ELSEVIER

Comput. Methods Appl. Mech. Engrg. 188 (2000) 455–471

Computer methods in applied mechanics and engineering

www.elsevier.com/locate/cma

High-order local artificial boundary conditions for problems in unbounded domains ☆

Weizhu Bao *, Houde Han

Department of Applied Mathematics, Tsinghua University, Beijing 100084, People's Republic of China

Received 24 July 1998

Abstract

In this paper we present error estimates for the finite element approximation of Poisson and modified Helmholtz equations outside an obstacle or in a semi-infinite strip in the plane. The finite element approximation is formulated on a bounded domain using a local approximate artificial boundary condition. In fact there is a sequence of local approximate boundary conditions for a given artificial boundary. Our error estimates are based on the mesh size and the location of the artificial boundary. The numerical stability and robustness of the method are discussed. Numerical experiments are presented to demonstrate the performance of the method and our error estimates.

1. Introduction

In this paper we consider two types of unbounded domain problems that are exterior problems and problems in semi-infinite strips. Exterior problems are associated with the infinite space outside an obstacle (Fig. 1). Problems in semi-infinite strips are encountered in application involving flow through wave guide (Fig. 2) or flow around an obstacle in a channel (Fig. 3) The corresponding unbounded domain is denoted by Ω.

We consider a linear elliptic second-order boundary value problem in two dimensions, with each of the three setups shown in Figs. 1–3. The boundary Γ_i is decomposed into two disjoint parts, Γ_{D} where a Dirichlet boundary condition is given, and Γ_N where a Neumann boundary condition is given. The statement of the problem is:

$$-\nabla\cdot\kappa(x)\nabla u(x)+\beta(x)u(x)=f(x),\quad \text{in } \Omega, \tag{1.1}$$

$$u=g,\quad \text{on } \Gamma_{\mathrm{D}}, \tag{1.2}$$

$$\frac{\partial u}{\partial n}=k,\quad \text{on } \Gamma_N, \tag{1.3}$$

$$\text{Some conditions at infinity}, \tag{1.4}$$

$$\text{If } \Omega \text{ is a strip: some boundary conditions on } \Gamma_{\mathrm{U}} \text{ and } \Gamma_{\mathrm{L}}; \tag{1.5}$$

where $u(x)$ is the unknown function, $f(x),\ \kappa(x)\geqslant 1,\ \beta(x)\geqslant\beta_0\geqslant 0,\ g$ and k are given functions satisfying: $\mathrm{supp}(f)$, $\mathrm{supp}(\kappa-1)$ and $\mathrm{supp}(\beta-\beta_0)$ are compact and $\kappa(x),\ \beta(x)\in L^\infty(\Omega)$.

☆ This work was supported partly by the Climbing Program of National Key Project of Foundation and the National Natural Science Foundation of China. Computation was supported by the State Key Lab. of Scientific and Engineering Computing in China.

* Corresponding author.

0045-7825/00/$ - see front matter
PII: S 0 0 4 5 - 7 8 2 5 (9 9) 0 0 1 8 8 - 7

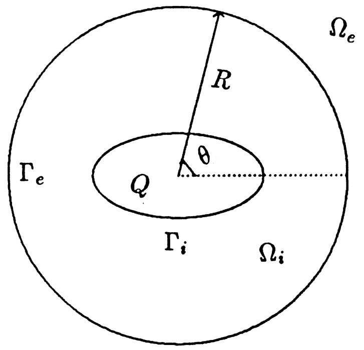

Fig. 1.

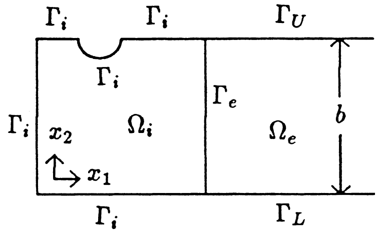

Fig. 2.

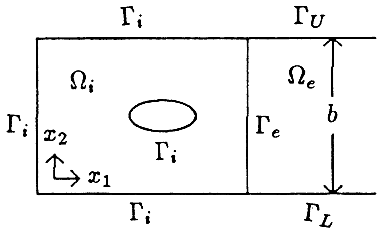

Fig. 3.

In order to bound the physical domain of the problem, we introduce an artificial boundary Γ_e which divides the unbounded domain Ω into two parts, Ω_i and Ω_e, such that $\text{supp}(f)\cup\text{supp}(\kappa-1)\cup\text{supp}(\beta-\beta_0)\cup\Gamma_i\subset\Omega_i$ (see Figs. 1–3). In an exterior problem it is a circle and in a problem in semi-infinite strip it is a segment.

If Ω_e is a strip (Figs. 2 and 3), then the boundary condition (1.5) on $\Gamma_{\rm U}$ and $\Gamma_{\rm L}$ is either the Dirichlet condition $u=0$ or the Neumann condition $\partial u/\partial x_2=0$. Let b be the width of the channel Ω_e. We introduce a Cartesian coordinate system (x_1,x_2), such that the ray $\Gamma_{\rm L}$ coincides with the x_1 axis and Γ_e is the segment $\{(d,x_2)\ :\ 0\leqslant x_2\leqslant b\}$. In the case of an exterior problem (Fig. 1), we use a polar coordinate system (r,θ), such that Γ_e is the circle $\{(R,\theta)\ :\ 0\leqslant\theta\leqslant 2\pi\}$. Furthermore let $R_0=\inf\{r\ :\ (r,\theta)\in\Gamma_i\cup\text{supp}(f)\cup\text{supp}(\kappa-1)\cup\text{supp}(\beta-\beta_0)\}$ and $\Gamma_0=\{(R_0,\theta)\ :\ 0\leqslant\theta\leqslant 2\pi\}$ in the case of an exterior problem and $d_0=\inf\{x_1\ :\ (x_1,x_2)\in\Gamma_i\cup\text{supp}(f)\cup\text{supp}(\kappa-1)\cup\text{supp}(\beta-\beta_0)\}$ and $\Gamma_0=\{(d_0,x_2)\ :\ 0\leqslant x_2\leqslant b\}$ in the case of a problem in a semi-infinite strip.

Let $|\Gamma_{\rm D}|$ denote the length of the curve $\Gamma_{\rm D}$. The condition at infinity (1.4) is as follows: In an exterior problem or problems in semi-infinite strips with Neumann boundary condition at $\Gamma_{\rm U}$ and $\Gamma_{\rm L}$ when $\beta_0>0$ or $|\Gamma_{\rm D}|=0$ and $\beta(x)\equiv 0$ or problems in semi-infinite strips with Dirichlet boundary condition at $\Gamma_{\rm U}$ and $\Gamma_{\rm L}$, the solution u is required to varnish at infinity; in an exterior problem or problems in semi-infinite strips with the Neumann boundary condition at $\Gamma_{\rm U}$ and $\Gamma_{\rm L}$ when $\beta_0=0$ and $|\Gamma_{\rm D}|\neq 0$ or $\beta(x)\not\equiv\beta_0=0$ and $|\Gamma_{\rm D}|=0$, the solution u is required to be bounded at infinity.

There are several methods solving boundary value problems in unbounded domains [6]. One of the most popular methods is to introduce an artificial boundary Γ_e and set up artificial boundary conditions on it. Then the original problem is reduced to a boundary value problem in a bounded computational domain Ω_i. Then a numerical approximation of the original problem in Ω_i can be obtained by solving the reduced problem. In the last two decades, many authors have worked on this subject for various problems by different techniques (see [3,4,8], [9–17,19] and references therein).

Recently Givoli et al. [7] also analyze the boundary value problem (1.1)–(1.5). But the condition at infinity (1.4) is not well proposed in some cases, such as in the case of $\beta_0 = 0$ and $|\Gamma_D| \neq 0$ in an exterior problem the solution u can not be required varnish at infinity in two dimensions. Otherwise there maybe no solution for the problem.

In the above works, several authors also gave error estimates for the numerical solution. But their error estimates only depend on the mesh size of a partition of Ω_i and the approximate artificial boundary condition. How does the error depend on the location of the artificial boundary Γ_e is unknown? But this is a very interesting problem for engineers because this kind of error estimates can be used in choosing the bounded computational domain Ω_i. In this paper, we will provide error estimates for the finite element approximation of the problem (1.1)–(1.5) using a local artificial boundary condition, which depends on not only the mesh size but also the location of the artificial boundary Γ_e.

2. Local artificial boundary conditions at Γ_e

In order to derive local artificial boundary conditions at Γ_e for the problem (1.1)–(1.5), we analyze the problem in Ω_e. The first case we considered is the exterior problem with $\beta_0 = 0$ and $\Gamma_D \neq 0$. By separation of variables we find that the general solution which satisfies (1.1) in the domain Ω_e and the condition at infinity mentioned in Section 1 is:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{R}{r}\right)^n (a_n \cos n\theta + b_n \sin n\theta), \tag{2.1}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\theta)\cos n\theta \,\mathrm{d}\theta, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\theta)\sin n\theta \,\mathrm{d}\theta, \quad n \geqslant 0. \tag{2.2}$$

We differentiate (2.1) with respect to r and set $r = R$, to obtain

$$\frac{\partial u(R,\theta)}{\partial r} = -\frac{1}{\pi R}\sum_{n=1}^{\infty} n \int_0^{2\pi} u(R,\phi)\cos n(\theta-\phi) \,\mathrm{d}\phi \stackrel{\Delta}{=} Lu(R,\theta), \tag{2.3}$$

where L is a bounded operator from the Sobolev space $H^{1/2}(\Gamma_e)$ to $H^{-1/2}(\Gamma_e)$ [5]. This is the desired exact boundary condition on Γ_e.

In a similar way, we get the exact boundary conditions for all the other cases considered. They are summarized in Table 1. There $K_n(r)$ is the modified Bessel function of the second kind. A prime after a summation sign indicates that a factor 1/2 multiplies the term with $n = 0$.

Now we will design local artificial boundary conditions for the problem (1.1)–(1.5). The first case we consider again is the exterior problem with $\beta_0 = 0$ and $\Gamma_D \neq 0$. We consider a solution u on Γ_e which only consists of the first N harmonics. Thus

$$u(R,\theta) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos n\theta + b_n \sin n\theta), \tag{2.4}$$

where the a_n and b_n are constants (Fourier coefficients, see (2.2)). Substituting (2.4) into (2.3) and using the orthogonality of the coses and sines, we get

Table 1
Non-local exact boundary conditions for different problems

Exterior problem $\beta_0 = 0$	$\frac{\partial u}{\partial r}(R,\theta) = -\frac{1}{\pi R}\sum_{n=0}^{\infty} Z_n \int_0^{2\pi} u(R,\theta')\cos n(\theta-\theta')\,\mathrm{d}\theta'$
Exterior problem $\beta_0 > 0$	$\frac{\partial u}{\partial r}(R,\theta) = -\frac{1}{\pi R}\sum_{n=0}^{\infty'} Z_n \int_0^{2\pi} u(R,\theta')\cos n(\theta-\theta')\,\mathrm{d}\theta'$
Semi-infinite strip $u = 0$ on Γ_{U} and Γ_{L}	$\frac{\partial u}{\partial x_1}(d,x_2) = -\frac{2\pi}{b^2}\sum_{n=1}^{\infty} Z_n \int_0^{b} u(d,x_2')\sin\frac{n\pi x_2}{b}\sin\frac{n\pi x_2'}{b}\,\mathrm{d}x_2'$
Semi-infinite strip $\frac{\partial U}{\partial x_2} = 0$ on Γ_{U} and Γ_{L}	$\frac{\partial u}{\partial x_1}(d,x_2) = -\frac{2\pi}{b^2}\sum_{n=0}^{\infty'} Z_n \int_0^{b} u(d,x_2')\cos\frac{n\pi x_2}{b}\cos\frac{n\pi x_2'}{b}\,\mathrm{d}x_2'$

Case	Exterior problem	Semi-infinite strip $u = 0$ on Γ_{U} and Γ_{L}	Semi-infinite strip $(\partial u/\partial x_2 = 0)$ on Γ_{U} and Γ_{L}
$\beta_0 = 0$ $\lvert\Gamma_{\mathrm{D}}\rvert \neq 0$	$Z_0 = 0 \quad Z_n = n$	$Z_n = n$	$Z_0 = 0 \quad Z_n = n$
$\beta_0 = 0$ $\lvert\Gamma_{\mathrm{D}}\rvert = 0$	$Z_0 = 1 \quad Z_n = n$	$Z_n = n$	$Z_0 = 1 \quad Z_n = n$
$\beta_0 > 0$	$Z_n = \frac{-R\beta_0^{1/2}K_n'(\beta_0^{1/2}R)}{K_n(\beta_0^{1/2}R)}$	$Z_n = \frac{b}{\pi}\sqrt{\beta_0 + \frac{n^2\pi^2}{b^2}}$	$Z_n = \frac{b}{\pi}\sqrt{\beta_0 + \frac{n^2\pi^2}{b^2}}$

$$\frac{\partial u}{\partial r}(R,\theta) = -\frac{1}{R}\sum_{n=1}^{N} Z_n(a_n\cos n\theta + b_n\sin n\theta), \quad Z_n = n,\ n = 1,2,\ldots,N. \tag{2.5}$$

It is desired to find a linear differential operator L_N, which does not depend on n, such that

$$L_N[1] = 0, \quad Z_n\cos n\theta = L_N[\cos n\theta], \quad Z_n\sin n\theta = L_N[\sin n\theta],\ 1 \leqslant n \leqslant N. \tag{2.6}$$

With such an operator at hand and noting (2.4), the equality (2.5) can be written

$$\begin{aligned} \frac{\partial u}{\partial r}(R,\theta) &= -\frac{1}{R}\sum_{n=1}^{N}[a_n L_N(\cos n\theta) + b_n L_N(\sin n\theta)] \\ &= -\frac{1}{R}L_N\left[\sum_{n=1}^{N}(a_n\cos n\theta + b_n\sin n\theta)\right] = -\frac{1}{R}L_N[u(R,\theta)]. \end{aligned} \tag{2.7}$$

The above formula is a local boundary condition on Γ_e which is exact for all solutions consisting of at most the first N harmonics. Noting the fact

$$\frac{\mathrm{d}^{2m}}{\mathrm{d}\theta^{2m}}\cos n\theta = (-1)^m n^{2m}\cos n\theta \qquad \frac{\mathrm{d}^{2m}}{\mathrm{d}\theta^{2m}}\sin n\theta = (-1)^m n^{2m}\sin n\theta \quad \text{for } m \geqslant 0, \quad n \geqslant 1, \tag{2.8}$$

we can assume the operator L_N has the following form:

$$L_N[u(R,\theta)] = -\frac{1}{R}\sum_{m=1}^{N}(-1)^m\alpha_m^{(N)}\frac{\partial^{2m}}{\partial\theta^{2m}}u(R,\theta). \tag{2.9}$$

Inserting (2.4) into (2.9), we obtain

$$\sum_{m=1}^{N} n^{2m}\alpha_m^{(N)} = Z_n \equiv n, \quad n = 1,2,\ldots,N. \tag{2.10}$$

The linear system (2.10) for the coefficients $\alpha_m^{(N)}$ can be written in a matrix form, say

$$\begin{pmatrix} 1^2 & 1^4 & \ldots & 1^{2N} \\ 2^2 & 2^4 & \ldots & 2^{2N} \\ \vdots & \vdots & \ddots & \vdots \\ N^2 & N^4 & \ldots & N^{2N} \end{pmatrix} \begin{pmatrix} \alpha_1^{(N)} \\ \alpha_2^{(N)} \\ \vdots \\ \alpha_N^{(N)} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ N \end{pmatrix}. \tag{2.11}$$

The linear system (2.11) has a unique solution for any $N \in \mathbb{N}$ because the determinant of its coefficient matrix is a Vandermonde's and not equal zero. Thus we obtain a series of local artificial boundary conditions at Γ_e for the exterior problem with $\beta_0 = 0$ and $|\Gamma_D| \neq 0$

$$\frac{\partial u}{\partial r}(R,\theta) = L_N[u(R,\theta)] \equiv -\frac{1}{R}\sum_{m=1}^{N}(-1)^m \alpha_m^{(N)} \frac{\partial^{2m}}{\partial \theta^{2m}} u(R,\theta). \tag{2.12}$$

In a similar way, we get local artificial boundary conditions for all the other cases considered. They are summarized in Table 2.

Find the $\alpha_m^{(N)}$ by solving the system

$$\sum_{m=1}^{N} n^{2m}\alpha_m^{(N)} = Z_n \equiv n, \quad n = 1,2,\ldots,N, \tag{2.13}$$

the $\beta_m^{(N)}$ by solving the system

$$\sum_{m=0}^{N-1} n^{2m}\beta_m^{(N)} = Z_n, \quad n = 1,2,\ldots,N \tag{2.14}$$

Table 2
Local artificial boundary conditions for different problems

Exterior problem $\beta_0 = 0$ and $\|\Gamma_D\| \neq 0$	$\frac{\partial u}{\partial r}(R,\theta) = -\frac{1}{R}\sum_{m=1}^{N}(-1)^m \alpha_m^{(N)} \frac{\partial^{2m} u(R,\theta)}{\partial \theta^{2m}}$
Exterior problem $\beta_0 = 0$ and $\|\Gamma_D\| = 0$	$\frac{\partial u}{\partial r}(R,\theta) = -\frac{1}{R}\sum_{m=0}^{N-1}(-1)^m \beta_m^{(N)} \frac{\partial^{2m} u(R,\theta)}{\partial \theta^{2m}}$
Exterior problem $\beta_0 > 0$	$\frac{\partial u}{\partial r}(R,\theta) = -\frac{1}{R}\sum_{m=0}^{N}(-1)^m \gamma_m^{(N)} \frac{\partial^{2m} u(R,\theta)}{\partial \theta^{2m}}$
Semi-infinite strip $u = 0$ on Γ_U and Γ_L	$\frac{\partial u}{\partial x_1}(d,x_2) = -\sum_{m=0}^{N-1}\left(\frac{b}{\pi}\right)^{2m-1} \beta_m^{(N)} \frac{\partial^{2m} u(d,x_2)}{\partial x_1^{2m}}$
Semi-infinite strip $(\beta_0 > 0)$ $\partial u/\partial x_2 = 0$ on Γ_U and Γ_L	$\frac{\partial u}{\partial x_1}(d,x_2) = -\sum_{m=0}^{N}\left(\frac{b}{\pi}\right)^{2m-1} \gamma_m^{(N)} \partial x_1^{2m}$
Semi-infinite strip $(\beta_0 = 0)$ and $\|\Gamma_D\| \neq 0)$ $\partial u/\partial x_2 = 0$ on Γ_U and Γ_L	$\frac{\partial u}{\partial x_1}(d,x_2) = -\sum_{m=1}^{N}\left(\frac{b}{\pi}\right)^{2m-1} \alpha_m^{(N)} \frac{\partial^{2m} u(d,x_2)}{\partial x_1^{2m}}$
Semi-infinite strip $(\beta_0 = 0$ and $\|\Gamma_D\| = 0)$ $\partial u/\partial x_2 = 0$ on Γ_U and Γ_L	$\frac{\partial u}{\partial x_1}(d,x_2) = -\sum_{m=0}^{N}\left(\frac{b}{\pi}\right)^{2m-1} \beta_m^{(N)} \frac{\partial^{2m} u(d,x_2)}{\partial x_1^{2m}}$

and the $\gamma_m^{(N)}$ by solving the system

$$\sum_{m=0}^{N-1} n^{2m}\gamma_m^{(N)} = Z_n, \quad n = 0, 1, \ldots, N, \tag{2.15}$$

where Z_n is shown in Table 1. In Tables 3–5, the coefficients $\alpha_m^{(N)}, \beta_m^{(N)}$ and $\gamma_m^{(N)}$ are given for the first four or five local artificial boundary conditions.

For future reference we write the local artificial boundary conditions given in Table 2 as

$$\frac{\partial u}{\partial n}(x) = -L_N u(x) \equiv -\sum_{m=n_0}^{\tilde{N}} (-1)^m \bar{\alpha}_m^{(N)} \frac{\partial^{2m} u}{\partial \tau^{2m}}, \quad x \in \Gamma_e, \tag{2.16}$$

where the coefficients n_0 and $\tilde{N}$ are indicated in Table 6, $\partial/\partial n$ is the normal derivative at Γ_e and $\partial/\partial\tau$ is the tangential derivative at Γ_e. The local artificial boundary condition (2.16) is exact for all solution consisting of at most the first N harmonics in each problem considered.

Table 3
The coefficients $\alpha_m^{(N)}$ in the first five local artificial boundary conditions

	$\alpha_1^{(N)}$	$\alpha_2^{(N)}$	$\alpha_3^{(N)}$	$\alpha_4^{(N)}$	$\alpha_5^{(N)}$
$N=1$	1				
$N=2$	7/6	−1/6			
$N=3$	74/60	−15/60	1/60		
$N=4$	533/420	−43/144	11/360	−1/1008	
$N=5$	3881/3780	−214/643	71/1728	−13/6048	1/25920

Table 4
The coefficients $\beta_m^{(N)}$ in the first four local artificial boundary conditions

	$\beta_0^{(N)}$	$\beta_1^{(N)}$	$\beta_2^{(N)}$	$\beta_3^{(N)}$
$N=1$	Z_1 (1)			
$N=2$	$\frac{4Z_1}{3}-\frac{Z_2}{3}$ (2/3)	$-\frac{Z_1}{3}+\frac{Z_2}{3}$ (1/3)		
$N=3$	$\frac{3Z_1}{2}-\frac{3Z_2}{5}+\frac{Z_3}{10}$ (3/5)	$-\frac{13Z_1}{24}+\frac{2Z_2}{3}-\frac{Z_3}{8}$ (5/12)	$\frac{Z_1}{24}-\frac{Z_2}{15}+\frac{Z_3}{40}$ (−1/60)	
$N=4$	$\frac{-8Z_1}{5}-\frac{4Z_2}{5}+\frac{8Z_3}{35}-\frac{Z_4}{35}$ (4/7)	$-\frac{61Z_1}{90}+\frac{169Z_2}{180}-\frac{3Z_3}{10}+\frac{7Z_4}{180}$ (41/90)	$\frac{29Z_1}{360}-\frac{13Z_2}{90}+\frac{3Z_3}{40}-\frac{Z_4}{90}$ (−1/36)	$-\frac{Z_1}{360}+\frac{Z_2}{180}-\frac{Z_3}{280}+\frac{Z_4}{1260}$ (1/1260)

Table 5
The coefficients $\gamma_m^{(N)}$ in the first four local artificial boundary conditions

	$\gamma_0^{(N)}$	$\gamma_1^{(N)}$	$\gamma_2^{(N)}$	$\gamma_3^{(N)}$
$N=0$	Z_0			
$N=1$	Z_0	$Z_1 - Z_0$		
$N=2$	Z_0	$-\frac{5Z_0}{4}+\frac{4Z_1}{3}-\frac{Z_2}{12}$	$\frac{Z_0}{4}-\frac{Z_1}{3}+\frac{Z_2}{12}$	
$N=3$	Z_0	$-\frac{49Z_0}{36}+\frac{3Z_1}{2}$	$\frac{7Z_0}{18}-\frac{13Z_1}{24}$	$-\frac{Z_0}{36}+\frac{Z_1}{24}$
		$-\frac{3Z_2}{20}+\frac{Z_3}{90}$	$+\frac{Z_2}{6}-\frac{Z_3}{72}$	$-\frac{Z_2}{60}+\frac{Z_3}{360}$

Table 6
The coefficients of n_0 and $\tilde{N}$ for different problems

Case	Exterior problem	Semi-infinite strip $u=0$ on Γ_U and Γ_L	Semi-infinite strip $(\partial u/\partial x_2)=0$ on Γ_U and Γ_L
$\beta_0=0$ $\lvert\Gamma_D\rvert\neq 0$	$n_0=1 \quad \tilde{N}=N$	$n_0=0 \quad \tilde{N}=N-1$	$n_0=1 \quad \tilde{N}=N$
$\beta_0=0$ $\lvert\Gamma_D\rvert=0$	$n_0=0 \quad \tilde{N}=N-1$	$n_0=0 \quad \tilde{N}=N-1$	$n_0=0 \quad \tilde{N}=N-1$
$\beta_0>0$	$n_0=0 \quad \tilde{N}=N$	$n_0=0 \quad \tilde{N}=N-1$	$n_0=0 \quad \tilde{N}=N$

3. The finite element approximation

In this section we consider the finite element approximation of the problem (1.1)–(1.3) in Ω_i with the local artificial boundary condition (2.16). The first case we considered is the exterior problem with $\beta_0 = 0$ and $|\Gamma_D| \neq 0$. The other cases can be dealt similarly. Let $H^m(\Omega_i)$ and $H^s(\Gamma_e)$ be usual Sobolev spaces on Ω_i and Γ_e with integer m and real number s [1]. Suppose

$$H_g^1(\Omega_i) = \{v \in H^1(\Omega_i) : v = g \text{ on } \Gamma_D\},$$
$$H_*^1(\Omega_i) = \{v \in H^1(\Omega_i) : v = 0 \text{ on } \Gamma_D\}.$$

Then the weak form of the problem (1.1)–(1.3) with the exact boundary condition (2.3) on Γ_e is

$$\begin{aligned} &(\mathrm{P}) \quad \text{Find} \quad u \in H_g^1(\Omega_i) \qquad \text{such that} \\ &a(u,v) + b(u,v) = f(v), \quad \forall v \in H_*^1(\Omega_i), \end{aligned} \tag{3.1}$$

where

$$a(u,v) = \int_{\Omega_i} \kappa(x)\nabla v(x)\cdot\nabla u(x)\,\mathrm{d}x + \int_{\Omega_i} \beta(x)u(x)v(x)\,\mathrm{d}x, \quad \forall u,v \in H^1(\Omega_i), \tag{3.2}$$

$$b(u,v) = \sum_{m=1}^{\infty}\frac{m}{\pi}\int_0^{2\pi}\int_0^{2\pi}\cos m(\theta-\theta')u(R,\theta)v(R,\theta')\,\mathrm{d}\theta\,\mathrm{d}\theta', \quad \forall u,v \in H^1(\Omega_i), \tag{3.3}$$

$$f(v) = \int_{\Omega_i} f(x)v(x)\,\mathrm{d}x + \int_{\Gamma_N}\kappa k v\,\mathrm{d}s, \quad \forall v \in H^1(\Omega_i). \tag{3.4}$$

We note that the symmetric bilinear form $a(\cdot,\cdot)$ is bounded on $H^1(\Omega_i)\times H^1(\Omega_i)$ and coercive on $H^1_*(\Omega_i)\times H^1_*(\Omega_i)$, i.e. there exist positive constants M_1, M_2 such that

$$|a(w,v)| \leqslant M_1\|w\|_{1,\Omega_i}\cdot\|v\|_{1,\Omega_i}, \quad \forall w,v\in H^1(\Omega_i), \tag{3.5}$$

$$M_2\|v\|^2_{1,\Omega_i} \leqslant a(v,v), \quad \forall v\in H^1_*(\Omega_i). \tag{3.6}$$

Furthermore it is easy to show, see [16], that the symmetric bilinear forms $b(\cdot,\cdot)$ is bounded on $H^1(\Omega_i)\times H^1(\Omega_i)$, i.e. there exists a positive constant M_3 such that

$$|b(w,v)| \leqslant M_3\|w\|_{1,\Omega_i}\|v\|_{1,\Omega_i}, \quad \forall w,v\in H^1(\Omega_i); \tag{3.7}$$

and in addition for all $v\in H^1(\Omega_i): 0\leqslant b(v,v)$.

It follows immediately from the above inequality, (3.5)–(3.7) that the variational problems (P) is well-posed. Note that the well-posedness of (P) implies immediately the well-posedness of the original problem (1.1)–(1.5).

In order to construct the weak form of the problem (1.1)–(1.3) in Ω_i with the local artificial boundary condition (2.12), we define two sets

$$V_g = \{v\in H^1(\Omega_i)\ :\ v|_{\Gamma_e}\in H^{\tilde N}(\Gamma_e) \qquad v=g \text{ on } \Gamma_{\rm D}\}\subset H^1_g(\Omega_i),$$
$$V_* = \{v\in H^1(\Omega_i)\ :\ v|_{\Gamma_e}\in H^{\tilde N}(\Gamma_e) \qquad v=0 \text{ on } \Gamma_{\rm D}\}\subset H^1_*(\Omega_i).$$

Let

$$b_N(u,v) = -\int_{\Gamma_e} vL_N u\ {\rm d}s = \sum_{m=1}^{N}\alpha_m^{(N)}\int_0^{2\pi}\frac{\partial^m u(R,\theta)}{\partial\theta^m}\frac{\partial^m v(R,\theta)}{\partial\theta^m}\ {\rm d}\theta. \tag{3.8}$$

Then the weak form of problem (1.1)–(1.3) with the local artificial boundary condition (2.12) on Γ_e is:

$$\begin{aligned}&(\mathrm{P}_N)\quad \text{Find}\quad u_N\in V_g \qquad \text{such that}\\ &a(u_N,v)+b_N(u_N,v)=f(v),\quad \forall v\in V_*.\end{aligned} \tag{3.9}$$

From (3.5) and (3.6), the well-posedness of the problem (P_N) depends on the property of the bilinear form $b_N(u,v)$. For any $u,v\in V_0$, we expand $u|_{\Gamma_e}$ and $v|_{\Gamma_e}$ in Fourier series, namely

$$u(R,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta), \tag{3.10}$$

$$v(R,\theta)=\frac{c_0}{2}+\sum_{n=1}^{\infty}(c_n\cos n\theta+d_n\sin n\theta), \tag{3.11}$$

where a_n, b_n are defined in (2.2) and

$$c_n=\frac{1}{\pi}\int_0^{2\pi}v(R,\theta)\cos n\theta\ {\rm d}\theta,\quad d_n=\frac{1}{\pi}\int_0^{2\pi}v(R,\theta)\sin n\theta\ {\rm d}\theta,\ n\geqslant 0.$$

Substituting (3.10) and (3.11) into (3.8) and using the orthogonality of the coses and sines, we have that

$$b_N(u,v)=\pi\sum_{n=1}^{\infty}\left(\sum_{m=1}^{N}n^{2m}\alpha_m^{(N)}\right)(a_nc_n+b_nd_n)\equiv\pi\sum_{n=1}^{\infty}\sigma_n^{(N)}(a_nc_n+b_nd_n). \tag{3.12}$$

Thus the property of $b_N(u,v)$ depends on the property of $\sigma_n^{(N)}$.

Table 3 shows $\alpha_m^{(N)}$ is positive for odd m, and is negative for even $m>0$ for $1\leqslant N\leqslant 5$. This property can be demonstrated numerically for $1\leqslant N\leqslant 20$. Thus we have that

$$\alpha_1^{(N)}>0\quad \alpha_N^{(N)}=\begin{cases}>0 & N \text{ is odd},\\ <0 & N \text{ is even},\quad 1\leqslant N\leqslant 20.\end{cases} \tag{3.13}$$

From the engineering application point of view, the parameter N in (2.12) is always less than 10. Therefore in this paper we assume $N \leqslant 20$ in (2.12). Then for the $\sigma_n^{(N)}$ we have the following lemma.

Lemma 3.1. *If* $1 \leqslant N \leqslant 20$ *is odd, then*

$$\sigma_n^{(N)} \geqslant n \quad \forall n \geqslant 1 \quad \lim_{n\to+\infty} \frac{\sigma_n^{(N)}}{n^{2N}} = \alpha_N^{(N)} > 0. \tag{3.14}$$

If $1 \leqslant N \leqslant 20$ *is even, then*

$$\sigma_n^{(N)} < 0 \quad \text{when } n \text{ is sufficient large} \quad \lim_{n\to+\infty} \frac{\sigma_n^{(N)}}{n^{2N}} = \alpha_N^{(N)} < 0. \tag{3.15}$$

Proof. We set a polynomial function whose degree is $2N$, say

$$\gamma_N(t) = \sum_{m=1}^{N} \alpha_m^{(N)} t^{2m} - t. \tag{3.16}$$

Since $\gamma_N''(t)$ is an even polynomial function whose degree is $2N-2$ and $\gamma_N''(0) = 2\alpha_1^{(N)} > 0$ for $1 \leqslant N \leqslant 20$ by noting (3.13), we know that $\gamma_N''(t) = 0$ has at most $N-1$ non-negative roots. Thus $\gamma_N(t) = 0$ has at most $N+1$ non-negative roots. From (3.16) and (2.10), we know that $t = 0, 1, 2, \ldots, N$ are roots of $\gamma_N(t) = 0$. Thus for $1 \leqslant N \leqslant 20$, we have that

$$\gamma_N(t) \neq 0 \quad \forall t > N \quad \lim_{t\to+\infty} \frac{\gamma_N(t)}{t^{2N}} = \alpha_N^{(N)}. \tag{3.17}$$

Then the desired inequalities (3.14) and (3.15) follows immediately from (3.17) and (3.13).

It follows immediately from (3.5), (3.6), (3.12), (3.14), (3.15) that the variational problems (P_N) is well-posed in the case of odd $1 \leqslant N \leqslant 20$ or $N = 0$ and it is not well-posed in the case of even $0 < N \leqslant 20$.

Now we replace V_* and V_g by two finite dimensional subsets, $V_*^h \subset V_*$ and $V_g^h \subset V_g$ in which h is the mesh size [2]. A family of such subsets were introduced by Givoli et al. [7]. Then the problem (P_N) is approximated:

$$\begin{aligned} &(\mathrm{P}_N^h) \text{ Find } u_N^h \in V_g^h \qquad \text{such that} \\ &a(u_N^h, v^h) + b_N(u_N^h, v^h) = f(v^h) \quad \forall v^h \in V_*^h. \end{aligned} \tag{3.18}$$

From the above discussion it is straightforward to check that the problem (P_N^h) is well-posed for odd $1 \leqslant N \leqslant 20$ or $N = 0$ and is not well-posed for even $0 < N \leqslant 20$.

In the finite element method, the domain Ω_i is discretized into a finite number of element domains, and the sets V_g^h and V_*^h are spanned by piecewise polynomial 'shape functions' which obey certain rules. In practice, all the finite element calculation is performed on the element level. Let Ω_i^e be the domain of element e, $\partial\Omega_i^e$ be the boundary of element e, and let $\Gamma_e^e = \Gamma_e \cap \partial\Omega_i^e$. In addition, we define the element-level counterparts of the forms (3.2), (3.4) and (3.8), namely

$$a^e(u, v) = \int_{\Omega_i^e} \kappa(x)\nabla v(x) \cdot \nabla u(x)\, \mathrm{d}x + \int_{\Omega_i^e} \beta(x)u(x)v(x)\, \mathrm{d}x \stackrel{\Delta}{=} (v^e)^T \mathbb{K}_a^e u^e, \tag{3.19}$$

$$b_N^e(u, v) = \sum_{m=1}^{N} \alpha_m^{(N)} \int_{\Gamma_e^e} \frac{\partial^m u(R,\theta)}{\partial\theta^m} \frac{\partial^m v(R,\theta)}{\partial\theta^m}\, \mathrm{d}\theta \stackrel{\Delta}{=} (v^e)^T \mathbb{K}_b^e u^e, \tag{3.20}$$

$$f^e(v) = \int_{\Omega_i^e} f(x)v(x)\, \mathrm{d}x + \int_{\Gamma_N \cap \partial\Omega_i^e} \kappa k v\, \mathrm{d}s \stackrel{\Delta}{=} (v^e)^T F^e, \tag{3.21}$$

where $\mathbb{K}_a^e$ and $\mathbb{K}_b^e$ are the element-level stiffness matrices associated with the differential operator and the local artificial boundary condition at Γ_e, respectively; F^e is element-level load vector; u^e and v^e are element-level vectors of unknown function u_N^h and test function v^h, respectively. It is obvious that maximum values of the elements in $\mathbb{K}_a^e$, in $\mathbb{K}_b^e$ and in F^e are $O(1)$, $O(1/h^{2N-1})$ and $O(1)$, respectively. Thus if we use a same order numerical integral formula to calculate the elements in the stiffness matrices, the error leaded by the numerical integration for some elements in $\mathbb{K}_b^e$ is approximately $1/h^{2N-1}$ times that of the elements in $\mathbb{K}_a^e$. Furthermore the higher order local artificial boundary condition used the worse property of the stiffness matrix of the problem (P_N^h). This was observed in our numerical experiments. The reason is the constant $C_N^{(1)}$ in the coercive inequality (4.6) depending on N. The larger N the smaller $C_N^{(1)}$. Thus the problem (P_N^h) is not rebust when N is large.

It is possible to repeat the analysis for other cases considered and show that it leads to the same conclusions.

4. Error Estimates

In this section, we present error estimates for the finite element approximate problem (P_N^h). The first case again we considered is the exterior problem with $\beta_0 = 0$ and $|\Gamma_\mathrm{D}| \neq 0$. Without lose of generality, we can assume $g = 0$. As we know from §3, the problem (P_N^h) is well-posed when $1 \leqslant N \leqslant 20$ is odd or $N = 0$. Thus in this section we always assume $1 \leqslant N \leqslant 20$ is odd or $N = 0$.

In [7], an error estimate was also proposed for the finite element approximation of problem (1.1)–(1.5). In their proof, they assumed (see (7.13) in [7])

$$|b_N(u,v)| \leqslant M\|u\|_{1,\Omega_i} \cdot \|v\|_{1,\Omega_i}, \quad \forall u,v \in V_*, \tag{4.1}$$

where M is a constant. This assumption is not always true when $N \geqslant 1$ (see (4.2) with $r = R$ and (4.6) in the following).

To cope with the local artificial boundary condition (2.12), we recall an equivalent definition of Sobolev space $H^s(\Gamma_r)$ for any real number s [18]:

$$v \in H^s(\Gamma_r) \Longleftrightarrow v(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) \quad \text{and}$$
$$\frac{\pi c_0^2}{2} + \sum_{n=1}^{\infty}\pi(1+n^2)^s(a_n^2+b_n^2) < \infty, \tag{4.2}$$

where $\Gamma_r = \{(r,\theta) \ : \ 0 \leqslant \theta \leqslant 2\pi\}(r > 0)$. Thus we use the following semi-norm on Γ_r:

$$|v|_{s,\Gamma_r} = \left[\sum_{n=1}^{\infty}\pi n^{2s}(a_n^2+b_n^2)\right]^{1/2}. \tag{4.3}$$

Therefore by the Poincaré inequality [1], (3.5), (3.6) and (4.3) with $r = R$, we assign the following norm on V_*:

$$\|v\|_* = \left[a(v,v) + |v|_{N,\Gamma_R}^2\right]^{1/2}, \quad \forall v \in V_*. \tag{4.4}$$

From the discussion in Section 3 and (4.3), we have the following lemma.

Lemma 4.1. *There exist positive constants $C_N^{(1)}$ and $C_N^{(2)}$ depending on N ($1 \leqslant N \leqslant 20$ is odd) such that*

$$|b_N(u,v)| \leqslant C_N^{(2)}|u|_{N,\Gamma_R} \cdot |v|_{N,\Gamma_R}, \quad \forall u,v \in V_*, \tag{4.5}$$
$$C_N^{(1)}|v|_{N,\Gamma_R}^2 \leqslant b_N(v,v), \quad \forall v \in V_*. \tag{4.6}$$

Proof. For any odd $1 \leqslant N \leqslant 20$, noting (3.14), we know that there exist positive constants $C_N^{(1)}$ and $C_N^{(2)}$ such that

$$C_N^{(1)} n^{2N} \leqslant \sigma_n^{(N)} = \sum_{m=1}^{N} n^{2m} \alpha_m^{(N)} \leqslant C_N^{(2)} n^{2N}, \quad n = 1, 2, 3, \ldots \tag{4.7}$$

Thus the desired inequalities (4.5) and (4.6) is a combination of (4.7), (3.12), (3.10), (3.11), (4.2) with $r = R$, (4.3) and the Schwaz inequality.

Lemma 4.2. *Suppose $u \in V_g$ be a solution of the exterior problem* (1.1)–(1.5). *Then we have that*

$$|b(u, v) - b_N(u, v)| \leqslant \left(1 + C_N^{(2)}\right)\left(\frac{R_0}{R}\right)^{N+1} |u|_{N,\Gamma_{R_0}} \cdot |u|_{N,\Gamma_R}, \quad \forall v \in V_*. \tag{4.8}$$

Proof. Assume that

$$u(R_0, \theta) = \frac{\bar{a}_0}{2} + \sum_{m=1}^{\infty} (\bar{a}_m \cos m\theta + \bar{b}_m \sin m\theta), \tag{4.9}$$

$$v(R, \theta) = \frac{c_0}{2} + \sum_{m=1}^{\infty} (c_m \cos m\theta + d_m \sin m\theta). \tag{4.10}$$

Noting that u satisfies the Laplace equation in the domain $\{x \ : \ |x| > R_0\}$, we get

$$u(r, \theta) = \frac{\bar{a}_0}{2} + \sum_{m=1}^{\infty} \left(\frac{R_0}{r}\right)^m (\bar{a}_m \cos m\theta + \bar{b}_n \sin m\theta), \quad R_0 < r, \ 0 \leqslant \theta \leqslant 2\pi. \tag{4.11}$$

Setting $r = R$ in (4.11), we have

$$u(R, \theta) = \frac{\bar{a}_0}{2} + \sum_{m=1}^{\infty} \left(\frac{R_0}{R}\right)^m (\bar{a}_m \cos m\theta + \bar{b}_m \sin m\theta), \quad 0 \leqslant \theta \leqslant 2\pi. \tag{4.12}$$

Inserting (4.12) and (4.10) into the forms of $b(u, v)$ (see (3.3)) and $b_N(u, v)$ (see (3.8)), noting (4.3), (3.12), (4.5), (4.4), (2.10), we obtain:

$$\begin{aligned}
&|b(u, v) - b_N(u, v)| \\
&\quad = \left| \sum_{n=N+1}^{\infty} \pi\left(n - \sigma_n^{(N)}\right) \left(\frac{R_0}{R}\right)^n (\bar{a}_n c_n + \bar{b}_n d_n) \right| \\
&\quad \leqslant \sum_{n=N+1}^{\infty} \pi |n - \sigma_n^{(N)}| \left(\frac{R_0}{R}\right)^n (|\bar{a}_n c_n| + |\bar{b}_n d_n|) \\
&\quad \leqslant \left(1 + C_N^{(2)}\right)\left(\frac{R_0}{R}\right)^{N+1} \sum_{n=N+1}^{\infty} \pi n^{2N} (|\bar{a}_n c_n| + |\bar{b}_n d_n|) \\
&\quad \leqslant \left(1 + C_N^{(2)}\right)\left(\frac{R_0}{R}\right)^{N+1} \left[\sum_{n=N+1}^{\infty} \pi n^{2N} (\bar{a}_n^2 + \bar{b}_n^2)\right]^{1/2} \left[\sum_{n=N+1}^{\infty} \pi n^{2N} (c_n^2 + d_n^2)\right]^{1/2} \\
&\quad \leqslant \left(1 + C_N^{(2)}\right)\left(\frac{R_0}{R}\right)^{N+1} |u|_{N,\Gamma_{R_0}} \cdot |v|_{N,\Gamma_R} \leqslant \left(1 + C_N^{(2)}\right)\left(\frac{R_0}{R}\right)^{N+1} |u|_{N,\Gamma_{R_0}} \cdot \|v\|_*.
\end{aligned} \tag{4.13}$$

Combining the above two Lemmas, we have the following error estimate:

Theorem 4.1. *Let u be the solution of the problem* (P) *and u_N^h be the solution of the problem* (P_N^h). *Suppose $f \in L^2(\Omega_i)$ and $u|_{\Gamma_{R_0}} \in H^N(\Gamma_{R_0})$. Then we have that*

$$\|u-u_N^h\|_* \leqslant C_N\left[\inf_{v^h\in V_*^h}\|u-v^h\|_* + \left(\frac{R_0}{R}\right)^{N+1}|u|_{N,\Gamma_{R_0}}\right], \tag{4.14}$$

where C_N is a generic constant independent on h and R.

Proof. Let $e := u-u_N^h, e^v := v^h - u$ and $e^h := v^h - u_N^h$. From (P) and (P_N^h) we have that

$$a(e,v^h) + b(u,v^h) - b_N(u_N^h, v^h) = 0 \quad \forall v^h \in V_*^h. \tag{4.15}$$

From (3.2), (4.15) with $v^h = e^h$, (4.5), (4.6), (4.8), (4.13), we have that

$$\begin{aligned}\min\{1, C_N^{(1)}\}\|e^h\|_*^2 &\leqslant a(e^h,e^h) + b_N(e^h,e^h)\\ &= a(e^v,e^h) + b_N(e^v,e^h) + a(e,e^h) + b_N(e,e^h)\\ &= a(e^v,e^h) + b_N(e^v,e^h) + b_N(u,e^h) - b(u,e^h)\\ &\leqslant \left(1+C_N^{(2)}\right)\left[\|e^v\|_*\cdot\|e^h\|_* + \frac{R_0^{N+1}}{R^{N+1}}|u|_{N,\Gamma_{R_0}}|\cdot\|e^h\|_*\right].\end{aligned} \tag{4.16}$$

Thus

$$\|e^h\|_* \leqslant C_N\left[\|e^v\|_* + \left(\frac{R_0}{R}\right)^{N+1}|u|_{N,\Gamma_{R_0}}\right], \quad \forall v^h \in V_*^h. \tag{4.17}$$

Then the desired result (4.14) follows from (4.17) and the triangle inequality.

If we suppose $u \in H^{p+1}(\Omega_i), u|_{\Gamma_R} \in H^{p+N}(\Gamma_R), u|_{\Gamma_{R_0}} \in H^N(\Gamma_{R_0})$ and the interpolation error of V_*^h approximate to V_* [2,7] is

$$\inf_{v^h\in V_*^h}\|u-v^h\|_* \leqslant C_N h^p[|u|_{p+1,\Omega_i} + |u|_{p+N,\Gamma_R}]. \tag{4.18}$$

Then combining (4.17), (4.14) and the poincaré inequality [1], we get

$$\begin{aligned}\|u-u_N^h\|_{1,\Omega_0} &\leqslant C_0|u-u_N^h|_{1,\Omega_0} \leqslant C_0|u-u_N^h|_{1,\Omega_i} \leqslant C_0\|u-u_N^h\|_*\\ &\leqslant C_N\left[h^p(|u|_{p+1,\Omega_i} + |u|_{p+N,\Gamma_R}) + \left(\frac{R_0}{R}\right)^{N+1}|u|_{N,\Gamma_{R_0}}\right],\end{aligned} \tag{4.19}$$

where $\Omega_0 = \{x \in \Omega_i \ : \ |x| < R_0\}$ and C_0 is a constant independent on h, R and N.

Table 7
Error estimates for different problems using local artificial boundary conditions

EP $\beta_0 = 0$	$\lvert u-u_N^h\rvert_{1,\Omega_i} \leqslant C_{\tilde N}\left[h^P\left(\lvert u\rvert_{p+1,\Omega_i} + \lvert u\rvert_{p+\tilde N,\Gamma_R}\right) + \left(\frac{R_0}{R}\right)^{N+1}\lvert u\rvert_{\tilde N,\Gamma_{R_0}}\right]$
EP $\beta_0 > 0$	$\lVert u-u_N^h\rVert_{1,\Omega_i} \leqslant C_N\left[h^P\left(\lvert u\rvert_{p+1,\Omega_i} + \lvert u\rvert_{p+N,\Gamma_R}\right) + \frac{\left(1+R\beta_0^{1/2}\right)K_{N+1}\left(R\beta_0^{1/2}\right)}{K_{N+1}\left(R_0\sqrt{\beta_0}\right)}\lvert u\rvert_{N,\Gamma_{R_0}}\right]$
PSS $\beta_0 = 0$	$\lvert u-u_N^h\rvert_{0,\Omega_i} \leqslant C_{\tilde N}\left[h^P\left(\lvert u\rvert_{p+1,\Omega_i} + \lvert u\rvert_{p+\tilde N,\Gamma_R}\right) + e^{-(d-d_0)(N+1)}\lvert u\rvert_{\tilde N,\Gamma_{R_0}}\right]$
PSS $\beta_0 > 0$	$\lVert u-u_N^h\rVert_{1,\Omega_i} \leqslant C_{\tilde N}\left[h^P\left(\lvert u\rvert_{p+1,\Omega_i} + \lvert u\rvert_{p+\tilde N,\Gamma_R}\right) + e^{-(d-d_0)(N+1)}\lvert u\rvert_{\tilde N,\Gamma_{R_0}}\right]$

In a similar way, we get error estimates for all other cases considered. They are summarized in Table 7. There EP and PSS denote 'Exterior Problems' and 'Problems in Semi-infinite Strip' respectively.

5. Numerical experiments

In this section we present the numerical experiments which demonstrate the performance of the error estimate (4.19). In our computation continuous piecewise bilinear elements were used throughout the domain Ω_i, except in the single layer elements adjacent to the artificial boundary Γ_e. There, special finite elements, $C^{2,1}_{\Gamma_e}$ (which was introduced by Givoli et al. [7] and has $C^2(\Gamma_e)$ regularity at Γ_e), were used. That is to say, $p = 1$ in the interpolation error (4.18) [2,7].

Example 5.1 (*An exterior problem of Poisson equation*). We consider Poisson equation in the planar domain outside a circular obstacle of a radius $a = 0.5$ (see Fig. 1). The problem is governed by the following boundary value problem:

$$-\Delta u = f \quad \text{in} \quad \Omega = \{(r,\theta) \mid 0.5 < r \quad 0 \leqslant \theta \leqslant 2\pi\}, \tag{5.1}$$

$$u(0.5,\theta) = 0.4925 + 0.5 \ln \frac{20 + 4\sin\theta}{5 - \sin\theta} \quad 0 \leqslant \theta \leqslant 2\pi, \tag{5.2}$$

$$u \ \text{ is bounded} \quad r \to +\infty, \tag{5.3}$$

where

$$f(r,\theta) = \begin{cases} 8 - 16r^2 & 0.5 < r < 1.0,\ 0 \leqslant \theta \leqslant 2\pi, \\ 0 & 1.0 \leqslant r,\ 0 \leqslant \theta \leqslant 2\pi. \end{cases}$$

This problem has an exact solution:

$$(r,\theta) = \begin{cases} 0.5 \ln \dfrac{4r^2 + 2r\sin\theta + 0.25}{r^2 - 0.5r\sin\theta + 0.0625} + (r^2 - 1)^2 & 0.5 \leqslant r \leqslant 1.0, \quad 0 \leqslant \theta \leqslant 2\pi, \\ 0.5 \ln \dfrac{4r^2 + 2r\sin\theta + 0.25}{r^2 - 0.5r\sin\theta + 0.0625} & 1.0 \leqslant r,\ 0 \leqslant \theta \leqslant 2\pi. \end{cases} \tag{5.4}$$

First we introduce a circular artificial boundary Γ_e of radius $R = 1.0$. On Γ_e we apply the local artificial boundary condition (2.12) with $N = 0, 1, 2, 3$. In the annular computational domain Ω_i, we use four meshes respectively. The first mesh consists of 2 radial layers of elements, with 20 quadrilateral elements in each layer. We denote it as 2×20 ($h = 0.31416$). The other three meshes are 4×40 ($h = 0.15708$), 8×80 ($h = 0.07854$) and 16×160 ($h = 0.03927$). Table 8 shows the error, $| u - u_N^h |_{1,\Omega_i}$, for $N = 0, 1, 2, 3$.

For comparison, let $\bar{u}_N^h$ denote the finite element approximation on the domain Ω_i using the local artificial boundary conditions designed by Givoli et al. (see Table 3 in [7]). Table 9 shows the error, $| u - \bar{u}_N^h |_{1,\Omega_i}$, for $N = 1, 2, 3, 4$. Second we test the effect of the location of the artificial boundary Γ_e. Let $\Omega_R = \{(r,\theta) \mid 0.5 = R_0 < r < R \quad 0 \leqslant \theta \leqslant 2\pi\}$ denote the bounded computational domain with the artificial boundary Γ_R. We choose $R = 1.0, 1.5, 2.0, 2.5, 3.0$ respectively. The corresponding meshes we used were

Table 8
Errors for the exterior problem using local conditions (2.12)

Mesh	$h = 0.31416$	$h = 0.15708$	$h = 0.07854$	$h = 0.03927$
$N = 0$	0.8583	0.7397	0.7011	0.6907
$N = 1$	0.5173	0.2756	0.1406	0.0719
$N = 2$	0.5461	0.3329	0.2362	0.2036
$N = 3$	0.5492	0.3382	0.2432	0.2114

Table 9
Errors for the exterior problem using Givoli's conditions

Mesh	$h = 0.31416$	$h = 0.15708$	$h = 0.07854$	$h = 0.03927$
$N = 1$	1.0078	0.9002	0.8665	0.8575
$N = 2$	0.8439	0.7164	0.6747	0.6634
$N = 3$	1.5217	1.6489	1.8645	2.0622
$N = 4$	1.4908	1.6422	1.8643	2.0624

8×40, 16×40, 24×40, 32×40 and 40×40. That is to say, each computational domain has a mesh with the fixed mesh size $h = 0.07854$. Let u_N^R denotes the finite element approximation of the problem on the domain Ω_R with the corresponding mesh by using the local artificial boundary condition (2.12) on the artificial boundary Γ_R and u_∞^R corresponds to the finite element solution on Ω_R using a non-local artificial boundary condition (say, (2.3) with the first 101 terms in the right-hand side) at Γ_R. Fig. 4 shows the errors $E_R := | u_\infty^R - u_N^R |_{1,\Omega_R}$ for different R. We make several observations from the results:

(a) The Neumann condition ($N = 0$) is inferior to all the high-order local artificial boundary conditions.

(b) The first high-order local condition ($N = 1$) gives very good results as in this case the constants $C_N^{(1)} = 1$ (in (4.6)), $C_N^{(2)} = 1$ (in (4.5)) and $C_N = 0.5$ (in (4.19)).

(c) $N = 2$ and $N = 3$ give worse results than $N = 1$. For $N = 2$, the reason is that (P_2^h) is unstable. For $n = 3$, the reason is that the second term in the right-hand side of (4.19) dominates the error in the present case $R = R_0$.

(d) Givoli's conditions give bad results in this example. The reason is that this example doesn't satisfy the condition: $u \to 0$ when $r \to +\infty$, which was desired in [7].

Example 5.2 (*A problem of Poisson equation in semi-infinite strip*). The problem is governed by the following boundary value problem:

$$-\Delta u = f, \quad \text{in } \Omega = \{(x_1, x_2) \mid 0.0 < x_1 \quad 0 < x_2 < b\}, \tag{5.5}$$

$$u(0.0, x_2) = b + 0.25 + \sum_{m=1}^{50} \frac{1}{m^2} \cos \frac{m\pi x_2}{b}, \quad 0 \leqslant x_2 \leqslant b, \tag{5.6}$$

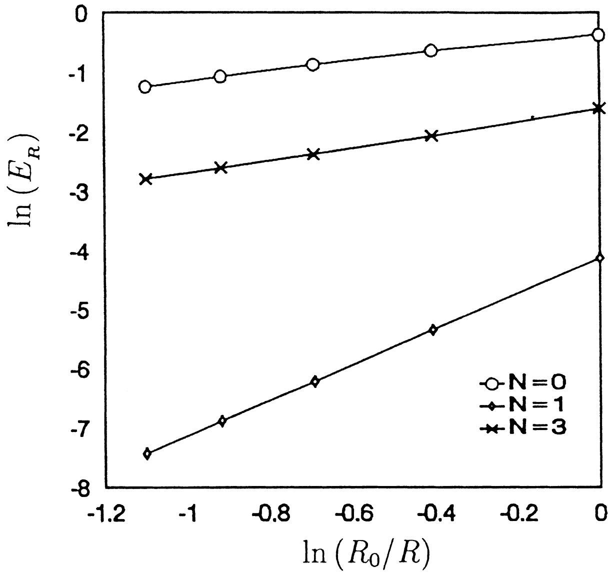

Fig. 4.

Table 10
Errors for the problem in a semi-infinite strip using local conditions

Mesh	$h = 0.25$	$h = 0.125$	$h = 0.0625$	$h = 0.03125$
$N = 0$	0.5628	0.5289	0.5182	0.5149
$N = 1$	0.2332	0.1326	0.0827	0.0590
$N = 2$	1.8045	1.0824	0.9480	0.9114
$N = 3$	0.2775	0.2066	0.1800	0.1707

Table 11
Errors for the problem in a semi-infinite strip using Givoli's conditions

Mesh	$h = 0.31416$	$h = 0.15708$	$h = 0.07854$	$h = 0.03927$
$N = 1$	2.1719	2.1614	2.1586	2.1578
$N = 2$	1.6678	1.6554	1.6519	1.6509
$N = 3$	3.2161	3.8261	4.4371	4.9270
$N = 4$	3.1879	3.8104	4.4301	4.9245

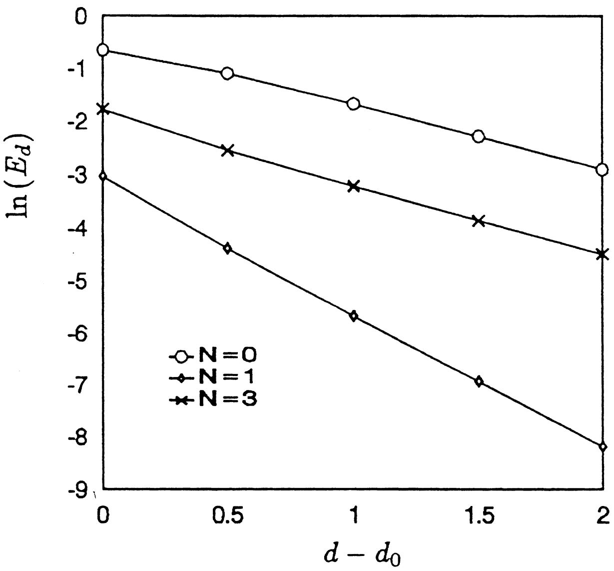

Fig. 5.

$$u \text{ is bounded}, \quad x_1 \to +\infty; \tag{5.7}$$

where

$$f(x) = \begin{cases} -2 & 0.0 < x_1 < 0.5, \quad 0 < x_2 < b, \\ 0 & 0.5 \leqslant x_1 < +\infty, \quad 0 < x_2 < b. \end{cases}$$

This problem has the exact solution:

$$u(x) = \begin{cases} b + \sum_{m=1}^{50} \frac{1}{m^2} e^{-m\pi x_1/b} \cos\frac{m\pi x_2}{b} + (x_1 - 0.5)^2 & 0 \leqslant x_1, \quad 0 \leqslant x_2 \leqslant b, \\ b + \sum_{m=1}^{50} \frac{1}{m^2} e^{-m\pi x_1/b} \cos\frac{m\pi x_2}{b} & 0 \leqslant x_1, \quad 0 \leqslant x_2 \leqslant b. \end{cases} \tag{5.8}$$

We take $b = 2.5$ and first introduce a segment $\Gamma_d = \{(d, x_2) \mid 0 \leqslant x_2 \leqslant b\}$ with $d = 0.5$ as an artificial boundary. On Γ_d we apply the local artificial boundary conditions shown in Table 2 with $N = 0, 1, 2, 3$ Four uniform meshes $2 \times 10, 4 \times 20, 8 \times 40$ and 16×80 were used corresponding to the mesh size $h = 0.25$, 0.125, 0.0625, 0.03125 of the bounded computational domain $\Omega_d = \{x \in \mathbb{R}^2 \mid 0 < x_1 < d \;\; 0 < x_2 < b\}$ respectively. Table 10 shows the error, $\mid u - u_N^h \mid_{1,\Omega_i}$, for $N = 0, 1, 2, 3$.

For comparison, let $\bar{u}_N^h$ denote the finite element approximation on the domain Ω_d using the local artificial boundary conditions designed by Givoli et al. (see Table 3 in [7]). Table 11 shows the error, $|u - \bar{u}_N^h|_{1,\Omega_d}$, for $N = 1, 2, 3, 4$. To test the effect of the location of the artificial boundary Γ_d, we choose $d = 0.5, 1.0, 1.5, 2.0, 2.5$ respectively. The uniform meshes we used for these bounded computational domains with the fixed mesh size $h = 0.0625$. That is to say, the corresponding meshes are $8 \times 40, 16 \times 40, 24 \times 40$, 32×40 and 40×40 for the domain Ω_d respectively. Let u_N^d and u_∞^d denote the similar meaning corresponding to example 1. Fig. 5 shows the errors $E_d := |u_\infty^d - u_N^d|_{1,\Omega_d}$ for different d. We can make the same observations as those in Example 1 from the results. Furthermore the instability of the condition corresponding to $N = 2$ is more obvious in this example. It is manifested by the fact that this condition generates a larger error than the lower-order conditions $N = 0$ and $N = 1$ and higher-order condition $N = 3$.

6. Conclusions

We have derived a series of local artificial boundary conditions for solving problems in unbounded domains in the plane. The finite element formulation is presented in a bounded computational domain using the local condition at a given artificial boundary. Error estimates for the finite element approximation are developed, which depends on the mesh size h and the location of the artificial boundary Γ_e (say, R or d). Numerical experiments for Poisson's equation outside an obstacle and in a semi-infinite strip demonstrate the error estimate. From our numerical results, we can make several conclusions:

1. In general, even-order ($N \geqslant 2$) local artificial boundary conditions can not be used in practice as it leads to instability.
2. The first high-order ($N = 1$) local artificial boundary condition is very useful in practice. This condition is very simple and easy to be used in programming. To cope with this condition, one can use standard 'conforming' finite elements and need not introduce special finite elements adjacent to the artificial boundary Γ_e. Furthermore one can get better accuracy by changing the location of the artificial boundary Γ_e (say, enlarge R or d). Thus we recommend engineers to use this condition.
3. Higher odd-order ($N \geqslant 3$) local artificial boundary conditions can be used in practice. One must introduce special elements adjacent to the artificial boundary Γ_e which posses $C^{\tilde{N}-1}(\Gamma_e)$ regularity, such as the $C_B^{k,p}$ elements introduced by Givoli et al. [7]. Another drawback is that one must be very careful in calculating the element-level stiffness matrix $\mathbb{K}_b^e$ as it is very sensitive to numerical integral formula.

References

[1] R.A. Adams, Sobolev Spaces, 1975.
[2] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[3] B. Engquist, A. Majda, Absorbing boundary conditions for the numerical simulation of waves, Math. Comp. 31 (1977) 629–651.
[4] K. Feng, Asymptotic radiation conditions for reduced wave equations, J. Comput. Math. 2 (1984) 130–138.
[5] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations, Springer-Verlag, Berlin, 1977.
[6] D. Givoli, Numerical Methods for Problems in Infinite Domains, Elsevier, Amsterdam, 1992.
[7] D. Givoli, I. Patlashenko, J. Keller, High-order boundary conditions and finite elements for infinite domains, Comput. Methods Appl. Mech. Eng. 143 (1997) 13–39.
[8] C.I. Goldstein, A finite element method for solving Helmholtz type equations in waveguides and other unbounded domains, Math. Comp. 39 (1982) 309–324.
[9] T.M. Hagstrom, H.B. Keller, Exact boundary conditions at artificial boundary for partial differential equations in cylinders, SIAM J. Math. Anal. 17 (1986) 322–341.
[10] T.M. Hagstrom, H.B. Keller, Asymptotic boundary conditions and numerical methods for nonlinear elliptic problems on unbounded domains, Math. Comp. 48 (1987) 449–470.

[11] L. Halpern, M. Schatzman, Artificial boundary conditions for incompressible viscous flows, SIAM J. Math. Anal. 20 (1989) 308–353.
[12] H. Han, W. Bao, An artificial boundary condition for two-dimensional incompressible viscous flows using the method of lines, Int. J. Numer. Methods Fluids 22 (1996) 483–493.
[13] H. Han, W. Bao, An artificial boundary condition for the incompressible viscous flows in a no-slip channel, J. Comput. Math. 13 (1995) 51–63.
[14] H. Han, J. Lu, W. Bao, A discrete artificial boundary condition for steady incompressible viscous flows in a no-slip channel using a fast iterative method, J. Comput. Phys. 114 (1994) 201–208.
[15] H. Han, W. Bao, T. Wang, Numerical simulation for the problem of infinite elastic foundation, Comput. Methods Appl. Mech. Eng. 147 (1997) 369–385.
[16] H. Han, X. Wu, Approximation of infinite boundary condition and its application to finite element method, J. Comput. Math. 3 (1985) 179–192.
[17] H. Han, X. Wu, The approximation of exact boundary condition at an artificial boundary for linear elastic equation and its application, Math. Comp. 59 (1992) 21–27.
[18] J.L. Lions, Optimal control of systems governed by partial differential equations, Springer-Verlag, Berlin and Heidelberg, 1971.
[19] D. Yu, Approximation of boundary conditions at infinity for a harmonic equation, J. Comput. Math. 3 (1985) 220–227.