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Artificial boundary conditions for incompressible Navier–Stokes equations: A well-posed result ☆

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Abstract

Numerical simulation of two-dimensional incompressible viscous flows around an obstacle is considered. Two horizontal straight line artificial boundaries are introduced and the original flow is approximated by a flow in an infinite channel with slip boundary condition on the wall. Then two vertical segment artificial boundaries are introduced and a series of approximate artificial boundary conditions on them are derived by imposing the continuity of velocity and the normal stress. Thus the original problem is reduced to a problem defined in a bounded computational domain. The well-posedness of the reduced problem is proved. The finite element approximation of this problem is given and error estimates are obtained. Furthermore numerical examples show the accuracy and efficiency of our artificial boundary conditions. © 2000 Published by Elsevier Science S.A. All rights reserved.

Keywords: Navier–Stokes equations; Incompressible viscous flows; Artificial boundary; Artificial boundary conditions

1. Introduction

Let Ω_i be a bounded domain in $\mathbb{R}^2$, with a simple closed curve boundary. Consider the Navier–Stokes (N–S) equations in the exterior domain $\mathbb{R}^2 \setminus \bar{\Omega}_i$, under Dirichlet boundary conditions:

$$(u \cdot \bigtriangledown)u + \bigtriangledown p = \nu \bigtriangleup u \quad \text{in } \mathbb{R}^2 \setminus \bar{\Omega}_i, \tag{1.1}$$

$$\bigtriangledown \cdot u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{\Omega}_i, \tag{1.2}$$

$$u|_{\partial\Omega_i} = 0, \tag{1.3}$$

$$u(x) \to u_\infty \equiv (a, 0)^{\mathrm{T}} \quad \text{when } r = \sqrt{x_1^2 + x_2^2} \to +\infty, \tag{1.4}$$

where $u = (u_1, u_2)^{\mathrm{T}}$ is the velocity, p the pressure, $\nu > 0$ the kinematic viscosity, $a > 0$ a constant and $x = (x_1, x_2)^{\mathrm{T}}$ is the coordinate.

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The boundary value problem (1.1)–(1.4) describes motion of a steady, viscous incompressible fluid around an obstacle of shape Ω_i, with no movement of the fluid particles on the boundary of Ω_i ("no-slip boundary condition"). In finding numerical solutions of the above problem, one often introduces artificial boundaries and sets up artificial boundary conditions on them. Then the original problem is reduced to a problem defined in a bounded computational domain. In the last two decades, many authors have designed artificial boundary conditions on a given artificial boundary for solving the N–S equations in an unbounded domain, see [3–6,10–13,15–17,20] and the references therein.

The layout of this paper is as follows. In Section 2 we introduce two horizontal straight lines and two vertical segments as artificial boundary and approximate N–S equations by Oseen equations in the region far from the obstacle. In Section 3 we design artificial boundary conditions using the continuity of velocity and the normal stress and reduce the original problem to a problem in a bounded computational domain. In Section 4, we prove a well-posed result for the reduced problem. In Section 5 we introduce the finite element approximation and establish an error estimate. Finally in Section 6 we report on some numerical results to show the accuracy of our artificial boundary conditions.

2. Navier–Stokes equations and their approximation in exterior domain

Taking a constant $L > 0$, such that $\bar{\Omega}_i \subset \Omega \equiv \mathbb{R}^2 \times (0, L)$, then problem (1.1)–(1.4) is approximated by the following problem when the constant L is sufficiently large [7,20]:

$$(u \cdot \nabla)u + \nabla p = \nu \triangle u \quad \text{in } \Omega \setminus \bar{\Omega}_i, \tag{2.1}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \setminus \bar{\Omega}_i, \tag{2.2}$$

$$u|_{\partial\Omega_i} = 0, \tag{2.3}$$

$$u_2|_{x_2=0,L} = 0, \quad \sigma_{12}|_{x_2=0,L} \equiv \nu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\Bigg|_{x_2=0,L} = 0, \quad -\infty < x_1 < +\infty, \tag{2.4}$$

$$u(x) \to u_\infty \quad \text{when } x_1 \to \pm\infty, \tag{2.5}$$

where σ_{12} is the tangential stress on the wall. The boundary condition (2.4) is called slip boundary condition and is equivalent to the following condition:

$$\frac{\partial u_1}{\partial x_2}\Bigg|_{x_2=0,L} = u_2|_{x_2=0,L} = 0, \quad -\infty < x_1 < +\infty. \tag{2.6}$$

Taking two constants $b < c$, such that $\Omega_i \subset (b, c) \times (0, L)$, then Ω is divided into three parts Ω_b, Ω_T and Ω_c by the artificial boundaries Γ_b and Γ_c (see Fig. 1) with

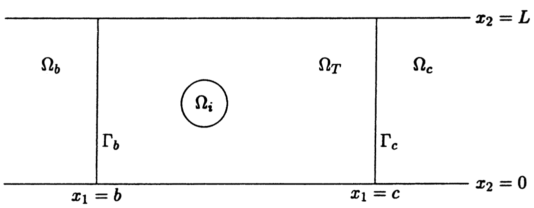

Fig. 1.

$$\Gamma_b = \{x \in \mathbb{R}^2 \mid x_1 = b,\ 0 \leqslant x_2 \leqslant L\},$$
$$\Gamma_c = \{x \in \mathbb{R}^2 \mid x_1 = c,\ 0 \leqslant x_2 \leqslant L\},$$
$$\Omega_b = \{x \in \mathbb{R}^2 \mid -\infty < x_1 < b,\quad 0 < x_2 < L\},$$
$$\Omega_T = \{x \in \mathbb{R}^2 \mid b < x_1 < c,\ 0 < x_2 < L\} \setminus \bar{\Omega}_i,$$
$$\Omega_c = \{x \in \mathbb{R}^2 \mid c < x_1 < +\infty,\ 0 < x_2 < L\}.$$

When $|b|$ and c are sufficiently large, the velocity u of problem (2.1)–(2.5) in the domain Ω_c (or Ω_b) is almost constant vector u_∞. Thus we can linearize N–S equations (2.1) and (2.2) in the domain Ω_c, namely the solution (u, p) of problem (2.1)–(2.5) approximately satisfies the following problem [13,20]:

$$a\frac{\partial u}{\partial x_1} + \bigtriangledown p = \nu \triangle u \quad \text{in } \Omega_c, \tag{2.7}$$

$$\bigtriangledown \cdot u = 0 \quad \text{in } \Omega_c, \tag{2.8}$$

$$\left.\frac{\partial u_1}{\partial x_2}\right|_{x_2=0,L} = u_2|_{x_2=0,L} = 0, \quad c \leqslant x_1 < +\infty, \tag{2.9}$$

$$u(x) \to u_\infty \quad \text{when } x_1 \to +\infty. \tag{2.10}$$

In [2], the author derived a general solution of the above problem (2.7)–(2.10) using the method of separation of variables under the assumption $\lim_{x_1 \to +\infty} p(x) = p_\infty \equiv 0$:

$$u_1(x) = a + \sum_{m=1}^{\infty}\left[a_m e^{-(m\pi/L)(x_1-c)} - \frac{m\pi}{L\lambda^-(m)} b_m e^{\lambda^-(m)(x_1-c)}\right]\cos\frac{m\pi x_2}{L}, \tag{2.11}$$

$$u_2(x) = \sum_{m=1}^{\infty}\left[a_m e^{-(m\pi/L)(x_1-c)} + b_m e^{\lambda^-(m)(x_1-c)}\right]\sin\frac{m\pi x_2}{L}, \tag{2.12}$$

$$p(x) = -a\sum_{m=1}^{\infty} a_m e^{-(m\pi/L)(x_1-c)}\cos\frac{m\pi x_2}{L}, \tag{2.13}$$

where

$$\lambda^-(m) = \frac{a - \sqrt{a^2 + 4\nu^2 m^2\pi^2/L^2}}{2\nu}, \quad m = 1, 2, \ldots,$$

and $a_1, b_1, a_2, b_2, \ldots$ are any constants.

3. Artificial boundary conditions on Γ_c

Let $\varepsilon(u) = (\varepsilon_{ij}(u))_{2\times 2}$ and $\sigma(u,p) = (\sigma_{ij}(u,p))_{2\times 2}$ denote the rate of strain and stress tensors, respectively. We have

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,2 \tag{3.1}$$

and

$$\sigma_{ij}(u,p) = -p\delta_{ij} + 2\nu\varepsilon_{ij}(u), \quad i,j = 1,2, \tag{3.2}$$

where δ_{ij} is the Kronecker Delta whose properties are:

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Moreover let $\sigma_n^+ = (\sigma_{n_1}^+, \sigma_{n_2}^+)^{\mathrm{T}}$ denote the normal stress on the artificial boundary Γ_c, thus

$$\sigma_{n_1}^+ = n_1\sigma_{11}(u,p) + n_2\sigma_{12}(u,p) = \sigma_{11}(u,p) = \left(-p + 2v\frac{\partial u_1}{\partial x_1}\right)\bigg|_{\Gamma_c}, \tag{3.3}$$

$$\sigma_{n_2}^+ = n_1\sigma_{21}(u,p) + n_2\sigma_{22}(u,p) = \sigma_{21}(u,p) = v\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\bigg|_{\Gamma_c}, \tag{3.4}$$

where $n = (n_1, n_2)^{\mathrm{T}} = (1,0)^{\mathrm{T}}$ is the outward normal vector on Γ_c. Similarly $\sigma_n^- = (\sigma_{n_1}^-, \sigma_{n_2}^-)^{\mathrm{T}}$ is such that

$$\sigma_{n_1}^- = \left(-p + 2v\frac{\partial u_1}{\partial x_1} - \frac{1}{2}(u_1 - a)^2\right)\bigg|_{\Gamma_c}, \tag{3.5}$$

$$\sigma_{n_2}^- = v\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) - \frac{1}{2}(u_1 - a)u_2\bigg|_{\Gamma_c}. \tag{3.6}$$

We now use the transmission conditions

$$u(c^-, x_2) = u(c^+, x_2), \quad 0 \leqslant x_2 \leqslant L, \tag{3.7}$$

$$\sigma_n^- = \sigma_n^+, \quad 0 \leqslant x_2 \leqslant L \tag{3.8}$$

to obtain artificial boundary conditions on Γ_c for problem (2.1)–(2.5). Substituting (2.11)–(2.13) with $x_1 = c$ into (3.3) and (3.4), we get

$$\sigma_{n_1}^+ = \sum_{m=1}^{\infty}\left[\left(a - \frac{2vm\pi}{L}\right)a_m - \frac{2vm\pi}{L}b_m\right]\cos\frac{m\pi x_2}{L}, \tag{3.9}$$

$$\sigma_{n_2}^+ = v\sum_{m=1}^{\infty}\left[-\frac{2m\pi}{L}a_m + \left(\lambda^-(m) + \frac{m^2\pi^2}{L^2\lambda^-(m)}\right)b_m\right]\sin\frac{m\pi x_2}{L}. \tag{3.10}$$

From (2.11) and (2.12) with $x_1 = c$ and (3.9) and (3.10), a computation shows:

$$\begin{aligned}\sigma_{n_1}^+ = \sum_{m=1}^{\infty}\Bigg[&\frac{2v(-m\pi + L\lambda^-(m))}{L^2}\int_0^L u_1(c,x_2)\cos\frac{m\pi x_2}{L}\,\mathrm{d}x_2 \\ &- \frac{2vm\pi(m\pi + L\lambda^-(m))}{L^3\lambda^-(m)}\int_0^L u_2(c,x_2)\sin\frac{m\pi x_2}{L}\,\mathrm{d}x_2\Bigg]\cos\frac{m\pi x_2}{L} \\ \equiv T_1(u),&\end{aligned} \tag{3.11}$$

$$\begin{aligned}\sigma_{n_2}^+ = \sum_{m=1}^{\infty}\Bigg[&\frac{-2v(m\pi + L\lambda^-(m))}{L^2}\int_0^L u_1(c,x_2)\cos\frac{m\pi x_2}{L}\,\mathrm{d}x_2 \\ &+ \frac{2v(-m\pi + L\lambda^-(m))}{L^2}\int_0^L u_2(c,x_2)\sin\frac{m\pi x_2}{L}\,\mathrm{d}x_2\Bigg]\sin\frac{m\pi x_2}{L} \\ \equiv T_2(u).&\end{aligned} \tag{3.12}$$

Therefore problem (2.1)–(2.5) (say (1.1)–(1.4)) is approximated by the following problem in the bounded computational domain Ω_T:

$$(u\cdot\nabla)u + \nabla p = v\,\triangle\,u \quad \text{in } \Omega_T, \tag{3.13}$$

$$\nabla\cdot u = 0 \quad \text{in } \Omega_T, \tag{3.14}$$

$$u|_{\partial\Omega_i} = 0, \tag{3.15}$$

$$\left.\frac{\partial u_1}{\partial x_2}\right|_{x_2=0,L} = u_2|_{x_2=0,L} = 0, \quad b \leqslant x_1 \leqslant c, \tag{3.16}$$

$$u = u_\infty \quad \text{on } \Gamma_b, \tag{3.17}$$

$$\sigma_n^- = T(u) \equiv \begin{pmatrix} T_1(u) \\ T_2(u) \end{pmatrix} \quad \text{on } \Gamma_c. \tag{3.18}$$

Let

$$T_1^N(u) = \sum_{m=1}^{N} \left[\frac{2v(-m\pi + L\lambda^-(m))}{L^2} \int_0^L u_1(c,x_2) \cos\frac{m\pi x_2}{L}\,\mathrm{d}x_2 - \frac{2vm\pi(m\pi + L\lambda^-(m))}{L^3\lambda^-(m)} \int_0^L u_2(c,x_2) \sin\frac{m\pi x_2}{L}\,\mathrm{d}x_2 \right] \cos\frac{m\pi x_2}{L}, \tag{3.19}$$

$$T_2^N(u) = \sum_{m=1}^{N} \left[\frac{-2v(m\pi + L\lambda^-(m))}{L^2} \int_0^L u_1(c,x_2) \cos\frac{m\pi x_2}{L}\,\mathrm{d}x_2 + \frac{2v(-m\pi + L\lambda^-(m))}{L^2} \int_0^L u_2(c,x_2) \sin\frac{m\pi x_2}{L}\,\mathrm{d}x_2 \right] \sin\frac{m\pi x_2}{L}, \tag{3.20}$$

$$T^N(u) = \begin{pmatrix} T_1^N(u) \\ T_2^N(u) \end{pmatrix} \quad N = 0, 1, 2, \ldots,$$

with

$$T^0(u) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then we get a sequence of approximate artificial boundary conditions at Γ_c, namely

$$\sigma_n^- = T^N(u), \quad N = 0, 1, 2, \ldots, \tag{3.21}$$

where $N = 0$ corresponds to the stress-free boundary condition which is often used in engineering literature. Hence problem (2.1)–(2.5) (say (1.1)–(1.4)) can be approximated by the following sequence problems, $N = 0, 1, 2, \ldots$, in the bounded computational domain Ω_T:

$$(u \cdot \bigtriangledown)u + \bigtriangledown p = v \bigtriangleup u \quad \text{in } \Omega_T, \tag{3.22}$$

$$\bigtriangledown \cdot u = 0 \quad \text{in } \Omega_T, \tag{3.23}$$

$$\left.\frac{\partial u_1}{\partial x_2}\right|_{x_2=0,L} = u_2|_{x_2=0,L} = 0, \quad b \leqslant x_1 \leqslant c, \tag{3.24}$$

$$u|_{\partial\Omega_i} = 0, \tag{3.25}$$

$$u = u_\infty \quad \text{on } \Gamma_b, \tag{3.26}$$

$$\sigma_n^- = T^N(u) \quad \text{on } \Gamma_c. \tag{3.27}$$

In a similar way, we can derive approximate artificial boundary conditions at the artificial boundary Γ_b.

In the following section we will show that the boundary value problems (3.13)–(3.18) and (3.22)–(3.27) are well-posed for appropriate kinematic viscosity v.

4. The solutions of the reduced problems

Let $H^m(\Omega_T)$ and $H^s(\Gamma_c)$ denote the usual Sobolev spaces on the domain Ω_T and the boundary Γ_c, with integer m and real number s [1]. Furthermore let

$$\Gamma_1 = \{x \in \mathbb{R}^2 \mid x_2 = 0,\ b \leqslant x_1 \leqslant c\} \cup \{x \in \mathbb{R}^2 \mid x_2 = L,\ b \leqslant x_1 \leqslant c\},$$
$$\Gamma_i = \partial\Omega_i,$$
$$M = \{v \in H^1(\Omega_T) \times H^1(\Omega_T) \mid v|_{\Gamma_b} = u_\infty,\ v|_{\Gamma_i} = 0,\ v_2|_{\Gamma_1} = 0\},$$
$$V = \{v \in H^1(\Omega_T) \times H^1(\Omega_T) \mid v|_{\Gamma_b \cup \Gamma_i} = 0,\ v_2|_{\Gamma_1} = 0\},$$

with norm $\|v\|_V^2 = |v_1|_{1,\Omega_T}^2 + |v_2|_{1,\Omega_T}^2$,

$$W = L^2(\Omega_T) \text{ with norm } \|q\|_W = \|q\|_{L^2(\Omega_T)}.$$

Then the boundary value problem (3.13)–(3.18) is equivalent to the following variational problem:

Find $(u,p) \in M \times W$ such that

$$A(u,v) + A_0(u,v) + A_1(u,v) + A_2(u,u,v) + B(v,p) = f(v) \quad \forall v \in V, \tag{4.1}$$
$$B(u,q) = 0 \quad \forall q \in W, \tag{4.2}$$

where

$$A(u,v) = 2\nu \int_{\Omega_T} \sum_{i,j=1}^{2} \varepsilon_{ij}(u) \cdot \varepsilon_{ij}(v)\, \mathrm{d}x \equiv 2\nu \int_{\Omega_T} \varepsilon(u) : \varepsilon(v)\, \mathrm{d}x, \tag{4.3}$$

$$A_0(u,v) = \frac{a}{2} \int_0^L [2u_1(c,x_2)v_1(c,x_2) + u_2(c,x_2)v_2(c,x_2)]\, \mathrm{d}x_2, \tag{4.4}$$

$$A_2(u,v,w) = \frac{1}{2} \int_{\Omega_T} \{[(u \cdot \nabla)v] \cdot w - [(u \cdot \nabla)w] \cdot v\}\, \mathrm{d}x$$
$$= \frac{1}{2} \int_{\Omega_T} \sum_{i,j=1}^{2} u_i \left(\frac{\partial v_j}{\partial x_i} w_j - \frac{\partial w_j}{\partial x_i} v_j \right) \mathrm{d}x, \tag{4.5}$$

$$B(v,q) = -\int_{\Omega_T} q \nabla \cdot v\, \mathrm{d}x, \tag{4.6}$$

$$f(v) = \frac{a^2}{2} \int_0^L v_1(c,x_2)\, \mathrm{d}x_2, \tag{4.7}$$

$$\begin{aligned} A_1(u,v) = & -\int_{\Gamma_c} \sigma_n^- \cdot v\, \mathrm{d}x_2 = -\int_{\Gamma_c} T(u) \cdot v\, \mathrm{d}x_2 \\ = & \sum_{m=1}^{\infty} \Bigg[\frac{2\nu(m\pi - L\lambda^-(m))}{L^2} \int_0^L u_1(c,x_2) \cos\frac{m\pi x_2}{L}\, \mathrm{d}x_2 \int_0^L v_1(c,x_2) \cos\frac{m\pi x_2}{L}\, \mathrm{d}x_2 \\ & + \frac{2\nu m\pi(m\pi + L\lambda^-(m))}{L^3\lambda^-(m)} \int_0^L u_2(c,x_2) \sin\frac{m\pi x_2}{L}\, \mathrm{d}x_2 \int_0^L v_1(c,x_2) \cos\frac{m\pi x_2}{L}\, \mathrm{d}x_2 \\ & + \frac{2\nu(m\pi + L\lambda^-(m))}{L^2} \int_0^L u_1(c,x_2) \cos\frac{m\pi x_2}{L}\, \mathrm{d}x_2 \int_0^L v_2(c,x_2) \sin\frac{m\pi x_2}{L}\, \mathrm{d}x_2 \\ & + \frac{2\nu(m\pi - L\lambda^-(m))}{L^2} \int_0^L u_2(c,x_2) \sin\frac{m\pi x_2}{L}\, \mathrm{d}x_2 \int_0^L v_2(c,x_2) \sin\frac{m\pi x_2}{L}\, \mathrm{d}x_2 \Bigg]. \end{aligned} \tag{4.8}$$

Furthermore let

$$\begin{aligned}A_1^N(u,v) = & -\int_{\Gamma_c} T^N(u)\cdot v \,\mathrm{d}x_2 \\ = & \sum_{m=1}^{N}\Bigg[\frac{2\nu(m\pi - L\lambda^-(m))}{L^2}\int_0^L u_1(c,x_2)\cos\frac{m\pi x_2}{L}\,\mathrm{d}x_2\int_0^L v_1(c,x_2)\cos\frac{m\pi x_2}{L}\,\mathrm{d}x_2 \\ & + \frac{2\nu m\pi(m\pi + L\lambda^-(m))}{L^3\lambda^-(m)}\int_0^L u_2(c,x_2)\sin\frac{m\pi x_2}{L}\,\mathrm{d}x_2\int_0^L v_1(c,x_2)\cos\frac{m\pi x_2}{L}\,\mathrm{d}x_2 \\ & + \frac{2\nu(m\pi + L\lambda^-(m))}{L^2}\int_0^L u_1(c,x_2)\cos\frac{m\pi x_2}{L}\,\mathrm{d}x_2\int_0^L v_2(c,x_2)\sin\frac{m\pi x_2}{L}\,\mathrm{d}x_2 \\ & + \frac{2\nu(m\pi - L\lambda^-(m))}{L^2}\int_0^L u_2(c,x_2)\sin\frac{m\pi x_2}{L}\,\mathrm{d}x_2\int_0^L v_2(c,x_2)\sin\frac{m\pi x_2}{L}\,\mathrm{d}x_2\Bigg]. \qquad (4.9)\end{aligned}$$

Then the boundary value problem (3.22)–(3.27) is equivalent to the following variational problem:

Find $(u_N, p_N) \in M \times W$ such that

$$A(u_N,v) + A_0(u_N,v) + A_1^N(u_N,v) + A_2(u_N,u_N,v) + B(v,p_N) = f(v) \quad \forall v\in V, \qquad (4.10)$$

$$B(u_N,q) = 0 \quad \forall q \in W. \qquad (4.11)$$

From Körn's inequality [19], we have that the following holds.

Lemma 4.1. *The bilinear form $A(u,v)$ is symmetric, bounded and coercive on $V\times V$, namely there are two positive constants α_0 and β_0 such that*

$$|A(u,v)| \leqslant \alpha_0\|u\|_V\cdot\|v\|_V \quad \forall u,v\in V, \qquad (4.12)$$

$$A(v,v) \geqslant \nu\beta_0\|v\|_V^2 \quad \forall v \in V. \qquad (4.13)$$

Lemma 4.2. *The bilinear form $B(v,q)$ is bounded on $V\times W$ and satisfies the Babuška–Brezzi (B–B) condition* [8], *namely there exist positive constants α_1 and β_1, such that*

$$|B(v,q)| \leqslant \alpha_1\|v\|_V\cdot\|q\|_W \quad \forall v\in V, \quad q\in W, \qquad (4.14)$$

$$\sup_{v\in V\setminus\{0\}}\frac{B(v,q)}{\|v\|_V} \geqslant \beta_1\|q\|_W \quad \forall q\in W. \qquad (4.15)$$

Proof. (i) There exist $q_0\in L^2(\Omega_T)$ and $w^0\in V$, such that

$$\operatorname{div} w^0 = q_0, \quad \int_{\Omega_T} q_0\,\mathrm{d}x \neq 0. \qquad (4.16)$$

(ii) For any $q\in L^2(\Omega_T)$, we have that

$$q = q^* + \tilde{\alpha}q_0 \quad \text{with} \quad \tilde{\alpha} = \frac{\int_{\Omega_T} q\,\mathrm{d}x}{\int_{\Omega_T} q_0\,\mathrm{d}x}. \qquad (4.17)$$

Hence $q^*\in L_0^2(\Omega_T)\equiv\{q \mid q\in L^2(\Omega_T) \text{ and } \int_{\Omega_T} q\,\mathrm{d}x = 0\}$ and

$$\|q^*\|_W \leqslant c\|q\|_W. \qquad (4.18)$$

By virtue of Corollary 2.4 [8, p. 24] there exists an element $w^*\in V$, such that

$$\operatorname{div} w^* = q^* \quad \|w^*\|_V \leqslant c\|q^*\|_M. \qquad (4.19)$$

Let $w_q = w^* + \tilde{\alpha} w^0$, then

$$\operatorname{div} w_q = \operatorname{div} w^* + \tilde{\alpha} \operatorname{div} w^0 = q^* + \tilde{\alpha} q^0 = q \tag{4.20}$$

and

$$\|w_q\|_V \leqslant \|w^*\|_V + |\tilde{\alpha}| \cdot \|w^0\|_V \leqslant c\|q\|_M. \tag{4.21}$$

(iii) Then the B–B condition (4.15) follows from (ii) immediately.

Lemma 4.3. *The bilinear forms $A_0(u,v) + A_1(u,v)$ and $A_0(u,v) + A_1^N(u,v)$ are bounded on $V \times V$, i.e., there is a constant $\alpha_2 > 0$, such that*

$$|A_0(u,v) + A_1(u,v)| \leqslant \alpha_2 \|u\|_V \cdot \|v\|_V \quad \forall u,\ v \in V, \tag{4.22}$$

$$|A_0(u,v) + A_1^N(u,v)| \leqslant \alpha_2 \|u\|_V \cdot \|v\|_V \quad \forall u,\ v \in V. \tag{4.23}$$

Furthermore

$$A_0(v,v) + A_1(v,v) \geqslant A_0(v,v) + A_1^N(v,v) \geqslant 0 \quad \forall v \in V,\ N = 0,1,2,\ldots \tag{4.24}$$

Proof. For any $u, v \in V$, we know that $u_1|_{\Gamma_c}$ and $v_1|_{\Gamma_c}$ belong to $H^{1/2}(\Gamma_c)$, $u_2|_{\Gamma_c}$ and $v_2|_{\Gamma_c}$ belong to $H_0^{1/2}(\Gamma_c)$ by the trace theorem [1]. Let

$$u_1(c,x_2) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x_2}{L} \qquad a_m = \frac{2}{L}\int_0^L u_1(c,x_2) \cos\frac{m\pi x_2}{L}\, \mathrm{d}x_2, \tag{4.25}$$

$$u_2(c,x_2) = \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x_2}{L} \qquad b_m = \frac{2}{L}\int_0^L u_2(c,x_2) \sin\frac{m\pi x_2}{L}\, \mathrm{d}x_2, \tag{4.26}$$

$$v_1(c,x_2) = \frac{\tilde{a}_0}{2} + \sum_{m=1}^{\infty} \tilde{a}_m \cos\frac{m\pi x_2}{L} \qquad \tilde{a}_m = \frac{2}{L}\int_0^L v_1(c,x_2) \cos\frac{m\pi x_2}{L}\, \mathrm{d}x_2, \tag{4.27}$$

$$v_2(c,x_2) = \sum_{m=1}^{\infty} \tilde{b}_m \sin\frac{m\pi x_2}{L} \qquad \tilde{b}_m = \frac{2}{L}\int_0^L v_2(c,x_2) \sin\frac{m\pi x_2}{L}\, \mathrm{d}x_2. \tag{4.28}$$

Then by the equivalent norm on the spaces $H^{1/2}(\Gamma_c)$ and $H_0^{1/2}(\Gamma_c)$ [18], we know that there exists a constant α_3 such that

$$\left[\sum_{m=1}^{\infty} m\left(a_m^2 + b_m^2\right)\right]^{1/2} \leqslant \alpha_3 \|u\|_V \quad \left[\sum_{m=1}^{\infty} m\left(\tilde{a}_m^2 + \tilde{b}_m^2\right)\right]^{1/2} \leqslant \alpha_3 \|v\|_V. \tag{4.29}$$

Substituting (4.25)–(4.28) into (4.8) and (4.9), we have that

$$\begin{aligned} A_1(u,v) = \sum_{m=1}^{\infty} \Bigg[& \frac{\nu(m\pi - L\lambda^-(m)) a_m \tilde{a}_m}{2} + \frac{\nu m\pi(m\pi + L\lambda^-(m)) b_m \tilde{a}_m}{2L\lambda^-(m)} + \frac{\nu(m\pi + L\lambda^-(m)) a_m \tilde{b}_m}{2} \\ & + \frac{\nu(m\pi - L\lambda^-(m)) b_m \tilde{b}_m}{2} \Bigg]. \end{aligned} \tag{4.30}$$

Since $0 < -L\lambda^-(m) \leqslant m\pi$, $\forall m \in \mathbb{N}$, and

$$\lim_{m\to+\infty} \frac{m\pi}{-L\lambda^-(m)} = 1,$$

it follows that there is a constant $c_1 > 0$, such that

$$|A_1(u,v)| \leqslant c_1 \sum_{m=1}^{\infty} m(|a_m\tilde{a}_m| + |b_m\tilde{a}_m| + |a_m\tilde{b}_m| + |b_m\tilde{b}_m|)$$
$$\leqslant c_1 \left[\sum_{m=1}^{\infty} m\left(a_m^2 + b_m^2\right)\right]^{1/2} \cdot \left[\sum_{m=1}^{\infty} m\left(\tilde{a}_m^2 + \tilde{b}_m^2\right)\right]^{1/2} \leqslant c_2 \|u\|_V \cdot \|v\|_V, \quad (4.31)$$

where $c_2 = c_1 \cdot \alpha_3^2$. Thus the inequality (4.22) follows from (4.31) and the trace theorem [1]. Furthermore

$$A_1(v,v) = \frac{v}{2} \sum_{m=1}^{\infty} \left[(m\pi - L\lambda^-(m))(\tilde{a}_m^2 + \tilde{b}_m^2) + \frac{(m\pi + L\lambda^-(m))^2}{L\lambda^-(m)} \tilde{a}_m\tilde{b}_m\right]$$
$$\geqslant \frac{v}{4} \sum_{m=1}^{\infty} \left[2m\pi - 2L\lambda^-(m) + \frac{(m\pi + L\lambda^-(m))^2}{L\lambda^-(m)}\right] \left(\tilde{a}_m^2 + \tilde{b}_m^2\right)$$
$$= \sum_{m=1}^{\infty} \left(m\pi v - \frac{aL}{4}\right) \left(\tilde{a}_m^2 + \tilde{b}_m^2\right) \geqslant -\frac{a}{2} \sum_{m=1}^{\infty} \frac{L}{2} \left(\tilde{a}_m^2 + \tilde{b}_m^2\right)$$
$$\geqslant -\frac{a}{2} \int_{\Gamma_c} v(c,x_2) \cdot v(c,x_2) \, \mathrm{d}x_2. \quad (4.32)$$

Therefore

$$A_0(v,v) + A_1(v,v) \geqslant \frac{a}{2} \int_{\Gamma_c} v \cdot v \, \mathrm{d}x_2 - \frac{a}{2} \int_{\Gamma_c} v \cdot v \, \mathrm{d}x_2 = 0 \quad \forall v \in V. \quad (4.33)$$

In a similar way, we get the inequality (4.23) and the other part of (4.24). Hence the proof is completed.

On the other hand, if u is a solution of problem (4.1) and (4.2) and $u|_{\Gamma_c} \in H^2(\Gamma_c) \times H^2(\Gamma_c)$, then

$$u_1|_{\Gamma_c} \in \left\{ w \in H^2(0,L), \left.\frac{\partial w}{\partial z}\right|_{z=0,L} = 0 \right\}$$

and

$$u_2|_{\Gamma_c} \in \left\{ w \in H^2(0,L), w|_{z=0,L} = 0 \right\}.$$

Thus we have that [18]

$$\left[\sum_{m=1}^{\infty} m^4\left(a_m^2 + b_m^2\right)\right]^{1/2} \leqslant \alpha_4 \|v\|_{2,\Gamma_c}, \quad (4.34)$$

where α_4 is a constant. Hence

$$\left|A_1(u,v) - A_1^N(u,v)\right| = \left| \sum_{m=N+1}^{\infty} \left[\frac{v(m\pi - L\lambda^-(m))a_m\tilde{a}_m}{2} + \frac{vm\pi(m\pi + L\lambda^-(m))b_m\tilde{a}_m}{2L\lambda^-(m)} + \frac{v(m\pi + L\lambda^-(m))a_m\tilde{b}_m}{2} + \frac{v(m\pi - L\lambda^-(m))b_m\tilde{b}_m}{2} \right] \right|$$
$$\leqslant c_1 \left[\sum_{m=N+1}^{\infty} m\left(a_m^2 + b_m^2\right)\right]^{1/2} \cdot \left[\sum_{m=N+1}^{\infty} m\left(\tilde{a}_m^2 + \tilde{b}_m^2\right)\right]^{1/2}$$
$$\leqslant \frac{c_1}{(N+1)^{3/2}} \left[\sum_{m=N+1}^{\infty} m^4\left(a_m^2 + b_m^2\right)\right]^{1/2} \cdot \left[\sum_{m=N+1}^{\infty} m\left(\tilde{a}_m^2 + \tilde{b}_m^2\right)\right]^{1/2}$$
$$\leqslant \frac{c_2}{(N+1)^{3/2}} \|u\|_{2,\Gamma_c} \cdot \|v\|_V, \quad (4.35)$$

where c_2 is a constant independent of N, u and v. Thus we obtain the following result.

Lemma 4.4. *If u is a solution of the problem* (4.1) *and* (4.2) *and $u|_{\Gamma_c} \in H^2(\Gamma_c) \times H^2(\Gamma_c)$, then the following holds*:

$$\left|A_1(u,v) - A_1^N(u,v)\right| \leqslant \frac{c}{(N+1)^{3/2}} \|u\|_{2,\Gamma_c} \cdot \|v\|_V \quad \forall v \in V, \tag{4.36}$$

where c is a constant independent of N, u, v and p.

Lemma 4.5. *The trilinear form $A_2(u,v,w)$ is bounded on $V \times V \times V$, namely there is a constant α_5 such that* [8]

$$|A_2(u,v,w)| \leqslant \alpha_5 \|u\|_V \cdot \|v\|_V \cdot \|w\|_V \quad \forall u,v,w \in V. \tag{4.37}$$

Furthermore

$$A_2(u,v,v) = 0 \quad \forall u,v \in V. \tag{4.38}$$

By the trace theorem [1], we know that $f(v)$ is a bounded functional on V, i.e., there exists a constant α_6 such that

$$|f(v)| \leqslant \alpha_6 \|v\|_V \quad \forall v \in V. \tag{4.39}$$

Let $V_a = \{v \in M \mid \operatorname{div} v = 0\}$ and

$$\Phi = \sup_{u,v,w \in V \setminus \{0\}} \frac{A_2(u,v,w)}{\|u\|_V \cdot \|v\|_V \cdot \|w\|_V}, \tag{4.40}$$

$$g(u^{(0)}; v) = f(v) - A(u^{(0)}, v) - A_0(u^{(0)}, v) - A_1(u^{(0)}, v) - A_2(u^{(0)}, u^{(0)}, v), \tag{4.41}$$

$$\rho(u^{(0)}) = \sup_{v \in V} \frac{A_2(v, u^{(0)}, v)}{\|v\|_V^2} \quad \forall u^{(0)} \in M, \tag{4.42}$$

$$\|g(u^{(0)})\|_{V'} = \sup_{v \in V} \frac{g(u^{(0)}; v)}{\|v\|_V}, \tag{4.43}$$

$$v_0 = \inf_{u^{(0)} \in V_a} \quad \rho(u^{(0)}) + [\Phi \cdot \|g(u^{(0)})\|_{V'}]^{1/2}. \tag{4.44}$$

Combining Lemmas 4.1–4.5 and Theorems 1.2 and 1.4 in Chapter IV of [8], we obtain

Theorem 4.1. *The variational problem* (4.1) *and* (4.2) *has at least one solution $(u,p) \in M \times W$ and problem* (4.10) *and* (4.11) *has at least one solution $(u_N, p_N) \in M \times W$ for $N = 0, 1, 2, \ldots$*

Theorem 4.2. *For $v\beta_0 > v_0$, the variational problem* (4.1) *and* (4.2) *has a unique solution $(u,p) \in M \times W$ and problem* (4.10) *and* (4.11) *has a unique solution $(u_N, p_N) \in M \times W$ for $N = 0, 1, 2, \ldots$ Furthermore if $u|_{\Gamma_c} \in H^2(\Gamma_c) \times H^2(\Gamma_c)$, we have the following error bound*:

$$\|u - u_N\|_V + \|p - p_N\|_W \leqslant \frac{c}{(N+1)^{3/2}} \|u\|_{2,\Gamma_c}, \tag{4.45}$$

where c is a generic constant independent of u, p and N.

Proof. Combining Lemmas 4.1–4.5 and the technique used in Theorems 2.4 in Chapter IV of [8], we know that problem (4.1) and (4.2) has a unique solution $(u,p) \in M \times W$ and problem (4.10) and (4.11) has a unique solution $(u_N, p_N) \in M \times W$ for $N = 0, 1, 2, \ldots$

Let $u^{(0)} \in V_a$. Taking $v = w := u - u^{(0)} \in V$ in (4.1) and $q = p$ in (4.2), noting (4.13), (4.24), (4.38), (4.41), (4.42) and (4.43), we obtain

$$
\begin{aligned}
\nu\beta_0\|w\|_V^2 &\leqslant A(w,w) \leqslant A(w,w) + A_0(w,w) + A_1(w,w) + A_2(u,w,w)\\
&= A(u,w) + A_0(u,w) + A_1(u,w) + A_2(u,u,w) - A(u^{(0)},w) - A_0(u^{(0)},w)\\
&\quad - A_1(u^{(0)},w) - A_2(u,u^{(0)},w)\\
&= f(w) - A(u^{(0)},w) - A_0(u^{(0)},w) - A_1(u^{(0)},w) - A_2(u^{(0)},u^{(0)},w) - A_2(w,u^{(0)},w)\\
&\leqslant \left\|g(u^{(0)})\right\|_{V'}\|w\|_V + \rho(u^{(0)})\|w\|_V^2.
\end{aligned}
\tag{4.46}
$$

Thus we have that

$$
\left\|u - u^{(0)}\right\|_V = \|w\|_V \leqslant \frac{\|g(u^{(0)})\|_{V'}}{\nu\beta_0 - \rho(u^{(0)})} \quad \forall u^{(0)} \in V_{\bar{a}} = \{w \in V_a \mid \nu\beta_0 > \rho(w)\}. \tag{4.47}
$$

Combining the above inequality, the triangle inequality and Lemma 2.3 [8, p. 287], we have that

$$
\|u\|_V \leqslant \inf_{u^{(0)}\in V_{\bar{a}}} \left[\left\|u^{(0)}\right\|_V + \frac{\left\|g(u^{(0)})\right\|_{V'}}{\nu\beta_0 - \rho(u^{(0)})}\right] \leqslant \inf_{u^{(0)}\in V_{\bar{a}}} \left[\left\|u^{(0)}\right\|_V + \frac{\left\|g(u^{(0)})\right\|_{V'}}{\nu\beta_0}\right]. \tag{4.48}
$$

In a similar way, we obtain

$$
\left\|u_N - u^{(0)}\right\|_V \leqslant \frac{\left\|g(u^{(0)})\right\|_{V'}}{\nu\beta_0 - \rho(u^{(0)})} \quad \forall u^{(0)} \in V_{\bar{a}} \quad \|u_N\|_V \leqslant \inf_{u^{(0)}\in V_{\bar{a}}} \left[\left\|u^{(0)}\right\|_V + \frac{\left\|g(u^{(0)})\right\|_{V'}}{\nu\beta_0}\right]. \tag{4.49}
$$

Let $e_u = u - u_N$, $e_p = p - p_N$. Subtracting (4.1) and (4.2) from (4.10) and (4.11), we obtain

$$
\begin{aligned}
&A(e_u,v) + A_0(e_u,v) + A_1^N(e_u,v) + A_2(e_u,u,v) + A_2(u_N,e_u,v) + B(v,e_p)\\
&\quad = A_1^N(u,v) - A_1(u,v) \quad \forall v \in V,
\end{aligned}
\tag{4.50}
$$

$$
B(e_u,q) = 0 \quad \forall q \in W. \tag{4.51}
$$

Taking $v = e_u$ in (4.50) and $q = e_p$ in (4.51), noting (4.13), (4.24), (4.38), (4.36), (4.40), (4.42) and (4.47), we obtain

$$
\begin{aligned}
\nu\beta_0\|e_u\|_V^2 &\leqslant A(e_u,e_u) \leqslant A(e_u,e_u) + A_0(e_u,e_u) + A_1^N(e_u,e_u)\\
&= A_1^N(u,e_u) - A_1(u,e_u) - A_2(e_u,u-u^{(0)},e_u) - A_2(e_u,u^{(0)},e_u)\\
&\leqslant \frac{c}{(N+1)^{3/2}}\|u\|_{2,\Gamma_c}\cdot\|e_u\|_V + \Phi\|e_u\|_V^2\cdot\left\|u-u^{(0)}\right\|_V + \rho(u^{(0)})\|e_u\|_V^2\\
&\leqslant \frac{c}{(N+1)^{3/2}}\|u\|_{2,\Gamma_c}\cdot\|e_u\|_V + \frac{\Phi\left\|g(u^{(0)})\right\|_{V'}}{\nu\beta_0 - \rho(u^{(0)})}\|e_u\|_V^2 + \rho(u^{(0)})\|e_u\|_V^2 \quad \forall u^{(0)} \in V_{\bar{a}}.
\end{aligned}
\tag{4.52}
$$

Combining (4.52) and (4.44) and Lemma 2.3 [8, p. 287], we have that

$$
\|e_u\| \leqslant \inf_{u^{(0)}\in V_{\bar{a}}} \frac{\nu\beta_0 - \rho(u^{(0)})}{[\nu\beta_0 - \rho(u^{(0)})]^2 - \Phi\|g(u^{(0)})\|_{V'}}\cdot\frac{c}{(N+1)^{3/2}}\|u\|_{2,\Gamma_c} \leqslant \frac{c}{(N+1)^{3/2}}\|u\|_{2,\Gamma_c}. \tag{4.53}
$$

From Eq. (4.50), noting (4.12), (4.23), (4.36), (4.48), (4.49), (4.40) and (4.53), we have that

$$
\begin{aligned}
\left|B(v,e_p)\right| &= \left|A_1^N(u,v) - A_1(u,v) - A(e_u,v) - A_0(e_u,v) - A_1^N(e_u,v) - A_2(e_u,u,v) - A_2(u_N,e_u,v)\right|\\
&\leqslant \left[\frac{c}{(N+1)^{3/2}}\|u\|_{2,\Gamma_c}\|v\|_V + \left(\alpha_0 + \alpha_2 + \Phi\|u\|_V + \Phi\|u_N\|_V\right)\|e_u\|_V\right]\cdot\|v\|_V\\
&\leqslant \frac{\bar{c}}{(N+1)^{3/2}}\|u\|_{2,\Gamma_c}\cdot\|v\|_V \quad \forall v \in V.
\end{aligned}
\tag{4.54}
$$

Combining the above inequality and (4.15), we obtain

$$\|e_p\|_W = \|p - p_{_N}\|_M \leqslant \frac{1}{\beta_1} \sup_{v\in V\setminus\{0\}} \frac{B(v, e_p)}{\|v\|_V} \leqslant \frac{\bar{c}}{\beta_1 (N+1)^{3/2}} \|u\|_{2,\Gamma_c}. \tag{4.55}$$

Then the inequality (4.45) follows from (4.53) and (4.55) immediately.

5. The finite element approximation

Let $\mathscr{T}^h$ be a regular partition of the domain Ω_T. Suppose M^h, V^h and W^h are finite element subsets of M, V and W. Furthermore, we also assume that they are the optimally compatible, i.e., V^h and W^h should satisfy the following conditions [14]:

(a) The errors $\inf_{v^h\in V^h} \|u - v^h\|_V$ and $\inf_{q^h\in W^h} \|p - q^h\|_W$ have the same order in h, i.e., there is a constant α, such that

$$\inf_{v^h\in V^h} \left\|u - v^h\right\|_V \leqslant \alpha h^m |u|_{m+1,\Omega_T} \qquad \inf_{q^h\in W^h} \left\|p - q^h\right\|_W \leqslant \alpha h^m |p|_{m,\Omega_T}. \tag{5.1}$$

(b) There exists a constant β independent of h, such that

$$\sup_{v^h\in V^h\setminus\{0\}} \frac{B(v^h, q)}{\|v^h\|_V} \geqslant \beta \|q\|_M \quad \forall q \in W^h. \tag{5.2}$$

Then the finite element approximation of problem (4.10) and (4.11) is:

Find $(u^h_{_N}, p^h_{_N}) \in M^h \times W^h$ such that

$$A(u^h_{_N}, v) + A_0(u^h_{_N}, v) + A^N_1(u^h_{_N}, v) + A_2(u^h_{_N}, u^h_{_N}, v) + B(v, p^h_{_N}) = f(v) \quad \forall v \in V^h, \tag{5.3}$$

$$B(u^h_{_N}, q) = 0 \quad \forall q \in W^h. \tag{5.4}$$

Theorem 5.1. *Assume the hypothesis of Theorem* 4.2. *Then problem* (5.3) *and* (5.4) *has a unique solution* $(u^h_{_N}, p^h_{_N}) \in M^h \times W^h$.

The proof of this theorem is similar to the proof of Theorem 4.2. It is omitted here.

Theorem 5.2. *Assume the hypothesis of Theorem* 4.2. *Let* (u, p) *be the unique solution of problem* (4.1)–(4.11), (u^h_N, p^h_N) *the unique solution of problem* (5.3) *and* (5.4). *Assume* $u \in H^{m+1}(\Omega_T) \times H^{m+1}(\Omega_T)$, $u|_{\Gamma_c} \in H^2(\Gamma_c) \times H^2(\Gamma_c)$ and $p \in H^m(\Omega_T)$. *Then we have the following error estimate*:

$$\left\|u - u^h_{_N}\right\|_V + \left\|p - p^h_{_N}\right\|_W \leqslant ch^m[|u|_{m+1,\Omega_T} + |p|_{m,\Omega_T}] + \frac{\bar{c}}{(N+1)^{3/2}} \|u\|_{2,\Gamma_c}, \tag{5.5}$$

where c, $\bar{c}$ *are generic constants independent of* u, p, h *and* N.

Proof. For the errors $\|u_{_N} - u^h_{_N}\|_V$ and $\|p_N - p^h_N\|_W$, by a standard technique of mixed finite element method [8] we have that

$$\left\|u_{_N} - u^h_{_N}\right\|_V + \|p_N - p^h_N\|_W \leqslant c\left[\inf_{v^h\in M^h} \left\|u_{_N} - v^h\right\|_V + \inf_{q^h\in W^h} \left\|p_{_N} - q^h\right\|_W\right]. \tag{5.6}$$

Combining (5.6) and the triangle inequality, we obtain

$$\begin{aligned}\|u-u_N^h\|_V+\|p-p_N^h\|_W &\leqslant \|u_N-u_N^h\|_V+\|p_N-p_N^h\|_W+\|u-u_N\|_V+\|p-p_N\|_W\\ &\leqslant c\left[\inf_{v^h\in M^h}\|u_N-v^h\|_V+\inf_{q^h\in W^h}\|p_N-q^h\|_W\right]+\|u-u_N\|_V+\|p-p_N\|_W\\ &\leqslant c\left[\inf_{v^h\in M^h}\|u-v^h\|_V+\inf_{q^h\in W^h}\|p-q^h\|_W\right]+(c+1)\left[\|u-u_N\|_V+\|p-p_N\|_W\right].\end{aligned} \tag{5.7}$$

Then the error estimate (5.5) follows from (5.7) and (4.45) immediately.

From the error estimate (5.5), we can see that the error consists of two parts, one of them is from the finite element approximation, the other is from the approximate artificial boundary condition (3.21). For example, for the Taylor/Hood (say, $P2/P1$) element [8] the error estimate (5.5) holds with $m=2$.

6. Numerical example

To show the accuracy of our boundary conditions (3.21), we apply them to compute flows around different obstacles. In our computations, the Taylor/Hood ($P2/P1$) element is used.

Example 1. The flow around a rectangular cylinder obstacle. The obstacle Ω_i is defined by the domain

$$\Omega_i=\left\{x\in\mathbb{R}^2 \mid 0.8<x_1<1.2,\ -\frac{l}{10}<x_2<\frac{l}{10}\right\}.$$

Then the bounded computational domain Ω_T is given by

$$\Omega_T=\left\{x\in\mathbb{R}^2 \mid b<x_1<c,\ -l<x_2<l\right\}\setminus\bar{\Omega}_i.$$

We take $b=0$, $c=2.8$, $l=0.5$, $a=1.0$.

As Ω_i is a rectangular, we assume symmetry of the flow and consider only the upper half-domain of Ω_T. Thus the following slip boundary condition is posed on the boundary $\Gamma=\{(x_1,0.0)\mid b\leqslant x_1\leqslant 0.8$ or $1.2\leqslant x_1\leqslant c\}$:

$$\frac{\partial u_1(x)}{\partial x_2}=u_2(x)=0,\quad x\in\Gamma. \tag{6.1}$$

Three meshes were used in our computation. Fig. 2 shows the partition $\mathcal{T}^h$ for mesh A of Ω_T. Mesh B was generated by dividing each triangle in mesh A into four equal smaller triangles. Mesh C was generated similarly from mesh B. To test the effect of the approximate artificial boundary condition (3.21), let (u_∞^h,p_∞^h) denote the solution of problem (5.3) and (5.4) with $N=N^*$ sufficiently large. In our computation we take $N^*=50$. Let (u_N^h,p_N^h) denote the solution of problem (5.3) and (5.4).

Tables 1 and 2 show the maximum error of $u_\infty^h-u_N^h$ over mesh points and $\|u_\infty^h-u_N^h\|_{0,\Omega_T}$, $\|u_\infty^h-u_N^h\|_{1,\Omega_T}$, $\|p_\infty^h-p_N^h\|_{0,\Omega_T}$ on mesh B with different kinematic viscosity ν. Furthermore Fig. 3 shows the velocity field on mesh C with $\nu=0.01$. Figs. 4–6 show contours of the velocity and pressure on mesh C with $\nu=0.01$. Fig. 7 shows the velocity field on mesh C with $\nu=0.002$.

From Tables 1 and 2 and Figs. 3–6, we can see that the approximate artificial boundary condition (3.21) is very effective for the N–S equations and more accurate than the stress-free boundary condition (i.e.,

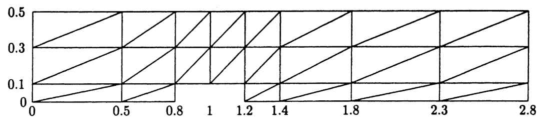

Fig. 2. Mesh A.

Table 1
The effect of artificial boundary conditions ($\nu = 0.01$)

Error	$N=0$	$N=2$	$N=4$	$N=6$	$N=10$
$\max \lvert u^h_\infty - u^h_N \rvert$	2.5470E-02	9.3415E-04	3.9685E-04	2.4541E-04	1.3980E-04
$\lVert u^h_\infty - u^h_N \rVert_{0,\Omega_T}$	3.6260E-03	1.3577E-04	3.0273E-05	2.4014E-05	6.8662E-06
$\lVert u^h_\infty - u^h_N \rVert_{1,\Omega_T}$	0.4639	2.2811E-02	8.5664E-03	7.5692E-03	3.9018E-03
$\lVert p^h_\infty - p^h_N \rVert_{0,\Omega_T}$	8.8937E-04	6.7310E-05	5.8346E-06	4.5829E-06	2.3533E-06

Table 2
The effect of artificial boundary conditions ($\nu = 0.002$)

Error	$N=0$	$N=2$	$N=4$	$N=6$	$N=10$
$\max \lvert u^h_\infty - u^h_N \rvert$	0.1093	2.1962E-2	1.2245E-2	6.1613E-3	1.9766E-3
$\lVert u^h_\infty - u^h_N \rVert_{0,\Omega_T}$	3.0789E-2	3.1106E-3	1.0901E-3	6.9587E-4	7.2452E-5
$\lVert u^h_\infty - u^h_N \rVert_{1,\Omega_T}$	3.3880	0.4341	0.1479	0.1062	1.7403E-2
$\lVert p^h_\infty - p^h_N \rVert_{0,\Omega_T}$	4.8924E-2	1.2379E-3	1.8367E-4	1.0127E-4	5.5941E-6

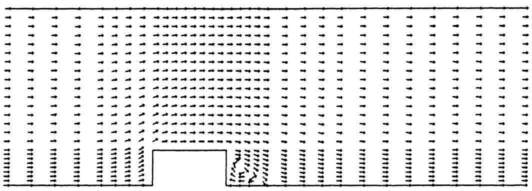

Fig. 3. The velocity field ($\nu = 0.01$).

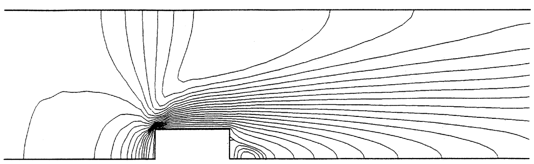

Fig. 4. The contour of the first component of the velocity ($\nu = 0.01$).

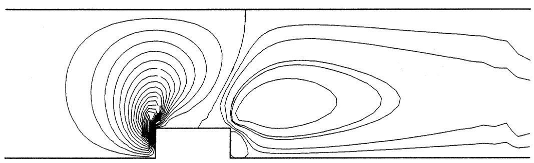

Fig. 5. The contour of the second component of the velocity ($\nu = 0.01$).

$N = 0$ in (3.21)) which is often used in engineering literatures. The results also suggest that only a few terms in the bilinear form $A_1(u, v)$ (see (4.8)) are needed in order to get good accuracy.

Here we remark on the computational cost associated with our artificial boundary conditions. Compared with the stress-free boundary condition, the additional computational cost consists of two parts: (a) Computing the stiff matrix related to the bilinear form $A_1(u, v)$ (see (4.10) and (4.9)). (b) The extra computational work needed for the solution of the linear system due to the inclusion of the artificial boundary condition. Since the integrands in $A_1(u, v)$ only involve simple trigonometric functions, the cost of part (a) is negligible by evaluating these integral explicitly. The nonlocality is only within the nodes which

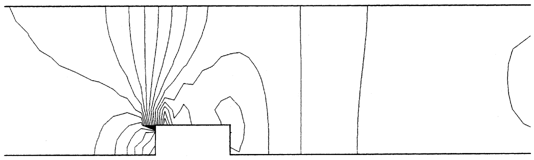

Fig. 6. The contour of the pressure ($\nu = 0.01$).

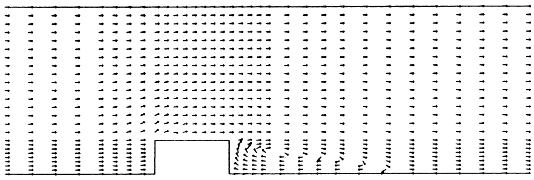

Fig. 7. The velocity field ($\nu = 0.002$).

are on the artificial boundary. The number of these nodes is far less than the number of the total nodes in real large-scale applications. Furthermore the contribution of our artificial boundary conditions to the global stiff matrix does not degrade the sparseness of the global stiff matrix. Thus the cost of part (b) is far less than the total cost in the computation and is independent of the value of N, the number of terms of the artificial boundary conditions. One can find detail discussions in [9] for similar cases. In our case, the additional computational cost with using the high-order artificial boundary conditions is no more than 5% of the cost with using the stress-free boundary condition. But the solutions are much more accurate.

Another issue is the effect of locations of the artificial boundary. We set

$$\Omega_0 = \{x \in \mathbb{R}^2 \mid b < x_1 < c_0,\ -l < x_2 < l\} \setminus \bar{\Omega}_i,$$

and take $b = 0$, $c_0 = 2.8$, $l = 0.5$, $a = 1.0$. Let (u^h_∞, p^h_∞) denote the solution of the problem (5.3) and (5.4) with $N = N^*$ and $c = 4.8$. Tables 3 and 4 show the maximum error of $u^h_\infty - u^h_N$ over mesh points in Ω_0 and $\|u^h_\infty - u^h_N\|_{0,\Omega_0}$, $\|u^h_\infty - u^h_N\|_{1,\Omega_0}$, $\|p^h_\infty - p^h_N\|_{0,\Omega_0}$ on mesh B with different locations of the artificial boundary for $\nu = 0.1$.

Tables 3 and 4 show that the numerical solutions become more accurate when one chooses the artificial boundary farther from the obstacle. In order to obtain the same accuracy, the computational domain with

Table 3
The effect of locations of the artificial boundary with stress-free boundary condition

Error	$\max \lvert u^h_\infty - u^h_0 \rvert$	$\|u^h_\infty - u^h_0\|_{0,\Omega_0}$	$\|u^h_\infty - u^h_0\|_{1,\Omega_0}$	$\|p^h_\infty - p^h_N\|_{0,\Omega_0}$
$c = 2.8$	4.2834E-3	1.5870E-3	0.1718	2.2026E-3
$c = 3.8$	2.0641E-4	2.7787E-7	7.4609E-4	2.9615E-7
$c = 4.8$	9.9610E-6	5.3598E-9	1.0354E-4	3.4761E-9

Table 4
The effect of locations of the artificial boundary with artificial boundary conditions of $N = 10$

Error	$\max \lvert u^h_\infty - u^h_N \rvert$	$\|u^h_\infty - u^h_N\|_{0,\Omega_0}$	$\|u^h_\infty - u^h_N\|_{1,\Omega_0}$	$\|p^h_\infty - p^h_N\|_{0,\Omega_0}$
$c = 2.8$	5.5765E-5	8.4865E-6	4.2229E-3	5.5325E-5
$c = 3.8$	2.6316E-6	5.4682E-9	1.0457E-4	1.2806E-7
$c = 4.8$	1.6880E-8	5.3583E-9	1.0351E-4	3.4420E-9

stress-free boundary condition needs to be much larger than that with our high-order artificial boundary conditions.

Example 2. The flow around a quadrilateral cylinder obstacle. The obstacle Ω_i is defined by the domain

$$\Omega_i = \left\{ x \in \mathbb{R}^2 \mid 0.8 < x_1 < 1.2,\ -\frac{1}{2}(0.2 - |x - 1.0|) < x_2 < \frac{1}{2}(0.2 - |x - 1.0|) \right\}.$$

Then the bounded computational domain Ω_T is given by

$$\Omega_T = \left\{ x \in \mathbb{R}^2 \mid b < x_1 < c,\ -l < x_2 < l \right\} \setminus \bar{\Omega}_i.$$

We take $b = 0$, $c = 2.8$, $l = 0.5$, $a = 1.0$. The meshes used in this example are similar as those in Example 1. Figs. 8 and 9 show the velocity fields for this example with $\nu = 0.01$ and $\nu = 0.002$ by using our approximate artificial boundary condition (3.21) with $N = N^*$.

Example 3. The flow around a circular cylinder obstacle. The obstacle Ω_i is defined by the domain

$$\Omega_i = \left\{ x \in \mathbb{R}^2 \mid \sqrt{(x_1 - 1.0)^2 + x_2^2} < 0.2 \right\}.$$

Then the bounded computational domain Ω_T is given by

$$\Omega_T = \left\{ x \in \mathbb{R}^2 \mid b < x_1 < c,\ -l < x_2 < l \right\} \setminus \bar{\Omega}_i.$$

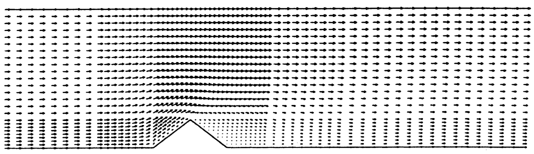

Fig. 8. The velocity field for Example 2 ($\nu = 0.01$).

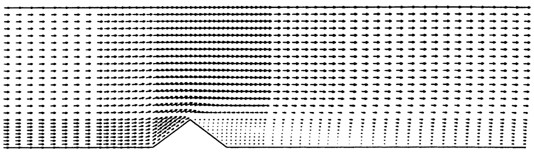

Fig. 9. The velocity field for Example 2 ($\nu = 0.002$).

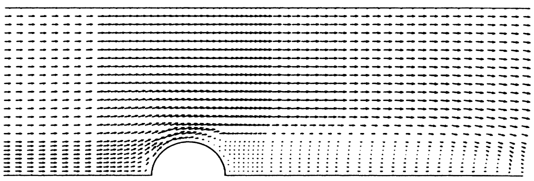

Fig. 10. The velocity field for Example 3 ($\nu = 0.01$).

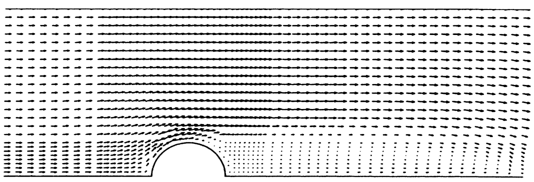

Fig. 11. The velocity field for Example 3 ($\nu = 0.002$).

We take $b = 0$, $c = 2.8$, $l = 0.5$, $a = 1.0$. The meshes used in this example are similar as those in Example 1. Figs. 10 and 11 show the velocity fields for this example with $\nu = 0.01$ and $\nu = 0.002$ by using our approximate artificial boundary condition (3.21) with $N = N^*$. These results are coincide with the results in [7] which were obtained in a large domain.

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