MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Lecture 4: Representability of recursive objects

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The aim of this lecture is to show the representability of recursive objects in R(exp). Our representations are formulated in terms of semantic entailment.

Definition. An $\mathcal{L}_{\mathrm{A}}(\exp)$ theory T *semantically entails* an $\mathcal{L}_{\mathrm{A}}(\exp)$ sentence σ if all models of T satisfy σ. In this case, we write $T \models \sigma$.

Remark 4.1. In view of Proposition 2.6(2), a subset $S \subseteq \mathbb{N}^k$ is r.e. if and only if there exists $\theta(x_1, x_2, \dots, x_k) \in \Sigma_1$ such that $S = \{(a_1, a_2, \dots, a_k) \in \mathbb{N}^k : \mathbb{N} \models \theta(\bar{a})\}$.

Corollary 4.2 (Σ_1 semi-representability of r.e. sets in R(exp))**.** For all Σ_1 formulas $\theta(x_1, x_2, \dots, x_k)$,

$$\{(m_1, m_2, \dots, m_k) \in \mathbb{N}^k : \mathbb{N} \models \theta(\bar{m})\} = \{(m_1, m_2, \dots, m_k) \in \mathbb{N}^k : \mathrm{R}(\exp) \models \theta(\underline{\bar{m}})\}.$$

Hence R(exp) is Σ_1 *complete*, i.e., it semantically entails all Σ_1 sentences true in $\mathbb{N}$.

Proof. The $\subseteq$ direction is Proposition 3.6(2). For the $\supseteq$ direction, if $\bar{m} \in \mathbb{N}$ such that $\mathrm{R}(\exp) \models \theta(\underline{\bar{m}})$, then $\mathbb{N} \models \theta(\bar{m})$ because $\mathbb{N} \models \mathrm{R}(\exp)$ and $\underline{\bar{m}}^M = \bar{m}$. □

Corollary 4.2 asserts semi-representability, i.e., the falsehood of the defining formula corresponds to the unprovability of a formula, not the provability of its negation. This has the disadvantage that semi-representability may be destroyed when one strengthens the theory. To get outright representability, we use the following observation, which is the first place where we invoke (RComp).

Observation 4.3. Let $M \models \mathrm{R}(\exp)$. Then $n < a$ whenever $n \in \mathbb{N}$ and $a \in M \setminus \mathbb{N}$.

Proof. Suppose $n \not< a$. Then either $a = n$ or $a < n$ by (RComp). In either case, we know $a \in \mathbb{N}$, in view of (RInit). □

Instead of recursive set, we represent recursive functions in R(exp). Intuitively, these are functions that can be computed by algorithms. In view of the correspondence between programs and formulas given by Proposition 2.6, we do not need to mention programs in the definition of recursive functions at all.

Definition. A function $F\colon \mathbb{N}^k \to \mathbb{N}$ is *recursive* if there is $\theta(x_1, x_2, \dots, x_k, y) \in \Sigma_1$ such that for all $\bar{m}, n \in \mathbb{N}$,

$$n = F(\bar{m}) \quad \Leftrightarrow \quad \mathbb{N} \models \theta(\bar{m}, n).$$

Notation. $\bigwedge_{j=1}^{\ell} \theta_j = \theta_1 \wedge \theta_2 \wedge \dots \wedge \theta_\ell$.

Theorem 4.4 (Σ_1 representability of recursive functions over R(exp); Mostowski, R. Robinson, Tarski)**.** Let F be a recursive function $\mathbb{N}^k \to \mathbb{N}$. Then there is a Σ_1 formula $\rho(x_1, x_2, \dots, x_k, y)$ that *represents* F over R(exp), i.e., for all $m_1, m_2, \dots, m_k \in \mathbb{N}$,

$$\mathrm{R}(\exp) \models \rho(\underline{\bar{m}}, \underline{F(\bar{m})}) \wedge \forall y\ \big(\rho(\underline{\bar{m}}, y) \to y = \underline{F(\bar{m})}\big).$$

While the first conjunct in the displayed line above provides positive information about the function F, the second conjunct provides negative information: if $\bar{m}, n \in \mathbb{N}$ with $n \neq F(\bar{m})$, then $\mathrm{R}(\exp) \models \neg\rho(\underline{\bar{m}}, \underline{n})$, which is stronger than having $\mathrm{R}(\exp) \not\models \rho(\underline{\bar{m}}, \underline{n})$

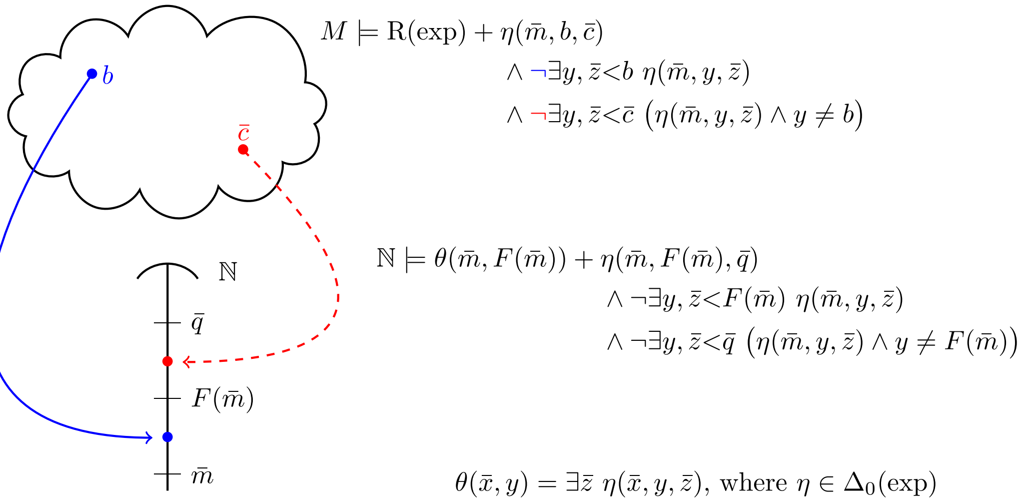

Figure 4.1: Forcing the witnesses $b, \bar{c}$ to be in $\mathbb{N}$ by comparing them with $F(\bar{m}), \bar{q}$

Proof. Use the recursiveness of F to find $\eta(\bar{x}, y, z_1, z_2, \dots, z_\ell) \in \Delta_0(\exp)$ such that for all $\bar{m}, n \in \mathbb{N}$,

$$n = F(\bar{m}) \quad \Leftrightarrow \quad \mathbb{N} \models \exists \bar{z}\ \eta(\bar{m}, n, \bar{z}).$$

We verify that $\rho(\bar{x}, y) = \exists \bar{z}\ \chi(\bar{x}, y, \bar{z})$ satisfies the requirements, where $\chi(\bar{x}, y, \bar{z})$ is defined to be the $\Delta_0(\exp)$ formula

$$\eta(\bar{x}, y, \bar{z}) \wedge \forall y'{<}y\ \neg\exists \bar{z}'{<}y\ \eta(\bar{x}, y', \bar{z}') \\ \wedge \bigwedge_{j=1}^{\ell} \forall y'{<}z_j\ \big(\exists \bar{z}'{<}z_j\ \eta(\bar{x}, y', \bar{z}') \rightarrow y' = y\big).$$

Fix $\bar{m} \in \mathbb{N}$ and $M \models \mathrm{R}(\exp)$.

Use the choice of η to find $q_1, q_2, \dots, q_\ell \in \mathbb{N} \models \eta(\bar{m}, F(\bar{m}), \bar{q})$. Then

$$\mathbb{N} \models \eta(\bar{m}, F(\bar{m}), \bar{q}) \wedge \forall y'{<}F(\bar{m})\ \neg\exists \bar{z}'\ \eta(\bar{m}, y', \bar{z}') \\ \wedge \bigwedge_{j=1}^{\ell} \forall y'{<}q_j\ \big(\exists \bar{z}'\ \eta(\bar{m}, y', \bar{z}') \rightarrow y' = F(\bar{m})\big)$$

by the choice of η again because the image of $\bar{m}$ under F is unique. In particular,

$$\mathbb{N} \models \eta(\bar{m}, F(\bar{m}), \bar{q}) \wedge \forall y'{<}F(\bar{m})\ \neg\exists \bar{z}'{<}F(\bar{m})\ \eta(\bar{m}, y', \bar{z}') \\ \wedge \bigwedge_{j=1}^{\ell} \forall y'{<}q_j\ \big(\exists \bar{z}'{<}q_j\ \eta(\bar{m}, y', \bar{z}') \rightarrow y' = F(\bar{m})\big)$$

These properties of $\bar{m}, F(\bar{m}), \bar{q}$ transfer from $\mathbb{N}$ to M by $\Delta_0(\exp)$ absoluteness (i.e., Proposition 3.6). Hence $M \models \chi(\bar{m}, F(\bar{m}), \bar{q})$, implying $M \models \rho(\bar{m}, F(\bar{m}))$ by the definition of ρ.

Take $b \in M \models \rho(\bar{m}, b)$. Find $\bar{c} \in M \models \chi(\bar{m}, b, \bar{c})$, or unravelling the definition of χ,

$$M \models \eta(\bar{m}, b, \bar{c}) \wedge \forall y'{<}b\ \forall \bar{z}'{<}b\ \neg\eta(\bar{m}, y', \bar{z}') \\ \wedge \bigwedge_{j=1}^{\ell} \forall y'{<}c_j\ \forall \bar{z}'{<}c_j\ \big(\eta(\bar{m}, y', \bar{z}') \rightarrow y' = b\big).$$

Recall from the previous paragraph that $M \models \eta(\bar{m}, F(\bar{m}), \bar{q})$, where $F(\bar{m}), \bar{q} \in \mathbb{N}$. Thus $b \in \mathbb{N}$ by the second conjunct above and Observation 4.3. If we have $j \in \{1, 2, \dots, \ell\}$ with $c_j \notin \mathbb{N}$, then the third conjunct above and Observation 4.3 tell us $b = F(\bar{m})$. So suppose $\bar{c} \in \mathbb{N}$. Then $\mathbb{N} \models \eta(\bar{m}, b, \bar{c})$ by $\Delta_0(\exp)$ absoluteness. Hence the choice of η implies $b = F(\bar{m})$. $\square$

At first glance, it is not clear why we should compare the y witnesses with the $\bar{z}$ witnesses in the proof above, but we will use this type of witness comparison trick again.

The theory R(exp) consists of infinitely many sentences which seem to describe the arithmetic operations completely. One may suspect that this power to represent recursive objects originates from, or even requires, this infinitude. However, this suspicion is not true.

Proposition 4.5. $\mathrm{Q(exp)} \models \mathrm{R(exp)}$, where Q(exp) is the $\mathcal{L}_{\mathrm{A}}(\exp)$ theory consisting of the following sentences.

(Q1) $1 = 0 + 1$.

(QS$_0$) $\forall x\ \big(x \neq 0 \leftrightarrow \exists y\ (x = y + 1)\big)$.

(QS$_1$) $\forall x, y\ (x + 1 = y + 1 \to x = y)$.

(Q+$_0$) $\forall x\ (x + 0 = x)$.

(Q+$_1$) $\forall x, y\ (x + (y + 1) = (x + y) + 1)$.

(Q×$_0$) $\forall x\ (x \times 0 = 0)$.

(Q×$_1$) $\forall x, y\ (x \times (y + 1) = (x \times y) + x)$.

(Qexp$_0$) $2^0 = 1$.

(Qexp$_1$) $\forall x\ (2^{x+1} = 2^x + 2^x)$.

(Q<) $\forall x, y\ \big(x < y \leftrightarrow x \neq y \wedge \exists z\ (z + x = y)\big)$.

Proof. Long but elementary. So let us show only $\mathrm{Q(exp)} \models (\mathrm{R}+)$. Fix $M \models \mathrm{Q(exp)}$. We show by induction on $n \in \mathbb{N}$ that

$$M \models \underline{m} + \underline{n} = \underline{m+n} \quad \text{for all } m, n \in \mathbb{N}.$$

Base step $n = 0$. Let $m \in \mathbb{N}$. Then

$$\begin{aligned}
\underline{m}^M +^M \underline{0}^M &= \underline{m}^M +^M 0^M && \text{as } \underline{0} = 0;\\
&= \underline{m}^M && \text{by (Q+}_0\text{)};\\
&= (\underline{m+0})^M && \text{as } m + 0 = m.
\end{aligned}$$

Induction step. Let $m, n \in \mathbb{N}$ such that $M \models \underline{m} + \underline{n} = \underline{m+n}$. Then

$$\begin{aligned}
\underline{m}^M +^M (\underline{n+1})^M &= \underline{m}^M +^M (\underline{n}^M +^M 1^M) && \text{as } \underline{n+1} = \underline{n} + 1;\\
&= (\underline{m}^M +^M \underline{n}^M) +^M 1^M && \text{by (Q+}_1\text{)};\\
&= (\underline{m+n})^M +^M 1^M && \text{by the induction hypothesis};\\
&= (\underline{m+n+1})^M && \text{as } \underline{m+n+1} = \underline{m+n} + 1.
\end{aligned}$$

□

Assignment 4.6. Show that $\mathrm{Q(exp)} \models \forall x, y\ (x + y = 0 \to x = 0 \wedge y = 0)$. [4 points]

In the next lecture, we will see a clever application of Theorem 4.4, which will form the basis of our proof (and many other proofs) of the Incompleteness Theorems.