MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Lecture 8: The First Incompleteness Theorem

Tin Lok Wong

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The aim of this lecture is to prove the First Incompleteness Theorem.

We saw in the previous lecture from the Henkinization Lemma and the Model Construction Theorem that every consistent theory has a model. These enable us to finish the proof of Gödel's Completeness Theorem. Our proof is essentially that of Henkin's.

Completeness Theorem (Gödel)**.** Let Φ be a set of $\mathcal{L}_\mathrm{A}(\exp)$ formulas and θ be an $\mathcal{L}_\mathrm{A}(\exp)$ formula. If $\Phi \models \theta$, then $\Phi \vdash \theta$.

Proof. Suppose $\Phi \nvdash \theta$. Then $\Phi + \neg\theta \nvdash \bot$ by (RAA). Apply the change of variables $\mathsf{v}_i \mapsto \mathsf{v}_{2i}$ if needed, we may assume that infinitely many variables do not appear free in $\Phi + \neg\theta$. Use the Henkinization Lemma to extend $\Phi + \neg\theta$ to a set of $\mathcal{L}_\mathrm{A}(\exp)$ formulas Φ^* which is consistent, Henkinized, and which decides all $\mathcal{L}_\mathrm{A}(\exp)$ formulas. Then the Model Construction Theorem gives an $\mathcal{L}_\mathrm{A}(\exp)$ structure M with elements $a_0, a_1, \ldots$ such that $M \models \varphi(a_0, a_1, \ldots, a_\ell)$ for all $\varphi(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_\ell) \in \Phi^*$. In particular, we know $M \models \varphi(a_0, a_1, \ldots, a_\ell)$ for all $\varphi(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_\ell) \in \Phi$, and $M \models \neg\theta(\bar{a})$. Hence $\Phi \not\models \theta$. □

It follows from Soundness and Completeness that semantic and syntactic entailments coincide. This is a fundamental theorem in mathematical logic which has countlessly many applications. Here we give one example.

Example 8.1. Let $m, q \in \mathbb{N}$. To show

$$\mathrm{R}(\exp) \vdash \forall x\ \big(\underline{m} \not< x \vee \underline{q} \not< x \rightarrow x \leqslant \underline{\max\{m,q\}}\big),$$

it suffices to establish semantic entailment by the Completeness Theorem. Let $a \in M \models \mathrm{R}(\exp)$ such that $M \models m \not< a \vee q \not< a$. Then

$$\begin{aligned}
& M \models a < m \vee a = m \vee a < q \vee a = q && \text{by (RComp);}\\
\therefore\quad & M \models a = 0 \vee a = 1 \vee \cdots \vee a = m-1 \vee a = m &&\\
& \qquad \vee a = 0 \vee a = 1 \vee \cdots \vee a = q-1 \vee a = q && \text{by (RInit);}\\
\therefore\quad & M \models a \leqslant \max\{m,q\} && \text{by (R<).}
\end{aligned}$$

Recall that, by the finitary nature of proofs, we can restrict our attention to proofs in which only finitely many formulas appear. Such proofs can be arithmetized in a way similar to how we arithmetized formulas in Lecture 5.

Definition. We rewrite each proof

$$\pi \quad = \quad \{\varphi_0^1, \varphi_0^2, \ldots, \varphi_0^{k_0}\} \vdash \theta_0, \quad \{\varphi_1^1, \varphi_1^2, \ldots, \varphi_1^{k_1}\} \vdash \theta_1, \quad \ldots, \quad \{\varphi_\ell^1, \varphi_\ell^2, \ldots, \varphi_\ell^{k_\ell}\} \vdash \theta_\ell$$

as

$$\varphi_0^1, \varphi_0^2, \ldots, \varphi_0^{k_0} \vdash \theta_0, , \varphi_1^1, \varphi_1^2, \ldots, \varphi_1^{k_1} \vdash \theta_1, \ldots, , \varphi_\ell^1, \varphi_\ell^2, \ldots, \varphi_\ell^{k_\ell} \vdash \theta_\ell$$

and view it as a natural number $\ulcorner\pi\urcorner$ written in hexadecimal representation via Table 8.1. We call $\ulcorner\pi\urcorner$ the *Gödel number* of π.

Symbol	,	(	)	v	$\top$	=	$\neg$	$\vee$	$\exists$	0	1	+	$\times$	exp	<	$\vdash$
Digit	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Table 8.1: An extension of Table 5.1 for the coding of proofs

According to the definition above, one proof may have more than one Gödel numbers because, while permuting the elements of a set $\{\varphi_j^1, \varphi_j^2, \ldots, \varphi_j^{k_j}\}$ appearing in a proof does not change the proof, it changes the Gödel number. This will not cause any problem for our purposes.

Given a string of symbols, one can algorithmically check whether or not it is proof, and which assumptions it invokes. Therefore, if the set of assumptions we are allowed to make is algorithmically checkable, then we have an algorithm to decide whether or not a proof is valid.

Definition. A set of $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas Φ is *recursive* if $\{\ulcorner\varphi\urcorner : \varphi \in \Phi\}$ is recursive.

Recursive nature of proofs. If Φ is a recursive set of $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas, then

$$\{(\ulcorner\pi\urcorner, \ulcorner\theta\urcorner) : \theta \text{ is an } \mathcal{L}_{\mathrm{A}}(\exp) \text{ formula and } \pi \text{ is a proof of } \Phi \vdash \theta\}$$

is recursive.

Proof. First, one can check algorithmically whether or not a natural number is the Gödel number of a proof because the deduction rules can be checked algorithmically. Second, one can verify algorithmically whether or not the final sequent $\Phi_\ell \vdash \theta_\ell$ in a proof satisfies $\Phi_\ell \subseteq \Phi$ because Φ is recursive. Hence the set in question is recursive by the Church–Turing Thesis. □

Although the correctness of proofs is algorithmically checkable, to come up with a proof is often a more difficult task.

Assignment 8.2. Use the Church–Turing Thesis and the recursive nature of proofs to show that if Φ is a recursive set of $\mathcal{L}_{\mathrm{A}}(\exp)$ formulas, then

$$\{\ulcorner\theta\urcorner : \theta \text{ is an } \mathcal{L}_{\mathrm{A}}(\exp) \text{ formula and } \Phi \vdash \theta\}$$

is r.e. [5 points]

Putting everything together, we obtain the First Incompleteness Theorem. We are going to prove a version stronger than the one described in Lecture 5. We will cook up a new notion of provability for this purpose; see Figure 8.1. The witness comparison trick involved is reminiscent of our proof of the Σ_1 representability of recursive functions in R(exp) in Lecture 4.

First Incompleteness Theorem (Rosser)**.** Let T be a recursive consistent $\mathcal{L}_{\mathrm{A}}(\exp)$ theory. If $T \vdash \mathrm{R}(\exp)$, then one can find a Π_1 sentence σ such that $T \nvdash \sigma$ and $T \nvdash \neg\sigma$.

Proof. Use the recursive nature of proofs and the Church–Turing Thesis to find $\Delta_0(\exp)$ formulas $\alpha(x, y, z_1, z_2, \ldots, z_k), \alpha'(x, y, z'_1, z'_2, \ldots, z'_\ell)$ such that for all $m, n \in \mathbb{N}$,

(1) $\mathbb{N} \models \exists \bar{z}\ \alpha(m, n, \bar{z})$ if and only if n is the Gödel number of an $\mathcal{L}_{\mathrm{A}}(\exp)$ sentence σ and m is the Gödel number of a proof of $T \vdash \sigma$; and

(2) $\mathbb{N} \models \exists \bar{z}'\ \alpha'(m, n, \bar{z}')$ if and only if n is the Gödel number of an $\mathcal{L}_{\mathrm{A}}(\exp)$ sentence σ and m is the Gödel number of a proof of $T \vdash \neg\sigma$.

Let $\varphi(y)$ be a Π_1 formula equivalent to $\neg\exists x, \bar{z}\ \beta(x, y, \bar{z})$ given by Lemma 5.2, where $\beta(x, y, \bar{z})$ is

$$\alpha(x, y, \bar{z}) \wedge \forall x', \bar{z}' {<} x\ \neg\alpha'(x', y, \bar{z}') \wedge \bigwedge_{i=1}^{k} \forall x', \bar{z}' {<} z_i\ \neg\alpha'(x', y, \bar{z}').$$

Apply the Diagonal Lemma to find a Π_1 sentence σ such that

$$\mathrm{R}(\exp) \vdash \sigma \leftrightarrow \varphi(\underline{\sigma}).$$

Suppose $T \vdash \sigma$. Let m be the Gödel number of a proof of $T \vdash \sigma$. Use (1) to find $\bar{q} \in M \models \alpha(m, \ulcorner\sigma\urcorner, \bar{q})$.

$$\mathrm{R(exp)} \vdash \sigma \leftrightarrow \neg\exists \text{proof } x \text{ of } T \vdash \sigma \text{ such that } \neg\exists \text{proof } x' < x \text{ of } T \vdash \neg\sigma. \qquad (*)$$

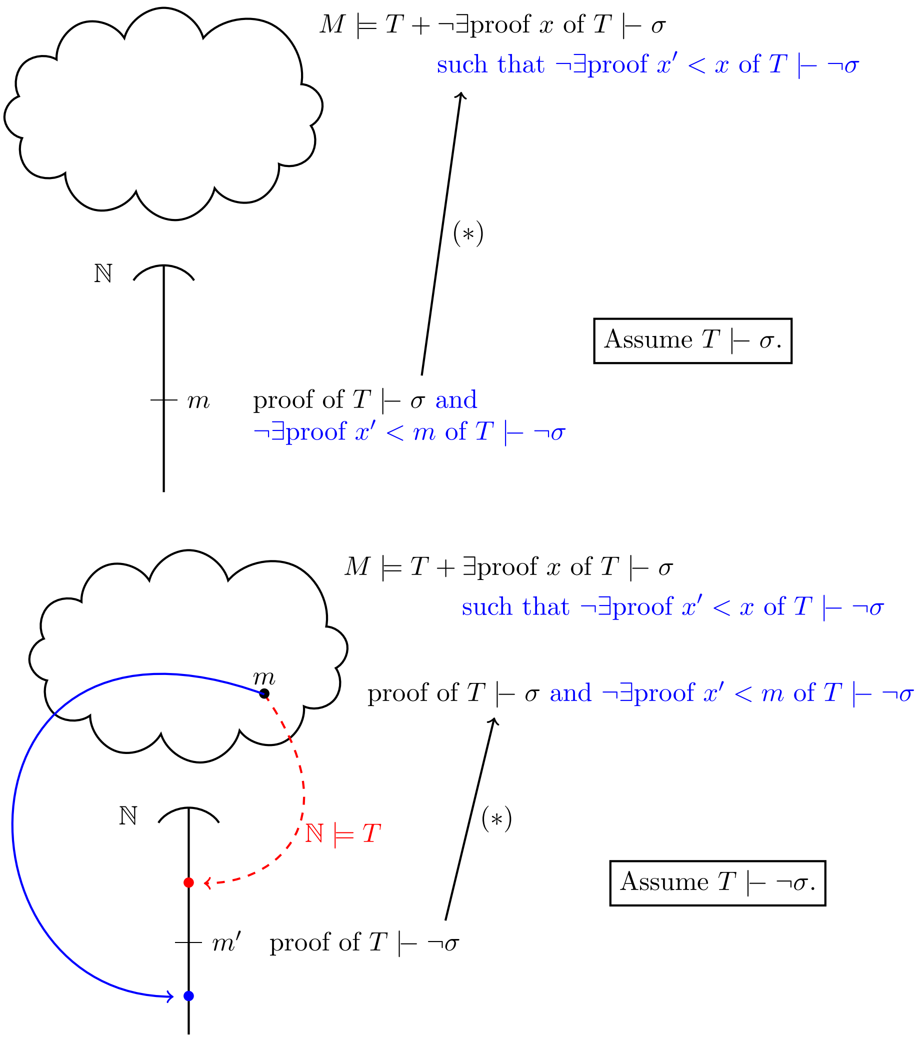

Figure 8.1: Adjusting Gödel's argument for Rosser's theorem

Case 1: Suppose we have $m', \bar{q}' < \max\{m, \bar{q}\}$ such that $\mathbb{N} \models \alpha'(m', \ulcorner\sigma\urcorner, \bar{q}')$. Then (2) says m' is the Gödel number of a proof of $T \vdash \neg\sigma$. Hence $T \vdash \bot$ by $(\bot)$.

Case 2: Suppose $\mathbb{N} \models \forall x', \bar{z}' {<} \max\{m, \bar{q}\}\ \neg\alpha'(x', \ulcorner\sigma\urcorner, \bar{z}')$. Then

$$\begin{array}{lll} & \mathbb{N} \models \beta(m, \ulcorner\sigma\urcorner, \bar{q}) & \text{by the definition of } \beta; \\ \therefore & \mathbb{N} \models \exists x, \bar{z}\ \beta(x, \ulcorner\sigma\urcorner, \bar{z}) & \text{by the truth definition;} \\ \therefore & \mathrm{R(exp)} \vdash \exists x, \bar{z}\ \beta(x, \underline{\sigma}, \bar{z}) & \text{by the } \Sigma_1 \text{ completeness of R(exp);} \\ \therefore & \mathrm{R(exp)} \vdash \neg\varphi(\underline{\sigma}) & \text{by the choice of } \varphi; \\ \therefore & \mathrm{R(exp)} \vdash \neg\sigma & \text{by the choice of } \sigma; \\ \therefore & T \vdash \neg\sigma & \text{as } T \vdash \mathrm{R(exp)}; \\ \therefore & T \vdash \bot & \text{by } (\bot). \end{array}$$

Suppose $T \vdash \neg\sigma$. On the one hand, as $T \vdash \mathrm{R(exp)}$, this implies $T \vdash \neg\varphi(\underline{\sigma})$ by the choice of σ. Hence the choice of φ tells us

$$T \vdash \exists x, \bar{z}\ \beta(x, \underline{\sigma}, \bar{z}). \tag{$\dagger$}$$

On the other hand, let m' be the Gödel number of a proof of $T \vdash \neg\sigma$. Use (2) to find $q'_1, q'_2, \dots, q'_\ell \in \mathbb{N} \models \alpha'(m', \ulcorner\sigma\urcorner, \bar{q}')$ Then $\mathrm{R(exp)} \vdash \alpha'(\underline{m'}, \underline{\sigma}, \underline{\bar{q}'})$ by the Σ_1 completeness of R(exp). So, as $T \vdash \mathrm{R(exp)}$, we derive from ($\dagger$) that $T \vdash \exists x, \bar{z}\ \beta(x, \underline{\sigma}, \bar{z}) \wedge \alpha'(\underline{m'}, \underline{\sigma}, \underline{\bar{q}'})$, i.e.,

$$T \vdash \exists x, \bar{z} \left(\begin{array}{l} \alpha(x, \underline{\sigma}, \bar{z}) \wedge \forall x', \bar{z}' {<} x\ \neg\alpha'(x', \underline{\sigma}, \bar{z}') \\ \wedge \bigwedge_{i=1}^{k} \forall x', \bar{z}' {<} z_i\ \neg\alpha'(x', \underline{\sigma}, \bar{z}') \wedge \alpha'(\underline{m'}, \underline{\sigma}, \underline{\bar{q}'}) \end{array} \right).$$

As one can readily see, the x and the z_i's between the big brackets above must satisfy

$$(\underline{m'} \not< x \vee \underline{q'_1} \not< x \vee \underline{q'_2} \not< x \vee \cdots \vee \underline{q'_\ell} \not< x) \wedge \bigwedge_{i=1}^{k} (\underline{m'} \not< z_i \vee \underline{q'_1} \not< z_i \vee \underline{q'_2} \not< z_i \vee \cdots \vee \underline{q'_\ell} \not< z_i).$$

Hence $T \vdash \exists x, \bar{z} {\leqslant} \underline{\max\{m', \bar{q}'\}}\ \alpha(x, \underline{\sigma}, \bar{z})$ by Example 8.1. If T proves the negation of the same sentence, then already $T \vdash \bot$ by $(\bot)$. So suppose not. Then, as $T \vdash \mathrm{R(exp)}$,

$$\begin{array}{lll} & \mathbb{N} \not\models \neg\exists x, \bar{z} {\leqslant} \underline{\max\{m', \bar{q}'\}}\ \alpha(x, \ulcorner\sigma\urcorner, \bar{z}) & \text{by the } \Sigma_1 \text{ completeness of R(exp);} \\ \therefore & \mathbb{N} \models \exists x, \bar{z} {\leqslant} \underline{\max\{m', \bar{q}'\}}\ \alpha(x, \ulcorner\sigma\urcorner, \bar{z}) & \text{by the truth definition;} \\ \therefore & \mathbb{N} \models \exists x, \bar{z}\ \alpha(x, \ulcorner\sigma\urcorner, \bar{z}) & \text{trivially.} \end{array}$$

Use the truth definition to find $m \in \mathbb{N} \models \exists \bar{z}\ \alpha(m, \ulcorner\sigma\urcorner, \bar{z})$. Then (1) tells us that m is the Gödel number of a proof of $T \vdash \sigma$. Hence $T \vdash \bot$ by $(\bot)$. □

The First Incompleteness Theorem asserts the so-called *essential incompleteness* of R(exp), i.e., not only R(exp) fails to decide all sentences, but no matter how one extends R(exp), as long as the extension is consistent and recursive, some sentence will remain undecided in this extension. Moreover, this incompleteness phenomenon occurs already at the Π_1 level, where the sentences are relatively simple.