INTERPRETING WEAK KÖNIG'S LEMMA USING THE ARITHMETIZED COMPLETENESS THEOREM

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ABSTRACT. We present a previously unpublished proof of the conservativity of WKL_0 over $\text{I}\Sigma_1$ using the Arithmetized Completeness Theorem, which, in particular, constitutes an ω-interpretation of WKL_0 in $\text{I}\Sigma_1$. We also show that WKL_0^* is interpretable in $\text{I}\Delta_0 + \exp$.

$\text{I}\Sigma_1$ and WKL_0 are prominent theories in the study of arithmetic and the foundations of mathematics [3, 8]. The intimate connections between these theories can best be seen from a fundamental and influential theorem of Leo Harrington's which states that every countable model of $\text{I}\Sigma_1$ expands to a model of WKL_0. This theorem, in particular, implies WKL_0 is conservative over $\text{I}\Sigma_1$.

In this short note, we demonstrate how (several strengthenings due independently to Hájek [2] and Avigad [1] of) Harrington's theorem can be established simply by putting together a number of standard facts in the literature. While the usual proof of Harrington's theorem is based on tree forcing [8, Section IX.2], our approach uses instead a version of the Arithmetized Completeness Theorem, which can be proved by a forcing argument in which the conditions are binary strings of bounded lengths.

The notation in this paper is more or less standard. Unless otherwise stated, 'definable' means 'parametrically definable'. We assume familiarity with Hájek–Pudlák [3] and some acquaintance with Simpson [8]. Nevertheless, let us first briefly review some basic definitions and facts.

The language for first-order arithmetic $\mathcal{L}_I$ has symbols $0, 1, +, \times, \leqslant$. A *bounded quantifier* is one of the form $Qx{\leqslant}t$, where $Q \in \{\forall, \exists\}$ and t is a term not involving x. An $\mathcal{L}_I$ formula is Δ_0 if all the quantifiers it contains are bounded. Formulas of the form $\exists \bar{v} \; \varphi(\bar{v}, \bar{x})$ where $\varphi \in \Delta_0$ are called Σ_1. A formula θ is Δ_1 if, modulo logical equivalence, both θ and $\neg\theta$ are Σ_1. The closure of Σ_1 under Boolean operations and bounded quantification is denoted $\Delta_0(\Sigma_1)$. Axiomatize $\text{I}\Sigma_1$ by the theory of the non-negative parts of discretely ordered rings (commonly referred to as PA^-) and the induction scheme for Σ_1 formulas. Define $\text{I}\Delta_0$ similarly. The theory $\text{B}\Sigma_1$ consists of $\text{I}\Delta_0$ and the Σ_1 *collection scheme*, which asserts that

$$\forall \bar{z} \; \forall a \; \left(\forall x{\leqslant}a \; \exists y \; \varphi(x, y, \bar{z}) \rightarrow \exists b \; \forall x{\leqslant}a \; \exists y{\leqslant}b \; \varphi(x, y, \bar{z}) \right)$$

whenever $\varphi \in \Sigma_1$. We denote by exp a fixed $\mathcal{L}_{\mathrm{I}}$ sentence expressing the totality of $x \mapsto 2^x$ over $\mathrm{I}\Delta_0$. It is commonly known [3, Sections I.1(c), IV.1(f), and V.1(a)] that $\mathrm{I}\Sigma_1 \vdash \mathrm{B}\Sigma_1 + \exp$ and $\mathrm{B}\Sigma_1 \nvdash \exp$. Within $\mathrm{I}\Delta_0 + \exp$, the usual operations on sequences and syntactical objects are well behaved [3, Section V.3].

The language for second-order arithmetic $\mathcal{L}_{\mathrm{II}}$ has a first-order sort for numbers and a second-order sort for sets of numbers. We write $\mathcal{L}_{\mathrm{II}}$ structures as pairs $(M, \mathscr{X})$, where M is the universe for the first-order sort, and $\mathscr{X}$ is the universe for the second-order sort. The language $\mathcal{L}_{\mathrm{II}}$ has a copy of $\mathcal{L}_{\mathrm{I}}$ on the first-order sort, and a membership relation linking the two sorts. All $\mathcal{L}_{\mathrm{II}}$ structures are required to satisfy the axiom of extensionality. The definitions of $\Sigma_0^0, \Sigma_1^0, \Delta_1^0, \mathrm{I}\Sigma_0^0, \mathrm{I}\Sigma_1^0, \dots$ are analogous to those of $\Delta_0, \Sigma_1, \Delta_1, \mathrm{I}\Delta_0, \mathrm{I}\Sigma_1, \dots$, except that we now allow the appearance of second-order variables. *Weak König's Lemma* asserts that every unbounded 0–1 tree has an unbounded branch. The theory WKL_0^* consists of $\mathrm{I}\Sigma_0^0 + \exp$, the Δ_1^0 comprehension scheme, and Weak König's Lemma. Define $\mathrm{WKL}_0 = \mathrm{WKL}_0^* + \mathrm{I}\Sigma_1^0$. The following theorem shows that WKL_0 is *conservative* over $\mathrm{I}\Sigma_1$, i.e., every $\mathcal{L}_{\mathrm{I}}$ sentence provable in WKL_0 is already provable in $\mathrm{I}\Sigma_1$.

Theorem 1 (Hájek, Avigad)**.** Every $M \models \mathrm{I}\Sigma_1$ expands to $(M, \mathscr{X}) \models \mathrm{WKL}_0$.

Proof. First, it is well known [3, Corollary I.4.34(2)] that $\mathrm{I}\Sigma_1$ proves the consistency of $\mathrm{I}\Delta_0 + \exp$. Second, the Low Arithmetized Completeness Theorem in Hájek–Pudlák [3, Theorem I.4.27] tells us that, provably in $\mathrm{I}\Sigma_1$, every Δ_1-definable consistent theory has a definable model all of whose Σ_1^0 properties are $\Delta_0(\Sigma_1)$-definable. These two facts together yield $K \models \mathrm{I}\Delta_0 + \exp$ properly end extending M such that whenever $\bar{c} \in K$ and θ is an $\mathcal{L}_{\mathrm{I}}$ formula,

$(*)$ $\{\bar{x} \in M : K \models \theta(\bar{v}, \bar{x}, \bar{c})$ for some $\bar{v} \in M\}$ is $\Delta_0(\Sigma_1)$-definable in M.

As is usual in Arithmetized Completeness Theorem constructions (for example, that in Paris [6, page 254]), the embedding $M \to K$ hinges on the observation that every $a \in M$ is represented by a term $\breve{a}$ in the $\mathcal{L}_{\mathrm{I}}$ in M. The image of this embedding is closed downwards because $\mathrm{I}\Delta_0 + \exp \vdash \forall x \leqslant \breve{a} \bigvee_{i \leqslant a} x = \breve{\imath}$ for all a, provably in $\mathrm{I}\Sigma_1$.

Let us say that an element $c \in K$ *codes* a subset $A \subseteq M$ if

$$A = \{i \in M : i\text{th digit in the binary expansion of } c \text{ is } 1\}.$$

Denote the collection of all subsets of M coded in K by $\mathrm{Cod}(K/M)$. A standard overspill argument [9, Theorem 4.8] shows $(M, \mathrm{Cod}(K/M)) \models \mathrm{WKL}_0^*$. For the sake of completeness, we include a sketch of this argument here. Consider Δ_1^0 comprehension. If $A \subseteq M$ that is Δ_1^0-definable in $(M, \mathrm{Cod}(K/M))$, then its Δ_1^0-ness overspills into $K \setminus M$, producing a bound which enables K to see A in a Δ_0 way. Bounded Δ_0 comprehension, available in K from $\mathrm{I}\Delta_0 + \exp$, then gives what we want. Next, if B is an unbounded 0–1 tree in $(M, \mathrm{Cod}(K/M))$, then its code contains a node whose length is in $K \setminus M$ by overspill, and the predecessors of such a node trace an unbounded branch in B. This shows Weak König's Lemma in $(M, \mathrm{Cod}(K/M))$.

It remains to prove $(M, \mathrm{Cod}(K/M)) \models \mathrm{I}\Sigma_1^0$. Notice for each $\varphi \in \Sigma_0^0$, we can find $\psi \in \Delta_0$, simply by replacing set parameters by their codes, such that if $\bar{A} \in \mathrm{Cod}(K/M)$ coded by $\bar{c} \in K$, then for all $\bar{x} \in M$,

$$(M, \mathrm{Cod}(K/M)) \models \varphi(\bar{x}, \bar{A}) \quad \Leftrightarrow \quad K \models \psi(\bar{x}, \bar{c}).$$

Hence, every Σ_1^0 property of $(M, \mathrm{Cod}(K/M))$ translates to a $\Delta_0(\Sigma_1)$ property of M by $(*)$. Since $\mathrm{I}\Sigma_1$ proves the induction scheme for $\Delta_0(\Sigma_1)$ formulas [3, Lemma I.2.14], this implies $(M, \mathrm{Cod}(K/M)) \models \mathrm{I}\Sigma_1^0$, as required. □

A similar proof was independently discovered by F. Félix Lara-Martín [personal communication]. It solves the second part of Problem 1 in Paris [6]. Analogous arguments for higher levels of the arithmetic hierarchy can be found in Paris's paper. The original proofs by Hájek and Avigad both go via an ω-interpretation of WKL_0 in $\mathrm{I}\Sigma_1$. The reader may consult Section III.1(a) of Hájek–Pudlák [3] for background information about interpretations.

Definition. An interpretation of an $\mathcal{L}_{\mathrm{II}}$ theory in an $\mathcal{L}_{\mathrm{I}}$ theory is an *ω-interpretation* if its restriction to the first-order sort is the identity interpretation.

Essentially, an ω-interpretation of an $\mathcal{L}_{\mathrm{II}}$ theory T in an $\mathcal{L}_{\mathrm{I}}$ theory T_0 is a uniform recipe for expanding every $M \models T_0$ to $(M, \mathscr{X}) \models T$. Hájek's ω-interpretation [2, Section 3] employs a notion of 'very low' sets with respect to which Weak König's Lemma is true. Avigad's ω-interpretation [1] involves a formalization of the usual forcing proof. While the former of these is slightly cleaner, the latter has the advantage of being applicable also to general models $(M, \mathscr{X}) \models \mathrm{I}\Sigma_1^0$, not only the ones in which $\mathscr{X}$ is finite. Upon closer inspection, one sees that our proof is uniform enough to give rise to an ω-interpretation too. Notice our ω-interpretation, unlike theirs, is iteration-free.

Theorem 2 (Hájek, Avigad)**.** There is an ω-interpretation of WKL_0 in $\mathrm{I}\Sigma_1$.

Proof. By looking into the proof of the Low Arithmetized Completeness Theorem, the reader can verify that the definition (of the elementary diagram) of K in our proof of Theorem 1 does not depend on M. Since every element of K codes a set in $\mathrm{Cod}(K/M)$, we can use the definition of K as the interpretation of the second-order sort. The membership relation is interpreted accordingly. (We do not have a choice on how the other symbols are interpreted.) □

As shown by Simpson and Smith [9, Theorem 4.6], an analogue of Theorem 1 holds at the $\mathrm{B}\Sigma_1 + \exp$ level: every *countable* model $M \models \mathrm{B}\Sigma_1 + \exp$ expands to $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$. It is not known whether the countability condition can be omitted here. In particular, it is not known whether there is an ω-interpretation of WKL_0^* in $\mathrm{B}\Sigma_1 + \exp$. What we can prove here is that WKL_0^* is interpretable in $\mathrm{B}\Sigma_1 + \exp$. Since $\mathrm{B}\Sigma_1 + \exp$ is interpretable in $\mathrm{I}\Delta_0 + \exp$ [2, Theorem 2.4], this is equivalent to interpretability in $\mathrm{I}\Delta_0 + \exp$. It answers Question (5) in Hájek [2].

Corollary 3. There is an interpretation of WKL_0^* in $\mathrm{I}\Delta_0 + \exp$.

Proof. We only describe how to make a model of WKL_0^* from a model of $\mathrm{I}\Delta_0 + \exp$. Modulo the finite axiomatizability of $\mathrm{I}\Sigma_1$ [3, Theorem I.2.52], it is straightforward to check that this description gives rise to an interpretation.

Take any $M \models \mathrm{I}\Delta_0 + \exp$. If $M \models \mathrm{I}\Sigma_1$, then apply Theorem 2. Suppose $M \not\models \mathrm{I}\Sigma_1$. Follow Kołodziejczyk–Yokoyama [5, Lemma 9] to find a parameter-free Σ_1-definable proper cut I of M that is closed under $x \mapsto 2^x$. Then $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}_0^*$ as in our proof of Theorem 1. This is the model we want. □

Having known the conservativity of a stronger theory T over a weaker theory T_0, the natural question is then whether T helps make proofs of theorems of T_0 significantly shorter. Ignjatovic [4, Chapter 3] observed that ω-interpretations can be utilized to obtain negative answers to such questions.

Definition. Let T_0 be (an axiomatization of) a theory, and T be an extension of T_0, possibly in a bigger language. We say T has *at most polynomial speed-up* over T_0 if there exists $n \in \mathbb{N}$ such that for every proof from T with ℓ symbols of a theorem of T_0, there exists a proof from T_0 with at most $\ell^n + n$ symbols of the same conclusion.

It does not matter whether we mean sequence-proofs or tree-proofs here because these two notions are polynomially related to each other [7, Theorem 4.1]. With sequence-proofs in mind, it is easy to see that for every finitely axiomatized $\mathcal{L}_{\mathbb{II}}$ theory T, if one has an ω-interpretation of T in an $\mathcal{L}_{\mathrm{I}}$ theory T_0, then T has at most polynomial speed-up over T_0.

Corollary 4 (Hájek, Avigad)**.** WKL_0 has at most polynomial speed-up over $\mathrm{I}\Sigma_1$.

Proof. Since WKL_0 is finitely axiomatizable [8, Lemma VIII.2.10], this follows directly from Theorem 2. $\qquad\square$

We do not know whether WKL_0^* has at most polynomial speed-up over $\mathrm{I}\Delta_0 + \exp$.

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