MA5219 LOGIC AND FOUNDATION OF MATHEMATICS I

Outline of my solutions to the assignments

Tin Lok Wong

Last edited: 13 November, 2018

Assignment 1.6. Let $\theta(w)$ be $w > 0 \wedge \forall y{\leqslant}w\ \big(\text{divides}(y,w) \wedge \text{prime}(y) \rightarrow y = 1+1\big)$.

Assignment 2.7. Let $k = 0$, so that both $\bar{x}$ and $\bar{a}$ are empty. Let θ be the Π_1 formula $\forall y\ (y = y)$. Then $\mathbb{N} \models \forall y\ (y = y)$ by the truth definition because $a = a$ for every $a \in \mathbb{N}$. The definition of $[\![\ldots]\!]$ tells us $[\![\forall y\ (y = y)]\!] = [\![\neg\exists y\ \neg(y = y)]\!] = \texttt{true}$ if and only if $[\![\exists y\ \neg(y = y)]\!] = \texttt{false}$. However, it also tells us that $[\![\exists y\ \neg(y = y)]\!] \neq \texttt{false}$. Thus $[\![\forall y\ (y = y)]\!] \neq \texttt{true}$.

Assignment 3.7. Let σ be the Π_1 sentence $\forall x, y\ (x \times y = y \times x)$. Then $\mathbb{N} \models \sigma$ since multiplication in $\mathbb{N}$ is commutative. However, the model $\mathbb{N}^\infty$ of $\mathrm{R}(\exp)$ from Example 3.1 does not satisfy σ because $0 \times \infty = \infty \neq 0 = \infty \times 0$ in $\mathbb{N}^\infty$.

Assignment 4.6. Let $a, b \in M \models \mathrm{Q}(\exp)$.

Consider first the case when $b = 0$. If $a + b = 0$, then $0 = a + b = a + 0 = a$ by $(\mathrm{Q}{+_0})$. So $M \models a + b = 0 \rightarrow a = 0 \wedge b = 0$.

Next consider the case when $b \neq 0$. Use (QS_0) to find $c \in M \models b = c + 1$. Then

$$\begin{aligned} a + b = a + (c+1) &= (a + c) + 1 && \text{by } (\mathrm{Q}{+_1}); \\ &\neq 0 && \text{by } (\mathrm{QS}_0). \end{aligned}$$

Hence $M \models a + b = 0 \rightarrow a = 0 \wedge b = 0$ trivially because the hypothesis is not true.

Assignment 5.3. First, Lemma 5.2(1) gives us a Π_1 formula φ which is semantically equivalent to $\neg\psi$. Then apply the Diagonal Lemma to φ to find a Π_1 sentence σ such that

$$\mathrm{R}(\exp) \models \sigma \leftrightarrow \neg\psi(\underline{\sigma}).$$

As $M \models \mathrm{R}(\exp)$, we know $M \models \sigma \leftrightarrow \neg\psi(\underline{\sigma})$. By the truth definition, either $M \models \sigma$ or $M \models \neg\sigma$.

- If $M \models \sigma$, then $M \models \neg\psi(\ulcorner\sigma\urcorner)$.
- If $M \models \neg\sigma$, then $M \models \psi(\ulcorner\sigma\urcorner)$.

Either way, we have $M \not\models \sigma \leftrightarrow \psi(\ulcorner\sigma\urcorner)$.

Assignment 6.3.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{Q}(\exp) \vdash \forall x\ \forall y\ (x + (y+1) = (x+y)+1)}\ (\text{asn})}{\mathrm{Q}(\exp) \vdash \forall y\ (0 + (y+1) = (0+y)+1)}\ (\overline{\forall}\mathrm{R})}{\mathrm{Q}(\exp) \vdash 0 + (0+1) = (0+0)+1}\ (\overline{\forall}\mathrm{R}) \qquad \dfrac{\dfrac{}{\mathrm{Q}(\exp) \vdash \forall x\ (x + 0 = x)}\ (\text{asn})}{\mathrm{Q}(\exp) \vdash 0 + 0 = 0}\ (\overline{\forall}\mathrm{R})}{\mathrm{Q}(\exp) \vdash 0 + (0+1) = 0+1}\ (\text{Leibniz})$$

Assignment 7.1. Suppose $[c_i] = t^M(\overline{[c]})$ and $[c_j] = s^M(\overline{[c]})$. Then for all $k \in \mathbb{N}$,

$$\begin{aligned} & [c_k] = (t+s)^M(\overline{[c]}) \\ \Leftrightarrow\quad & [c_k] = t^M(\overline{[c]}) +^M s^M(\overline{[c]}) = [c_i] +^M [c_j] && \text{by the definition of term evaluation;} \\ \Leftrightarrow\quad & \Phi^* \vdash \mathsf{v}_k = \mathsf{v}_i + \mathsf{v}_j && \text{by the definition of } +^M; \\ \Leftrightarrow\quad & \Phi^* \vdash \mathsf{v}_k = t(\bar{\mathsf{v}}) + s(\bar{\mathsf{v}}) && \text{by the equality rules} \end{aligned}$$

because $\Phi^* \vdash \mathsf{v}_i = t(\bar{\mathsf{v}})$ and $\Phi^* \vdash \mathsf{v}_j = s(\bar{\mathsf{v}})$ by hypothesis.

Assignment 8.2. In view of the Church–Turing Thesis, it suffices to describe an algorithm which on input $n \in \mathbb{N}$ returns `true` if and only if

$$n \in \{\ulcorner\theta\urcorner : \theta \text{ is an } \mathcal{L}_\mathrm{A}(\exp) \text{ formula and } \Phi \vdash \theta\}.$$

(1) Suppose $n \in \mathbb{N}$ is given as input.

(2) Check whether n is the Gödel number of an $\mathcal{L}_\mathrm{A}(\exp)$ formula. If the answer is no, then the algorithm can already return `false` and stop.

(3) Suppose $n = \ulcorner\theta\urcorner$, where θ is an $\mathcal{L}_\mathrm{A}(\exp)$ formula.

(4) For $m = 0, 1, 2, \ldots$, successively check whether m is the Gödel number of a proof of $\Phi \vdash \theta$. This can be done algorithmically by the recursive nature of proofs because Φ is recursive. If any of these checks gives a positive answer, then return `true`.

Assignment 9.3. Let $\varphi(y)$ be the $\Delta_0(\exp)$ formula

$$\exists z{\leqslant}y\ \left(2^{4(2z)} \leqslant y \wedge y < 2^{4(2z+1)}\right)$$

so that for all $\mathcal{L}_\mathrm{A}(\exp)$ sentences σ,

$$\mathbb{N} \models \varphi(\underline{\sigma}) \quad \Leftrightarrow \quad \text{the number of symbols in } \sigma \text{ is odd under our Gödel numbering.}$$

Let σ and τ be the $\mathcal{L}_\mathrm{A}(\exp)$ sentences $\top$ and $(\neg\top)$ respectively. On the one hand, we know $\mathbb{N} \models \top$ and $\mathbb{N} \not\models \neg\top$ by the truth definition. So $\mathbb{N} \models \sigma$ and $\mathbb{N} \not\models \tau$. On the other hand, we know $\mathbb{N} \models \varphi(\underline{\sigma})$ and $\mathbb{N} \not\models \varphi(\underline{\tau})$ because σ consists of 1 symbol and τ consists of 4 symbols. Combining the two, we see that $\mathbb{N}$ satisfies

$$\sigma \leftrightarrow \varphi(\underline{\sigma}), \quad \tau \leftrightarrow \varphi(\underline{\tau}), \quad \text{and} \quad \neg(\sigma \leftrightarrow \tau).$$

Since all these sentences are $\Delta_0(\exp)$, they must be provable in $\mathrm{R}(\exp)$ by Corollary 4.2.

Assignment 10.2.

$$\begin{aligned}&\exists y_1, y_2{\leqslant}s\ \exists i_1, i_2{<}\mathrm{len}(s)\\ &\quad\left(\begin{aligned}&i_1 > j \wedge i_2 > j \wedge (s)''_{i_1} = y_1 \wedge (s)''_{i_2} = y_2\\ &\wedge \mathrm{antet}(y_1) \mathrel{\dot{=}} \mathrm{antet}(y_2) \wedge \mathrm{antet}(y_2) \mathrel{\dot{=}} \mathrm{antet}(x)\\ &\wedge \exists a{\leqslant}y_2\ \exists b{\leqslant}x\ \left(\mathrm{succt}(y_1) = \underline{\neg}\,\hat{}\,a\,\hat{}\,\underline{\vee}\,\hat{}\,b \wedge \mathrm{succt}(y_2) = a \wedge \mathrm{succt}(x) = b\right)\end{aligned}\right)\end{aligned}$$

Assignment 11.6. Fix $t, t' \in M \models \mathrm{I}\Delta_0(\exp)$. We will frequently use $(\exp\times)$ below without explicitly mentioning it.

Let $j \in M \models j < \mathrm{len}(t')$. Recall $(s, i) \mapsto \langle s\rangle_i$ is a total function on M^2. Use the definition of $\langle t'\rangle_j$ to find $z, w \in M$ such that $t' = 2^{4(j+1)}z + 2^{4j}\langle t'\rangle_j + w$ and $w < 2^{4j}$. Then

$$\begin{aligned} t \,\hat{;}\, t' &= 2^{4(\mathrm{len}(t')+1)}t + t' && \text{by the definition of } t \,\hat{;}\, t';\\ &= 2^{4(\mathrm{len}(t')+1)}t + 2^{4(j+1)}z + 2^{4j}\langle t'\rangle_j + w\\ &= 2^{4(j+1)}(2^{4(\mathrm{len}(t')-j)}t + z) + 2^{4j}\langle t'\rangle_j + w.\end{aligned}$$

Thus $\langle t \,\hat{;}\, t'\rangle_j = \langle t'\rangle_j$ by the definition of $\langle t \,\hat{;}\, t'\rangle_j$.

The definition of $t \,\hat{;}\, t'$ and the definition of len tell us $t \,\hat{;}\, t' = 2^{4(\mathrm{len}(t')+1)}t + 2^{4\,\mathrm{len}(t')} \times 0 + t'$ and $t' < 2^{4\,\mathrm{len}(t')}$ respectively. Hence $\langle t \,\hat{;}\, t'\rangle_{\mathrm{len}(t')} = 0$ by the definition of $\langle t \,\hat{;}\, t'\rangle_{\mathrm{len}(t')}$.

Let $i \in M \models i < \mathrm{len}(t)$. Recall $(s, j) \mapsto \langle s\rangle_j$ is a total function on M^2. Use the definition of $\langle t\rangle_i$ to find $z, w \in M$ such that $t = 2^{4(i+1)}z + 2^{4i}\langle t\rangle_i + w$ and $w < 2^{4i}$. Then

$$\begin{aligned} t \,\hat{;}\, t' &= 2^{4(\mathrm{len}(t')+1)}t + t' && \text{by the definition of } t \,\hat{;}\, t';\\ &= 2^{4(\mathrm{len}(t')+1)}(2^{4(i+1)}z + 2^{4i}\langle t\rangle_i + w) + t'\\ &= 2^{4(\mathrm{len}(t')+i+2)}z + 2^{4(\mathrm{len}(t')+i+1)}\langle t\rangle_i + (2^{4(\mathrm{len}(t')+1)}w + t').\end{aligned}$$

If $w=0$, then

$$\begin{aligned} 2^{4(\operatorname{len}(t')+1)}w+t' &= t' < 2^{4\operatorname{len}(t')} && \text{by the definition of len;}\\ &< 2^{4(\operatorname{len}(t')+i+1)} && \text{by (<+S), (×/<), and (exp/<),} \end{aligned}$$

and thus $\langle t\mathbin{\widehat{;}}t'\rangle_{\operatorname{len}t'+i+1}=\langle t\rangle_i$ by the definition of $\langle t\mathbin{\widehat{;}}t'\rangle_{\operatorname{len}t'+i+1}$. So suppose $w\neq 0$. Then the definition of $\operatorname{len}(w)$ tells us $2^{4(\operatorname{len}(w)-1)}\leqslant w<2^{4i}$. Thus $\operatorname{len}(w)-1<i$ by (lin), (exp/<) and (irrefl). This implies $\operatorname{len}(w)\leqslant i$ in view of (+/<) and (<S). Now

$$\begin{aligned} 2^{4(\operatorname{len}(t')+1)}w+t' &= w\mathbin{\widehat{;}}t' && \text{by the definition of } w\mathbin{\widehat{;}}t';\\ &< 2^{4\operatorname{len}(w\widehat{;}t')} && \text{by the definition of len;}\\ &= 2^{4(\operatorname{len}(w)+\operatorname{len}(t')+1)} && \text{by Lemma 11.5(1), as } w\neq 0;\\ &\leqslant 2^{4(\operatorname{len}(t')+i+1)} && \text{by (+/<) and (exp/<), as } \operatorname{len}(w)\leqslant i. \end{aligned}$$

Hence $\langle t\mathbin{\widehat{;}}t'\rangle_{\operatorname{len}t'+i+1}=\langle t\rangle_i$ by the definition of $\langle t\mathbin{\widehat{;}}t'\rangle_{\operatorname{len}t'+i+1}$ again. (One student found an alternative proof of $2^{4(\operatorname{len}(t')+1)}w+t'<2^{4(\operatorname{len}(t')+i+1)}$ using the fact that $w<2^{4i}$ implies $w+1\leqslant 2^{4i}$. This avoids Lemma 11.5(1) and hence the case-splitting.)

Assignment 12.7. Fix $\bar{a}\in M$. Reason informally in M.

Suppose $(\theta\vee\eta)(\bar{a})$ holds. Then either $\theta(\bar{a})$ holds or $\eta(\bar{a})$ holds by the truth definition. If $\theta(\bar{a})$ holds, then the first conjunct in (1) tells us $T\vdash\theta(\underline{\bar{a}})$, and so $T\vdash(\theta\vee\eta)(\underline{\bar{a}})$ by ($\vee_0$). If $\eta(\bar{a})$ holds, then the first conjunct in (2) tells us $T\vdash\eta(\underline{\bar{a}})$, and so $T\vdash(\theta\vee\eta)(\underline{\bar{a}})$ by ($\vee_1$).

Suppose $\neg(\theta\vee\eta)(\underline{\bar{a}})$ holds. Then both $\neg\theta(\underline{\bar{a}})$ and $\neg\eta(\underline{\bar{a}})$ hold by the truth definition. So the second conjunct in (1) and the second conjunct in (2) imply $T\vdash\neg\theta(\underline{\bar{a}})$ and $T\vdash\neg\eta(\underline{\bar{a}})$ respectively. Hence $T\vdash\neg(\theta\vee\eta)(\underline{\bar{a}})$ by ($\neg\vee$).

Assignment 13.3. First, suppose $(m_1,m_2,\ldots,m_k)\in S$. Then, by the choice of α and α',

$$\mathbb{N}\models\exists z\ \alpha(\bar{m},z)\wedge\forall z'\ \neg\alpha'(\bar{m},z').$$

This quickly implies $\mathbb{N}\models\exists z\ \big(\alpha(\underline{\bar{m}},z)\wedge\forall z'{<}z\ \neg\alpha'(\underline{\bar{m}},z')\big)$. So

$$\mathrm{R}(\exp)\vdash\exists z\ \big(\alpha(\underline{\bar{m}},z)\wedge\forall z'{<}z\ \neg\alpha'(\underline{\bar{m}},z')\big)$$

by the Σ_1 completeness of R(exp).

Next, suppose $(m_1,m_2,\ldots,m_k)\in\mathbb{N}^k\setminus S$. Then, by the choice of α and α',

$$\mathbb{N}\models\forall z\ \neg\alpha(\bar{m},z)\wedge\exists z'\ \alpha'(\bar{m},z').$$

Fix $q'\in\mathbb{N}\models\alpha'(\bar{m},q')$. To show $\mathrm{R}(\exp)\vdash\neg\exists z\ \big(\alpha(\underline{\bar{m}},z)\wedge\forall z'{<}z\ \neg\alpha'(\underline{\bar{m}},z')\big)$, we take $q\in M\models\mathrm{R}(\exp)+\alpha(\bar{m},q)$, and prove $M\models\exists z'{<}q\ \alpha'(\bar{m},z')$. Notice that if $n\in\mathbb{N}$, then $\mathbb{N}\models\neg\alpha(\bar{m},n)$ by the first conjunct in the displayed line above, and so the $\Delta_0(\exp)$ absoluteness between M and $\mathbb{N}$ implies $M\models\neg\alpha(\bar{m},n)$. Thus $q\notin\mathbb{N}$. Observation 4.3 then tells us $q'<q$. Applying the $\Delta_0(\exp)$ absoluteness between M and $\mathbb{N}$ again, we see that $M\models\alpha'(\bar{m},q')$. Hence q' witnesses $M\models\exists z'{<}q\ \alpha'(\bar{m},z')$, as required.

Assignment 14.7. As $K\subseteq M$, the Diagram Lemma implies $M\models\mathrm{Diag}(K)$. If $K\models\theta(\bar{a})$, then

$$\begin{aligned} & \theta(\bar{a})\in\mathrm{Diag}(K) && \text{by the definition of Diag}(K);\\ \therefore\quad & M\models\theta(\bar{a}) && \text{as } M\models\mathrm{Diag}(K). \end{aligned}$$

Conversely, if $M\models\theta(\bar{a})$, then

$$\begin{aligned} & M\not\models\neg\theta(\bar{a}) && \text{by the truth definition;}\\ \therefore\quad & \neg\theta(\bar{a})\notin\mathrm{Diag}(K) && \text{as } M\models\mathrm{Diag}(K);\\ \therefore\quad & K\not\models\neg\theta(\bar{a}) && \text{by the definition of Diag}(K);\\ \therefore\quad & K\models\theta(\bar{a}) && \text{by the truth definition.} \end{aligned}$$

Here we used the fact that both θ and $\neg\theta$ are quantifier-free.

Assignment 15.2. To get the extension M of K we want, it suffices to show

$$T + \mathrm{Diag}(K) \nvdash \bot$$

in view of the Diagram Lemma. Note that $\mathrm{Diag}(K)$ is closed under conjunction. So we can apply Remark 15.1. Take $\alpha(\bar{y}) \in \forall_0(\mathcal{L})$ and $\bar{b} \in K$ such that $\alpha(\bar{b}) \in \mathrm{Diag}(K)$. Then

	$K \models \alpha(\bar{b})$	by the definition of $\mathrm{Diag}(K)$;
$\therefore$	$K \models \exists \bar{y}\ \alpha(\bar{y})$	by the truth definition;
$\therefore$	$\big(\forall \bar{y}\ \neg\alpha(\bar{y})\big) \notin \forall_1\text{-Th}(T)$	as $K \models \forall_1\text{-Th}(T)$;
$\therefore$	$T \nvdash \forall \bar{y}\ \neg\alpha(\bar{y})$	by the definition of $\forall_1\text{-Th}(T)$, as $\forall \bar{y}\ \neg\alpha(\bar{y})$ is an $\forall_1$ sentence;
$\therefore$	$T + \exists \bar{y}\ \alpha(\bar{y}) \nvdash \bot$	by logic;
$\therefore$	$T + \alpha(\bar{b}) \nvdash \bot$	as T does not mention $\bar{b}$, and $\bar{y}$ do not appear free in T.

Assignment 16.11. If $\alpha = \mathrm{lcm}\{\alpha_\ell : \ell < L\}$, then the formula below has the required properties:

$$\bigvee_{j<K} \Big(\bigwedge_{k<K} s_j(\bar{x}) \leqslant s_k(\bar{x}) \wedge \bigvee_{\rho=1}^{\alpha} \bigwedge_{\ell<L} s_j(\bar{x}) - \underline{\rho} \equiv t_\ell(\bar{x}) \mod \alpha_\ell \Big).$$

Assignment 17.8. Suppose (ii) fails. Let $m \in \mathbb{N}$ and $F \colon \mathbb{N} \to \mathbb{N}$ witness the failure of (ii). Notice that if some $j \in \mathbb{N}$ makes $a_j = 0$, then $0 = a_{j+1} = a_{j+2} = \cdots$ and so $\lim_{i\to\infty} a_i = 0$, which is not the case by the choice of m and F. So every a_j is nonzero. This implies, for every $j \in \mathbb{N}$,

$$\mathrm{Sub}^{F(j)}_{F(j+1)}(a_j) = a_{j+1} + 1 > a_{j+1},$$

and hence by Lemma 17.6(1) and (5),

$$\mathrm{Sub}^{F(j)}_{\omega}(a_j) = \mathrm{Sub}^{F(j+1)}_{\omega}(\mathrm{Sub}^{F(j)}_{F(j+1)}(a_j)) > \mathrm{Sub}^{F(j+1)}_{\omega}(a_{j+1}).$$

This shows $\{\mathrm{Sub}^{F(j)}_{\omega}(a_j) : j \in \mathbb{N}\}$ is a nonempty subset of ε_0 with no least element. So (i) fails.

Assignment 18.1. Fix $M \models \mathrm{PA}$. The base case is trivial because $M \models \forall x\ (x \not< 0)$. For the induction step, let $a \in M$ such that

$$M \models \forall x{<}a\ \exists y\ \eta(x, y) \to \exists b\ \forall x{<}a\ \exists y{<}b\ \eta(x, y).$$

Suppose $M \models \forall x{<}a+1\ \exists y\ \eta(x, y)$. Then

$$M \models \forall x{<}a\ \exists y\ \eta(x, y) \wedge \exists y\ \eta(a, y).$$

The first conjunct, together with the induction hypothesis, gives $b_0 \in M \models \forall x{<}a\ \exists y{<}b_0\ \eta(x, y)$. The second disjunct gives $b_1 \in M \models \eta(a, b_1)$. If $b = \max\{b_0, b_1 + 1\}$, then

$$M \models \forall x{<}a+1\ \exists y{<}b\ \eta(x, y).$$

Assignment 19.4. If $t^{\mathbb{N}} = 0$, then use

$$\dfrac{\dfrac{\vdash t = 0}{t \neq 0 \vdash}\ \neg\mathrm{L}}{t \neq 0 \vdash \exists w\ (t = w + 1)}\ \mathrm{wR}$$

If $t^{\mathbb{N}} \neq 0$, then invoke (QS_0) in $\mathbb{N}$ to find $n \in \mathbb{N} \models t = \underline{n} + 1$, and use

$$\dfrac{\dfrac{\vdash t = \underline{n} + 1}{\vdash \exists w\ (t = w + 1)}\ \exists\mathrm{R}}{t \neq 0 \vdash \exists w\ (t = w + 1)}\ \mathrm{wL}$$

Assignment 20.8. Apply the following operations to π_1 to obtain $\tilde{\pi}$.

(1) Remove all the direct ancestors of the leftmost θ in the end-sequent.

(2) The tree of sequents obtained may not be a **Nat**-proof because perhaps

$$\mathrm{wL}\,\frac{\Gamma \vdash \Delta}{\theta, \Gamma \vdash \Delta} \quad \overset{(1)}{\mapsto} \quad \begin{array}{c}\Gamma \vdash \Delta \\ \hdashline \Gamma \vdash \Delta\end{array}$$

in which case remove one of the two sequents $\Gamma \vdash \Delta$ in the result here. The remaining steps remain compliant with the axioms and the deduction rules of **Nat**. For instance, if $\theta = \beta(t_0, t_1, \ldots, t_{\ell-1})$, then perhaps

$$\beta(\bar{t}) \vdash \beta(\bar{s}) \quad \overset{(1)}{\mapsto} \quad \vdash \beta(\bar{s})$$

which is an axiom in **Nat** since $\beta(\bar{t}) \vdash \beta(\bar{s})$ being an axiom in **Nat** implies $t_i^{\mathbb{N}} = s_i^{\mathbb{N}}$ for all $i < \ell$, and thus $\mathbb{N} \models \beta(\bar{s})$. Note that $\theta \vdash$ is not an axiom in **Nat** because $\mathbb{N} \models \theta$.

Assignment 21.3. Since $M \succcurlyeq K$, it suffices to show $K \models \forall y\ \exists x{\geqslant}y\ \theta(x)$. For every $b \in K$,

	$a \geqslant b$	as $M \supseteq_{\mathrm{e}} K$ and $a \in M \setminus K$;
$\therefore$	$M \models \exists x{\geqslant}b\ \theta(x)$	as $M \models \theta(a)$;
$\therefore$	$K \models \exists x{\geqslant}b\ \theta(x)$	as $M \succcurlyeq K$.