MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Examination

Tin Lok Wong

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Answers different from those listed here can also be correct.

Question 1. Which of the following sentences are true? Which of them are false? Briefly justify your assertions. In a negative case, it suffices to provide a counterexample.

(a) Fix a recursive Gödel numbering of a language $\mathcal{L}$. Let K, M be $\mathcal{L}$ structures. If K is an $\mathcal{L}$ substructure of M, and $\mathrm{Th}(K)$ is not recursive, then $\mathrm{Th}(M)$ is not recursive.

(b) Fix a recursive Gödel numbering of a language $\mathcal{L}$. Let M be an $\mathcal{L}$ structure. Let K be the *reduct* of M to a language $\mathcal{L}_0$, i.e.,

- K is an $\mathcal{L}_0$ structure with the same universe as M;
- all symbols in $\mathcal{L}_0$ are in $\mathcal{L}$; and
- the interpretations of the symbols in $\mathcal{L}_0$ are the same in K and in M.

If $\mathrm{Th}(K)$ is not recursive, then $\mathrm{Th}(M)$ is not recursive.

(c) Let $\theta(x)$ be an $\mathcal{L}_\mathrm{A}(\exp)$ formula. If $\mathrm{PA} \vdash \theta(\underline{n})$ for all $n \in \mathbb{N}$, then $\mathrm{PA} \vdash \forall x\ \theta(x)$.

(d) PA proves neither $\mathrm{Con}(\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA}))$ nor $\neg\mathrm{Con}(\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA}))$.

(e) $\mathrm{PA} \vdash \Box_{\mathrm{PA}}(\underline{\sigma \vee \tau}) \rightarrow \big(\Box_{\mathrm{PA}}\, \underline{\sigma} \vee \Box_{\mathrm{PA}}\, \underline{\tau}\big)$ for all $\mathcal{L}_\mathrm{A}(\exp)$ sentences σ, τ.

My solution. (a) False: consider $K = (\mathbb{N}, 0, 1, +, \times, <)$ and $M = (\mathbb{R}, 0, 1, +, \times, <)$.

(b) True: from $\mathrm{Th}(M)$ one can compute $\mathrm{Th}(K)$.

(c) False: let $\theta(x)$ express 'x is not the Gödel number of a proof of $\mathrm{PA} \vdash \bot$'.

(d) True:

(1) $\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA}) \nvdash \bot$ by the Second Incompleteness Theorem applied to PA;
(2) $\mathrm{PA}+\neg\mathrm{Con}(\mathrm{PA}) \nvdash \mathrm{Con}(\mathrm{PA}+\neg\mathrm{Con}(\mathrm{PA}))$ by (1) and the Second Incompleteness Theorem applied to $\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA})$;
(3) $\mathrm{PA} \nvdash \mathrm{Con}(\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA}))$ by (2);
(4) $\mathrm{PA} \nvdash \neg\mathrm{Con}(\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA}))$ as $\mathbb{N} \models \mathrm{PA} + \mathrm{Con}(\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA}))$ by (1).

(e) False: let $\sigma = \mathrm{Con}(\mathrm{PA})$ and $\tau = \neg\mathrm{Con}(\mathrm{PA})$. □

Question 2. Fix a recursive consistent $\mathcal{L}_\mathrm{A}(\exp)$ theory $T \supseteq \mathrm{R}(\exp)$. Let $\alpha(x, y, z), \alpha'(x, y, z)$ be $\Delta_0(\exp)$ formulas such that for all $m, n \in \mathbb{N}$,

(1) $\mathbb{N} \models \exists z\ \alpha(m, n, z)$ if and only if n is the Gödel number of an $\mathcal{L}_\mathrm{A}(\exp)$ sentence σ and m is the Gödel number of a proof of $T \vdash \sigma$; and

(2) $\mathbb{N} \models \exists z'\ \alpha'(m', n, z')$ if and only if n is the Gödel number of an $\mathcal{L}_\mathrm{A}(\exp)$ sentence σ and m' is the Gödel number of a proof of $T \vdash \neg\sigma$.

Let $\nabla(y)$ be the Σ_1 formula $\exists x, z\ \beta(x,y,z)$, where $\beta(x,y,z)$ is

$$\alpha(x,y,z) \wedge \forall x', z'{<}x\ \neg\alpha'(x',y,z') \wedge \forall x', z'{<}z\ \neg\alpha'(x',y,z').$$

Take any $\mathcal{L}_\mathrm{A}(\exp)$ sentence σ.

(a) Show that if $T \vdash \sigma$, then $T \vdash \nabla(\underline{\sigma})$.

(b) Suppose $T \vdash \neg\sigma$. Fill in the blanks in the following proof of $T \vdash \neg\nabla(\underline{\sigma})$.

> Let m' be the Gödel number of a proof of $T \vdash \neg\sigma$. Apply (2) to find $q' \in \mathbb{N} \models \alpha'(m', \ulcorner\sigma\urcorner, q')$. Take $m, q \in M \models T$. Suppose
>
> $$M \models \forall x', z'{<}m\ \neg\alpha'(x', \ulcorner\sigma\urcorner, z') \wedge \underline{\qquad\text{(i)}\qquad}.$$
>
> Note that $M \models \alpha'(m', \ulcorner\sigma\urcorner, q')$ by $\underline{\qquad\qquad\text{(ii)}\qquad\qquad}$. So
>
> $$(m' \not< m \text{ or } q' \not< m) \quad\text{and}\quad \underline{\qquad\text{(iii)}\qquad}.$$
>
> Thus $m, q \in \underline{\text{(iv)}}$. Now, since $\underline{\ \text{(v)}\ }$, we know $T \nvdash \sigma$. So $\underline{\qquad\text{(vi)}\qquad}$ by $\underline{\text{(vii)}}$. This implies $\mathbb{N} \models \neg\alpha(m, \ulcorner\sigma\urcorner, q)$, and hence $\underline{\qquad\text{(viii)}\qquad}$ by the $\Delta_0(\exp)$ absoluteness between M and $\mathbb{N}$.

My solution. (a) Let m be the Gödel number of a proof of $T \vdash \sigma$. On the one hand, apply (1) to find $q \in \mathbb{N} \models \alpha(m, \ulcorner\sigma\urcorner, q)$. On the other hand, since $T \nvdash \bot$, we know $T \nvdash \neg\sigma$. So (2) implies $\mathbb{N} \models \forall x', z'\ \neg\alpha'(x', \ulcorner\sigma\urcorner, z')$. Combining the two, we see that $\mathbb{N} \models \beta(m, \ulcorner\sigma\urcorner, q)$. This implies $\mathbb{N} \models \nabla(\underline{\sigma})$ and thus $T \vdash \nabla(\underline{\sigma})$ by the Σ_1 completeness of $\mathrm{R}(\exp)$.

(b) (i) $\forall x', z'{<}q\ \neg\alpha'(x', \ulcorner\sigma\urcorner, z')$

(ii) the $\Delta_0(\exp)$ absoluteness between M and $\mathbb{N}$

(iii) $(m' \not< q$ or $q' \not< q)$

(iv) $\mathbb{N}$

(v) $T \nvdash \bot$

(vi) $\mathbb{N} \models \forall x, z\ \neg\alpha(x, \ulcorner\sigma\urcorner, z)$

(vii) (1)

(viii) $M \models \neg\alpha(m, \ulcorner\sigma\urcorner, q)$ □

Question 3. Let φ, ψ be $\mathcal{L}_\mathrm{A}(\exp)$ sentences satisfying

$$\mathrm{R}(\exp) \vdash \varphi \to \psi \quad\text{and}\quad \mathrm{R}(\exp) \nvdash \psi \to \varphi. \qquad (*)$$

Find an $\mathcal{L}_\mathrm{A}(\exp)$ sentence θ such that

(i) $\mathrm{R}(\exp) \vdash \varphi \to \theta$ and $\mathrm{R}(\exp) \vdash \theta \to \psi$;

(ii) $\mathrm{R}(\exp) \nvdash \theta \to \varphi$ and $\mathrm{R}(\exp) \nvdash \psi \to \theta$.

Justify your answer. You may use without proof the following facts.

(1) Suppose $(*)$ holds. Then an $\mathcal{L}_\mathrm{A}(\exp)$ sentence θ satisfies (i) if and only if it is equivalent over $\mathrm{R}(\exp)$ to

$$\psi \wedge (\varphi \vee \sigma)$$

for some $\mathcal{L}_\mathrm{A}(\exp)$ sentence σ.

(2) For all $\mathcal{L}_{\mathrm{A}}(\exp)$ sentences χ, ξ, ζ and all $\mathcal{L}_{\mathrm{A}}(\exp)$ theories T,

(A)	$T \vdash \xi \to \zeta$	if and only if	$T \vdash \xi \to \xi \wedge \zeta$;
(B)	$T \vdash \xi \to \zeta$	if and only if	$T \vdash \zeta \vee \xi \to \zeta$;
(C)	$T \vdash \xi \to \zeta$	if and only if	$T + \xi + \neg\zeta \vdash \bot$;
(D)	$T \vdash \xi \to \zeta$	if and only if	$T + \xi \vdash \zeta$;
(E)	$T \vdash \xi \to \zeta$	if and only if	$T + \neg\zeta \vdash \neg\xi$;
(F)	$T \vdash \chi \wedge \xi \to \zeta$	if and only if	$T + \chi \vdash \xi \to \zeta$;
(G)	$T \vdash \xi \vee \zeta$	if and only if	$T + \neg\xi \vdash \zeta$.

Hint: apply (2)(C) to (∗), then use the First Incompleteness Theorem.

My solution. As $\mathrm{R}(\exp) \nvdash \psi \to \varphi$, we know $\mathrm{R}(\exp) + \psi + \neg\varphi \nvdash \bot$ by (2)(C). Use the First Incompleteness Theorem to find an $\mathcal{L}_{\mathrm{A}}(\exp)$ sentence σ such that

$$\mathrm{R}(\exp) + \psi + \neg\varphi \nvdash \sigma \quad \text{and} \quad \mathrm{R}(\exp) + \psi + \neg\varphi \nvdash \neg\sigma.$$

Let $\theta = \psi \wedge (\varphi \vee \sigma)$. Then (i) holds by (1). For (ii), note

	$\mathrm{R}(\exp) + \psi + \neg\varphi \nvdash \neg\sigma$	by the choice of σ;
$\therefore$	$\mathrm{R}(\exp) + \psi \nvdash \sigma \to \varphi$	by (2)(E);
$\therefore$	$\mathrm{R}(\exp) + \psi \nvdash \varphi \vee \sigma \to \varphi$	by (2)(B);
$\therefore$	$\mathrm{R}(\exp) \nvdash \psi \wedge (\varphi \vee \sigma) \to \varphi$	by (2)(F);
$\therefore$	$\mathrm{R}(\exp) \nvdash \theta \to \varphi$	by the definition of θ.

Similarly,

	$\mathrm{R}(\exp) + \psi + \neg\varphi \nvdash \sigma$	by the choice of σ;
$\therefore$	$\mathrm{R}(\exp) + \psi \nvdash \varphi \vee \sigma$	by (2)(G);
$\therefore$	$\mathrm{R}(\exp) \nvdash \psi \to (\varphi \vee \sigma)$	by (2)(D);
$\therefore$	$\mathrm{R}(\exp) \nvdash \psi \to \psi \wedge (\varphi \vee \sigma)$	by (2)(A);
$\therefore$	$\mathrm{R}(\exp) \nvdash \psi \to \theta$	by the definition of θ. □

Question 4. Let T be an $\mathcal{L}_{\mathrm{A}}(\exp)$ theory extending $\mathrm{R}(\exp)$ and $\Box(y)$ be an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula satisfying the following derivability conditions for all $\mathcal{L}_{\mathrm{A}}(\exp)$ sentences σ, τ:

(N) if $T \vdash \sigma$, then $T \vdash \Box\underline{\sigma}$;

(IN) $T \vdash \Box\underline{\sigma} \to \Box\underline{\Box\underline{\sigma}}$;

(□D) $T \vdash \Box\underline{(\sigma \to \tau)} \to (\Box\underline{\sigma} \to \Box\underline{\tau})$.

(a) Explain why there is an $\mathcal{L}_{\mathrm{A}}(\exp)$ sentence σ such that $T \vdash \sigma \leftrightarrow \Box(\underline{\neg\sigma})$. If you invoke the Diagonal Lemma, then please specify to which formula it is applied. You may use without proof that some recursive function $F\colon \mathbb{N} \to \mathbb{N}$ maps $\ulcorner\sigma\urcorner \mapsto \ulcorner\neg\sigma\urcorner$ for every $\mathcal{L}_{\mathrm{A}}(\exp)$ sentence σ.

(b) Let σ be an $\mathcal{L}_{\mathrm{A}}(\exp)$ sentence satisfying $T \vdash \sigma \leftrightarrow \Box(\underline{\neg\sigma})$. Show that $T \vdash \bot$ if and only if $T \vdash \neg\sigma$.

My solution. (a) Let $\rho(x, y)$ be an $\mathcal{L}_{\mathrm{A}}(\exp)$ formula which represents the recursive function F over $\mathrm{R}(\exp)$, i.e., for every $m \in \mathbb{N}$,

$$\mathrm{R}(\exp) \vdash \rho(\underline{m}, \underline{F(m)}) \wedge \forall y\ \big(\rho(\underline{m}, y) \to y = \underline{F(m)}\big).$$

Apply the Diagonal Lemma to the formula $\exists y\ \big(\rho(x, y) \wedge \Box(y)\big)$ to find an $\mathcal{L}_{\mathrm{A}}(\exp)$ sentence σ such that

$$\mathrm{R}(\exp) \vdash \sigma \leftrightarrow \exists y\ \big(\rho(\underline{\sigma}, y) \wedge \Box(y)\big).$$

Then $\mathrm{R}(\exp) \vdash \sigma \leftrightarrow \Box(\underline{\neg\sigma})$ because $\rho(x, y)$ represents F over $\mathrm{R}(\exp)$, and $F(\ulcorner\sigma\urcorner) = \ulcorner\neg\sigma\urcorner$.

(b) If $T \vdash \bot$, then $T \vdash \neg\sigma$ by (RAA). Conversely, suppose $T \vdash \neg\sigma$. On the one hand, this implies $T \vdash \Box(\underline{\neg\sigma})$ by (N). On the other hand, this implies $T \vdash \neg\Box(\underline{\neg\sigma})$ by the choice of σ. Hence, we deduce via ($\bot$) that $T \vdash \bot$. $\square$

Question 5. Consider the structure $\mathbb{Z} = (\mathbb{Z}, 0, 1, +, -, <)$ in the language $\mathcal{L}_{\text{DOG}}$. Follow the steps below to show that if a subset of $\mathbb{Z}$ is defined by a quantifier-free $\mathcal{L}_{\text{DOG}}$ formula, then it is the union of finitely many (finite, semi-infinite, or infinite) intervals, i.e., it is of the form

$$\bigcup_{i \leqslant n} \{x \in \mathbb{Z} : a_i < x < b_i\}$$

where $n \in \mathbb{N}$, each $a_i \in \mathbb{Z} \cup \{-\infty\}$, and each $b_i \in \mathbb{Z} \cup \{+\infty\}$. The proof is by induction on the defining formula. Let $\theta(x)$ be a quantifier-free $\mathcal{L}_{\text{DOG}}$ formula and

$$S = \{x \in \mathbb{Z} : \mathbb{Z} \models \theta(x)\}.$$

(a) Suppose $\theta(x)$ is $\lambda x = \mu$, where $\lambda, \mu \in \mathbb{Z}$. Show that S is the union of finitely many intervals.

(b) Suppose $\theta(x)$ is $\lambda x > \mu$, where $\lambda, \mu \in \mathbb{Z}$. Show that S is the union of finitely many intervals.

(c) Suppose $\theta(x)$ is $\neg\eta(x)$, and

$$S' := \{x \in \mathbb{Z} : \mathbb{Z} \models \eta(x)\} = \bigcup_{i \leqslant n} \{x \in \mathbb{Z} : a_i < x < b_i\},$$

where $n \in \mathbb{N}$, each $a_i \in \mathbb{Z} \cup \{-\infty\}$, and each $b_i \in \mathbb{Z} \cup \{+\infty\}$. Show that S is the union of finitely many intervals.

(d) Suppose $\theta(x) = \eta_1(x) \vee \eta_2(x)$, where

$$S_1 := \{x \in \mathbb{Z} : \mathbb{Z} \models \eta_1(x)\} \quad \text{and} \quad S_2 := \{x \in \mathbb{Z} : \mathbb{Z} \models \eta_2(x)\}$$

are both unions of finitely many intervals. Explain why S is the union of finitely many intervals.

My solution. Note that the following are all intervals:

- $\varnothing = \{x \in \mathbb{Z} : 1 < x < 0\}$;
- $\mathbb{Z} = \{x \in \mathbb{Z} : -\infty < x < +\infty\}$;
- $\{\lambda\} = \{x \in \mathbb{Z} : \lambda - 1 < x < \lambda + 1\}$, whenever $\lambda \in \mathbb{Z}$.

(a) $S = \begin{cases} \varnothing, & \text{if } \lambda = 0 \neq \mu \text{ or if } \lambda \text{ does not divide } \mu; \\ \mathbb{Z}, & \text{if } \lambda = 0 = \mu; \\ \{\frac{\mu}{\lambda}\}, & \text{if } \lambda \neq 0 \text{ and } \lambda \text{ divides } \mu. \end{cases}$

(b) $S = \begin{cases} \varnothing, & \text{if } \lambda = 0 \leqslant \mu; \\ \mathbb{Z}, & \text{if } \lambda = 0 > \mu; \\ \{x \in \mathbb{Z} : \lfloor \frac{\mu}{\lambda} \rfloor < x < +\infty\}, & \text{if } \lambda > 0; \\ \{x \in \mathbb{Z} : -\infty < x < \lceil \frac{\mu}{\lambda} \rceil\}, & \text{if } \lambda < 0. \end{cases}$

(c) First, if $S' = \varnothing$, then $S = \mathbb{Z}$. Second, the union of two intersecting intervals is an interval: if $c, c' \in \mathbb{Z} \cup \{-\infty\}$ and $d, d' \in \mathbb{Z} \cup \{-\infty\}$ such that $c' < d$, then

$$\{x \in \mathbb{Z} : c < x < d\} \cup \{x \in \mathbb{Z} : c' < x < d'\} = \{x \in \mathbb{Z} : c < x < d'\},$$

where $+\infty \not< x \not< -\infty$ for any $x \in \mathbb{Z} \cup \{\pm\infty\}$. Therefore, without loss of generality, we may assume

$$a_0 < b_0 < a_1 < b_1 < \cdots < a_n < b_n.$$

Then $S = I \cup \{x \in \mathbb{Z} : b_0 \leqslant x \leqslant a_1\} \cup \{x \in \mathbb{Z} : b_1 \leqslant x \leqslant a_2\} \cup \cdots \cup \{x \in \mathbb{Z} : b_{n-1} \leqslant x \leqslant a_n\} \cup J$, where

$$I = \begin{cases} \{x \in \mathbb{Z} : -\infty < x \leqslant a_0\}, & \text{if } a_0 \in \mathbb{Z}; \\ \varnothing, & \text{otherwise,} \end{cases}$$

and

$$J = \begin{cases} \{x \in \mathbb{Z} : b_n \leqslant x < +\infty\}, & \text{if } b_n \in \mathbb{Z}; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Rewriting $u \leqslant x$ and $x \leqslant v$ as $u - 1 < x$ and $x < v + 1$ respectively gives the required result.

(d) Since both S_1 and S_2 are unions of finitely many intervals, so is $S = S_1 \cup S_2$. □