MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Lecture 7: Gödel's Completeness Theorem

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The main aim of this lecture is to prove that all consistent theories have models. This is an important special case of Gödel's Completeness Theorem.

Completeness Theorem (Gödel)**.** Let Φ be a set of $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas and θ be an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula. If $\Phi \models \theta$, then $\Phi \vdash \theta$.

The Completeness Theorem tells us that our choice of deduction rules is sufficient to capture semantic entailment. Hence, although semantic entailment is defined in terms of infinitary objects like structures, it is inherently finitary because it can be characterized in terms of proofs, which are finite, as we discussed in the previous lecture. The syntax is able to match the semantics here essentially because Tarski's inductive clauses for truth are finitary.

Here is the plan of our proof of the Completeness Theorem.

(A) Reduce to the case when $\theta = \bot$.

(B) Prove that if $\Phi \nvdash \bot$, then there is a Henkin set of formulas $\Phi^* \supseteq \Phi$.

(C) Build a model of a Henkin set of formulas Φ^*.

We show (B) and (C) in this lecture. The net effect of these two is that one has a model of Φ. Notice this model cannot satisfy $\bot$ by the truth definition. We defer the proof of (A) to the next lecture. Let us explain the terminology.

Definition. Let Φ be a set of $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas.

- Φ is *consistent* if $\Phi \nvdash \bot$.
- Φ is said to *decide all* $\mathcal{L}_{\mathrm{A}}(\exp)$ *formulas* if for all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas θ, either $\Phi \vdash \theta$ or $\Phi \vdash \neg\theta$.
- Φ is said to be *Henkinized* or have the *Henkin property* if for every $\mathcal{L}_{\mathrm{A}}(\exp)$ formula $\eta(\bar{x}, y)$, if $\Phi \vdash \exists y\ \eta(\bar{x}, y)$, then $\Phi \vdash \eta(\bar{x}, z)$ for some variable z.

As the reader can verify using (cut) and (RAA), a set of formulas is consistent if and only if it does not prove both θ and $\neg\theta$ for any formula θ. Hence consistency and deciding-all-formulas are dual to each other. Some authors use *completeness* to mean deciding-all-formulas. We would like to reserve the word *complete* for theories that decide all *sentences*. Roughly speaking, a set of formulas is Henkinized if and only if every existential statement it proves is witnessed by a variable.

To make a set of formulas decide all formulas, one puts in either θ or $\neg\theta$ for every formula θ. To Henkinize a set of formulas, one assigns a witness for any existential statement proved. Therefore, by brute force, one can extend any consistent set of formulas to one that is consistent, Henkinized, and that decides all formulas. This establishes (B).

Henkinization Lemma. Let Φ be a consistent set of $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas in which infinitely many variables do *not* appear free. Then Φ can be extended to a consistent, Henkinized set of $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas that decides all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas.

Proof. For notational convenience, we concentrate on the case when Φ is an $\mathcal{L}_{\mathrm{A}}(\exp)$ theory. Fix an enumeration $(\theta_j(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_j))_{j\in\mathbb{N}}$ of all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas. We will construct another sequence $(\varphi_j(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_{j+1}))_{j\in\mathbb{N}}$ by recursion, with the inductive assumption that at each stage $j \in \mathbb{N}$,

$$\Phi_j := \Phi \cup \{\varphi_i(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_{i+1}) : i < j\} \not\vdash \bot.$$

At the end, we will set $\Phi^* = \Phi \cup \{\varphi_i(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_{i+1}) : i \in \mathbb{N}\}$. The inductive assumption ensures that every finite subset of Φ^* is consistent. Thus $\Phi^* \not\vdash \bot$ by the Compactness Lemma.

Assume $(\varphi_i)_{i<j}$ is defined. Consider θ_j.

Case 1: Suppose $\Phi_j + \theta_j \vdash \bot$. Then $\Phi_j + \neg\theta_j \not\vdash \bot$ by (cut) and the inductive assumption. Set $\varphi_j = \neg\theta_j$. Notice $\Phi_{j+1} \vdash \neg\theta_j$ by (asn).

Case 2: Suppose $\Phi_j + \theta_j \not\vdash \bot$.

Case 2a: Suppose $\theta_j = \exists y\ \eta_j(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_j, y)$. Then $\Phi_j + \eta_j(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_j, \mathsf{v}_{j+1}) \not\vdash \bot$ by ($\exists$L) because v_{j+1} does not appear free in any element of Φ_j. Set $\varphi_j(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_{j+1}) = \eta_j(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_{j+1})$. Notice $\Phi_{j+1} \vdash \theta_j$ by (asn) and ($\exists$R).

Case 2b: Suppose θ_j is not of the form $\exists y\ \eta_j$. Then set $\varphi_j = \theta_j$. Notice $\Phi_{j+1} \vdash \theta_j$ by (asn).

By construction, the set Φ^* decides all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas. To show the Henkin property, suppose $\Phi^* \vdash \exists y\ \eta$, where η is an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula. Find $j \in \mathbb{N}$ such that $\theta_j = \exists y\ \eta$. We know $\varphi_j \neq \neg\theta_j$ in view of (asn) and ($\bot$) because $\Phi^* \not\vdash \bot$. Hence we are in Case 2a in step j. Our construction thus makes $\varphi_j = \eta(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_{j+1})$. So $\Phi^* \vdash \eta(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_{j+1})$ by (asn).

For the general case when some variables may appear free in Φ, instead of using the next variable v_{j+1} in Case 2a of the construction, pick one that has not yet appeared free. □

All the three properties defined above for sets of formulas are in a sense necessary when we want to build a model because of the $\neg$ clause and the $\exists$ clause in the truth definition. Now we show that these properties are also sufficient, thus establishing (C). The proof resembles the construction of a free object factored out by a set of relations in algebra. The Henkin property allows one to make true not only a set of relations (which is essentially a set of atomic formulas) but also a set of arbitrary formulas. The constituents of the model to be constructed are essentially the variables in the language.

Model Construction Theorem. Let Φ^* be a consistent Henkinized set of $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas which decides all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas. Then one can construct an $\mathcal{L}_{\mathrm{A}}(\exp)$ structure M with an enumeration $a_0, a_1, \ldots$ such that $M \models \varphi(a_0, a_1, \ldots, a_\ell)$ for all $\varphi(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_\ell) \in \Phi^*$.

Proof. Let $(c_j)_{j\in\mathbb{N}}$ one-to-one enumerate the elements of a countably infinite set C. Define a binary relation $\sim$ on C by setting

$$c_i \sim c_j \quad \Leftrightarrow \quad \Phi^* \vdash \mathsf{v}_i = \mathsf{v}_j$$

for all $i, j \in \mathbb{N}$. Then $\sim$ is an equivalence relation by (refl), (sym), and (tran). Let $M = C/{\sim} = \{[c_j] : j \in \mathbb{N}\}$, where $[c_j] = \{c_i : c_i \sim c_j\}$. Turn M into an $\mathcal{L}_{\mathrm{A}}(\exp)$ structure by setting

- $[c_j] = 0^M \quad \Leftrightarrow \quad \Phi^* \vdash \mathsf{v}_j = 0$;
- $[c_j] = 1^M \quad \Leftrightarrow \quad \Phi^* \vdash \mathsf{v}_j = 1$;
- $[c_k] = [c_i] +^M [c_j] \quad \Leftrightarrow \quad \Phi^* \vdash \mathsf{v}_k = \mathsf{v}_i + \mathsf{v}_j$;
- $[c_k] = [c_i] \times^M [c_j] \quad \Leftrightarrow \quad \Phi^* \vdash \mathsf{v}_k = \mathsf{v}_i \times \mathsf{v}_j$;
- $[c_k] = \exp^M([c_i]) \quad \Leftrightarrow \quad \Phi^* \vdash \mathsf{v}_k = \exp \mathsf{v}_i$; and
- $[c_i] <^M [c_j] \quad \Leftrightarrow \quad \Phi^* \vdash \mathsf{v}_i < \mathsf{v}_j$

for all $i, j, k \in \mathbb{N}$. The rest of the proof splits into three parts:

(i) verify that these operations on M are well-defined;

(ii) show by induction on t that for all $\mathcal{L}_{\mathrm{A}}(\exp)$ terms $t(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_\ell)$ and all $i \in \mathbb{N}$,

$$[c_i] = t^M([c_0], [c_1], \ldots, [c_\ell]) \quad \Leftrightarrow \quad \Phi^* \vdash \mathsf{v}_i = t(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_\ell);$$

(iii) show by induction on φ that for all $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas $\varphi(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_\ell)$,

$$M \models \varphi([c_0], [c_1], \ldots, [c_\ell]) \quad \Leftrightarrow \quad \Phi^* \vdash \varphi(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_\ell).$$

At the end, we can let $a_0, a_1, \ldots$ be $[c_0], [c_1], \ldots$ respectively.

We demonstrate how (i) can be proved by showing $+^M$ is well-defined.

Existence. Let $[c_i], [c_j] \in M$. Notice $\Phi^* \vdash \mathsf{v}_i + \mathsf{v}_j = \mathsf{v}_i + \mathsf{v}_j$ by (refl). So $\Phi^* \vdash \exists y\ (y = \mathsf{v}_i + \mathsf{v}_j)$ by ($\exists$R). Since Φ^* is Henkinized, we have a variable v_k such that $\Phi^* \vdash \mathsf{v}_k = \mathsf{v}_i + \mathsf{v}_j$. Then $[c_k] = [c_i] +^M [c_j]$ by the definition of $+^M$.

Uniqueness. Let $i, j, k, i', j', k' \in \mathbb{N}$ such that

$$[c_k] = [c_i] +^M [c_j] \quad \text{and} \quad [c_{k'}] = [c_{i'}] +^M [c_{j'}] \quad \text{and} \quad [c_i] = [c_{i'}] \quad \text{and} \quad [c_j] = [c_{j'}].$$

Unravelling the definitions, we know Φ^* proves

$$\mathsf{v}_k = \mathsf{v}_i + \mathsf{v}_j \quad \text{and} \quad \mathsf{v}_{k'} = \mathsf{v}_{i'} + \mathsf{v}_{j'} \quad \text{and} \quad \mathsf{v}_i = \mathsf{v}_{i'} \quad \text{and} \quad \mathsf{v}_j = \mathsf{v}_{j'}.$$

It follows from the equality rules that $\Phi^* \vdash \mathsf{v}_k = \mathsf{v}_{k'}$. Hence $[c_k] = [c_{k'}]$ by the definition of $\sim$.

For (ii), see Assignment 7.1.

Finally, we proceed to (iii). The two directions of the equivalence to be proven here are the reason why we often have two deduction rules for each logical connective.

Consider $\top$. The truth definition implies $M \models \top$, and ($\top$) implies $\Phi^* \vdash \top$.

Consider $t = s$, where t, s are $\mathcal{L}_{\mathrm{A}}(\exp)$ terms. Let $[c_i] = t^M(\overline{[c]})$ and $[c_j] = s^M(\overline{[c]})$. Then

$$\begin{aligned}
& M \models (t = s)(\overline{[c]}) && \\
\Leftrightarrow\quad & [c_i] = t^M(\overline{[c]}) = s^M(\overline{[c]}) = [c_j] && \text{by the truth definition;} \\
\Leftrightarrow\quad & \Phi^* \vdash \mathsf{v}_i = \mathsf{v}_j && \text{by the definition of } \sim; \\
\Leftrightarrow\quad & \Phi^* \vdash t(\bar{\mathsf{v}}) = s(\bar{\mathsf{v}}) && \text{by the equality rules,}
\end{aligned}$$

because $\Phi^* \vdash \mathsf{v}_i = t(\bar{\mathsf{v}})$ and $\Phi^* \vdash \mathsf{v}_j = s(\bar{\mathsf{v}})$ by (ii). The formula $t < s$ can be dealt with similarly.

Consider $\neg\varphi$, where φ is an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula. If $M \models \neg\varphi(\overline{[c]})$, then

$$\begin{aligned}
& M \not\models \varphi(\overline{[c]}) && \text{by the truth definition;} \\
\therefore\quad & \Phi^* \not\vdash \varphi(\bar{\mathsf{v}}) && \text{by the induction hypothesis;} \\
\therefore\quad & \Phi^* \vdash \neg\varphi(\bar{\mathsf{v}}) && \text{as } \Phi^* \text{ decides all } \mathcal{L}_{\mathrm{A}}(\exp) \text{ formulas.}
\end{aligned}$$

Conversely, if $M \not\models \neg\varphi(\overline{[c]})$, then

$$\begin{aligned}
& M \models \varphi(\overline{[c]}) && \text{by the truth definition;} \\
\therefore\quad & \Phi^* \vdash \varphi(\bar{\mathsf{v}}) && \text{by the induction hypothesis;} \\
\therefore\quad & \Phi^* \not\vdash \neg\varphi(\bar{\mathsf{v}}) && \text{by } (\bot), \text{ as } \Phi^* \not\vdash \bot.
\end{aligned}$$

Consider $\varphi \vee \psi$, where φ, ψ are $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas. If $M \models (\varphi \vee \psi)(\overline{[c]})$, then

$$
\begin{array}{lcccl}
 & M \models \varphi(\overline{[c]}) & \text{or} & M \models \psi(\overline{[c]}) & \text{by the truth definition;} \\
\therefore & \Phi^* \vdash \varphi(\bar{\mathsf{v}}) & \text{or} & \Phi^* \vdash \psi(\bar{\mathsf{v}}) & \text{by the induction hypothesis;} \\
\therefore & & \Phi^* \vdash (\varphi \vee \psi)(\bar{\mathsf{v}}) & & \text{by } (\vee).
\end{array}
$$

Conversely, if $M \not\models (\varphi \vee \psi)(\overline{[c]})$, then

$$
\begin{array}{lcccl}
 & M \not\models \varphi(\overline{[c]}) & \text{and} & M \not\models \psi(\overline{[c]}) & \text{by the truth definition;} \\
\therefore & \Phi^* \nvdash \varphi(\bar{\mathsf{v}}) & \text{and} & \Phi^* \nvdash \psi(\bar{\mathsf{v}}) & \text{by the induction hypothesis;} \\
\therefore & \Phi^* \vdash \neg\varphi(\bar{\mathsf{v}}) & \text{and} & \Phi^* \vdash \neg\psi(\bar{\mathsf{v}}) & \text{as } \Phi^* \text{ decides all } \mathcal{L}_{\mathrm{A}}(\exp) \text{ formulas;} \\
\therefore & & \Phi^* \vdash \neg(\varphi \vee \psi)(\bar{\mathsf{v}}) & & \text{by } (\neg\vee); \\
\therefore & & \Phi^* \nvdash (\varphi \vee \psi)(\bar{\mathsf{v}}) & & \text{by } (\bot), \text{ as } \Phi^* \nvdash \bot.
\end{array}
$$

Consider $\exists y\ \varphi$, where $\varphi(\bar{x}, y)$ is an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula. Suppose $M \models (\exists y\ \varphi)(\overline{[c]})$. Use the truth definition to find $[c_k] \in M \models \varphi(\overline{[c]}, [c_k])$. Then $\Phi^* \vdash \varphi(\bar{\mathsf{v}}, \mathsf{v}_k)$ by the induction hypothesis. Hence $(\exists\mathrm{R})$ implies $\Phi^* \vdash (\exists y\ \varphi)(\bar{\mathsf{v}})$.

Conversely, suppose $\Phi^* \vdash (\exists y\ \varphi)(\bar{\mathsf{v}})$. Since Φ^* is Henkinized, we get a variable v_k such that $\Phi^* \vdash \varphi(\bar{\mathsf{v}}, \mathsf{v}_k)$. Then $M \models \varphi(\overline{[c]}, c_k)$ by the induction hypothesis. Hence the truth definition implies $M \models (\exists y\ \varphi)(\overline{[c]})$. □

Assignment 7.1. Fill in part (ii) of the proof of the Model Construction Theorem by showing the following statement:

> if M is the structure defined there, and $t(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_\ell), s(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_\ell)$ are $\mathcal{L}_{\mathrm{A}}(\exp)$ terms such that for all $i, j \in \mathbb{N}$,
>
> $$[c_i] = t^M([c_0], [c_1], \ldots, [c_\ell]) \quad \Leftrightarrow \quad \Phi^* \vdash \mathsf{v}_i = t(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_\ell)$$
>
> and
>
> $$[c_j] = s^M([c_0], [c_1], \ldots, [c_\ell]) \quad \Leftrightarrow \quad \Phi^* \vdash \mathsf{v}_j = s(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_\ell),$$
>
> then for all $k \in \mathbb{N}$,
>
> $$[c_k] = (t+s)^M([c_0], [c_1], \ldots, [c_\ell]) \quad \Leftrightarrow \quad \Phi^* \vdash \mathsf{v}_k = (t+s)(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_\ell).$$

[6 points]

As we have shown in this lecture, every consistent theory is true in some structure. In the next lecture, we will see how this helps in establishing the Completeness Theorem and eventually the First Incompleteness Theorem.