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SINGULAR LIMITS OF KLEIN–GORDON–SCHRÖDINGER EQUATIONS TO SCHRÖDINGER–YUKAWA EQUATIONS*

WEIZHU BAO†, XUANCHUN DONG†, AND SHU WANG‡

Abstract. In this paper, we study analytically and numerically the singular limits of the nonlinear Klein–Gordon–Schrödinger (KGS) equations in $\mathbb{R}^d$ $(d = 1, 2, 3)$ both with and without a damping term to the nonlinear Schrödinger–Yukawa (SY) equations. By using the two-scale matched asymptotic expansion, formal limits of the solution of the KGS equations to the solution of the SY equations are derived with an additional correction in the initial layer. Then for general initial data, weak and strong convergence results are established for the formal limits to provide rigorous mathematical justification for the matched asymptotic approximation by using the weak compactness argument and the (modulated) energy method, respectively. In addition, for well-prepared initial data, optimal quadratic and linear convergence rates are obtained for the KGS equations both with and without the damping term, respectively, and for ill-prepared initial data, the optimal linear convergence rate is obtained. Finally, numerical results for the KGS equations are presented to confirm the asymptotic and analytic results.

Key words. Klein–Gordon–Schrödinger equations, Schrödinger–Yukawa equations, weak convergence, strong convergence, matched asymptotic expansion, convergence rate

AMS subject classifications. 35Q55, 35Q70, 65M70, 81Q05

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1. Introduction.

In this paper, we study analytically and numerically the singular limits of the following nonlinear Klein–Gordon–Schrödinger (KGS) equations describing a system of a conserved scalar nucleon interacting with a neutral scalar meson coupled though the Yukawa interaction [1, 3, 6, 18]:

$$i\,\partial_t\psi^\varepsilon(\mathbf{x},t) + \Delta\psi^\varepsilon(\mathbf{x},t) + \phi^\varepsilon(\mathbf{x},t)\psi^\varepsilon(\mathbf{x},t) = 0, \quad \mathbf{x}\in\mathbb{R}^d(d=1,2,3),\ t>0, \tag{1.1}$$

$$\varepsilon^2\partial_{tt}\phi^\varepsilon(\mathbf{x},t) + \varepsilon\gamma\partial_t\phi^\varepsilon(\mathbf{x},t) - \Delta\phi^\varepsilon(\mathbf{x},t) + \beta\phi^\varepsilon(\mathbf{x},t) - |\psi^\varepsilon(\mathbf{x},t)|^2 = 0, \tag{1.2}$$

with initial conditions

$$\psi^\varepsilon(\mathbf{x},0) = \psi_0^\varepsilon(\mathbf{x}), \quad \phi^\varepsilon(\mathbf{x},0) = \phi_0^\varepsilon(\mathbf{x}), \quad \partial_t\phi^\varepsilon(\mathbf{x},0) = \phi_1^\varepsilon(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^d. \tag{1.3}$$

Here the complex-valued unknown function $\psi^\varepsilon = \psi^\varepsilon(\mathbf{x},t)$ represents a scalar nucleon field, the real-valued unknown function $\phi^\varepsilon = \phi^\varepsilon(\mathbf{x},t)$ represents a scalar meson field, $\varepsilon > 0$ is a parameter inversely proportional to the speed of light, and $\gamma \geq 0$ and $\beta \geq 0$ are two constants. In fact, when $\varepsilon = 1$, $\gamma = 0$, and $\beta = 1$, the system (1.1)–(1.2) reduces to the standard KGS equations [6]. When $\gamma > 0$, a damping mechanism is added to the Klein–Gordon equation (1.2).

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†Department of Mathematics and Center for Computational Science and Engineering, National University of Singapore, Singapore 117543 (bao@math.nus.edu.sg, http://www.math.nus.edu.sg/~bao/; dong.xuanchun@nus.edu.sg). The first author acknowledges support by Ministry of Education of Singapore grants R-146-000-120-112 and R-158-000-002-112.

‡College of Applied Sciences, Beijing University of Technology, PingLeYuan 100, Chaoyang District, Beijing 100124, People's Republic of China (wangshu@bjut.edu.cn). This author acknowledges support by the NSFC (grant 10771009) and BSFC (grant 1082001) of China, the Educational Ministry of China (grant NCET-04-0203), and the Personal Ministry of China.

For the KGS equations (1.1)–(1.2) in the classical regime, i.e., $\varepsilon = O(1)$ and $\beta = O(1)$, a series of analytical and numerical studies were reported in the literature. In the analysis aspect, for instance, Guo [9] found global solutions to some problems, Hayashi and von Wahl [12] proved the existence of global strong solutions, Fukuda and Tsutsumi [6, 7, 8] showed the existence and uniqueness of global smooth solutions, and Guo and Miao [10] studied asymptotic behavior of the solution. Also, Ohta [23] established the stability of stationary states, Biler [4] studied attractors of the system, Guo and Li [11, 16] and Ozawa and Tsutsumi [26] studied attractors of the system and asymptotic smoothing effect of the solution, and Lu and Wang [17] found global attractors. In addition, for plane, solitary, and periodic wave solutions of the standard KGS equations, we refer to [5, 13, 30] and the references therein. In the numerics aspect, Xiang [32] proposed a conservative spectral method for discretizing the KGS equations and gave error estimates for this method, Zhang [33] studied a conservative finite difference method for the KGS equations in one dimension (1D), and Bao and Yang [3] designed an efficient and accurate time-splitting pseudospectral method for simulating the KGS equations and applied the method to study numerically wave motion and interaction in 1D, 2D, and 3D. On the other hand, for the KGS equations (1.1)–(1.2) in the singular limit regime, i.e., $0 < \varepsilon \ll 1$ and $\beta = O(1)$, very few analytical and numerical results are available in the literature, although some numerical results were reported in [3] for the case $\varepsilon \to 0$. The main aim of this paper is to provide analytical and numerical results for this singular limit of the KGS equations. We remark that, for the KGS equations (1.1)–(1.2) in the nonrelativistic regime, i.e., $0 < \varepsilon \ll 1$ and $\beta = O\left(\frac{1}{\varepsilon^2}\right)$, due to the fact that the Hamiltonian associated with the system is *not* uniformly bounded, the analysis and computation are quite difficult, and we will address these issues in future work. Therefore, throughout this paper we assume that $\beta = 1$.

Formally setting $\varepsilon \to 0$ in the KGS equations, one can get the following nonlinear Schrödinger–Yukawa (SY) equations [26]:

$$i\,\partial_t \psi^0(\mathbf{x},t) + \Delta\psi^0(\mathbf{x},t) + \phi^0(\mathbf{x},t)\psi^0(\mathbf{x},t) = 0, \tag{1.4}$$

$$-\Delta\phi^0(\mathbf{x},t) + \phi^0(\mathbf{x},t) - |\psi^0(\mathbf{x},t)|^2 = 0, \qquad \mathbf{x} \in \mathbb{R}^d, \quad t \ge 0, \tag{1.5}$$

with the initial condition

$$\psi^0(\mathbf{x},0) = \psi_0^{(l)}(\mathbf{x}) = \psi_0(\mathbf{x}) := \lim_{\varepsilon\to 0} \psi_0^\varepsilon(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{1.6}$$

where $\psi^0(\mathbf{x},t) = \lim_{\varepsilon\to 0} \psi^\varepsilon(\mathbf{x},t)$ and $\phi^0(\mathbf{x},t) = \lim_{\varepsilon\to 0} \phi^\varepsilon(\mathbf{x},t)$. Since there are three initial data, i.e., $\psi_0^\varepsilon(\mathbf{x})$, $\phi_0^\varepsilon(\mathbf{x})$, and $\phi_1^\varepsilon(\mathbf{x})$, to be chosen arbitrarily for the KGS equations (1.1)–(1.3), and there is only one initial data, i.e., $\psi_0(\mathbf{x})$, for the limiting SY equations (1.4)–(1.6), we first find the conditions satisfied by $\phi_0^{(l)}(\mathbf{x}) := \phi^0(\mathbf{x},0)$ and $\phi_1^{(l)}(\mathbf{x}) = \partial_t\phi^0(\mathbf{x},0)$ of the solution of the SY equations. From (1.5), we have

$$\phi^0(\mathbf{x},t) = (-\Delta + I)^{-1}\,|\psi^0(\mathbf{x},t)|^2, \qquad \mathbf{x} \in \mathbb{R}^d, \quad t \ge 0, \tag{1.7}$$

where I is the identity operator. Plugging (1.6) into (1.7) with $t = 0$, we get

$$\phi_0^{(l)}(\mathbf{x}) := \phi^0(\mathbf{x},0) = (-\Delta + I)^{-1}\,|\psi^0(\mathbf{x},0)|^2 = (-\Delta + I)^{-1}\,|\psi_0(\mathbf{x})|^2, \qquad \mathbf{x} \in \mathbb{R}^d. \tag{1.8}$$

In addition, multiplying (1.4) by $\bar\psi^0$ (here $\bar f$ denotes the conjugate of a function f), we obtain

$$i\,\bar\psi^0(\mathbf{x},t)\partial_t\psi^0(\mathbf{x},t) + \bar\psi^0(\mathbf{x},t)\Delta\psi^0(\mathbf{x},t) + \phi^0(\mathbf{x},t)|\psi^0(\mathbf{x},t)|^2 = 0, \qquad \mathbf{x} \in \mathbb{R}^d. \tag{1.9}$$

Subtracting (1.9) from its conjugate, we have

$$\begin{aligned}\partial_t|\psi^0(\mathbf{x},t)|^2 &= -i\left[\psi^0(\mathbf{x},t)\Delta\bar{\psi}^0(\mathbf{x},t) - \bar{\psi}^0(\mathbf{x},t)\Delta\psi^0(\mathbf{x},t)\right] \\ &= -i\nabla\cdot\left[\psi^0(\mathbf{x},t)\nabla\bar{\psi}^0(\mathbf{x},t) - \bar{\psi}^0(\mathbf{x},t)\nabla\psi^0(\mathbf{x},t)\right] \\ &= -\nabla\cdot\mathbf{j}^0(\mathbf{x},t), \qquad \mathbf{x}\in\mathbb{R}^d, \quad t\geq 0,\end{aligned} \tag{1.10}$$

where $\mathbf{j}^0$ is usually called current for the SY equations (1.4)–(1.5) and is defined as

$$\mathbf{j}^0(\mathbf{x},t) = i(\psi^0(\mathbf{x},t)\nabla\bar{\psi}^0(\mathbf{x},t) - \bar{\psi}^0(\mathbf{x},t)\nabla\psi^0(\mathbf{x},t)), \qquad \mathbf{x}\in\mathbb{R}^d, \quad t\geq 0. \tag{1.11}$$

Differentiating (1.7) with respect to t, noticing (1.10), we obtain

$$\begin{aligned}\partial_t\phi^0(\mathbf{x},t) &= (-\Delta+I)^{-1}\,\partial_t|\psi^0(\mathbf{x},t)|^2 \\ &= -\,(-\Delta+I)^{-1}\,\nabla\cdot\mathbf{j}^0(\mathbf{x},t), \qquad \mathbf{x}\in\mathbb{R}^d, \quad t\geq 0.\end{aligned} \tag{1.12}$$

Plugging (1.6) into (1.12) with $t=0$, noticing (1.11), we get

$$\phi_1^{(l)}(\mathbf{x}) := \partial_t\phi^0(\mathbf{x},0) = -\,(-\Delta+I)^{-1}\,\nabla\cdot\mathbf{j}_0(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^d, \tag{1.13}$$

where

$$\begin{aligned}\mathbf{j}_0(\mathbf{x}) := \mathbf{j}^0(\mathbf{x},0) &= i(\psi^0(\mathbf{x},0)\nabla\bar{\psi}^0(\mathbf{x},0) - \bar{\psi}^0(\mathbf{x},0)\nabla\psi^0(\mathbf{x},0)) \\ &= i(\psi_0(\mathbf{x})\nabla\bar{\psi}_0(\mathbf{x}) - \bar{\psi}_0(\mathbf{x})\nabla\psi_0(\mathbf{x})), \qquad \mathbf{x}\in\mathbb{R}^d.\end{aligned} \tag{1.14}$$

Based on the above results, for $m\geq 2$, we classify the initial data in (1.3) for the KGS equations into two typical types as follows:

(i) Well-prepared initial data for $\gamma=0$; i.e., there exist two constants $C_1>0$ and $\varepsilon_1>0$ such that for $0<\varepsilon\leq\varepsilon_1$,

$$\|\psi_0^\varepsilon - \psi_0^{(l)}\|_{H^m(\mathbb{R}^d)} + \|\phi_0^\varepsilon - \phi_0^{(l)}\|_{H^m(\mathbb{R}^d)} + \|\varepsilon(\phi_1^\varepsilon - \phi_1^{(l)})\|_{H^m(\mathbb{R}^d)} \leq C_1\,\varepsilon^2. \tag{1.15}$$

(ii) Well-prepared initial data for $\gamma>0$; i.e., there exist two constants $C_2>0$ and $\varepsilon_2>0$ such that for $0<\varepsilon\leq\varepsilon_2$,

$$\|\psi_0^\varepsilon - \psi_0^{(l)}\|_{H^m(\mathbb{R}^d)} + \|\phi_0^\varepsilon - \phi_0^{(l)}\|_{H^m(\mathbb{R}^d)} + \|\varepsilon(\phi_1^\varepsilon - \phi_1^{(l)})\|_{H^m(\mathbb{R}^d)} \leq C_2\,\varepsilon. \tag{1.16}$$

(iii) Ill-prepared initial data for $\gamma\geq 0$; i.e., there exist two constants $C_3>0$ and $\varepsilon_3>0$ such that for $0<\varepsilon\leq\varepsilon_3$,

$$\|\psi_0^\varepsilon - \psi_0^{(l)}\|_{H^m(\mathbb{R}^d)} \leq C_3\,\varepsilon, \quad \|\phi_0^\varepsilon - \phi_0^{(l)}\|_{H^m(\mathbb{R}^d)} + \|\varepsilon(\phi_1^\varepsilon - \phi_1^{(l)})\|_{H^m(\mathbb{R}^d)} = O(1). \tag{1.17}$$

For the well-prepared initial data (1.15) and (1.16) for $\gamma=0$ and $\gamma>0$, respectively, it is straightforward to derive the formal limits of the solution of the KGS equations to the solution of the SY equations. On the contrary, for the ill-prepared initial data (1.17), an initial layer correction must be added to the solution of the SY equations so that they can approximate the solution of the KGS equations in the singular limit regime, i.e., $0<\varepsilon\ll 1$. The aim of this paper is to provide rigorous mathematical justification for the formal limits of the solution of the KGS equations to the solution of the SY equations with and without an initial layer correction for well-prepared and ill-prepared initial data, respectively. Weak convergence results are established by using the weak compactness arguments, and strong convergence results are proved by using the (modulated) energy method [19, 20, 27]. Numerical results are reported to confirm the strong convergence results and convergence rates.

Below we state our main results.

THEOREM 1.1 (weak and strong convergence for weak solutions). *Suppose that the initial data in* (1.3) *for the KGS equations have the following regularity; i.e., there exists a positive constant C independent of ε such that*

$$\|(\psi_0^\varepsilon, \phi_0^\varepsilon)(\cdot)\|_{H^1(\mathbb{R}^d)} + \|\varepsilon\phi_1^\varepsilon(\cdot)\|_{L^2(\mathbb{R}^d)} \le C. \tag{1.18}$$

(i) *There exists a unique global weak solution $(\psi^\varepsilon, \phi^\varepsilon)$ of the KGS equations satisfying $\psi^\varepsilon, \phi^\varepsilon \in V := L^\infty(0,\infty; H^1(\mathbb{R}^d))$. In addition, when $\varepsilon \to 0$, a sequence of $(\psi^\varepsilon, \phi^\varepsilon)$ converges to $(\psi^0, \phi^0) \in V \times V$ weakly star, where (ψ^0, ϕ^0) is the unique weak solution of the SY equations* (1.4)–(1.6) *in the distribution sense.*

(ii) *Assume that $\psi_0^\varepsilon \to \psi_0$ in $H^1(\mathbb{R}^d)$, $\phi_0^\varepsilon \to \phi_0 = (-\Delta + I)^{-1}|\psi_0|^2$ in $H^1(\mathbb{R}^d)$, $\varepsilon\phi_1^\varepsilon \to 0$ in $L^2(\mathbb{R}^d)$ when $\varepsilon \to 0$, and the SY equations* (1.4)–(1.6) *admit a unique global weak solution $(\psi^0, \phi^0) \in V \times V$; then we have the strong convergence for the global weak solution $(\psi^\varepsilon, \phi^\varepsilon)$ of the KGS equations when $\varepsilon \to 0$,*

$$\psi^\varepsilon - \psi^0 \to 0, \qquad \phi^\varepsilon - \phi^0 \to 0 \quad \text{in} \quad L^2(0,T; H^1(\mathbb{R}^d)), \tag{1.19}$$

for any given $T > 0$.

The above results from the weak solutions for the KGS equations to the weak solutions for the SY equations do not imply any convergence rate on ε. For the strong solutions for the SY equations, we have the following convergence rate results.

THEOREM 1.2 ("optimal" convergence rates for strong solutions). *Let $m \ge 2$, and suppose that the initial data in* (1.3) *for the KGS equations satisfy*

$$\|(\psi_0^\varepsilon, \phi_0^\varepsilon)(\cdot)\|_{H^m(\mathbb{R}^d)} + \|\varepsilon\phi_1^\varepsilon(\cdot)\|_{H^m(\mathbb{R}^d)} \le C \tag{1.20}$$

for some positive constant C independent of ε; then the KGS equations (1.1)–(1.3) *admit a unique global smooth solution $(\psi^\varepsilon, \phi^\varepsilon) \in W \times W$ with $W = L^\infty(0,T; H^m(\mathbb{R}^d))$, and the SY equations* (1.4)–(1.6) *admit a unique global smooth solution $(\psi^0, \phi^0) \in W \times W$. Moreover, we have the following "optimal" convergence rates:*

(i) *For well-prepared initial data (i.e., the initial data in* (1.3) *satisfy* (1.15) *and* (1.16) *for $\gamma = 0$ and $\gamma > 0$, respectively) for any finite time interval $[0,T]$ with $T > 0$, there exist positive constants $C_1 > 0$ and $\varepsilon_0 = \varepsilon_0(T) > 0$ depending only upon the initial data ψ_0 such that for any $t \in [0,T]$ and $\varepsilon \in (0, \varepsilon_0]$,*

$$\|(\psi^\varepsilon(\cdot,t) - \psi^0(\cdot,t), \phi^\varepsilon(\cdot,t) - \phi^0(\cdot,t))\|_{L^\infty(0,T;H^m(\mathbb{R}^d))} \le C_1 \begin{cases} \varepsilon^2, & \gamma = 0, \\ \varepsilon, & \gamma > 0. \end{cases} \tag{1.21}$$

(ii) *For ill-prepared initial data (i.e., the initial data in* (1.3) *satisfy* (1.17)*), assume they also satisfy $\psi_0^{(l)}, \phi_0^\varepsilon - \phi_0^{(l)}, \phi_1^\varepsilon - \phi_1^{(l)} \in \mathcal{S}(\mathbb{R}^d) = \{u \in C^\infty(\mathbb{R}^d) \mid \|u\|_{k,\alpha} := \sup_{\mathbf{x}\in\mathbb{R}^d}(1+|\mathbf{x}|^2)^{k/2}|D^\alpha u(\mathbf{x})| < \infty$ for all $k \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^d\}$, which is the Schwartz space, and there exist positive constants δ_0 and $C_{k,\alpha}$, which are independent of ε such that*

$$\|\phi_0^\varepsilon - \phi_0^{(l)}\|_{k,\alpha} + \|\phi_1^\varepsilon - \phi_1^{(l)}\|_{k,\alpha} \le C_{k,\alpha}, \quad 0 \le \varepsilon \le \delta_0, \quad \forall k \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^d, \tag{1.22}$$

then for any finite time interval $[0,T]$ with $T > 0$, there exist positive constants $C_2 > 0$ and $\varepsilon_1 = \varepsilon_1(T) > 0$ depending only upon the initial data $(\psi_0, \phi_0^\varepsilon, \phi_1^\varepsilon)$ such that for *any* $t \in [0, T]$ *and* $\varepsilon \in (0, \varepsilon_1]$,

$$\|(\psi^\varepsilon(\cdot,t) - \psi^0(\cdot,t), \phi^\varepsilon(\cdot,t) - \phi^0(\cdot,t) - \phi_I^0(\cdot,t/\varepsilon))\|_{L^\infty(0,T;H^m(\mathbb{R}^d))}$$

$$\leq C_2 \begin{cases} \varepsilon^{1/2}, & \gamma = 0 \text{ and } d = 1, \\ \varepsilon, & \gamma > 0 \text{ or } d = 2, 3. \end{cases} \tag{1.23}$$

Here, $\phi_I^0(\mathbf{x}, \tau)$*, depending upon* ε*, where* $\tau = t/\varepsilon$ *is the fast time variable, is the solution of the following Klein–Gordon equation:*

$$\partial_{\tau\tau}\phi_I^0(\mathbf{x},\tau) + \gamma\partial_\tau\phi_I^0(\mathbf{x},\tau) - \Delta\phi_I^0(\mathbf{x},\tau) + \phi_I^0(\mathbf{x},\tau) = 0, \qquad \mathbf{x} \in \mathbb{R}^d,\ \tau > 0, \tag{1.24}$$

$$\phi_I^0(\mathbf{x},0) = \phi_0^\varepsilon(\mathbf{x}) - \phi_0^{(l)}(\mathbf{x}), \quad \partial_\tau\phi_I^0(\mathbf{x},0) = 0, \qquad \mathbf{x} \in \mathbb{R}^d. \tag{1.25}$$

REMARK 1.1. *The assumption that the initial error belongs to the Schwartz class is used for applying dispersive estimate to the Klein–Gordon equation following Nelson* [21, 22]*. In fact, this assumption can be relaxed by the following* [14]*: There exist positive constants* δ_0 *and* C_k *(*k *is sufficiently large) which are independent of* ε *such that*

$$\|\phi_0^\varepsilon - \phi_0^{(l)}\|_{W^{k,1}(\mathbb{R}^d)} + \|\phi_1^\varepsilon - \phi_1^{(l)}\|_{W^{k,1}(\mathbb{R}^d)} \leq C_k, \quad 0 \leq \varepsilon \leq \delta_0.$$

REMARK 1.2. *The role of introducing* $\phi_I^0(\mathbf{x},\tau)$*, which is a dispersive function in the fast time argument* τ*, is to recover the initial layer or time oscillations with length* $O(\varepsilon)$ *present in the KGS equations when initial data is ill-prepared. Such initial layer effects will be discussed in section* 2*.*

The paper is organized as follows. In section 2, we provide some preliminary results for the KGS equations, carry out formal limits of the solution of the KGS equations to the solution of SY equations, and find the correction in the initial layer. In section 3, we establish weak and strong convergence results for the weak solutions in Theorem 1.1 by using the weak compactness argument and the modulated energy techniques. In section 4, we prove "optimal" convergence rates for the strong solutions in Theorem 1.2 by using the energy method. In section 5, numerical results are reported to confirm our convergence results. Finally, some conclusions are drawn in section 6. Throughout the paper, C, C_I, C_1, and C_2 are generic constants independent of ε, and we adopt the standard notation of the Sobolev spaces and their norms.

2. Preliminaries and formal limits. In this section, we state some basic facts of the KGS equations (1.1)–(1.2) and the SY equations (1.4)–(1.5) and find the matched asymptotic approximations of the solution of the KGS equations (1.1)–(1.3) by using the singular perturbation techniques.

For the KGS equations (1.1)–(1.2), the *wave charge*

$$D^\varepsilon(t) := D(\psi^\varepsilon(\cdot,t)) = \int_{\mathbb{R}^d} |\psi^\varepsilon(\mathbf{x},t)|^2\, d\mathbf{x} \equiv \int_{\mathbb{R}^d} |\psi_0^\varepsilon(\mathbf{x})|^2\, d\mathbf{x} = D(\psi_0^\varepsilon), \quad t \geq 0, \tag{2.1}$$

is conserved for any $\gamma \geq 0$. In addition, when there is no damping term, i.e., $\gamma = 0$ in (1.2), the *Hamiltonian*

$$\begin{aligned} H^\varepsilon(t) &= \int_{\mathbb{R}^d} \left[\frac{1}{2}\left(|\nabla\phi^\varepsilon|^2 + |\phi^\varepsilon|^2 + \varepsilon^2|\partial_t\phi^\varepsilon|^2\right) + |\nabla\psi^\varepsilon|^2 - \phi^\varepsilon|\psi^\varepsilon|^2\right](\mathbf{x},t)\, d\mathbf{x} \\ &\equiv \int_{\mathbb{R}^d} \left[\frac{1}{2}\left(|\nabla\phi_0^\varepsilon|^2 + |\phi_0^\varepsilon|^2 + \varepsilon^2|\phi_1^\varepsilon|^2\right) + |\nabla\psi_0^\varepsilon|^2 - \phi_0^\varepsilon|\psi_0^\varepsilon|^2\right](\mathbf{x})\, d\mathbf{x} \\ &:= H^\varepsilon(0), \qquad t \geq 0, \end{aligned} \tag{2.2}$$

is also conserved, and when there is a damping term, i.e., $\gamma > 0$ in (1.2), we have

$$\frac{dH^\varepsilon(t)}{dt} = -\gamma\,\varepsilon \int_{\mathbb{R}^d} |\partial_t \phi^\varepsilon(\mathbf{x},t)|^2 \, d\mathbf{x} \le 0, \qquad t \ge 0, \tag{2.3}$$

which immediately implies that the Hamiltonian $H^\varepsilon(t)$ decreases when time t increases.

Similarly, the SY equations (1.4)–(1.5) have at least two invariants, which are the wave charge

$$D^0(t) := D(\psi^0(\cdot,t)) = \int_{\mathbb{R}^d} |\psi^0(\mathbf{x},t)|^2 \, d\mathbf{x} \equiv \int_{\mathbb{R}^d} |\psi_0(\mathbf{x})|^2 \, d\mathbf{x} = D(\psi_0), \quad t \ge 0, \tag{2.4}$$

and the *Hamiltonian*

$$\begin{aligned} H^0(t) &= \int_{\mathbb{R}^d} \left[\frac{1}{2}\left(|\nabla\phi^0|^2 + |\phi^0|^2\right) + |\nabla\psi^0|^2 - \phi^0|\psi^0|^2 \right](\mathbf{x},t) \, d\mathbf{x} \\ &= \int_{\mathbb{R}^d} \left[|\nabla\psi^0(\mathbf{x},t)|^2 - \frac{1}{2}|\psi^0(\mathbf{x},t)|^2 \, (-\Delta+I)^{-1}|\psi^0(\mathbf{x},t)|^2 \right] d\mathbf{x} \\ &\equiv \int_{\mathbb{R}^d} \left[\frac{1}{2}\left(|\nabla\phi_0(\mathbf{x})|^2 + |\phi_0(\mathbf{x})|^2\right) + |\nabla\psi_0(\mathbf{x})|^2 - \phi_0(\mathbf{x})|\psi_0(\mathbf{x})|^2 \right] d\mathbf{x} \\ &= \int_{\mathbb{R}^d} \left[|\nabla\psi_0(\mathbf{x})|^2 - \frac{1}{2}|\psi_0(\mathbf{x})|^2 \, (-\Delta+I)^{-1}|\psi_0(\mathbf{x})|^2 \right] d\mathbf{x} := H^0(0), \quad t \ge 0. \end{aligned} \tag{2.5}$$

In what follows we shall perform some formal limits. For the KGS equations (1.1)–(1.2) in the singular limit regime, i.e., $0 \le \varepsilon \ll 1$, we rewrite (1.2) as

$$-\Delta\phi^\varepsilon + \phi^\varepsilon = |\psi^\varepsilon(\mathbf{x},t)|^2 - \varepsilon^2\partial_{tt}\phi^\varepsilon - \varepsilon\gamma\partial_t\phi^\varepsilon, \qquad \mathbf{x} \in \mathbb{R}^d, \quad t \ge 0. \tag{2.6}$$

Solving (2.6), formally we get

$$\begin{aligned} \phi^\varepsilon &= (-\Delta+I)^{-1}|\psi^\varepsilon|^2 - (-\Delta+I)^{-1}\left[\varepsilon^2\partial_{tt}\phi^\varepsilon + \varepsilon\gamma\partial_t\phi^\varepsilon\right] \\ &= (-\Delta+I)^{-1}|\psi^\varepsilon|^2 + \begin{cases} O(\varepsilon^2), & \gamma = 0, \\ O(\varepsilon), & \gamma > 0, \end{cases} \qquad \mathbf{x} \in \mathbb{R}^d, \quad t \ge 0. \end{aligned} \tag{2.7}$$

In addition, multiplying (1.1) by $\bar{\psi}^\varepsilon$, we obtain

$$i\,\bar{\psi}^\varepsilon(\mathbf{x},t)\partial_t\psi^\varepsilon(\mathbf{x},t) + \bar{\psi}^\varepsilon(\mathbf{x},t)\Delta\psi^\varepsilon(\mathbf{x},t) + \phi^\varepsilon(\mathbf{x},t)|\psi^\varepsilon(\mathbf{x},t)|^2 = 0, \qquad \mathbf{x} \in \mathbb{R}^d. \tag{2.8}$$

Taking the imaginary part of (2.8), we get

$$\begin{aligned} \partial_t|\psi^\varepsilon(\mathbf{x},t)|^2 &= -i\left[\psi^\varepsilon(\mathbf{x},t)\Delta\bar{\psi}^\varepsilon(\mathbf{x},t) - \bar{\psi}^\varepsilon(\mathbf{x},t)\Delta\psi^\varepsilon(\mathbf{x},t)\right] \\ &= -i\nabla\cdot\left[\psi^\varepsilon(\mathbf{x},t)\nabla\bar{\psi}^\varepsilon(\mathbf{x},t) - \bar{\psi}^\varepsilon(\mathbf{x},t)\nabla\psi^\varepsilon(\mathbf{x},t)\right] \\ &= -\nabla\cdot\mathbf{j}^\varepsilon(\mathbf{x},t), \qquad \mathbf{x} \in \mathbb{R}^d, \quad t \ge 0, \end{aligned} \tag{2.9}$$

where $\mathbf{j}^\varepsilon$ is usually called current for the KGS equations (1.1)–(1.2) and is defined as

$$\mathbf{j}^\varepsilon(\mathbf{x},t) = i(\psi^\varepsilon(\mathbf{x},t)\nabla\bar{\psi}^\varepsilon(\mathbf{x},t) - \bar{\psi}^\varepsilon(\mathbf{x},t)\nabla\psi^\varepsilon(\mathbf{x},t)), \qquad \mathbf{x} \in \mathbb{R}^d, \quad t \ge 0. \tag{2.10}$$

Differentiating (2.7) with respect to t, noticing (2.9), formally we obtain

$$\begin{aligned} \partial_t\phi^\varepsilon &= (-\Delta+I)^{-1}\partial_t|\psi^\varepsilon|^2 - (-\Delta+I)^{-1}\left[\varepsilon^2\partial_{ttt}\phi^\varepsilon + \varepsilon\gamma\partial_{tt}\phi^\varepsilon\right] \\ &= -(-\Delta+I)^{-1}\nabla\cdot\mathbf{j}^\varepsilon + \begin{cases} O(\varepsilon^2), & \gamma = 0, \\ O(\varepsilon), & \gamma > 0, \end{cases} \quad \mathbf{x} \in \mathbb{R}^d, \quad t \ge 0. \end{aligned} \tag{2.11}$$

Based on the above formal limits results, for the well-prepared initial data in (1.15) and (1.16) for $\gamma = 0$ and $\gamma > 0$, respectively, when $\varepsilon \to 0$, formally we can see that the solution of the KGS equations (1.1)–(1.3) converges to the solution of the SY equations (1.4)–(1.6) quadratically and linearly when $\gamma = 0$ and $\gamma > 0$, respectively. On the contrary, for ill-prepared initial data in (1.17), an initial layer with length $O(\varepsilon)$ exists in the solution of the KGS equations (1.1)–(1.3) when $0 < \varepsilon \ll 1$. In order to recover the initial layer in the solution of KGS equations, let $\tau = t/\varepsilon$ be the fast time variable, and here we present two-scale matched asymptotic expansion. Using the standard perturbation analysis, we expand ψ^ε and ϕ^ε as

$$\psi^\varepsilon(\mathbf{x}, t) = \psi^0(\mathbf{x}, t) + \varepsilon\psi_I^1(\mathbf{x}, \tau) + \cdots, \qquad \mathbf{x} \in \mathbb{R}^d, \quad t \geq 0, \tag{2.12}$$

$$\phi^\varepsilon(\mathbf{x}, t) = \phi^0(\mathbf{x}, t) + \phi_I^0(\mathbf{x}, \tau) + \varepsilon\left[\phi^1(\mathbf{x}, t) + \phi_I^1(\mathbf{x}, \tau)\right] + \cdots. \tag{2.13}$$

Plugging (2.12) and (2.13) into the KGS (1.1)–(1.3) and collecting the leading order terms in the two scales, we get

$$\left[i\,\partial_t\psi^0 + \Delta\psi^0 + \psi^0\phi^0\right](\mathbf{x}, t) + \left[i\,\partial_\tau\psi_I^1(\mathbf{x}, \tau) + \psi^0(\mathbf{x}, 0)\phi_I^0(\mathbf{x}, \tau)\right] = O(\varepsilon), \tag{2.14}$$

$$\begin{aligned}&\left[-\Delta\phi^0 + \phi^0 - |\psi^0|^2\right](\mathbf{x}, t) + \left[\partial_{\tau\tau}\phi_I^0 + \gamma\partial_\tau\phi_I^0 - \Delta\phi_I^0 + \phi_I^0\right](\mathbf{x}, \tau)\\ &+ \varepsilon\left[\partial_{\tau\tau}\phi_I^1(\mathbf{x}, \tau) + \gamma\partial_\tau\phi_I^1(\mathbf{x}, \tau) - \Delta\phi_I^1(\mathbf{x}, \tau) + \phi_I^1(\mathbf{x}, \tau) - 2\mathrm{Re}(\psi^0(\mathbf{x}, 0)\bar{\psi}_I^1(\mathbf{x}, \tau))\right]\\ &+ \varepsilon\left[-\Delta\phi^1 + \phi^1 + \gamma\partial_t\phi^0\right](\mathbf{x}, t) = O(\varepsilon^2),\end{aligned} \tag{2.15}$$

$$\psi_0^\varepsilon(\mathbf{x}) = \psi^\varepsilon(\mathbf{x}, t = 0) = \psi^0(\mathbf{x}, t = 0) + \varepsilon\psi_I^1(\mathbf{x}, \tau = 0) + \cdots, \tag{2.16}$$

$$\begin{aligned}\phi_0^\varepsilon(\mathbf{x}) = \phi^\varepsilon(\mathbf{x}, t = 0) &= \phi^0(\mathbf{x}, t = 0) + \phi_I^0(\mathbf{x}, \tau = 0)\\ &+ \varepsilon\left[\phi^1(\mathbf{x}, t = 0) + \phi_I^1(\mathbf{x}, \tau = 0)\right] + \cdots,\end{aligned} \tag{2.17}$$

$$\begin{aligned}\phi_1^\varepsilon(\mathbf{x}) = \partial_t\phi^\varepsilon(\mathbf{x}, t = 0) &= \partial_t\phi^0(\mathbf{x}, t = 0) + \frac{1}{\varepsilon}\partial_\tau\phi_I^0(\mathbf{x}, \tau = 0)\\ &+ \left[\varepsilon\partial_t\phi^1(\mathbf{x}, t = 0) + \partial_\tau\phi_I^1(\mathbf{x}, \tau = 0)\right] + \cdots.\end{aligned} \tag{2.18}$$

Equating the leading order terms in (2.14)–(2.18) with respect to the slow variable t, we obtain that $(\psi^0(\mathbf{x}, t), \phi^0(\mathbf{x}, t))$ satisfies the SY equations (1.4)–(1.6). Similarly, equating the leading order terms in (2.15), (2.17), and (2.18) with respect to the fast variable τ, noticing (1.8) and (1.13), we obtain that $\phi_I^0(\mathbf{x}, \tau)$ satisfies the linear Klein–Gordon equation (1.24)–(1.25). In addition, from (2.14)–(2.18), we have that $\phi^1(\mathbf{x}, t)$ satisfies the Yukawa equation

$$-\Delta\phi^1(\mathbf{x}, t) + \phi^1(\mathbf{x}, t) = -\gamma\partial_t\phi^0(\mathbf{x}, t), \qquad \mathbf{x} \in \mathbb{R}^d, \quad t \geq 0, \tag{2.19}$$

$\psi_I^1(\mathbf{x}, \tau)$ satisfies the following ordinary differential equation (ODE),

$$i\,\partial_\tau\psi_I^1(\mathbf{x}, \tau) = -\psi^0(\mathbf{x}, 0)\phi_I^0(\mathbf{x}, \tau), \qquad \mathbf{x} \in \mathbb{R}^d, \quad \tau > 0, \tag{2.20}$$

$$\psi_I^1(\mathbf{x}, \tau = 0) = 0, \qquad \mathbf{x} \in \mathbb{R}^d, \tag{2.21}$$

and $\phi_I^1(\mathbf{x}, \tau)$ satisfies the following linear Klein–Gordon equation,

$$\left[\partial_{\tau\tau}\phi_I^1 + \gamma\partial_\tau\phi_I^1 - \Delta\phi_I^1 + \phi_I^1\right](\mathbf{x}, \tau) = 2\mathrm{Re}(\psi^0(\mathbf{x}, 0)\bar{\psi}_I^1(\mathbf{x}, \tau)), \ \mathbf{x} \in \mathbb{R}^d, \ \tau > 0, \tag{2.22}$$

$$\phi_I^1(\mathbf{x}, 0) = -\phi^1(\mathbf{x}, t = 0), \quad \partial_\tau\phi_I^1(\mathbf{x}, 0) = \phi_1^\varepsilon(\mathbf{x}) - \phi_1^{(l)}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{2.23}$$

with $\phi_1^{(l)}$ given as in (1.13).

For the Klein–Gordon equations (1.24)–(1.25) and (2.22)–(2.23), when $\gamma = 0$, by applying the results for the standard Klein–Gordon equations in Nelson [21, 22], and, respectively, when $\gamma > 0$, by applying the standard energy method, we have the following dispersion estimate.

PROPOSITION 2.1 (see [14, 15, 21, 22, 28]). *Let $\alpha \in \mathbb{N}_0^d$ and $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$. Assume that $\phi_0^\varepsilon - \phi_0^{(l)}, \phi_1^\varepsilon - \phi_1^{(l)} \in \mathcal{S}(\mathbb{R}^d)$ satisfy (1.22); then the Klein–Gordon equations (1.24)–(1.25) and (2.22)–(2.23) and the ODE (2.20)–(2.21) have global smooth solutions ϕ_I^0, ϕ_I^1, and ψ_I^1, respectively. In addition, there exist positive constants C and κ, depending only upon γ and the initial data ψ_0, $\phi_0^\varepsilon - \phi_0^{(l)}$, and $\phi_1^\varepsilon - \phi_1^{(l)}$, such that*

$$\|D^\alpha \phi_I^0(\cdot,\tau)\|_{L^\infty(\mathbb{R}^d)} \le C \begin{cases} 1/(1+\tau)^{d/2}, & \gamma = 0, \\ e^{-\kappa\,\tau}, & \gamma > 0, \end{cases} \qquad 0 \le \tau < \infty, \tag{2.24}$$

and, for $\gamma \ge 0$,

$$\|D^\alpha(\phi_I^0, \phi_I^1, \psi_I^1)(\cdot,\tau)\|_{L^2(\mathbb{R}^d)} \le C, \qquad 0 \le \tau < \infty. \tag{2.25}$$

Finally, we recall the global existence result about the smooth solution to the SY equations (1.4)–(1.6) and the Yukawa equation (2.19).

PROPOSITION 2.2 (see [15, 28]). *Let $\alpha \in \mathbb{N}_0^d$ and assume that $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$; then the SY equations (1.4)–(1.6) have a unique global smooth solution $(\psi^0, \phi^0) \in [C^k([0,\infty); H^m(\mathbb{R}^d))]^2$ for any positive integers k and m. In addition, for any finite time interval $[0,T]$ with $T > 0$, there exists a positive constant $C = C(T) > 0$ depending only upon T and the initial data ψ_0 such that for any integer $s > 0$,*

$$\|\partial_t^s D^\alpha(\psi^0, \phi^0, \phi^1)(\mathbf{x},t)\|_{L^\infty(0,T;H^m(\mathbb{R}^d))} \le C(T), \qquad 0 \le t \le T. \tag{2.26}$$

3. Proof of convergence results in Theorem 1.1. In this section, we prove the weak and strong convergence results of weak solutions in Theorem 1.1 by using weak convergence arguments and modulated energy techniques motivated by [19]. Assume that $\gamma \ge 0$ is a fixed constant. To do this, we need to obtain some uniform bounds about the quantities $\nabla\psi^\varepsilon$, ϕ^ε, $\varepsilon\partial_t\phi^\varepsilon$, $\nabla\phi^\varepsilon$, etc.

Denote $H_0^\varepsilon(t)$ by

$$H_0^\varepsilon(t) = \int_{\mathbb{R}^d} \left[\frac{1}{2}\left(|\nabla\phi^\varepsilon|^2 + |\phi^\varepsilon|^2 + \varepsilon^2|\partial_t\phi^\varepsilon|^2\right) + |\nabla\psi^\varepsilon|^2\right](\mathbf{x},t)\,d\mathbf{x}, \quad t \ge 0,$$

which includes the first four terms of the integrand in $H^\varepsilon(t)$; then there exist two positive constants C_1 and C_2 such that

$$C_1 H_0^\varepsilon(t) - C_2\|\psi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^{\frac{2(8-d)}{4-d}} \le H^\varepsilon(t) \le C_1\, H_0^\varepsilon(t) + C_2\|\psi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^{\frac{2(8-d)}{4-d}}, \quad t \ge 0. \tag{3.1}$$

It suffices to prove the inequality

$$\left|\int_{\mathbb{R}^d} \left[\phi^\varepsilon|\psi^\varepsilon|^2\right](\mathbf{x},t)d\mathbf{x}\right| \le \frac{1}{4}\|(\phi^\varepsilon, \nabla\phi^\varepsilon, \nabla\psi^\varepsilon)(\cdot,t)\|_{L^2(\mathbb{R}^d)}^2 + C\|\psi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^{\frac{2(8-d)}{4-d}}. \tag{3.2}$$

In fact, by using the Gagliado–Nirenberg inequality

$$\|u\|_{L^4(\mathbb{R}^d)} \le C\|u\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{4}}\,\|\nabla u\|_{L^2(\mathbb{R}^d)}^{\frac{d}{4}} \quad \forall u \in H^1(\mathbb{R}^d), \tag{3.3}$$

we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \left[\phi^\varepsilon|\psi^\varepsilon|^2\right](\mathbf{x},t)\, d\mathbf{x}\right| \\
&\le \|(\phi^\varepsilon|\psi^\varepsilon|)(\cdot,t)\|_{L^2(\mathbb{R}^d)}\ \|\psi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)} \\
&\le \|\phi^\varepsilon(\cdot,t)\|_{L^4(\mathbb{R}^d)}\ \|\psi^\varepsilon(\cdot,t)\|_{L^4(\mathbb{R}^d)}\ \|\psi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)} \\
&\le C\|\phi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{4}}\ \|\nabla\phi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^{\frac{d}{4}}\ \|\nabla\psi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^{\frac{d}{4}}\ \|\psi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^{2-\frac{d}{4}} \\
&\le \frac{1}{4}\|(\phi^\varepsilon,\nabla\phi^\varepsilon,\nabla\psi^\varepsilon)(\cdot,t)\|_{L^2(\mathbb{R}^d)}^2 + C\|\psi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^{\frac{2(8-d)}{4-d}}
\end{aligned}$$

with the help of Young's inequality.

According to the assumption (1.18) on the initial data, there exists a constant $C > 0$, independent of ε, such that

$$D(\psi_0) \le C, \qquad H^\varepsilon(0) \le C,$$

which, together with (2.1)–(2.3), yields that

$$D^\varepsilon(t) \le C, \qquad H^\varepsilon(t) \le C. \tag{3.4}$$

Using (3.4) and (3.1), we get, for all $t \ge 0$, that

$$\|(\psi^\varepsilon,\nabla\psi^\varepsilon,\phi^\varepsilon,\nabla\phi^\varepsilon,\varepsilon\partial_t\phi^\varepsilon)(\cdot,t)\|_{L^2(\mathbb{R}^d)} \le C, \tag{3.5}$$

which implies that the quantities $\|(\psi^\varepsilon,\phi^\varepsilon)\|_{L^\infty(0,\infty;H^1(\mathbb{R}^d))}$ and $\|\varepsilon\partial_t\phi^\varepsilon\|_{L^\infty(0,\infty;L^2(\mathbb{R}^d))}$ are bounded uniformly in ε. So, some subsequence of $(\psi^\varepsilon,\phi^\varepsilon,\varepsilon\partial_t\phi^\varepsilon)(\mathbf{x},t)$, also labelled by ε, has a weak limit (ψ^0,ϕ^0,w). More precisely, we have

$$\psi^\varepsilon \rightharpoonup \psi^0 \ \text{ weak star in } \ L^\infty(0,\infty;H^1(\mathbb{R}^d)), \tag{3.6}$$

$$\phi^\varepsilon \rightharpoonup \phi^0 \ \text{ weak star in } \ L^\infty(0,\infty;H^1(\mathbb{R}^d)), \tag{3.7}$$

which implies that $\partial_t\phi^\varepsilon$ converges as well (in the distribution sense); thus,

$$\varepsilon\partial_t\phi^\varepsilon \rightharpoonup w = 0 \ \text{ weak star in } \ L^\infty(0,\infty;L^2(\mathbb{R}^d)). \tag{3.8}$$

Moreover, since the embedding maps $u \mapsto u : H^1(\mathbb{R}^d) \hookrightarrow L^4(\mathbb{R}^d)$ and $(u,v) \mapsto u\,v : H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ are continuous, it follows from (3.6) and (3.7) that the quantities $\||\psi^\varepsilon|^2\|_{L^\infty(0,\infty;L^2(\mathbb{R}^d))}$ and $\|\phi^\varepsilon\psi^\varepsilon\|_{L^\infty(0,\infty;L^2(\mathbb{R}^d))}$ are bounded uniformly in ε too. So, by the weak compactness arguments, we obtain

$$\phi^\varepsilon\psi^\varepsilon \rightharpoonup z \ \text{ weak star in } L^\infty(0,\infty;L^2(\mathbb{R}^d)), \tag{3.9}$$

$$|\psi^\varepsilon|^2 \rightharpoonup g \ \text{ weak star in } \ L^\infty(0,\infty;L^2(\mathbb{R}^d)). \tag{3.10}$$

Also, (3.5) implies that the quantities $\|\Delta(\psi^\varepsilon,\phi^\varepsilon)\|_{L^\infty(0,\infty;H^{-1}(\mathbb{R}^d))}$ are bounded uniformly in ε, which, combined with (3.6) and (3.7), yields that

$$\Delta\psi^\varepsilon \rightharpoonup \Delta\psi^0 \ \text{ weak star in } L^\infty(0,\infty;H^{-1}(\mathbb{R}^d)), \tag{3.11}$$

$$\Delta\phi^\varepsilon \rightharpoonup \Delta\phi^0 \ \text{ weak star in } L^\infty(0,\infty;H^{-1}(\mathbb{R}^d)). \tag{3.12}$$

Taking into account (3.6), (3.9), and (3.11), equation (1.1) implies that

$$\|\partial_t\psi^\varepsilon\|_{L^\infty(0,\infty;H^{-1}(\mathbb{R}^d))} \text{ is bounded uniformly in } \varepsilon, \tag{3.13}$$

$$\partial_t\psi^\varepsilon \rightharpoonup \partial_t\psi^0 \text{ weak star in } L^\infty(0,\infty;H^{-1}(\mathbb{R}^d)). \tag{3.14}$$

Using the above results, the proof of Theorem 1.1(i) will be complete if we establish that

$$z = \psi^0\phi^0, \qquad g = |\psi^0|^2. \tag{3.15}$$

To prove (3.15), let $0 < T < \infty$ be any finite time, and let Ω be any bounded subdomain of $\mathbb{R}^d$. First, using (3.5) and (3.13), we have the uniform estimates

$$\|\psi^\varepsilon\|_{L^p(0,T;H^1(\Omega))} \leq C \text{ for any } 1 < p < \infty, \qquad \|\partial_t\psi^\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))} \leq C.$$

Then, noticing that the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ is compact, it follows from the Lions–Aubin lemma that, as $\varepsilon \to 0$,

$$\psi^\varepsilon \to \psi^0 \text{ strongly in } L^p(0,T;L^4(\Omega)), \qquad 1 < p < \infty, \tag{3.16}$$

and thus,

$$|\psi^\varepsilon|^2 \to |\psi^0|^2 \text{ strongly in } L^2(0,T;\Omega). \tag{3.17}$$

In fact, here we use

$$\begin{aligned}
&\int_0^T \int_\Omega \left| |\psi^\varepsilon|^2 - |\psi^0|^2(\mathbf{x},t) \right|^2 \, d\mathbf{x}dt \\
&\leq \int_0^T \left(\int_\Omega |\psi^\varepsilon - \psi^0|^4(\mathbf{x},t) \, d\mathbf{x} \right)^{1/2} \left(\int_\Omega \left| |\psi^\varepsilon| + |\psi^0| \right|^4 (\mathbf{x},t) \, d\mathbf{x} \right)^{1/2} dt \\
&\leq \int_0^T \left(\int_\Omega |\psi^\varepsilon - \psi^0|^4(\mathbf{x},t) \, d\mathbf{x} \right)^{1/2} dt \, \left(\sup_{0\leq t\leq T} \int_\Omega \left| |\psi^\varepsilon| + |\psi^0| \right|^4 (\mathbf{x},t) \, d\mathbf{x} \right)^{1/2} \\
&\leq C \sup_{0\leq t\leq T} \left(\|\psi^\varepsilon\|^2_{H^1(\mathbb{R}^d)} + \|\psi^0\|^2_{H^1(\mathbb{R}^d)} \right) \|\psi^\varepsilon - \psi^0\|^2_{L^2(0,T;L^4(\Omega))}.
\end{aligned}$$

Because the strong convergence implies the weak convergence, Ω is arbitrary, and the embedding $L^\infty(0,\infty;L^2(\mathbb{R}^d)) \hookrightarrow L^2(0,T;\Omega)$ is continuous, the second equality in (3.15) can be obtained from (3.10) and (3.17). For any $\xi \in C_0^\infty((0,T)\times\mathbb{R}^d)$ with the compact support set Ω, we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \left[\xi \left(\psi^\varepsilon\phi^\varepsilon - \psi^0\phi^0 \right) \right] (\mathbf{x},t) d\mathbf{x}dt \\
&= \int_0^T \int_{\mathbb{R}^d} \left[\left(\psi^\varepsilon - \psi^0 \right) \phi^\varepsilon \xi + \psi^0 \left(\phi^\varepsilon - \phi^0 \right) \xi \right] (\mathbf{x},t) d\mathbf{x}dt,
\end{aligned} \tag{3.18}$$

which, together with (3.7) and (3.16), yields the first equality in (3.15) due to $\xi\phi^\varepsilon \in L^2(0,T;L^{\frac{4}{3}}(\Omega))$ uniformly in ε and $\xi\psi^0 \in L^1(0,T;H^{-1}(\mathbb{R}^d))$. Combining all of the above weak convergence arguments and using (3.6)–(3.15), we complete the proof of the weak convergence result in Theorem 1.1(i).

Now we prove the strong convergence of the weak solution in Theorem 1.1(ii). Letting $0 < T < \infty$, using the conservation of wave charges (2.1) and (2.4), and noticing L^2 weak convergence of ψ^ε and (3.13), when $\varepsilon \to 0$, for $0 \leq t \leq T$, we obtain

$$\begin{aligned}
\|(\psi^\varepsilon - \psi^0)(\cdot,t)\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left[\psi^\varepsilon\overline{\psi^\varepsilon} + \psi^0\overline{\psi^0} - \psi^\varepsilon\overline{\psi^0} - \overline{\psi^\varepsilon}\psi^0 \right] (\mathbf{x},t) d\mathbf{x} \\
&= \|\psi^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d)} - \|\psi^0(\cdot,t)\|^2_{L^2(\mathbb{R}^d)} - 2\mathrm{Re} \int_{\mathbb{R}^d} \left[(\psi^\varepsilon - \psi^0)\overline{\psi^0} \right] (\mathbf{x},t) d\mathbf{x} \\
&= \|\psi_0^\varepsilon\|^2_{L^2(\mathbb{R}^d)} - \|\psi_0\|^2_{L^2(\mathbb{R}^d)} - 2\mathrm{Re} \int_{\mathbb{R}^d} \left[(\psi^\varepsilon - \psi^0)\overline{\psi^0} \right] (\mathbf{x},t) d\mathbf{x} \to 0.
\end{aligned} \tag{3.19}$$

From (2.3), (2.2), and (2.5), we have for $0 \le t \le T$,

$$\begin{aligned} H^\varepsilon(0) - H^0(0) &= H^\varepsilon(t) - H^0(t) + \gamma\varepsilon \int_0^t \int_{\mathbb{R}^d} |\partial_\tau \phi(\mathbf{x},\tau)|^2 d\mathbf{x} d\tau \\ &= \int_{\mathbb{R}^d} \left[\frac{1}{2} \left(|\nabla(\phi^\varepsilon - \phi^0)|^2 + |\phi^\varepsilon - \phi^0|^2 + \varepsilon^2 |\partial_t \phi^\varepsilon|^2 \right) + |\nabla(\psi^\varepsilon - \psi^0)|^2 \right] (\mathbf{x}, t)\, d\mathbf{x} \\ &\quad + \gamma\varepsilon \int_0^t \int_{\mathbb{R}^d} |\partial_\tau \phi(\mathbf{x},\tau)|^2 d\mathbf{x} d\tau + \int_{\mathbb{R}^d} \left[\phi^0 |\psi^0|^2 - \phi^\varepsilon |\psi^\varepsilon|^2 \right] (\mathbf{x}, t)\, d\mathbf{x} \\ &\quad + \mathrm{Re} \int_{\mathbb{R}^d} \left[\nabla(\phi^\varepsilon - \phi^0)\nabla\phi^0 + 2(\phi^\varepsilon - \phi^0)\phi^0 + 2\nabla(\psi^\varepsilon - \psi^0)\nabla\overline{\psi^0} \right] (\mathbf{x}, t)\, d\mathbf{x}. \end{aligned} \tag{3.20}$$

From the assumptions, noticing (2.2) and (2.5), we have

$$H^\varepsilon(0) \to H^0(0), \qquad \varepsilon \to 0. \tag{3.21}$$

Using (3.17) and (3.7), noticing $\phi^\varepsilon, |\psi^0|^2 \in L^\infty(0,\infty; L^2(\Omega))$, when $\varepsilon \to 0$, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \left(|\psi^\varepsilon|^2 \phi^\varepsilon - |\psi^0|^2 \phi^0 \right) (\mathbf{x}, t) d\mathbf{x} \\ &= \int_{\mathbb{R}^d} \left[\left(|\psi^\varepsilon|^2 - |\psi^0|^2 \right) \phi^\varepsilon + |\psi^0|^2 \left(\phi^\varepsilon - \phi^0 \right) \right] (\mathbf{x}, t) d\mathbf{x} \to 0, \quad 0 \le t \le T. \end{aligned} \tag{3.22}$$

By the weak H^1 convergence for $(\phi^\varepsilon, \psi^\varepsilon)$, the last term in the right-hand side of (3.20) goes to zero when $\varepsilon \to 0$. Combining the above arguments, noticing $\gamma \ge 0$, and taking $\varepsilon \to 0$ in (3.20), we obtain the result (1.19), and the proof is complete. □

4. Proof of convergence rates in Theorem 1.2. In this section, we prove the "optimal" convergence rates for strong solutions in Theorem 1.2 by using the energy method, the decay estimates in (2.24) for the Klein–Gordon equations, and the regularities. We notice that a similar problem for the singular limits of the Zakharov equations to the nonlinear Schrödinger equation was studied in [24, 25, 27] by different methods.

Let $(\psi^\varepsilon(\mathbf{x},t), \phi^\varepsilon(\mathbf{x},t))$ be the solution of the KGS equations (1.1)–(1.3), let $(\psi^0(\mathbf{x},t), \phi^0(\mathbf{x},t))$ the solution of the SY equations (1.4)–(1.6), let $\phi_I^0(\mathbf{x},\tau)$ (with $\tau = t/\varepsilon$ the fast time variable) and $\phi_I^1(\mathbf{x},\tau)$ be the solutions of the Klein–Gordon equations (1.24)–(1.25) and (2.22)–(2.23), respectively, let $\phi^1(\mathbf{x},t)$ be the solution of the Yukawa equation (2.19), and let $\psi_I^1(\mathbf{x},\tau)$ be the solution of the ODE (2.20)–(2.21). Denote the "error" function as

$$\psi_R^\varepsilon(\mathbf{x},t) = \psi^\varepsilon(\mathbf{x},t) - \psi^0(\mathbf{x},t) - \varepsilon\psi_I^1(\mathbf{x},t/\varepsilon), \qquad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \tag{4.1}$$

$$\phi_R^\varepsilon(\mathbf{x},t) = \phi^\varepsilon(\mathbf{x},t) - \phi^0(\mathbf{x},t) - \phi_I^0(\mathbf{x},t/\varepsilon) - \varepsilon\left[\phi^1(\mathbf{x},t) + \phi_I^1(\mathbf{x},t/\varepsilon)\right]. \tag{4.2}$$

Subtracting (1.4)–(1.6) from (1.1)–(1.3), noticing (1.24), (1.25), (2.19)–(2.23), (4.1), and (4.2), we have

$$i\partial_t\psi_R^\varepsilon + \Delta\psi_R^\varepsilon + \phi^\varepsilon\psi_R^\varepsilon + (\psi^0 + \varepsilon\psi_I^1)\phi_R^\varepsilon = g_I^\varepsilon, \tag{4.3}$$

$$\varepsilon^2\partial_{tt}\phi_R^\varepsilon + \varepsilon\gamma\partial_t\phi_R^\varepsilon - \Delta\phi_R^\varepsilon + \phi_R^\varepsilon - |\psi_R^\varepsilon|^2 - 2\mathrm{Re}(\bar{\psi}_R^\varepsilon(\psi^0 + \varepsilon\psi_I^1)) = h^\varepsilon, \tag{4.4}$$

with initial conditions

$$\psi_R^\varepsilon(\mathbf{x},0) = \psi_0^\varepsilon(\mathbf{x}) - \psi_0(\mathbf{x}), \quad \phi_R^\varepsilon(\mathbf{x},0) = 0, \quad \partial_t\phi_R^\varepsilon(\mathbf{x},0) = 0, \tag{4.5}$$

where the function $(g_I^\varepsilon, h^\varepsilon) = (g_I^\varepsilon, h^\varepsilon)(\mathbf{x},t)$ is given by

$$g_I^\varepsilon(x,t) = -\varepsilon\left[\Delta\psi_I^1 + (\psi^0+\varepsilon\psi_I^1)(\phi^1+\varepsilon\phi_I^1) + \psi_I^1(\phi^0+\phi_I^0)\right] - (\psi^0 - \psi^0(\mathbf{x},0))\phi_I^0, \tag{4.6}$$

$$h^\varepsilon(\mathbf{x},t) = -\varepsilon^2\left[\partial_{tt}\phi^0 + \gamma\partial_t\phi^1 - |\psi_I^1|^2 + \varepsilon\partial_{tt}\phi^1\right] + 2\varepsilon \mathrm{Re}((\psi^0 - \psi^0(\mathbf{x},0))\bar{\psi_I^1}). \tag{4.7}$$

In the following, we give the details of the proof of Theorem 1.2 only for the case of $\gamma = 0$. In this case, $\phi^1(\mathbf{x},t) \equiv 0$. For the case of $\gamma > 0$, the proof can be carried out in a similar manner and thus is omitted here for brevity. The key point in the proof is to establish the energy inequality of Gronwall's type for some kind of ε-weighted Sobolev's energy functional, which yields the uniform a priori estimates with respect to ε. Denote

$$f_d(\varepsilon) = \begin{cases} \varepsilon, & d=1, \\ \varepsilon^2, & d=2,3. \end{cases} \tag{4.8}$$

We divide the proof into five steps as follows.

Step 1. Find L^2-estimates of $\psi_R^\varepsilon(\mathbf{x},t)$.

Define $\chi_1(t) = \|\psi_R^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d)}$, $H_1(t) = \frac{1}{2}\|(\phi_R^\varepsilon, \nabla\phi_R^\varepsilon, \varepsilon\partial_t\phi_R^\varepsilon)(\cdot,t)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla\psi_R^\varepsilon\|^2_{L^2(\mathbb{R}^d)}$, and $H_2(t) = H_1(t) - \int_{\mathbb{R}^d}(\phi_R^\varepsilon|\psi_R^\varepsilon|^2)(\mathbf{x},t)d\mathbf{x}$.

Multiplying (4.3) by $\bar{\psi}_R^\varepsilon$ and then integrating over $\mathbb{R}^d$ with respect to $\mathbf{x}$, integrating by parts, and taking the imaginary part, we get, for any given $0 < T < \infty$,

$$\begin{aligned}\frac{d}{dt}\chi_1(t) = -2\,\mathrm{Im}\int_{\mathbb{R}^d}\bar{\psi}_R^\varepsilon\,&\left[\varepsilon\left(\Delta\psi_I^1 + (\psi^0+\varepsilon\psi_I^1)(\phi^1+\varepsilon\phi_I^1) + \psi_I^1(\phi^0+\phi_I^0)\right)\right.\\ &\left.+ (\psi^0 - \psi^0(\mathbf{x},0))\phi_I^0 + \phi_R^\varepsilon(\psi^0+\varepsilon\psi_I^1)\right](\mathbf{x},t)d\mathbf{x}, \qquad 0\le t\le T.\end{aligned} \tag{4.9}$$

Using the Cauchy–Schwarz inequality, (2.24), (2.25), the boundedness (2.26) of ψ^0, and the fact that $\phi^1 = 0$, we get

$$\begin{aligned}&-2\,\mathrm{Im}\int_{\mathbb{R}^d}\bar{\psi}_R^\varepsilon\left[\phi_R^\varepsilon(\psi^0+\varepsilon\psi_I^1) + (\psi^0-\psi^0(\mathbf{x},0))\phi_I^0\right.\\ &\qquad\qquad \left.+\,\varepsilon\left(\Delta\psi_I^1 + (\psi^0+\varepsilon\psi_I^1)(\phi^1+\varepsilon\phi_I^1) + \psi_I^1(\phi^0+\phi_I^0)\right)\right](\mathbf{x},t)d\mathbf{x}\\ &\le C\left[\chi_1(t) + H_1(t) + \int_{\mathbb{R}^d}|\psi^0(\mathbf{x},t) - \psi^0(\mathbf{x},0)|^2|\phi_I^0(\mathbf{x},t/\varepsilon)|^2 d\mathbf{x}\right] + C_I\varepsilon^2.\end{aligned} \tag{4.10}$$

Plugging (4.10) into (4.9), we have

$$\frac{d}{dt}\chi_1(t) \le C\left[\chi_1(t) + H_1(t) + \int_{\mathbb{R}^d}|\psi^0(\mathbf{x},t) - \psi^0(\mathbf{x},0)|^2|\phi_I^0(\mathbf{x},t/\varepsilon)|^2 d\mathbf{x}\right] + C_I\varepsilon^2. \tag{4.11}$$

Integrating (4.11) over $[0,t]$ with respect to t, we obtain, for $0\le t\le T$, that

$$\begin{aligned}\chi_1(t) \le \chi_1(0) &+ C\int_0^t[\chi_1(s) + H_1(s)]ds\\ &+ C\int_0^t\int_{\mathbb{R}^d}|\psi^0(\mathbf{x},s) - \psi^0(\mathbf{x},0)|^2|\phi_I^0(\mathbf{x},s/\varepsilon)|^2 d\mathbf{x}ds + C_I\varepsilon^2.\end{aligned} \tag{4.12}$$

Using the decay estimate in (2.24) for the Klein–Gordon equations, noticing (4.8), we get

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \left|\psi^0(\mathbf{x},s)-\psi^0(\mathbf{x},0)\right|^2 \left|\phi_I^0(\mathbf{x},s/\varepsilon)\right|^2 d\mathbf{x}ds \\
&\le \int_0^t \int_{\mathbb{R}^d} \left|\partial_t\psi^0(\mathbf{x},\theta s)\ s\right|^2 \left|\phi_I^0(\mathbf{x},s/\varepsilon)\right|^2 d\mathbf{x}ds \\
&\le \int_{\mathbb{R}^d} \sup_{0\le s\le t} |\partial_t\psi^0(\mathbf{x},s)|^2 d\mathbf{x} \int_0^t s^2 \|\phi_I^0(\cdot,s/\varepsilon)\|^2_{L^\infty(\mathbb{R}^d)} ds \\
&\le C\varepsilon^3 \int_0^{t/\varepsilon} s^2 \|\phi_I^0(\cdot,s)\|^2_{L^\infty(\mathbb{R}^d)} ds \le C\varepsilon^3 \int_0^{t/\varepsilon} \frac{s^2}{(1+s)^d} ds \le C\ f_d(\varepsilon)
\end{aligned} \tag{4.13}$$

for sufficiently small $\varepsilon > 0$ and for some $\theta \in (0,1)$ independent of ε. Plugging (4.13) into (4.12), we get

$$\chi_1(t) \le \chi_1(0) + C\int_0^t [\chi_1(s) + H_1(s)]ds + C_I(f_d(\varepsilon)+\varepsilon^2). \tag{4.14}$$

Step 2. Establish an entropy inequality based on the Hamiltonian $H_1(t)$.

Multiplying (4.3) by $-2\partial_t\bar{\psi_R^\varepsilon}$ and then integrating over $\mathbb{R}^d$ with respect to $\mathbf{x}$, integrating by parts, and taking the real part, we get, for any given $0 < T < \infty$,

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\psi_R^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d} \phi_R^\varepsilon\ \partial_t|\bar{\psi_R^\varepsilon}|^2 d\mathbf{x} - 2\mathrm{Re}\int_{\mathbb{R}^d} (\psi^0+\varepsilon\psi_I^1)\phi_R^\varepsilon \partial_t\bar{\psi_R^\varepsilon} d\mathbf{x} \\
&= 2\mathrm{Re}\int_{\mathbb{R}^d} \left(\phi^0+\phi_I^0+\varepsilon(\phi^1+\phi_I^1)\right)\psi_R^\varepsilon\partial_t\bar{\psi_R^\varepsilon} d\mathbf{x} \\
&\quad + 2\varepsilon\mathrm{Re}\int_{\mathbb{R}^d} \left(\Delta\psi_I^1 + (\psi^0+\varepsilon\psi_I^1)(\phi^1+\varepsilon\phi_I^1) + \psi_I^1(\phi^0+\phi_I^0)\right)\partial_t\bar{\psi_R^\varepsilon} d\mathbf{x} \\
&\quad + 2\mathrm{Re}\int_{\mathbb{R}^d} (\psi^0-\psi^0(\mathbf{x},0))\phi_I^0\partial_t\bar{\psi_R^\varepsilon} d\mathbf{x}, \qquad 0\le t\le T.
\end{aligned} \tag{4.15}$$

Similarly, multiplying (4.4) by $\partial_t\phi_R^\varepsilon$ and then integrating over $\mathbb{R}^d$ with respect to $\mathbf{x}$, integrating by parts, we get, for any given $0 < T < \infty$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\varepsilon\partial_t\phi_R^\varepsilon, \nabla\phi_R^\varepsilon, \phi_R^\varepsilon)\|^2_{L^2(\mathbb{R}^d)} - 2\mathrm{Re}\int_{\mathbb{R}^d} (\psi^0+\varepsilon\psi_I^1)\bar{\psi_R^\varepsilon}\partial_t\phi_R^\varepsilon d\mathbf{x} \\
&= \int_{\mathbb{R}^d} |\psi_R^\varepsilon|^2\partial_t\phi_R^\varepsilon d\mathbf{x} + 2\varepsilon\mathrm{Re}\int_{\mathbb{R}^d} (\psi^0-\psi^0(\mathbf{x},0))\bar{\psi_I^1}\ \partial_t\phi_R^\varepsilon\ d\mathbf{x} \\
&\quad + \varepsilon^2\int_{\mathbb{R}^d} (|\psi_I^1|^2 - \partial_{tt}\phi^0)\ \partial_t\phi_R^\varepsilon\ d\mathbf{x}, \qquad 0\le t\le T.
\end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16), we have

$$\begin{aligned}
&\frac{d}{dt}\left[H_2(t) - 2\mathrm{Re}\int_{\mathbb{R}^d}(\psi^0+\varepsilon\psi_I^1)\,\bar{\psi}_R^\varepsilon\,\phi_R^\varepsilon\,d\mathbf{x} - 2\varepsilon\mathrm{Re}\int_{\mathbb{R}^d}(\psi^0-\psi^0(\mathbf{x},0))\bar{\psi}_I^1\phi_R^\varepsilon\,d\mathbf{x}\right]\\
&= 2\varepsilon\mathrm{Re}\int_{\mathbb{R}^d}\left(\Delta\psi_I^1+(\psi^0+\varepsilon\psi_I^1)(\phi^1+\varepsilon\phi_I^1)+\psi_I^1(\phi^0+\phi_I^0)\right)\partial_t\bar{\psi}_R^\varepsilon d\mathbf{x}\\
&\quad + 2\mathrm{Re}\int_{\mathbb{R}^d}\left(\phi^0+\phi_I^0+\varepsilon(\phi^1+\phi_I^1)\right)\psi_R^\varepsilon\partial_t\bar{\psi}_R^\varepsilon d\mathbf{x} + 2\mathrm{Re}\int_{\mathbb{R}^d}(\psi^0-\psi^0(\mathbf{x},0))\phi_I^0\partial_t\bar{\psi}_R^\varepsilon d\mathbf{x}\\
&\quad - 2\mathrm{Re}\int_{\mathbb{R}^d}\left[(\partial_t\psi^0+\partial_\tau\psi_I^1)\bar{\psi}_R^\varepsilon\phi_R^\varepsilon\right]\,d\mathbf{x} - 2\varepsilon\mathrm{Re}\int_{\mathbb{R}^d}\partial_t\left((\psi^0-\psi^0(\mathbf{x},0))\bar{\psi}_I^1\right)\phi_R^\varepsilon\,d\mathbf{x}\\
&\quad + \varepsilon^2\int_{\mathbb{R}^d}(|\psi_I^1|^2-\partial_{tt}\phi^0)\partial_t\phi_R^\varepsilon d\mathbf{x}, \qquad 0\le t\le T.
\end{aligned}$$

Integrating the above equality over $[0,t]$ with respect to t, we get

$$\begin{aligned}
&H_2(t) - 2\mathrm{Re}\int_{\mathbb{R}^d}(\psi^0+\varepsilon\psi_I^1)\,\bar{\psi}_R^\varepsilon\,\phi_R^\varepsilon\,d\mathbf{x} - 2\varepsilon\mathrm{Re}\int_{\mathbb{R}^d}(\psi^0-\psi^0(\mathbf{x},0))\bar{\psi}_I^1\phi_R^\varepsilon\,d\mathbf{x}\\
&= 2\varepsilon\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left(\Delta\psi_I^1+(\psi^0+\varepsilon\psi_I^1)(\phi^1+\varepsilon\phi_I^1)+\psi_I^1(\phi^0+\phi_I^0)\right)\partial_s\bar{\psi}_R^\varepsilon d\mathbf{x}ds\\
&\quad + 2\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left(\phi^0+\phi_I^0+\varepsilon(\phi^1+\phi_I^1)\right)\psi_R^\varepsilon\partial_s\bar{\psi}_R^\varepsilon d\mathbf{x}ds\\
&\quad + 2\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}(\psi^0-\psi^0(\mathbf{x},0))\phi_I^0\partial_s\bar{\psi}_R^\varepsilon d\mathbf{x}ds - 2\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}(\partial_s\psi^0+\partial_\tau\psi_I^1)\bar{\psi}_R^\varepsilon\phi_R^\varepsilon\,d\mathbf{x}ds\\
&\quad - 2\varepsilon\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\partial_s\left((\psi^0-\psi^0(\mathbf{x},0))\bar{\psi}_I^1\right)\phi_R^\varepsilon\,d\mathbf{x}ds + \varepsilon^2\int_0^t\int_{\mathbb{R}^d}(|\psi_I^1|^2-\partial_{ss}\phi^0)\partial_s\phi_R^\varepsilon d\mathbf{x}ds\\
&\quad + H_2(0) - 2\mathrm{Re}\int_{\mathbb{R}^d}\left((\psi^0+\varepsilon\psi_I^1)\,\bar{\psi}_R^\varepsilon\,\phi_R^\varepsilon\right)(\mathbf{x},0)\,d\mathbf{x}\\
&\quad - 2\varepsilon\mathrm{Re}\int_{\mathbb{R}^d}\left((\psi^0-\psi^0(\mathbf{x},0))\bar{\psi}_I^1\phi_R^\varepsilon\right)(\mathbf{x},0)\,d\mathbf{x}, \qquad 0\le t\le T.
\end{aligned}$$

Now we estimate each term on the right-hand side of the above equality. For the three terms concerning $\partial_t\bar{\psi}_R^\varepsilon$ on the right-hand side of the above equality, we use (4.3) to control it. In fact, (4.3) can be rewritten as

$$\partial_t\bar{\psi}_R^\varepsilon = -i\Delta\bar{\psi}_R^\varepsilon - i\phi_R^\varepsilon\bar{\psi}_R^\varepsilon + J^\varepsilon(\mathbf{x},t), \tag{4.17}$$

where

$$J^\varepsilon(\mathbf{x},t) = -i\left[(\bar{\psi}^0+\varepsilon\bar{\psi}_I^1)\phi_R^\varepsilon\right] - i\left[(\phi^0+\phi_I^0+\varepsilon\phi_I^1)\bar{\psi}_R^\varepsilon\right] + i\bar{g}_I^\varepsilon$$

satisfies the estimate, for $|\alpha|\le m$, $m>\frac{d}{2}$,

$$\begin{aligned}
&\|\partial_{\mathbf{x}}^\alpha J^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^2 \le C\left[\left\|(\bar{\psi}^0+\varepsilon\bar{\psi}_I^1)\phi_R^\varepsilon\right\|_{H^m(\mathbb{R}^d)}^2 + \left\|(\phi^0+\phi_I^0+\varepsilon\phi_I^1)\bar{\psi}_R^\varepsilon\right\|_{H^m(\mathbb{R}^d)}^2\right]\\
&\quad +\varepsilon\left\|\Delta\bar{\psi}_I^1+(\bar{\psi}^0+\varepsilon\bar{\psi}_I^1)(\phi^1+\varepsilon\phi_I^1)+\bar{\psi}_I^1(\phi^0+\phi_I^0)\right\|_{H^m(\mathbb{R}^d)}^2\\
&\quad + \left\|\partial_{\mathbf{x}}^\alpha\left((\bar{\psi}^0-\bar{\psi}^0(\mathbf{x},0))\phi_I^0\right)\right\|_{L(\mathbb{R}^d)}^2\\
&\le C\left\|(\psi_R^\varepsilon,\phi_R^\varepsilon)(\cdot,t)\right\|_{H^m(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d}\left|\partial_{\mathbf{x}}^\alpha\left((\bar{\psi}^0(\mathbf{x},t)-\bar{\psi}^0(\mathbf{x},0))\phi_I^0(\mathbf{x},t/\varepsilon)\right)\right|^2 d\mathbf{x}\\
&\quad + C_I\varepsilon^2. \qquad (4.18)
\end{aligned}$$

Here we used the Cauchy–Schwarz inequality, the definition of g_I^ε, the properties of layer functions, and $\|fg\|_{H^m(\mathbb{R}^d)} \le \|f\|_{H^m(\mathbb{R}^d)}\|g\|_{H^m(\mathbb{R}^d)}$ for $m > \frac{d}{2}$.

Using (4.17), (4.18), (4.4), and the estimates (2.24) and (2.25), noticing the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
I_1 &:= 2\varepsilon \mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left(\Delta\psi_I^1 + (\psi^0+\varepsilon\psi_I^1)(\phi^1+\varepsilon\phi_I^1) + \psi_I^1(\phi^0+\phi_I^0)\right)\partial_s\bar{\psi}_R^\varepsilon d\mathbf{x}ds \\
&= 2\varepsilon \mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \nabla\left(\Delta\psi_I^1 + (\psi^0+\varepsilon\psi_I^1)(\phi^1+\varepsilon\phi_I^1) + \psi_I^1(\phi^0+\phi_I^0)\right)\cdot i\nabla\bar{\psi}_R^\varepsilon d\mathbf{x}ds \\
&\quad - 2\varepsilon \mathrm{Re}\int_0^t \int_{\mathbb{R}^d} i\left(\Delta\psi_I^1 + (\psi^0+\varepsilon\psi_I^1)(\phi^1+\varepsilon\phi_I^1) + \psi_I^1(\phi^0+\phi_I^0)\right)\phi^\varepsilon\bar{\psi}_R^\varepsilon d\mathbf{x}ds \\
&\quad + 2\varepsilon \mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left(\Delta\psi_I^1 + (\psi^0+\varepsilon\psi_I^1)(\phi^1+\varepsilon\phi_I^1) + \psi_I^1(\phi^0+\phi_I^0)\right)J^\varepsilon d\mathbf{x}ds \\
&\le C_I\varepsilon^2 + C\int_0^t [\chi_1(s)+H_1(s)]\,ds + \int_0^t\int_{\mathbb{R}^d}\left|(\psi^0(\mathbf{x},s)-\psi^0(\mathbf{x},0))\phi_I^0\left(\mathbf{x},\frac{s}{\varepsilon}\right)\right|^2 d\mathbf{x}ds \\
&\le C_I(f_d(\varepsilon)+\varepsilon^2) + C\int_0^t [\chi_1(s)+H_1(s)]\,ds,
\end{aligned}$$

with the aid of (4.13). Similarly, using (4.17), (4.18), (4.13), and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
I_2 &:= 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left(\phi^0+\phi_I^0+\varepsilon(\phi^1+\phi_I^1)\right)\psi_R^\varepsilon\partial_s\bar{\psi}_R^\varepsilon d\mathbf{x}ds \\
&= 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} (\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon\partial_s\bar{\psi}_R^\varepsilon d\mathbf{x}ds \\
&= 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} (\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon\left(-i\Delta\bar{\psi}_R^\varepsilon - i\phi_R^\varepsilon\bar{\psi}_R^\varepsilon + J^\varepsilon\right) d\mathbf{x}ds \\
&= 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} i\nabla\left((\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon\right)\cdot\nabla\bar{\psi}_R^\varepsilon d\mathbf{x}ds \\
&\quad + 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} (\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon J^\varepsilon d\mathbf{x}ds \\
&\le C_I(\varepsilon^2+f_d(\varepsilon)) + C\int_0^t[\chi_1(s)+H_1(s)]ds.
\end{aligned} \tag{4.19}$$

Similar to the estimate for I_1, we can control I_3 as follows:

$$\begin{aligned}
I_3 &:= 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} (\psi^0(\mathbf{x},t)-\psi^0(\mathbf{x},0))\phi_I^0\partial_s\bar{\psi}_R^\varepsilon\, d\mathbf{x}ds \\
&\le C\int_0^t [\chi_1(s)+H_1(s)]\,ds + C\int_0^t\int_{\mathbb{R}^d}\left|(\psi^0(\mathbf{x},s)-\psi^0(\mathbf{x},0))\phi_I^0\left(\mathbf{x},\frac{s}{\varepsilon}\right)\right|^2 d\mathbf{x}ds \\
&\quad + C\int_0^t\int_{\mathbb{R}^d}\left|\nabla\left((\psi^0(\mathbf{x},s)-\psi^0(\mathbf{x},0))\phi_I^0\left(\mathbf{x},\frac{s}{\varepsilon}\right)\right)\right|^2 d\mathbf{x}ds + C_I\varepsilon^2 \\
&\le C_I(f_d(\varepsilon)+\varepsilon^2) + C\int_0^t [\chi_1(s)+H_1(s)]\,ds.
\end{aligned} \tag{4.20}$$

Using (2.20), the estimates (2.24) and (2.25), and noticing the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
I_4 :=\ & 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} (\partial_s \psi^0 + \partial_\tau \psi_I^1)\bar{\psi}_R^\varepsilon \phi_R^\varepsilon \, d\mathbf{x}ds \\
& - 2\varepsilon \mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \partial_s \left((\psi^0 - \psi^0(\mathbf{x},0))\bar{\psi}_I^1 \right) \phi_R^\varepsilon \, d\mathbf{x}ds \\
=\ & 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} (\partial_s \psi^0 + i\psi^0(\mathbf{x},0)\phi_I^0)\bar{\psi}_R^\varepsilon \phi_R^\varepsilon \, d\mathbf{x}ds \\
& - 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} (\psi^0 - \psi^0(\mathbf{x},0)) i\bar{\psi}^0(\mathbf{x},0)\phi_I^0 \phi_R^\varepsilon \, d\mathbf{x}ds \\
& - 2\varepsilon \mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \partial_s \left(\psi^0 - \psi^0(\mathbf{x},0) \right) \bar{\psi}_I^1 \phi_R^\varepsilon \, d\mathbf{x}ds \\
\le\ & C_I (f_d(\varepsilon) + \varepsilon^2) + C\int_0^t [\chi_1(s) + H_1(s)]\, ds. \qquad (4.21)
\end{aligned}
$$

By using the Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
I_5 :=\ & \varepsilon^2 \int_0^t \int_{\mathbb{R}^d} \left(\left| \psi_I^1 \left(\mathbf{x}, \frac{s}{\varepsilon} \right) \right|^2 - \partial_{ss}\psi^0(\mathbf{x},s) \right) \partial_s \phi_R^\varepsilon(\mathbf{x},s) \, d\mathbf{x}ds \\
\le\ & \varepsilon^2 \int_{\mathbb{R}^d} \left[-\partial_{tt}\psi^0(\mathbf{x},t)\phi_R^\varepsilon(\mathbf{x},t) + \partial_{tt}\psi^0(\mathbf{x},0)\phi_R^\varepsilon(\mathbf{x},0) \right] \, d\mathbf{x} \\
& + \varepsilon^2 \int_0^t \int_{\mathbb{R}^d} \left(\partial_{sss}\psi^0 \phi_R^\varepsilon \right)(\mathbf{x},s) \, d\mathbf{x}ds + C\int_0^t H_1(s)\, ds + C_I \varepsilon^2 \\
\le\ & \delta \left[H_1(0) + H_1(t) \right] + C\left[\varepsilon^4 + \int_0^t H_1(s)\, ds \right] + C_I \varepsilon^2 \qquad (4.22)
\end{aligned}
$$

for some constant $\delta > 0$, independent of ε, to be chosen sufficiently small later. Plugging (4.19)–(4.22) into (4.17), we obtain, by the Cauchy–Schwarz inequality, that

$$
\begin{aligned}
& H_2(t) - 2\mathrm{Re}\int_{\mathbb{R}^d} (\psi^0 + \varepsilon\psi_I^1)\, \bar{\psi}_R^\varepsilon \, \phi_R^\varepsilon \, d\mathbf{x} - 2\varepsilon\mathrm{Re}\int_{\mathbb{R}^d} (\psi^0 - \psi^0(\mathbf{x},0))\bar{\psi}_I^1 \phi_R^\varepsilon \, d\mathbf{x} \\
& \le C H_2(0) + C\delta\, H_1(t) + C\left[H_1(0) + \chi_1(0) + \varepsilon^4 \right] + C_I(\varepsilon^2 + f_d(\varepsilon)) \\
& \quad + C\int_0^t [\chi_1(s) + H_1(s)]\, ds. \qquad (4.23)
\end{aligned}
$$

From the definition of $H_2(t)$, we know that there exists a positive constant $C > 0$ such that

$$
\begin{aligned}
& \frac{1}{4} H_1(t) - C\left[\chi_1(t) + (\chi_1(t))^{\frac{8-d}{4-d}} \right] - C_I \varepsilon^2 \\
& \le H_2(t) - 2\mathrm{Re}\int_{\mathbb{R}^d} (\psi^0 + \varepsilon\psi_I^1)\, \bar{\psi}_R^\varepsilon \, \phi_R^\varepsilon \, d\mathbf{x} - 2\varepsilon\mathrm{Re}\int_{\mathbb{R}^d} (\psi^0 - \psi^0(\mathbf{x},0))\bar{\psi}_I^1 \phi_R^\varepsilon \, d\mathbf{x} \\
& \le H_1(t) + C\left[\chi_1(t) + (\chi_1(t))^{\frac{8-d}{4-d}} \right] + C_I \varepsilon^2 \qquad (4.24)
\end{aligned}
$$

with the help of Young's inequality. In fact, here we use

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\left[-\phi_R^\varepsilon(\mathbf{x},t)\,|\psi_R^\varepsilon(\mathbf{x},t)|^2\right]\,d\mathbf{x}\right| \le \|\phi_R^\varepsilon\|_{L^4(\mathbb{R}^d)}\,\|\psi_R^\varepsilon\|_{L^4(\mathbb{R}^d)}\,\|\psi_R^\varepsilon\|_{L^2(\mathbb{R}^d)}\\
&\quad\le C\,\|\phi_R^\varepsilon\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{4}}\,\|\nabla\phi_R^\varepsilon\|_{L^2(\mathbb{R}^d)}^{\frac{d}{4}}\,\|\nabla\psi_R^\varepsilon\|_{L^2(\mathbb{R}^d)}^{\frac{d}{4}}\,\|\psi_R^\varepsilon\|_{L^2(\mathbb{R}^d)}^{2-\frac{d}{4}}\\
&\quad\le \frac{\delta}{2}\left[\|\phi_R^\varepsilon\|_{L^2(\mathbb{R}^d)}^2+\|\nabla\phi_R^\varepsilon\|_{L^2(\mathbb{R}^d)}^2+\|\nabla\psi_R^\varepsilon\|_{L^2(\mathbb{R}^d)}^2\right]+C_\delta\,\|\psi_R^\varepsilon\|_{L^2(\mathbb{R}^d)}^{\frac{2(8-d)}{4-d}}\\
&\quad\le \delta\,H_1(t)+C_\delta\,\|\psi_R^\varepsilon\|_{L^2(\mathbb{R}^d)}^{\frac{2(8-d)}{4-d}},\qquad d=1,2,3,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\left[-2\mathrm{Re}\left((\psi^0(\mathbf{x},t)+\varepsilon\psi_I^1)\bar{\psi_R^\varepsilon}(\mathbf{x},t)\right)\phi_R^\varepsilon(\mathbf{x},t)\right]d\mathbf{x}\right|\\
&\quad\le \frac{\delta}{2}\,\|\phi_R^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}+C_\delta\,\|\psi_R^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^2\\
&\quad\le \delta\,H_1(t)+C_\delta\,\|\psi_R^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^2
\end{aligned}$$

for some positive constant $\delta>0$ sufficiently small and independent of ε, and C_δ is a generic constant independent of ε. Combining (4.23) and (4.24) and choosing δ to be sufficiently small, we obtain

$$\begin{aligned}
H_1(t) &\le C\left[H_1(0)+\chi_1(0)+\chi_1(0)^{\frac{8-d}{4-d}}+\chi_1(t)+\chi_1(t)^{\frac{8-d}{4-d}}+\varepsilon^4\right]\\
&\qquad +C\int_0^t\left[\chi_1(s)+H_1(s)\right]ds+C_I(f_d(\varepsilon)+\varepsilon^2).
\end{aligned}\tag{4.25}$$

Step 3. Find H^m-estimates ($m\ge 2$) of $\psi_R^\varepsilon(\mathbf{x},t)$.

For the multi-index α satisfying $0<|\alpha|\le m$, define $\chi_2(t)=\|\partial_{\mathbf{x}}^\alpha\psi_R^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^2$, $H_3(t)=\|\nabla\partial_{\mathbf{x}}^\alpha\psi_R^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^2+\|\partial_{\mathbf{x}}^\alpha(\phi_R^\varepsilon,\nabla\phi_R^\varepsilon,\varepsilon\partial_t\phi_R^\varepsilon)(\cdot,t)\|_{L^2(\mathbb{R}^d)}^2$, and $H_4(t)=H_3(t)-2\int_{\mathbb{R}^d}[\partial_{\mathbf{x}}^\alpha|\psi_R^\varepsilon|^2\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon](\mathbf{x},t)d\mathbf{x}-4\mathrm{Re}\int_{\mathbb{R}^d}[\partial_{\mathbf{x}}^\alpha((\psi^0+\varepsilon\psi_I^1)\bar{\psi_R^\varepsilon})\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon](\mathbf{x},t)d\mathbf{x}$.

Taking $\partial_{\mathbf{x}}^\alpha$ of (4.3) and then multiplying by $\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}$, integrating over $\mathbb{R}^d$ with respect to $\mathbf{x}$, integrating by parts, and taking the imaginary part, we get, for any given $0<T<\infty$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\chi_2(t) &= -\,\mathrm{Im}\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\left((\psi^0+\varepsilon\psi_I^1)\phi_R^\varepsilon+(\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon\right)\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},t)\,d\mathbf{x}\\
&\quad -\,\mathrm{Im}\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha(\psi_R^\varepsilon\phi_R^\varepsilon)\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},t)d\mathbf{x}+\mathrm{Im}\int_{\mathbb{R}^d}\partial_{\mathbf{x}}^\alpha g_I^\varepsilon\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}d\mathbf{x},\quad 0\le t\le T.
\end{aligned}\tag{4.26}$$

We will estimate the three terms on the right-hand side of (4.26).

By Hölder's inequality, the Cauchy–Schwarz inequality, and Sobolev's embedding inequality, and using the boundedness of $\partial_{\mathbf{x}}^\alpha(\psi^0,\phi^0)$ and the initial layer functions

$\partial_{\mathbf{x}}^\alpha(\phi_I^0,\phi_I^1,\psi_I^1)$ (see (2.24), (2.25) and (2.26)), we get

$$
\begin{aligned}
I_6 &:= -\mathrm{Im}\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\left((\psi^0+\varepsilon\psi_I^1)\phi_R^\varepsilon+(\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon\right)\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},t)d\mathbf{x}\\
&= -\mathrm{Im}\int_{\mathbb{R}^d}\left[\left((\psi^0+\varepsilon\psi_I^1)\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon+(\phi^0+\phi_I^0+\varepsilon\phi_I^1)\partial_{\mathbf{x}}^\alpha\psi_R^\varepsilon\right)\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},t)d\mathbf{x}\\
&\quad -\mathrm{Im}\int_{\mathbb{R}^d}\left[\left(\partial_{\mathbf{x}}^\alpha\left((\psi^0+\varepsilon\psi_I^1)\phi_R^\varepsilon\right)-(\psi^0+\varepsilon\psi_I^1)\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right)\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},t)d\mathbf{x}\\
&\quad -\mathrm{Im}\int_{\mathbb{R}^d}\left[\left(\partial_{\mathbf{x}}^\alpha\left((\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon\right)-(\phi^0+\phi_I^0+\varepsilon\phi_I^1)\partial_{\mathbf{x}}^\alpha\psi_R^\varepsilon\right)\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},t)d\mathbf{x}\\
&\le C\left[\chi_2(t)+H_3(t)\right]+\|\partial_{\mathbf{x}}^\alpha((\psi^0+\varepsilon\psi_I^1)\phi_R^\varepsilon)-(\psi^0+\varepsilon\psi_I^1)\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\|_{L^2}\|\partial_{\mathbf{x}}^\alpha\psi_R^\varepsilon\|_{L^2}\\
&\quad +\|\partial_{\mathbf{x}}^\alpha((\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon)-(\phi^0+\phi_I^0+\varepsilon\phi_I^1)\partial_{\mathbf{x}}^\alpha\psi_R^\varepsilon\|_{L^2}\|\partial_{\mathbf{x}}^\alpha\psi_R^\varepsilon\|_{L^2}\\
&\le C\left[\chi_2(t)+H_3(t)++\left(\|\nabla(\psi^0+\varepsilon\psi_I^1)\|_{L^\infty}\|D_x^{m-1}\phi_R^\varepsilon\|_{L^2}\right.\right.\\
&\quad +\|\phi_R^\varepsilon\|_{L^\infty}\|D_x^m(\psi^0+\varepsilon\psi_I^1)\|_{L^2}+\|\nabla(\phi^0+\phi_I^0+\varepsilon\phi_I^1)\|_{L^\infty}\|D_x^{m-1}\psi_R^\varepsilon\|_{L^2}\\
&\quad \left.\left.+\|\psi_R^\varepsilon\|_{L^\infty}\|D_x^m(\phi^0+\phi_I^0+\varepsilon\phi_I^1)\|_{L^2}\right]\|\partial_{\mathbf{x}}^\alpha\psi_R^\varepsilon\|_{L^2}\right.\\
&\le C\left[\chi_2(t)+H_3(t)+\|(\phi_R^\varepsilon,\psi_R^\varepsilon)(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}\right].
\end{aligned}
$$

Thanks to the fact that $m\ge 2>\frac{d}{2}$, using the inequality

$$\|\partial_{\mathbf{x}}^\alpha(uv)\|_{L^2(\mathbb{R}^d)}\le C\|u\|_{H^m(\mathbb{R}^d)}\,\|v\|_{H^m(\mathbb{R}^d)},$$

we get

$$
\begin{aligned}
I_7 &:= -\mathrm{Im}\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\left(\psi_R^\varepsilon(\mathbf{x},t)\phi_R^\varepsilon(\mathbf{x},t)\right)\ \partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}(\mathbf{x},t)\right]d\mathbf{x}\\
&\le C\|\psi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}+\|\phi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}\ \|\psi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}. \qquad (4.27)
\end{aligned}
$$

Using the definition (4.6) of g_I^ε, by the Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
I_8 &:= \mathrm{Im}\int_{\mathbb{R}^d}\partial_{\mathbf{x}}^\alpha g_I^\varepsilon\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}d\mathbf{x}\\
&\le C\chi_2(t)+C\int_{\mathbb{R}^d}\left|\partial_{\mathbf{x}}^\alpha\left((\psi^0(\mathbf{x},t)-\psi^0(\mathbf{x},0))\phi_I^0(\mathbf{x},t/\varepsilon)\right)\right|^2d\mathbf{x}+C_I\varepsilon^2. \qquad (4.28)
\end{aligned}
$$

Plugging (4.27)–(4.28) into (4.26), we obtain

$$
\begin{aligned}
\frac{d}{dt}\chi_2(t) &\le C\int_{\mathbb{R}^d}\left|\partial_{\mathbf{x}}^\alpha\left((\psi^0(\mathbf{x},t)-\psi^0(\mathbf{x},0))\phi_I^0(\mathbf{x},t/\varepsilon)\right)\right|^2d\mathbf{x}+C_I\varepsilon^2\\
&\quad +C\|\phi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}\|\psi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}\\
&\quad +C\left[\chi_2(t)+H_3(t)+\|(\phi_R^\varepsilon,\psi_R^\varepsilon)(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}\right]. \qquad (4.29)
\end{aligned}
$$

Integrating (4.29) over $[0,t]$ with respect to t, we have

$$
\begin{aligned}
\chi_2(t) &\le C\int_0^t\int_{\mathbb{R}^d}\left|\partial_{\mathbf{x}}^\alpha\left((\psi^0(\mathbf{x},s)-\psi^0(\mathbf{x},0))\phi_I^0(\mathbf{x},s/\varepsilon)\right)\right|^2d\mathbf{x}ds\\
&\quad +C\int_0^t\left[\chi_2(s)+H_3(s)+\|(\phi_R^\varepsilon,\psi_R^\varepsilon)(\cdot,s)\|^2_{H^m(\mathbb{R}^d)}\right.\\
&\quad \left.+\|\phi_R^\varepsilon(\cdot,s)\|^2_{H^m(\mathbb{R}^d)}\|\psi_R^\varepsilon(\cdot,s)\|^2_{H^m(\mathbb{R}^d)}\right]ds+\chi_2(0)+C_I\varepsilon^2. \qquad (4.30)
\end{aligned}
$$

Using the decay rate in (2.24) for the Klein–Gordon equation, we have, for some $\theta_1 \in (0,1)$, that

$$\int_0^t \int_{\mathbb{R}^d} \left|\partial_{\mathbf{x}}^{\alpha}\left((\psi^0(\mathbf{x},s)-\psi^0(\mathbf{x},0))\phi_I^0(\mathbf{x},s/\varepsilon)\right)\right|^2 \, d\mathbf{x}ds$$
$$= \int_0^t \int_{\mathbb{R}^d} \left|\partial_{\mathbf{x}}^{\alpha}\left(\partial_s\psi^0(\mathbf{x},\theta_1 s)s\phi_I^0(\mathbf{x},s/\varepsilon)\right)\right|^2 \, d\mathbf{x}ds$$
$$\le C\sum_{\beta\le\alpha}\int_0^t \int_{\mathbb{R}^d} \left|\partial_{\mathbf{x}}^{\alpha-\beta}\partial_s\psi^0(\mathbf{x},\theta_1 s)s\partial_{\mathbf{x}}^{\beta}\phi_I^0(\mathbf{x},s/\varepsilon)\right|^2 \, d\mathbf{x}ds$$
$$\le C\sum_{\beta\le\alpha}\int_{\mathbb{R}^d}\sup_{0\le s\le t}|\partial_{\mathbf{x}}^{\alpha-\beta}\partial_s\psi^0(\mathbf{x},\theta_1 s)|^2 \, d\mathbf{x} \, \int_0^t \|s\partial_{\mathbf{x}}^{\beta}\phi_I^0(\cdot,s/\varepsilon)\|^2_{L^\infty(\mathbb{R}^d)} \, ds$$
$$\le C \, \sum_{\beta\le\alpha}\varepsilon^3\int_0^{t/\varepsilon}\|s\partial_{\mathbf{x}}^{\beta}\phi_I^0(\cdot,s)\|^2_{L^\infty(\mathbb{R}^d)}ds$$
$$\le C\,\varepsilon^3 \, \int_0^{t/\varepsilon}\frac{s^2}{(1+s)^d}ds \le C \, f_d(\varepsilon). \tag{4.31}$$

Substituting (4.31) into (4.30), we have

$$\chi_2(t) \le C\int_0^t \left[\chi_2(s)+H_3(s)+\|\phi_R^{\varepsilon}(\cdot,s)\|^2_{H^m(\mathbb{R}^d)}\|\psi_R^{\varepsilon}(\cdot,s)\|^2_{H^m(\mathbb{R}^d)}\right]ds$$
$$+C\int_0^t\|(\phi_R^{\varepsilon},\psi_R^{\varepsilon})(\cdot,s)\|^2_{H^m(\mathbb{R}^d)}ds+\chi_2(0)+C_I(\varepsilon^2+f_d(\varepsilon)). \tag{4.32}$$

Step 4. Establish an entropy inequality based on the Hamiltonian $H_3(t)$.

Taking $\partial_{\mathbf{x}}^{\alpha}$ of (4.3) with respect to $\mathbf{x}$ and then multiplying by $-2\partial_t\partial_{\mathbf{x}}^{\alpha}\bar{\psi_R^{\varepsilon}}$, integrating over $\mathbb{R}^d$ with respect to $\mathbf{x}$, integrating by parts, and taking the real part, we get, for any given $0<T<\infty$,

$$\frac{d}{dt}\|\nabla\partial_{\mathbf{x}}^{\alpha}\psi_R^{\varepsilon}(\cdot,t)\|^2_{L^2(\mathbb{R}^d)}$$
$$=2\mathrm{Re}\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^{\alpha}\left(\phi_R^{\varepsilon}\psi_R^{\varepsilon}\right)\partial_t\partial_{\mathbf{x}}^{\alpha}\bar{\psi_R^{\varepsilon}}\right](\mathbf{x},t)d\mathbf{x}-2\mathrm{Re}\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^{\alpha}g_I^{\varepsilon}\partial_t\partial_{\mathbf{x}}^{\alpha}\bar{\psi_R^{\varepsilon}}\right](\mathbf{x},t)d\mathbf{x}$$
$$+2\mathrm{Re}\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^{\alpha}\left((\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^{\varepsilon}+(\psi^0+\varepsilon\psi_I^1)\phi_R^{\varepsilon}\right)\,\partial_t\partial_{\mathbf{x}}^{\alpha}\bar{\psi_R^{\varepsilon}}\right](\mathbf{x},t)d\mathbf{x}. \tag{4.33}$$

Similarly, taking $\partial_{\mathbf{x}}^{\alpha}$ of (4.4) with respect to $\mathbf{x}$ and then multiplying by $\partial_t\partial_{\mathbf{x}}^{\alpha}\bar{\phi_R^{\varepsilon}}$, integrating over $\mathbb{R}^d$ with respect to $\mathbf{x}$, and integrating by parts, we get, for any given $0<T<\infty$,

$$\frac{d}{dt}\left[\|(\varepsilon\partial_t\partial_{\mathbf{x}}^{\alpha}\phi_R^{\varepsilon},\nabla\partial_{\mathbf{x}}^{\alpha}\phi_R^{\varepsilon},\partial_{\mathbf{x}}^{\alpha}\phi_R^{\varepsilon})\|^2_{L^2(\mathbb{R}^d)}-2\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^{\alpha}|\psi_R^{\varepsilon}|^2\partial_{\mathbf{x}}^{\alpha}\phi_R^{\varepsilon}\right](\mathbf{x},t)d\mathbf{x}\right.$$
$$\left.-4\mathrm{Re}\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^{\alpha}\left((\psi^0+\varepsilon\psi_I^1)\bar{\psi_R^{\varepsilon}}\right)\partial_{\mathbf{x}}^{\alpha}\phi_R^{\varepsilon}\right](\mathbf{x},t)d\mathbf{x}\right]$$
$$=-2\int_{\mathbb{R}^d}\left[\partial_t\partial_{\mathbf{x}}^{\alpha}|\psi_R^{\varepsilon}|^2\partial_{\mathbf{x}}^{\alpha}\phi_R^{\varepsilon}\right](\mathbf{x},t)d\mathbf{x}-2\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^{\alpha}h^{\varepsilon}\partial_t\partial_{\mathbf{x}}^{\alpha}\phi_R^{\varepsilon}\right](\mathbf{x},t)d\mathbf{x}$$
$$-4\,\mathrm{Re}\int_{\mathbb{R}^d}\left[\partial_t\partial_{\mathbf{x}}^{\alpha}\left((\psi^0+\varepsilon\psi_I^1)\bar{\psi_R^{\varepsilon}}\right)\partial_{\mathbf{x}}^{\alpha}\phi_R^{\varepsilon}\right](\mathbf{x},t)d\mathbf{x},\qquad 0\le t\le T. \tag{4.34}$$

Combining (4.33) and (4.34), we get

$$\begin{aligned}\frac{d}{dt}H_4(t) &= 2\mathrm{Re}\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\left(\phi_R^\varepsilon\psi_R^\varepsilon\right)\partial_t\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},t)d\mathbf{x} - 2\int_{\mathbb{R}^d}\left[\partial_t\partial_{\mathbf{x}}^\alpha|\psi_R^\varepsilon|^2\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},t)d\mathbf{x}\\ &+ 2\mathrm{Re}\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\left((\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon+(\psi^0+\varepsilon\psi_I^1)\phi_R^\varepsilon\right)\,\partial_t\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},t)d\mathbf{x}\\ &- 4\mathrm{Re}\int_{\mathbb{R}^d}\left[\partial_t\partial_{\mathbf{x}}^\alpha\left((\psi^0+\varepsilon\psi_I^1)\bar{\psi_R^\varepsilon}\right)\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},t)d\mathbf{x}\\ &- 2\mathrm{Re}\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha g_I^\varepsilon\partial_t\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},t)d\mathbf{x} - 2\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha h^\varepsilon\partial_t\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},t)d\mathbf{x}.\end{aligned}\tag{4.35}$$

Integrating (4.35) with respect to t, we get

$$\begin{aligned}H_4(t) &= 2\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\left(\phi_R^\varepsilon\psi_R^\varepsilon\right)\partial_s\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},s)d\mathbf{x}ds\\ &- 2\int_0^t\int_{\mathbb{R}^d}\left[\partial_s\partial_{\mathbf{x}}^\alpha|\psi_R^\varepsilon|^2\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds\\ &+ 2\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\left((\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon+(\psi^0+\varepsilon\psi_I^1)\phi_R^\varepsilon\right)\,\partial_s\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},s)d\mathbf{x}ds\\ &- 4\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\left((\psi^0+\varepsilon\psi_I^1)\partial_s\bar{\psi_R^\varepsilon}\right)\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds\\ &- 2\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha g_I^\varepsilon\partial_s\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},s)d\mathbf{x}ds - 2\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha h^\varepsilon\partial_s\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds\\ &- 4\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\left((\partial_s\psi^0+\partial_\tau\psi_I^1)\bar{\psi_R^\varepsilon}\right)\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds + H_4(0).\end{aligned}\tag{4.36}$$

Now we estimate each term in the right-hand side of (4.36).

First, thanks to (4.17), by the Cauchy–Schwarz inequality, Sobolev's embedding inequality, and the estimates (4.18), (4.31), we have

$$\begin{aligned}I_9 &:= 2\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\left(\phi_R^\varepsilon\psi_R^\varepsilon\right)\partial_s\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},s)d\mathbf{x}ds\\ &= 2\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\left(\phi_R^\varepsilon\psi_R^\varepsilon\right)\partial_{\mathbf{x}}^\alpha\left(-i\Delta\bar{\psi_R^\varepsilon}-i\phi_R^\varepsilon\bar{\psi_R^\varepsilon}+J^\varepsilon\right)\right](\mathbf{x},s)d\mathbf{x}ds\\ &= 2\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}i\left[\nabla\partial_{\mathbf{x}}^\alpha\left(\phi_R^\varepsilon\psi_R^\varepsilon\right)\cdot\nabla\partial_{\mathbf{x}}^\alpha\bar{\psi_R^\varepsilon}\right](\mathbf{x},s)d\mathbf{x}ds\\ &\quad+ 2\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\left(\phi_R^\varepsilon\psi_R^\varepsilon\right)\partial_{\mathbf{x}}^\alpha J^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds\\ &\le C\int_0^t\left[\|\nabla\partial_{\mathbf{x}}^\alpha(\phi_R^\varepsilon\psi_R^\varepsilon)\|_{L^2(\mathbb{R}^d)}^2+\|\nabla\partial_{\mathbf{x}}^\alpha\psi_R^\varepsilon\|_{L^2(\mathbb{R}^d)}^2\right](s)ds\\ &\quad+ C\int_0^t\left[\|\partial_{\mathbf{x}}^\alpha(\phi_R^\varepsilon\psi_R^\varepsilon)\|_{L^2(\mathbb{R}^d)}^2+\|\partial_{\mathbf{x}}^\alpha J^\varepsilon\|_{L^2(\mathbb{R}^d)}^2\right](s)ds\\ &\le C_I(\varepsilon^2+f_d(\varepsilon)) + C\int_0^t\left[\|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|_{H^{m+1}(\mathbb{R}^d)}^2+\|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|_{H^{m+1}(\mathbb{R}^d)}^4\right](s)ds.\end{aligned}$$

As in the above inequality, we have

$$
\begin{aligned}
I_{10} &:= -2\int_0^t \int_{\mathbb{R}^d} \left[\partial_s \partial_{\mathbf{x}}^\alpha |\psi_R^\varepsilon|^2 \partial_{\mathbf{x}}^\alpha \phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds \\
&\le C_I(\varepsilon^2 + f_d(\varepsilon)) + C\int_0^t \left[\|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|_{H^{m+1}(\mathbb{R}^d)}^2 + \|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|_{H^{m+1}(\mathbb{R}^d)}^4\right](s)ds
\end{aligned}
$$

and

$$
\begin{aligned}
I_{11} &:= 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left[\partial_{\mathbf{x}}^\alpha \left((\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon + (\psi^0+\varepsilon\psi_I^1)\phi_R^\varepsilon\right)\partial_s\partial_{\mathbf{x}}^\alpha \bar{\psi_R^\varepsilon}\right](\mathbf{x},s)d\mathbf{x}ds \\
&= 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left[\nabla\partial_{\mathbf{x}}^\alpha \left((\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon + (\psi^0+\varepsilon\psi_I^1)\phi_R^\varepsilon\right)(i\nabla\partial_{\mathbf{x}}^\alpha \bar{\psi_R^\varepsilon})\right](\mathbf{x},s)d\mathbf{x}ds \\
&\quad - 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left[\partial_{\mathbf{x}}^\alpha \left((\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon + (\psi^0+\varepsilon\psi_I^1)\phi_R^\varepsilon\right)\partial_{\mathbf{x}}^\alpha\left(i\phi_R^\varepsilon \bar{\psi_R^\varepsilon}\right)\right](\mathbf{x},s)d\mathbf{x}ds \\
&\quad + 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left[\partial_{\mathbf{x}}^\alpha \left((\phi^0+\phi_I^0+\varepsilon\phi_I^1)\psi_R^\varepsilon + (\psi^0+\varepsilon\psi_I^1)\phi_R^\varepsilon\right)\partial_{\mathbf{x}}^\alpha J^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds \\
&\le C\int_0^t \left[\|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|_{H^{m}(\mathbb{R}^d)}^4 + \|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|_{H^{m+1}(\mathbb{R}^d)}^2\right](s)ds \\
&\quad + C\int_0^t \int_{\mathbb{R}^d} \left|\partial_{\mathbf{x}}^\alpha J^\varepsilon(\mathbf{x},s)\right|^2 \, d\mathbf{x}ds \\
&\le C_I(\varepsilon^2 + f_d(\varepsilon)) + C\int_0^t \left[\|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|_{H^{m}(\mathbb{R}^d)}^4 + \|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|_{H^{m+1}(\mathbb{R}^d)}^2\right](s)ds.
\end{aligned}
$$

Using (4.17), (2.20), (4.18), and (4.31), we get

$$
\begin{aligned}
I_{12} &:= -4\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left[\partial_{\mathbf{x}}^\alpha \left((\psi^0+\varepsilon\psi_I^1)\partial_s \bar{\psi_R^\varepsilon}\right)\partial_{\mathbf{x}}^\alpha \phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds \\
&\quad - 4\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left[\partial_{\mathbf{x}}^\alpha \left((\partial_s\psi^0+\partial_\tau\psi_I^1)\bar{\psi_R^\varepsilon}\right)\partial_{\mathbf{x}}^\alpha \phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds \\
&= -4\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left[\partial_{\mathbf{x}}^\alpha \left((\psi^0+\varepsilon\psi_I^1)(-i\Delta\bar{\psi_R^\varepsilon} - i\phi_R^\varepsilon\bar{\psi_R^\varepsilon} + J^\varepsilon)\right)\partial_{\mathbf{x}}^\alpha \phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds \\
&\quad - 4\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left[\partial_{\mathbf{x}}^\alpha \left((\partial_s\psi^0 - i\psi^0(\mathbf{x},0)\phi_I^0)\bar{\psi_R^\varepsilon}\right)\partial_{\mathbf{x}}^\alpha \phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds \\
&= -4\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \Big[\partial_{\mathbf{x}}^\alpha \left((\psi^0+\varepsilon\psi_I^1)i\nabla\bar{\psi_R^\varepsilon}\right)\cdot\nabla\partial_{\mathbf{x}}^\alpha \phi_R^\varepsilon \\
&\qquad + \partial_{\mathbf{x}}^\alpha\left(\nabla(\psi^0+\varepsilon\psi_I^1)\cdot i\nabla\bar{\psi_R^\varepsilon}\right)\partial_{\mathbf{x}}^\alpha \phi_R^\varepsilon\Big](\mathbf{x},s)d\mathbf{x}ds \\
&\quad - 4\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left[\partial_{\mathbf{x}}^\alpha \left((\psi^0+\varepsilon\psi_I^1)(-i\phi_R^\varepsilon\bar{\psi_R^\varepsilon} + J^\varepsilon)\right)\partial_{\mathbf{x}}^\alpha \phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds \\
&\quad - 4\mathrm{Re}\int_0^t \int_{\mathbb{R}^d} \left[\partial_{\mathbf{x}}^\alpha \left((\partial_s\psi^0 - i\psi^0(\mathbf{x},0)\phi_I^0)\bar{\psi_R^\varepsilon}\right)\partial_{\mathbf{x}}^\alpha \phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds \\
&\le C_I(\varepsilon^2 + f_d(\varepsilon)) + C\int_0^t \left[\|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|_{H^{m}(\mathbb{R}^d)}^4 + \|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|_{H^{m+1}(\mathbb{R}^d)}^2\right](s)ds. \qquad (4.37)
\end{aligned}
$$

Finally, integrating by parts and noticing the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
I_{13} :=& -2\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha g_I^\varepsilon \partial_s\partial_{\mathbf{x}}^\alpha \bar{\psi_R^\varepsilon}\right](\mathbf{x},s)d\mathbf{x}ds - 2\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha h^\varepsilon \partial_s\partial_{\mathbf{x}}^\alpha \phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds\\
\le& \, C\int_0^t [\|(\phi_R^\varepsilon,\psi_R^\varepsilon)(\cdot,s)\|^2_{H^{m+1}(\mathbb{R}^d)} + H_3(s)]ds + C_I(\varepsilon^2+f_d(\varepsilon))\\
&- 2\varepsilon\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha((\psi^0-\psi^0(\mathbf{x},0))\bar{\psi_I^1})\partial_s\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds\\
&- \varepsilon^2\int_0^t\int_{\mathbb{R}^d}\left[\partial_{ss}\partial_{\mathbf{x}}^\alpha\phi^0\partial_s\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds\\
=& \, C\int_0^t [\|(\phi_R^\varepsilon,\psi_R^\varepsilon)(\cdot,s)\|^2_{H^{m+1}(\mathbb{R}^d)} + H_3(s)]ds + C_I(\varepsilon^2+f_d(\varepsilon))\\
&- 2\varepsilon\mathrm{Re}\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha((\psi^0-\psi^0(\mathbf{x},0))\bar{\psi_I^1})\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},t)d\mathbf{x}\\
&+ 2\varepsilon\mathrm{Re}\int_0^t\int_{\mathbb{R}^d}\left[\partial_{\mathbf{x}}^\alpha\partial_s((\psi^0-\psi^0(\mathbf{x},0))\bar{\psi_I^1})\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds\\
&- \varepsilon^2\int_{\mathbb{R}^d}\left[\partial_{tt}\partial_{\mathbf{x}}^\alpha\phi^0\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},t)d\mathbf{x} + \varepsilon^2\int_{\mathbb{R}^d}\left[\partial_{tt}\partial_{\mathbf{x}}^\alpha\phi^0\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},0)d\mathbf{x}\\
&+ \varepsilon^2\int_0^t\int_{\mathbb{R}^d}\left[\partial_{sss}\partial_{\mathbf{x}}^\alpha\phi^0\partial_{\mathbf{x}}^\alpha\phi_R^\varepsilon\right](\mathbf{x},s)d\mathbf{x}ds\\
\le& \, \delta H_3(t) + C\int_0^t [\|(\phi_R^\varepsilon,\psi_R^\varepsilon)(\cdot,s)\|^2_{H^{m+1}(\mathbb{R}^d)} + H_3(s)]ds\\
&+ C(H_3(0)+\varepsilon^4) + C_I(\varepsilon^2+f_d(\varepsilon)).
\end{aligned}\tag{4.38}
$$

Plugging (4.37)–(4.38) into (4.36), we obtain

$$
\begin{aligned}
H_4(t) \le& \, H_4(0) + \delta H_3(t) + C(H_3(0)+\varepsilon^4) + C_I(\varepsilon^2+f_d(\varepsilon))\\
&+ C\int_0^t\left[\|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|^4_{H^{m+1}(\mathbb{R}^d)} + \|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|^2_{H^{m+1}(\mathbb{R}^d)} + H_3(s)\right](s)ds.
\end{aligned}\tag{4.39}
$$

Similarly as in (4.24), we can deduce the relation between $H_3(t)$ and $H_4(t)$ as follows:

$$
\begin{aligned}
&\frac{1}{2}H_3(t) - C\left(1+\|\psi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}\right)\|\psi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)} \le H_4(t)\\
&\le 2H_3(t) + C\left(1+\|\psi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}\right)\|\psi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}.
\end{aligned}\tag{4.40}
$$

Combining (4.39) and (4.40), noticing (4.5), and taking δ to be sufficiently small, we get

$$
\begin{aligned}
H_3(t) \le& \, C(H_3(0)+\varepsilon^4) + C_I(\varepsilon^2+f_d(\varepsilon)) + C\left(1+\|\psi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}\right)\|\psi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}\\
&+ C\int_0^t\left[\|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|^4_{H^{m+1}(\mathbb{R}^d)} + \|(\phi_R^\varepsilon,\psi_R^\varepsilon)\|^2_{H^{m+1}(\mathbb{R}^d)} + H_3(s)\right](s)ds.
\end{aligned}\tag{4.41}
$$

Thus by defining $\chi_3(t) = \|\psi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}$ and $H_5(t) = \|(\phi_R^\varepsilon,\nabla\phi_R^\varepsilon,\varepsilon\partial_t\phi_R^\varepsilon)(\cdot,t)\|^2_{H^m(\mathbb{R}^d)} + \|\nabla\psi_R^\varepsilon(\cdot,t)\|^2_{H^m(\mathbb{R}^d)}$, it follows from (4.41) that

$$
\begin{aligned}
H_5(t) \le& \, C(H_5(0)+\varepsilon^4) + C_I(\varepsilon^2+f_d(\varepsilon)) + C(1+\chi_3(0))\chi_3(0) + C(1+\chi_3(t))\chi_3(t)\\
&+ C\int_0^t\left[\chi_3(s)+H_5(s)+\chi_3(s)^2+H_5(s)^2\right]ds.
\end{aligned}\tag{4.42}
$$

Step 5. Complete the proof of Theorem 1.2.

Introduce the energy function

$$\Gamma^\varepsilon(t) = \chi_1(t) + \chi_3(t) + H_1(t) + H_5(t). \tag{4.43}$$

Plugging (4.14), (4.25), (4.32), and (4.42) into (4.43) and noticing (4.5), we have

$$\Gamma^\varepsilon(t) \le \Gamma^\varepsilon(t)^{1+\kappa} + C\varepsilon^4 + C_I(\varepsilon^2 + f_d(\varepsilon)) + \int_0^t \left[\Gamma^\varepsilon(s) + \Gamma^\varepsilon(s)^{1+\kappa}\right] ds \tag{4.44}$$

for some constant $\kappa > 0$. By applying the same technique as in [31] and Gronwall's inequality to (4.44), noticing (1.15), (1.17), and the smallness of $\Gamma^\varepsilon(0)$ from the assumption put on the initial data, we immediately obtain the desired estimates in (1.21) and (1.23) with $\gamma = 0$. ∎

5. Numerical tests on convergence rates. In this section, we perform some numerical tests to study the convergence rates of the singular limits from KGS to SY equations. In order to do so, we solve numerically the KGS equations (1.1)–(1.3) by the phase space analytical solver + time-splitting spectral (PSAS+TSSP) method [3] and, respectively, the SY equations (1.4)–(1.6) by the TSSP method, proposed in [2]. The linear Klein–Gordon equations (1.24)–(1.25) are discretized in space by the Fourier spectral method and integrated in time analytically, which is the technique used in the PSAS+TSSP method dealing with the KG equation alone in the KGS equations. In our computations, initial conditions are chosen such that they decay to zero sufficiently fast as $\|\mathbf{x}\| \to \infty$. Moreover, we always choose mesh size $h > 0$ and time step $k > 0$ sufficiently small such that the discretization errors can be ignored. Let $(\psi^\varepsilon, \phi^\varepsilon)$ and (ψ^0, ϕ^0) be the numerical solutions of the KGS equations (1.1)–(1.3) and the SY equations (1.4)–(1.6), respectively, and let ϕ_I^0 be the numerical solution of the linear Klein–Gordon equation (1.24)–(1.25). To quantify the errors, for any $T > 0$, we define the following two error functions:

$$e_{\scriptscriptstyle W}(T) = \sup_{0\le t\le T} \{\|\psi^\varepsilon(\cdot,t) - \psi^0(\cdot,t)\|_{H^1(\mathbb{R}^d)} + \|\phi^\varepsilon(\cdot,t) - \phi^0(\cdot,t)\|_{H^1(\mathbb{R}^d)}\}$$

and

$$e_{\scriptscriptstyle I}(T) = \sup_{0\le t\le T} \left\{\|\psi^\varepsilon(\cdot,t) - \psi^0(\cdot,t)\|_{H^1(\mathbb{R}^d)} + \|\phi^\varepsilon(\cdot,t) - \phi^0(\cdot,t) - \phi_I^0(\cdot,t/\varepsilon)\|_{H^1(\mathbb{R}^d)}\right\}.$$

Example 1 (convergence rates study in 1D). We take $d = 1$ in (1.1)–(1.3), (1.4)–(1.6), and (1.24)–(1.25). The initial data ψ_0 in (1.6) is chosen as

$$\psi_0(x) = 3\mathrm{sech}(x)\, e^{-i\sqrt{3}x/4}, \qquad -\infty < x < \infty. \tag{5.1}$$

Let $\phi_0^{(l)}(x)$ and $\phi_1^{(l)}(x)$ be chosen as (1.8) and (1.13), respectively, with $\psi_0(x)$ as defined in (5.1). Then the initial data ψ_0^ε, ϕ_0^ε, and ϕ_1^ε in (1.3) and (1.25) are chosen as follows:

(i) Well-prepared initial data, i.e.,

$$\psi_0^\varepsilon(x) = \psi_0(x) + C\varepsilon^\alpha\, e^{-x^2/2}, \qquad -\infty < x < \infty, \tag{5.2}$$

$$\phi_0^\varepsilon(x) = \phi_0^{(l)}(x) + C\varepsilon^\alpha\, \mathrm{sech}(x), \quad \phi_1^\varepsilon(x) = \phi_1^{(l)}(x) + C\varepsilon^\alpha\, \mathrm{sech}(x), \tag{5.3}$$

where we take $\alpha = 2$ for $\gamma = 0$ and $\alpha = 1$ for $\gamma > 0$ in our computation.

Table 5.1
Convergence rate analysis for Example 1 with well-prepared initial data for $\gamma = 0$.

ε	1/4	1/8	1/16	1/32	1/64	1/128
$C = 0$	9.68E-1	2.30E-1	5.53E-2	1.39E-2	3.59E-3	8.89E-4
$C = 1$	2.63	6.18E-1	1.48E-1	3.69E-2	9.28E-3	2.44E-3

Table 5.2
Convergence rate analysis for Example 1 with well-prepared initial data for $\gamma = 1$.

ε	1/16	1/32	1/64	1/128	1/256	1/512
$C = 0$	1.11	5.99E-1	3.03E-1	1.53E-1	7.68E-2	3.85E-2
$C = 1$	2.67	1.35	6.76E-1	3.38E-1	1.69E-1	8.44E-2

Table 5.3
Convergence rate analysis for Example 1 with ill-prepared initial data for $\gamma = 0$.

ε	1/32	1/64	1/128	1/256	1/512	1/1024
$C = 0$	5.33E-1	2.66E-1	1.33E-1	6.67E-2	3.33E-2	1.66E-2
$C = 1$	6.26E-1	3.14E-1	1.57E-1	7.87E-2	3.93E-2	1.97E-2

Table 5.4
Convergence rate analysis for Example 1 with ill-prepared initial data for $\gamma = 1$.

ε	1/32	1/64	1/128	1/256	1/512	1/1024
$C = 0$	6.54E-1	3.28E-1	1.64E-1	8.19E-2	4.09E-2	2.05E-2
$C = 1$	7.29E-1	3.67E-1	1.84E-1	9.21E-2	4.61E-2	2.30E-2

(ii) Ill-prepared initial data, i.e.,

$$\psi_0^\varepsilon(x) = \psi_0(x) + C\varepsilon\, e^{-x^2/2}, \quad \phi_0^\varepsilon(x) = 0, \quad \phi_1^\varepsilon(x) = 0, \quad -\infty < x < \infty. \tag{5.4}$$

We solve the KGS and SY equations numerically on $[-120, 120]$ with mesh size $h = 1/8$ and time step $k = 0.00001$. Tables 5.1 and 5.2 show the errors $e_W(2) := e_W(T = 2)$ for well-prepared initial data with different ε and C for $\gamma = 0$ and $\gamma = 1$, respectively. Similarly, Tables 5.3 and 5.4 show the errors $e_I(2) := e_I(T = 2)$ for ill-prepared initial data with different ε and C for $\gamma = 0$ and $\gamma = 1$, respectively. We also plot out $\log(e_W)$ versus $\log(\varepsilon)$ and $\log(e_I)$ versus $\log(\varepsilon)$ in Figure 5.1 (left column).

From Tables 5.1–5.4 and Figure 5.1, we can draw the following conclusions for the convergence rates of the solution of the KGS equations (1.1)–(1.3) to the solution of the SY equations (1.4)–(1.6) in 1D: (i) For well-prepared initial data, they converge quadratically and linearly when $\gamma = 0$ (cf. Table 5.1) and $\gamma > 0$ (cf. Table 5.2), respectively, which agrees with the analytical results in (1.21) with $d = 1$; (ii) for ill-prepared initial data, they converge linearly for $\gamma \geq 0$ (cf. Tables 5.3 and 5.4). Note that in Theorem 1.2, only a half-order convergence rate is obtained for ill-prepared initial data in the absence of damping, i.e., $\gamma = 0$, in 1D. But, Table 5.3 shows that the convergence rate is still linear in this situation, just as the rate is when damping exists and initial data is ill-prepared. Thus, such numerical results indicate that the results in Theorem 1.2 can be improved. We remark here that this work is ongoing.

Example 2 (convergence rates study in 2D). We take $d = 2$ in (1.1)–(1.3), (1.4)–(1.6), and (1.24)–(1.25). The initial data ψ_0 in (1.6) is chosen as

$$\psi_0(x, y) = e^{-(2x^2+y^2)/2}, \qquad -\infty < x, y < \infty. \tag{5.5}$$

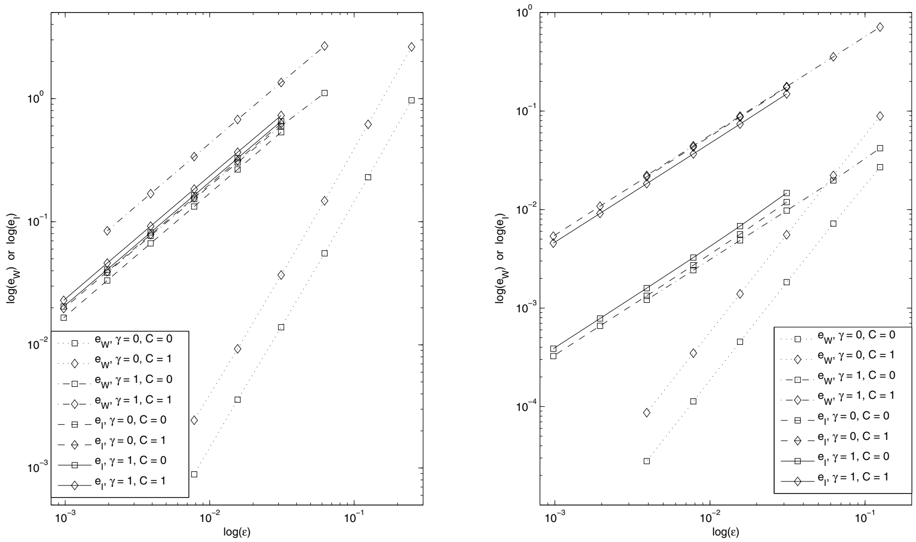

FIG. 5.1. *Convergence rate analysis in Example* 1 *(left) and Example* 2 *(right).*

Again, let $\phi_0^{(l)}(x,y)$ and $\phi_1^{(l)}(x,y)$ be chosen as (1.8) and (1.13), respectively, with $\psi_0(x,y)$ defined as in (5.5). Then the initial data ψ_0^ε, ϕ_0^ε, and ϕ_1^ε in (1.3) and (1.25) are chosen as follows:

(i) Well-prepared initial data, i.e.,

$$\psi_0^\varepsilon(x,y) = \psi_0(x,y) + C\varepsilon^\alpha \operatorname{sech}(x^2+y^2), \tag{5.6}$$

$$\phi_0^\varepsilon(x,y) = \phi_0^{(l)}(x,y) + C\varepsilon^\alpha \operatorname{sech}(x^2+y^2), \tag{5.7}$$

$$\phi_1^\varepsilon(x,y) = \phi_1^{(l)}(x,y) + C\varepsilon^\alpha \operatorname{sech}(x^2+y^2), \quad -\infty < x,\ y < \infty, \tag{5.8}$$

where we take $\alpha = 2$ for $\gamma = 0$ and $\alpha = 1$ for $\gamma > 0$ in our computation.

(ii) Ill-prepared initial data, i.e.,

$$\psi_0^\varepsilon(x,y) = \psi_0(x) + C\varepsilon \operatorname{sech}(x^2+y^2), \tag{5.9}$$

$$\phi_0^\varepsilon(x,y) = 0, \quad \phi_1^\varepsilon(x,y) = 0, \quad -\infty < x,\ y < \infty. \tag{5.10}$$

We solve the KGS and SY equations numerically on $[-40,40]^2$ with mesh size $h = 1/8$ and time step $k = 0.00002$. Tables 5.5 and 5.6 show the errors $e_W(1) := e_W(T=1)$ for well-prepared initial data with different ε and C for $\gamma = 0$ and $\gamma = 1$, respectively. Similarly, Tables 5.7 and 5.8 show the errors $e_I(1) := e_I(T=1)$ for ill-prepared initial data with different ε and C for $\gamma = 0$ and $\gamma = 1$, respectively. Meanwhile, $\log(e_W)$ versus $\log(\varepsilon)$ and $\log(e_I)$ versus $\log(\varepsilon)$ are plotted in Figure 5.1 (right column).

Table 5.5
Convergence rate analysis for Example 2 with well-prepared initial data for $\gamma = 0$.

ε	1/8	1/16	1/32	1/64	1/128	1/256
$C = 0$	2.69E-2	7.20E-3	1.83E-3	4.55E-4	1.13E-4	2.79E-5
$C = 1$	8.89E-2	2.22E-2	5.55E-3	1.39E-3	3.48E-4	8.67E-5

Table 5.6
Convergence rate analysis for Example 2 with well-prepared initial data for $\gamma = 1$.

ε	1/8	1/16	1/32	1/64	1/128	1/256
$C = 0$	4.19E-2	1.97E-2	9.75E-3	4.86E-3	2.42E-3	1.21E-3
$C = 1$	7.11E-1	3.55E-1	1.77E-1	8.87E-2	4.44E-2	2.22E-2

Table 5.7
Convergence rate analysis for Example 2 with ill-prepared initial data for $\gamma = 0$.

ε	1/32	1/64	1/128	1/256	1/512	1/1024
$C = 0$	1.19E-2	5.61E-3	2.72E-3	1.33E-3	6.59E-4	3.25E-4
$C = 1$	1.74E-1	8.66E-2	4.31E-2	2.15E-2	1.08E-2	5.37E-3

Table 5.8
Convergence rate analysis for Example 2 with ill-prepared initial data for $\gamma = 1$.

ε	1/32	1/64	1/128	1/256	1/512	1/1024
$C = 0$	1.47E-2	6.79E-3	3.25E-3	1.59E-3	7.84E-4	3.87E-4
$C = 1$	1.49E-1	7.37E-2	3.67E-2	1.83E-2	9.13E-3	4.56E-3

Again, from Tables 5.5–5.8 and Figure 5.1, we can draw similar conclusions for the convergence rates of the solution of the KGS equations (1.1)–(1.3) to the solution of the SY equations (1.4)–(1.6) in 2D as those in 1D. Also, such numerical results agree with the analytical results in (1.21) and (1.23) with $d = 2$.

6. Conclusions. We have studied the singular limits of the nonlinear Klein–Gordon–Schrödinger (KGS) equations both with and without a damping term to the Schrödinger–Yukawa equations both analytically and numerically. Along the analytical front, we first carried out the formal limits, found the initial layer correction, and classified the initial data as well-prepared and ill-prepared. Then for general initial data, we proved weak convergence results by using the weak compactness argument, and we proved strong convergence results from the weak solution of the KGS equations to the weak solution of the SY equations by modulated energy techniques. We also established strong convergence results for strong solutions and found the optimal convergence rates for well-prepared and ill-prepared initial data both with and without the damping term. On the numerical side, we solved the KGS equation and its limiting equations numerically by the efficient and accurate numerical methods proposed in [2, 3] for different types of initial data and different parameter regimes. Convergence rates were observed numerically for well-prepared and ill-prepared initial data, and they agreed with our analytical results.

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