MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Bibliographic and organizational remarks

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8 January, 2019

> Either mathematics is incompletable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine,
>
> or else there exist absolutely unsolvable diophantine problems of the type specified.
>
> Gödel (1951), Some basic theorems on the foundations of mathematics and their implications [9, page 310]

While general attributions were given in the text, precise references were omitted because of assessment arrangements. So we provide the references separately here. In addition, we include a few organizational remarks which may help the reader navigate between the lectures.

Our model of computation from Lecture 1 is non-standard. It is designed to match with the formulas in the language of arithmetic as demonstrated in Lecture 2. This connection between computation and arithmetic is well known. All appeals to the Church–Turing Thesis in this module can be replaced by explicit constructions of programs as in Lecture 10.

Our presentation of the Incompleteness Theorems originates from that in the Hájek–Pudlák book [12, Section III.2]. Two additional references are Kaye [18, Chapter 2] and Smoryński [40, Sections 2–4]. We include a symbol for exponentiation in the language to avoid having to define it for the Second Incompleteness Theorem; it is not needed for the First Incompleteness Theorem. For the sake of consistency (in the usual sense of the word), we made $\mathcal{L}_{\mathrm{A}}(\exp)$ our object language throughout, except in Lectures 13–16. Our arithmetization is similar to that in Wilkie–Paris [47, Section 4]. For Lecture 11, probably one can find better examples to illustrate the formalization of syntactic manipulations in $\mathrm{I}\Delta_0(\exp)$.

The ugly proof system introduced in Lecture 6 has no advantage at all, except that it happens to be what one needs to carry out the proof of the Completeness Theorem presented in Lecture 7. It is probably more appropriate to make Remark 22.7 already in Lecture 7.

The citations in Lecture 13 are incorporated into the bibliography below. Presburger's article on the decidability of arithmetic with addition is translated in Stansifer [43]. For Skolem's decidability result, see Smoryński [42, Section III.5]. The decidability results for $\mathbb{R}$ and $\mathbb{C}$ appeared in Tarski [44], although the mathematics involved was probably discovered already in the 1930s.

Abraham Robinson's work on the preservation theorem and model completeness can be found in almost every model theory text; see, for example, Hodges [14, Lemma 3.2.9]. This work is, however, not actually needed to show the recursiveness of $\mathrm{Th}(\mathbb{N}, 0, 1, +, <)$, as the reader can see from our proof in Lecture 16. Our proof of the quantifier-elimination theorem for Preburger Arithmetic originates from that in Rautenberg's book [35, Section 5.6] and in Stansifer's translation [43] of the original Presburger paper. More information can be found in Smoryński [42, Section 3.4] and Makowsky [25]. Smorynski's article [41] contains his application of Skolem's quantifier-elimination algorithm to Sylvester's theorem.

Our presentation of Goodstein's Theorem at the end of Lecture 17 follows that in Rathjen [34, Section 2]. The proof of the Cut-Elimination Theorem for **LK** in Lecture 19 is taken from Buss [4,

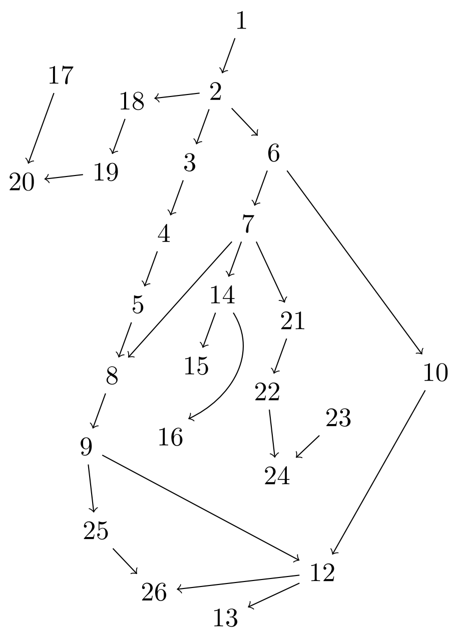

Figure 1: Major dependencies between lectures

Section 2.4]. Our proof of the consistency of **PA** in Lecture 20 is based on Rathjen [33], Hetzl [13], and Towsner [45]. The Cut-Elimination Theorem for **LK** is mathematically not needed for our consistency proof of **PA**. So it would be more economical to spend Lecture 19 on the analogous result for **Nat** instead.

The usual pairing function comes from Cantor [5, page 257]. Elementary end extensions of arbitrary models of PA were first constructed in Mac Dowell–Specker [24]; see Hodges [14, Sections 6.1 and 6.2] and Enayat–Mohsenipour [8, Theorem 1.2] for more references, including the contribution by Keisler. The connections between induction, regularity and elementary end extensions come from Paris–Kirby [29].

Indicators were introduced by Kirby–Paris [20]. Proposition 22.1 can be viewed as a variant of Theorem 4.3 in Parikh [27]. Theorem 22.8 can found in Kaye's book [18, Section 14.1]; see also Kirby's thesis [19, paragraph 5.4]. Note that the provability part of Theorem 22.8 is the only place where the Mac Dowell–Specker Theorem is used. So if one is only interested in unprovability, then one can omit Lecture 21 altogether.

Ramsey's Theorems have developed into a theory on their own in combinatorics [11]. The PA version of the Infinite Ramsey Theorem can be proved using the Mac Dowell–Specker Theorem; see Kossak–Schmerl [22, Theorem 2.2.16].

The equivalence between PH and the 1-consistency of PA was already observed in the original Paris–Harrington paper [28, Theorem 3.1]. The streamlined proofs presented in Lecture 24 come from Bovykin [2, Theorem 1]; cf. Kaye [18, Section 14.3]. The equivalence between Goodstein's theorem and the 1-consistency of PA can be found in the paper by Kirby and Paris [21, Theorem 4].

The Finite Incompleteness Theorem in Lecture 25 comes from Pudlák [31, Theorem 3.1]; see also Buss's article [3, Section 2.1]. In particular, a proof of the numeral version of the Diagonal Lemma with polynomial length bounds on page 124 can be found in Buss's article [3, proof of Theorem 4]. The connection with complexity theory was observed already in the Pudlák paper [31, Proposition 6.2]. For Hrubeš's theorem in Lecture 26, see Pudlák [32, Theorem 3.7].

In the examination, Questions 2 and 3 are presumably folklore; see Lindström [23, page 24] and Shavrukov–Visser [39, Section 2], for example. Question 4 was inspired by Theorem 1 in Jeroslow [17]. Question 5 originates from Makowsky's notes [25, page 25], in which it is used to demonstrate the existence of a Presburger formula that is not equivalent over Pres to any quantifier-free $\mathcal{L}_{\mathrm{DOG}}$ formula.

Towards the end of these lectures, a preprint by Chow [6] came to my attention. It does not only have the same (sub)title as my lectures, but also a large overlap in the choice of materials.

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Acknowledgements

I would like to thank Ziyuan Gao, Frank Stephan and Yue Yang for helpful discussions, support, and encouragement. I would also like to thank the students in my class for their participation, contribution, and feedback. Many of their questions and remarks are reflected in these notes.