^\infty$ is easily seen to satisfy various simple axioms for arithmetic. Let us set up some terminology to describe this.

Definition.

- An $\mathcal{L}_{\mathrm{A}}(\exp)$ *sentence* is an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula in which no variable appears free.
- An $\mathcal{L}_{\mathrm{A}}(\exp)$ *theory* is a set of $\mathcal{L}_{\mathrm{A}}(\exp)$ sentences.
- A *model* of an $\mathcal{L}_{\mathrm{A}}(\exp)$ theory T is an $\mathcal{L}_{\mathrm{A}}(\exp)$ structure M such that $M \models \sigma$ for all $\sigma \in T$. In this case, we write $M \models T$.
- If $n \in \mathbb{N}$, then $\underline{n}$ denotes the $\mathcal{L}_{\mathrm{A}}(\exp)$ term

$$0 + \underbrace{1 + 1 + \cdots + 1}_{n\text{-many 1's}}.$$

  These terms are called *numerals*.

Via these numerals, one can define a map $n \mapsto \underline{n}^M$ whenever M is an $\mathcal{L}_{\mathrm{A}}(\exp)$ structure. In the case when $M = \mathbb{N}^\infty$, this map is injective, preserves all arithmetic operations, and has image an initial segment of M. We extract these properties in the theory below.

Definition (Raphael Robinson)**.** Let $\mathrm{R}(\exp)$ be the $\mathcal{L}_{\mathrm{A}}(\exp)$ theory consisting of the following families of sentences, where $m, n \in \mathbb{N}$.

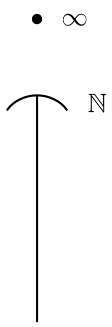

Figure 3.1: The $\mathcal{L}_{\mathrm{A}}(\exp)$ structure $\mathbb{N}^\infty$

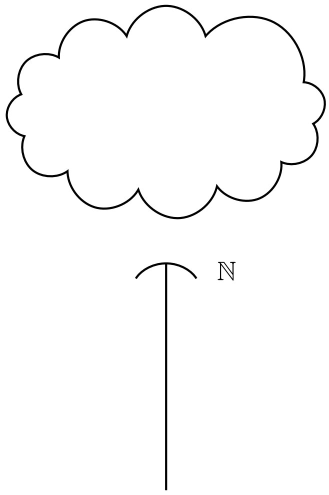

Figure 3.2: A general model of R(exp)

(R1) $\underline{1} = 1$.

(R+) $\underline{m} + \underline{n} = \underline{m+n}$.

(R×) $\underline{m} \times \underline{n} = \underline{m \times n}$.

(Rexp) $\exp(\underline{n}) = \underline{\exp(n)}$.

(R<) $\underline{m} < \underline{n}$ whenever $m < n$.

(R≠) $\underline{m} \neq \underline{n}$ whenever $m \neq n$.

(RInit) $\forall x{<}\underline{n}\ (x = \underline{0} \lor x = \underline{1} \lor \cdots \lor x = \underline{n-1})$.

(RComp) $\forall x\ (x < \underline{n} \lor x = \underline{n} \lor \underline{n} < x)$.

The empty disjunction is $\bot$ by definition.

Example 3.2. Both $\mathbb{N}$ and $\mathbb{N}^\infty$ are models of R(exp), but $\mathbb{R}$ is not because of (RInit).

R(exp) asserts that $+, \times, \exp$ are evaluated in the same way as in $\mathbb{N}$. Hence, since $\mathcal{L}_{\mathrm{A}}(\exp)$ terms are built up from $+, \times, \exp$, they are also evaluated in the same way as in $\mathbb{N}$.

Lemma 3.3. Let $M \models \mathrm{R}(\exp)$. For all $\mathcal{L}_{\mathrm{A}}(\exp)$ terms $t(x_1, x_2, \ldots, x_k)$ and all $n_1, n_2, \ldots, n_k \in \mathbb{N}$,

$$\left(\underline{t^{\mathbb{N}}(n_1, n_2, \ldots, n_k)}\right)^M = t^M(\underline{n_1}^M, \underline{n_2}^M, \ldots, \underline{n_k}^M).$$

Proof. We proceed by induction on the $\mathcal{L}_{\mathrm{A}}(\exp)$ term t.

Consider the variable x_i. By the definition of term evaluation,

$$\left(\underline{(x_i)^{\mathbb{N}}(n_1, n_2, \ldots, n_k)}\right)^M = \underline{n_i}^M = (x_i)^M(\underline{n_1}^M, \underline{n_2}^M, \ldots, \underline{n_k}^M).$$

Consider the constant symbol 0. By the definition of term evaluation,

$$\left(\underline{0^{\mathbb{N}}(\bar{n})}\right)^M = \underline{0}^M = 0^M = 0^M(\underline{\bar{n}}^M)$$

as $\underline{0} = 0$.

Consider the constant symbol 1.

$$\begin{aligned}
\left(\underline{1^{\mathbb{N}}(\bar{n})}\right)^M &= \underline{1}^M && \text{by the definition of term evaluation in } \mathbb{N};\\
&= 1^M && \text{by (R1)};\\
&= 1^M(\underline{\bar{n}}^M) && \text{by the definition of term evaluation in } M.
\end{aligned}$$

Consider the $\mathcal{L}_{\mathrm{A}}(\exp)$ term $t+s$.

$$\begin{aligned}
&\left(\underline{(t+s)^{\mathbb{N}}(\bar{n})}\right)^M \\
&= \left(\underline{t^{\mathbb{N}}(\bar{n}) + s^{\mathbb{N}}(\bar{n})}\right)^M && \text{by the definition of term evaluation in } \mathbb{N};\\
&= \left(\underline{t^{\mathbb{N}}(\bar{n})}\right)^M +^M \left(\underline{s^{\mathbb{N}}(\bar{n})}\right)^M && \text{by (R+)};\\
&= t^M(\underline{\bar{n}}^M) +^M s^M(\underline{\bar{n}}^M) && \text{by the induction hypothesis};\\
&= (t+s)^M(\underline{\bar{n}}^M) && \text{by the definition of term evaluation in } M.
\end{aligned}$$

Terms of the form $t \times s$ or $\exp(t)$ are dealt with in a similar way. □

By Lemma 3.3, it does not matter whether we evaluate a term in $\mathbb{N}$ or in a model of R(exp), as long as the objects involved are in $\mathbb{N}$. The same is true of the order $<$, but a small argument is needed to establish a converse to (R$<$).

Observation 3.4. Let $M \models \mathrm{R}(\exp)$. If $m, n \in \mathbb{N}$ with $M \models \underline{m} < \underline{n}$, then (RInit) gives $j < n$ such that $M \models \underline{m} = \underline{j}$, and thus $m = j < n$ by (R$\neq$).

It follows that there is copy of the standard model of arithmetic $\mathbb{N}$ at the bottom of every model of R(exp). In other words, we may view every model of R(exp) as an end extension of $\mathbb{N}$.

Convention 3.5. If $M \models \mathrm{R}(\exp)$, then we identify $\underline{n}^M$ with $n \in \mathbb{N}$.

Suppose we are running a $\Delta_0(\exp)$ program θ in a model $M \models \mathrm{R}(\exp)$ on inputs $\bar{a} \in \mathbb{N}$. The bounds for the **par-for** loops in θ are all evaluations of $\mathcal{L}_{\mathrm{A}}(\exp)$ terms on natural numbers, and thus they must all be in $\mathbb{N}$ by Lemma 3.3. It then follows from (RInit) that the program θ on input $\bar{a}$ is run entirely in the copy of $\mathbb{N}$ inside M; the part of M outside $\mathbb{N}$ is actually irrelevant. So the output of θ in M is the same as that in $\mathbb{N}$.

Now suppose we are running a Σ_1 program with a **par-while** loop in the model M. If one instance of the **par-while** loop in $\mathbb{N}$ returns `true`, then the **par-while** loop returns `true` in M because $\mathbb{N} \subseteq M$. The converse may not hold: there are **par-while** loops of which all `true` instances are in $M \setminus \mathbb{N}$.

The two paragraphs above explain why models of R(exp) have some idea of what programs output. In view of Proposition 2.6, we can formulate these in terms of $\Delta_0(\exp)$ and Σ_1 formulas.

Proposition 3.6. Let $M \models \mathrm{R}(\exp)$.

(1) ($\Delta_0(\exp)$ absoluteness between M and $\mathbb{N}$.) For all $\theta(\bar{x}) \in \Delta_0(\exp)$ and all $\bar{a} \in \mathbb{N}$,

$$\mathbb{N} \models \theta(\bar{a}) \quad \Leftrightarrow \quad M \models \theta(\bar{a}).$$

(2) For all $\theta(\bar{x}) \in \Sigma_1$ and all $\bar{a} \in \mathbb{N}$,

$$\mathbb{N} \models \theta(\bar{a}) \quad \Rightarrow \quad M \models \theta(\bar{a}).$$

Proof. We show (1) by induction on the $\Delta_0(\exp)$ formula θ.

Consider $\top$ and $\bar{a} \in \mathbb{N}$. The truth definition implies $\mathbb{N} \models \top(\bar{a})$ and $M \models \top(\bar{a})$.

Consider $t = s$ and $\bar{a} \in \mathbb{N}$, where $t(\bar{x}), s(\bar{x})$ are $\mathcal{L}_{\mathrm{A}}(\exp)$ terms.

$$\begin{aligned}
& \mathbb{N} \models (t = s)(\bar{a}) && \\
\Leftrightarrow\ & t^{\mathbb{N}}(\bar{a}) = s^{\mathbb{N}}(\bar{a}) && \text{by the truth definition;} \\
\Leftrightarrow\ & t^{M}(\bar{a}) = s^{M}(\bar{a}) && \text{by Lemma 3.3;} \\
\Leftrightarrow\ & M \models (t = s)(\bar{a}) && \text{by the truth definition.}
\end{aligned}$$

The case when the $\Delta_0(\exp)$ formula is $t < s$ is proved similarly.

Consider $\neg\theta$ and $\bar{a} \in \mathbb{N}$, where $\theta(\bar{x})$ is a $\Delta_0(\exp)$ formula.

$$\begin{aligned}
& \mathbb{N} \models \neg\theta(\bar{a}) && \\
\Leftrightarrow\ & \mathbb{N} \not\models \theta(\bar{a}) && \text{by the truth definition;} \\
\Leftrightarrow\ & M \not\models \theta(\bar{a}) && \text{by the induction hypothesis;} \\
\Leftrightarrow\ & M \models \neg\theta(\bar{a}) && \text{by the truth definition.}
\end{aligned}$$

Consider $\theta \vee \eta$ and $\bar{a}, \bar{b}, \bar{c} \in \mathbb{N}$ of appropriate lengths, where $\theta(\bar{x}, \bar{z}), \eta(\bar{y}, \bar{z})$ are $\Delta_0(\exp)$ formulas with $\{\bar{x}\} \cap \{\bar{y}\} = \varnothing$.

$$\begin{aligned}
& \mathbb{N} \models \theta(\bar{a}, \bar{c}) \vee \eta(\bar{b}, \bar{c}) && \\
\Leftrightarrow\ & \mathbb{N} \models \theta(\bar{a}, \bar{c}) \text{ or } \mathbb{N} \models \eta(\bar{b}, \bar{c}) && \text{by the truth definition;} \\
\Leftrightarrow\ & M \models \theta(\bar{a}, \bar{c}) \text{ or } M \models \eta(\bar{b}, \bar{c}) && \text{by the induction hypothesis;} \\
\Leftrightarrow\ & M \models \theta(\bar{a}, \bar{c}) \vee \eta(\bar{b}, \bar{c}) && \text{by the truth definition.}
\end{aligned}$$

Consider $\exists y{<}t\ \theta$ and $\bar{a} \in \mathbb{N}$, where $t(\bar{x})$ is an $\mathcal{L}_{\mathrm{A}}(\exp)$ term and $\theta(\bar{x}, y)$ is a $\Delta_0(\exp)$ formula. Suppose $\mathbb{N} \models \exists y{<}t(\bar{a})\ \theta(\bar{a}, y)$. Use the truth definition to find $b <^{\mathbb{N}} t^{\mathbb{N}}(\bar{a})$ in $\mathbb{N}$ such that $\mathbb{N} \models \theta(\bar{a}, b)$. Then $M \models \theta(\bar{a}, b)$ by the induction hypothesis, and $b <^{M} t^{M}(\bar{a})$ by (R<) and Lemma 3.3. Thus $M \models \exists y{<}t(\bar{a})\ \theta(\bar{a}, y)$ by the truth definition.

Conversely, suppose $M \models \exists y{<}t(\bar{a})\ \theta(\bar{a}, y)$. Use the truth definition to find $b <^{M} t^{M}(\bar{a})$ in M such that $M \models \theta(\bar{a}, b)$. Now $t^{M}(\bar{a}) = t^{\mathbb{N}}(\bar{a}) \in \mathbb{N}$ by Lemma 3.3, and so $b \in \mathbb{N}$ by (RInit). Hence $\mathbb{N} \models \theta(\bar{a}, b)$ by the induction hypothesis, and $b <^{\mathbb{N}} t^{\mathbb{N}}(\bar{a})$ by Observation 3.4. Thus $\mathbb{N} \models \exists y{<}t(\bar{a})\ \theta(\bar{a}, y)$ by the truth definition.

This completes the induction for (1). For (2), suppose $\mathbb{N} \models \exists \bar{y}\ \eta(\bar{a}, \bar{y})$, where $\eta \in \Delta_0(\exp)$. Use the truth definition to find $\bar{b} \in \mathbb{N} \models \eta(\bar{a}, \bar{b})$. Then $M \models \eta(\bar{a}, \bar{b})$ by (1). Thus $M \models \exists \bar{y}\ \eta(\bar{a}, \bar{y})$ by the truth definition. ☐

Assignment 3.7. Recall an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula is said to be Π_1 if it is of the form $\forall \bar{y}\ \theta(\bar{x}, \bar{y})$, where $\theta \in \Delta_0(\exp)$ and $\bar{y}$ is a possibly empty tuple of variables. Prove the existence of a model $M \models \mathrm{R}(\exp)$ and a sentence $\sigma \in \Pi_1$ such that $\mathbb{N} \models \sigma$ and $M \not\models \sigma$. [4 points]

Notice that Proposition 3.6(2) is not an equivalence. Hence models of $\mathrm{R}(\exp)$ may be wrong on which natural numbers are *not* elements of an r.e. set. Recursive sets are designed to overcome this problem, as we will see in the next lecture.