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# The discrete artificial boundary condition on a polygonal artificial boundary for the exterior problem of Poisson equation by using the direct method of lines ☆

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**Abstract**

The numerical simulation for the exterior problem of Poisson equation is considered. We introduced a polygonal artificial boundary $\Gamma_e$ and designed a discrete artificial boundary condition on it by using the direct method of lines. Then the original problem is reduced to a boundary value problem defined in a bounded computational domain with a polygonal boundary. The finite element approximation of this reduced boundary value problem is considered and it is proved that the finite element approximate problem is well posed. Furthermore numerical examples show that the discrete artificial boundary condition is very effective and more accurate than the Neumann boundary condition which is often used in engineering literatures. 

*Keywords:* Poisson equation; Polygonal artificial boundary; Discrete artificial boundary condition; Finite element approximation

## 1. Introduction

When computing the numerical solution of a boundary value problem of partial differential equations in an unbounded domain, one often introduces an artificial boundary to cut off the unbounded part of the domain and sets up an artificial boundary condition at the artificial boundary. Then the original problem is reduced to a boundary value problem defined in a bounded computational domain. In order to limit the computational cost, the bounded computational domain must not be too large. Then the artificial boundary condition must be a good approximation of the exact boundary condition on the artificial boundary. Thus the accuracy of the artificial boundary condition and the computational cost are closely related. Therefore designing an artificial boundary condition with high accuracy on a given artificial boundary has become an effective and important method for solving partial differential equations in an unbounded domain which arises in various fields of engineering, such as fluid flow around obstacles, coupling of structures with foundation, wave propagation and so on.

There are many authors who have worked on this subject for various problems by different techniques. For instance, Engquist and Majda [1] designed absorbing boundary conditions for the wave equation. Goldstein [2] presented the exact boundary condition and a sequence of its approximations at an artificial boundary for Helmholtz-type equation in waveguides. Feng [3] proposed the asymptotic radiation

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conditions for the reduced wave equation by using the asymptotic approximation of Hankel functions. Han and Wu [4,5] obtained the exact boundary conditions and a series of their approximations at an artificial boundary for the Laplace equation and the linear elastic system. The exact boundary condition at an artificial boundary for partial differential equations in an infinite cylinder was proposed by Hagstrom and Keller [6,7]. Shortly after, they used this technique to solve nonlinear problems. A family of artificial boundary conditions for unsteady Oseen equations in the velocity pressure formulation with small viscosity was developed by Halpern and Schatzman [8], which was then applied to unsteady Navier–Stokes equations. Nataf [9] designed an open boundary condition for a steady Oseen equation in streamfunction vorticity formulation, which is applied to viscous incompressible fluid flow around a body in a flat channel with slip boundary conditions on the wall. Hagstrom [10,11] proposed asymptotic boundary conditions at an artificial boundary for the simulation of time-dependent fluid flows. Han et al. [12] designed discrete artificial boundary conditions for incompressible viscous flows in an infinite channel by using a fast iterative method. Han and Bao [13,14] proposed discrete artificial boundary conditions for incompressible viscous flows in a channel by using the method of lines. Han et al. [15] developed artificial boundary conditions for the problem of infinite elastic foundation. Recently Ben-Porat and Givoli [16] considered an elliptic artificial boundary for the Laplace equation. One can find more references in Ref. [17].

Because of the restriction of the techniques they used, many authors mainly consider the regular artificial boundaries, such as circumferences, straight lines or a segment, as artificial boundaries in solving two-dimensional problems. As we know, it is very easy to implement discretizing a boundary value problem in a bounded domain with a polygonal boundary by using the finite element method [18]. Thus from the engineering point of view, it is natural to introduce a polygonal boundary as an artificial boundary for the problem in an unbounded domain. Thus how to design an artificial boundary condition with high accuracy at a given polygonal artificial boundary becomes an interesting open problem. In this paper, we propose a method to design a discrete artificial boundary condition at a given polygonal artificial boundary for the exterior problem of Poisson equation by using the direct method of lines. Then the problem is reduced to a boundary value problem defined in a bounded computational domain. Finite element approximation of the reduced problem is also considered. Furthermore numerical examples show that the discrete artificial boundary condition presented in this paper is very effective.

## 2. The discrete artificial boundary condition at a polygonal artificial boundary

Let $\Gamma_i$ be a bounded, simple and closed curve in $\mathbb{R}^2$ and $\Omega$ be the unbounded domain with boundary $\Gamma_i$. We consider the following model problem, the exterior problem of Poisson equation:

$$\Delta u = f \quad \text{in } \Omega, \tag{2.1}$$

$$u|_{\Gamma_i} = g, \tag{2.2}$$

$$u \text{ is bounded} \quad \text{when } r \to +\infty; \tag{2.3}$$

where $g$ is a given function on $\Gamma_i$, $f$ is a given function in $\Omega$ and we assume that its support is compact. This problem is defined in the unbounded domain $\Omega$. In Ref. [4], this problem was considered. They introduced a circumference as an artificial boundary and designed a series of artificial boundary conditions on it. Then the original problem (2.1)–(2.3) was reduced to a series of boundary value problems with different accuracy on a bounded computational domain. In this paper we introduce a polygonal artificial boundary $\Gamma_e$ in $\Omega$, then $\Omega$ is divided into two parts, the bounded part $\Omega_i$ and the unbounded part $\Omega_e = \Omega \setminus \bar{\Omega}_i$ (see Fig. 1). $\Gamma_e$ is given by

$$r = e(\theta) \quad 0 \leqslant \theta \leqslant 2\pi \tag{2.4}$$

and such that $\operatorname{supp} f \subset \Omega_i$.

If a suitable boundary condition of $u$ at $\Gamma_e$ is given, then we can consider the boundary value problem on the bounded domain $\Omega_i$. The goal of this section is to design the discrete artificial boundary condition at the

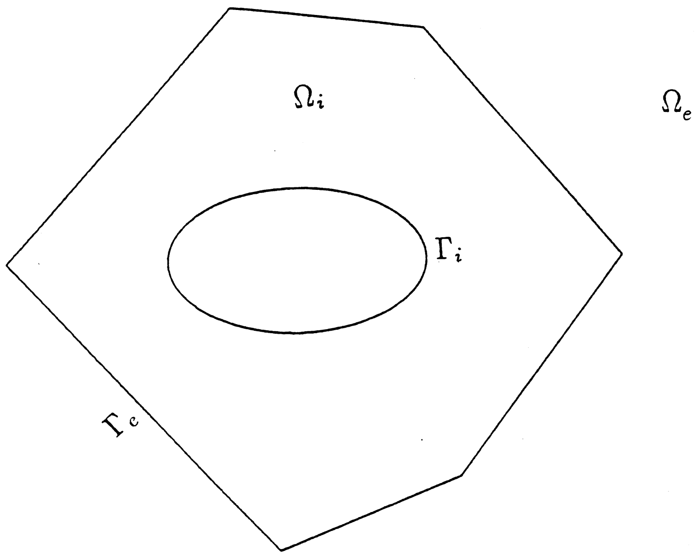


Fig. 1.

given polygonal artificial boundary $\Gamma_e$. We now consider the restriction of $u$, the solution of problem (2.1)–(2.3), in $\Omega_e$. Then we have that

$$\Delta u = 0 \quad \text{in } \Omega_e, \tag{2.5}$$

$$u|_{\Gamma_e} = u(e(\theta), \theta) \equiv u_0(\theta), \tag{2.6}$$

$$u \text{ is bounded} \quad \text{when } r \to +\infty. \tag{2.7}$$

Since the value $u|_{\Gamma_e}$ is unknown, the problem (2.5)–(2.7) cannot be solved independently. If $u|_{\Gamma_e} \in H^{1/2}(\Gamma_e)$ is given, then the problem (2.5)–(2.7) has a unique solution $u$ and we know $\partial u/\partial n|_{\Gamma_e} \in H^{-1/2}(\Gamma_e)$ [20], where $H^{\alpha}(\Gamma_e)$ denotes the usual Sobolov space on $\Gamma_e$ with real number $\alpha$ [18]. On the other hand, for any given $u_0 \in H^{1/2}(\Gamma_e)$, we know that the problem (2.5)–(2.7) has a unique solution $\tilde{u}$ [20]. By using the trace theorem [19], we know $\partial \tilde{u}/\partial n|_{\Gamma_e} \in H^{-1/2}(\Gamma_e)$. Hence if we define $\partial \tilde{u}/\partial n|_{\Gamma_e}$ as an image of $u_0$ from the space $H^{1/2}(\Gamma_e)$ to $H^{-1/2}(\Gamma_e)$, we obtain a bounded operator $K : H^{1/2}(\Gamma_e) \to H^{-1/2}(\Gamma_e)$, namely

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_e} = K(u|_{\Gamma_e}). \tag{2.8}$$

In fact the condition (2.8) is an exact boundary condition satisfied by the solution of the original problem (2.1)–(2.3). Hence the restriction of the solution of the problem (2.1)–(2.3) in $\Omega_i$ satisfies

$$\Delta u = f \quad \text{in } \Omega_i, \tag{2.9}$$

$$u|_{\Gamma_i} = g, \tag{2.10}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_e} = K(u|_{\Gamma_e}). \tag{2.11}$$

Unfortunately the bounded operator $K$ is unknown, thus the problem (2.9)–(2.11) cannot be solved directly. We now return to the problem (2.5)–(2.7) under the assumption $u|_{\Gamma_e}$ is given. As shown in Fig. 1, we assume that the polygonal artificial boundary $\Gamma_e$ has $n-1$ vertexes $\{a_i = (x_1^i, x_2^i),\ i = 1, 2, \ldots, n-1\}$ with

$$x_1^i = R_i \cos\theta_i \quad x_2^i = R_i \sin\theta_i \quad 1 \leqslant i \leqslant n-1. \tag{2.12}$$

Furthermore $a_n$ denotes the same vertex as $a_1$, but with the pole coordinate $(R_n = R_1,\ \theta_n = \theta_1 + 2\pi)$. For the ease of exposition, we assume that $\theta_1 = 0$. Then the rays $\{\theta = \theta_i,\ 1 \leqslant i \leqslant n-1\}$ divide $\Omega_e$ into $n-1$ parts

$$\Omega_e^i = \{x = (x_1, x_2) \mid x \in \Omega_e,\ \theta_i < \theta < \theta_{i+1}\} \quad 1 \leqslant i \leqslant n-1.$$

In the following our aim is to map the domain $\Omega_e$ into a strip and its boundary to a segment by a coordinate transformation. Thus on each subdomain $\Omega_e^i$, we introduce the mapping

$$x_1 = \frac{\rho_i e^{\rho} \cos\phi}{\sin(\phi - \alpha_i)} \quad \theta_i \leqslant \phi \leqslant \theta_{i+1} \quad 0 \leqslant \rho < +\infty \quad 1 \leqslant i \leqslant n-1,$$
$$x_2 = \frac{\rho_i e^{\rho} \sin\phi}{\sin(\phi - \alpha_i)} \quad \theta_i \leqslant \phi \leqslant \theta_{i+1} \quad 0 \leqslant \rho < +\infty \quad 1 \leqslant i \leqslant n-1, \tag{2.13}$$

with

$$\rho_i = \frac{x_1^{i+1} x_2^i - x_1^i x_2^{i+1}}{|\overline{a_i a_{i+1}}|} \equiv \frac{x_1^{i+1} x_2^i - x_1^i x_2^{i+1}}{\sqrt{(x_1^{i+1} - x_1^i)^2 + (x_2^{i+1} - x_2^i)^2}}, \tag{2.14}$$

$$\sin\alpha_i = \frac{x_2^{i+1} - x_2^i}{|\overline{a_i a_{i+1}}|} \equiv \frac{x_2^{i+1} - x_2^i}{\sqrt{(x_1^{i+1} - x_1^i)^2 + (x_2^{i+1} - x_2^i)^2}}, \tag{2.15}$$

$$\cos\alpha_i = \frac{x_1^{i+1} - x_1^i}{|\overline{a_i a_{i+1}}|} \equiv \frac{x_1^{i+1} - x_1^i}{\sqrt{(x_1^{i+1} - x_1^i)^2 + (x_2^{i+1} - x_2^i)^2}}. \tag{2.16}$$

It is straightforward to check that the transformation (2.13) maps $\Omega_e^i$ onto a semi-infinite strip (see Fig. 2):

$$\tilde{\Omega}_e^i = \{(\rho, \phi) \mid \theta_i < \phi < \theta_{i+1},\ 0 < \rho < +\infty\} \quad i = 1, 2, \ldots, n-1.$$

Then $\Omega_e$ is mapped onto $\tilde{\Omega}_e = \{(\rho, \phi) \mid 0 \leqslant \phi \leqslant 2\pi,\ 0 < \rho < +\infty\}$ and $\Gamma_e$ is mapped onto $\tilde{\Gamma}_e = \{(0, \phi) \mid 0 \leqslant \phi \leqslant 2\pi\}$. Moreover, in $\Omega_e^i (1 \leqslant i \leqslant n-1)$, we have that

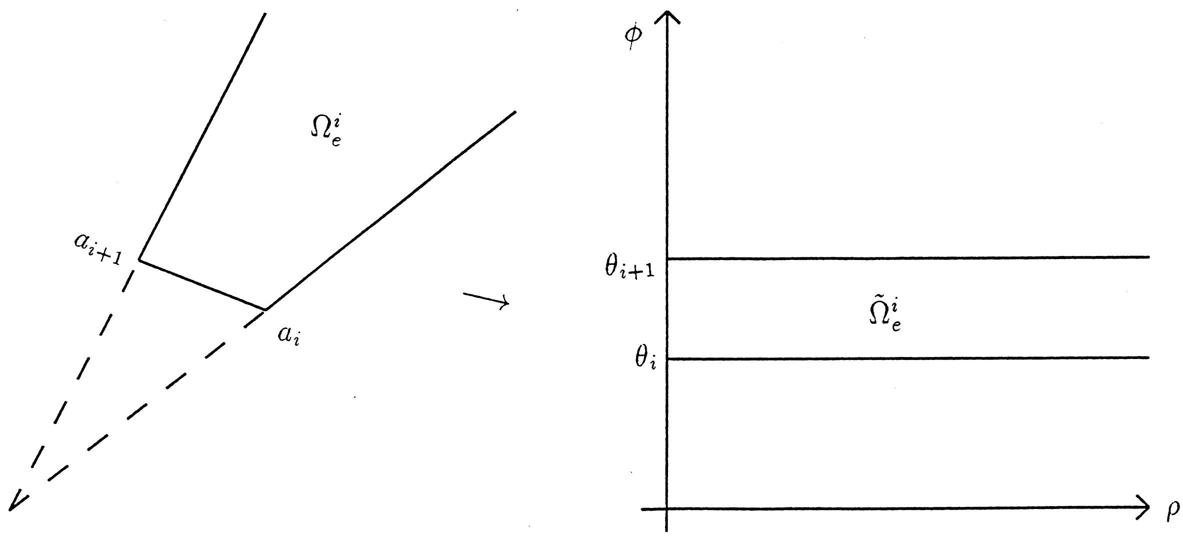


Fig. 2.

$$
\begin{aligned}
\frac{\partial}{\partial \rho} &= \frac{\rho_i e^{\rho} \cos\phi}{\sin(\phi-\alpha_i)} \frac{\partial}{\partial x_1} + \frac{\rho_i e^{\rho} \sin\phi}{\sin(\phi-\alpha_i)} \frac{\partial}{\partial x_2}, \\
\frac{\partial}{\partial \phi} &= -\frac{\rho_i e^{\rho} \cos\alpha_i}{\sin^2(\phi-\alpha_i)} \frac{\partial}{\partial x_1} - \frac{\rho_i e^{\rho} \sin\alpha_i}{\sin^2(\phi-\alpha_i)} \frac{\partial}{\partial x_2},
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x_1} &= -\rho_i^{-1} e^{-\rho} \left[ \sin\alpha_i \frac{\partial}{\partial \rho} + \sin\phi \sin(\phi-\alpha_i) \frac{\partial}{\partial \phi} \right], \\
\frac{\partial}{\partial x_2} &= \rho_i^{-1} e^{-\rho} \left[ \cos\alpha_i \frac{\partial}{\partial \rho} + \cos\phi \sin(\phi-\alpha_i) \frac{\partial}{\partial \phi} \right],
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial x_1^2} = \rho_i^{-2} e^{-2\rho} \Big[ & \sin^2\alpha_i \frac{\partial^2}{\partial \rho^2} + \sin 2\phi \sin^2(\phi-\alpha_i) \frac{\partial}{\partial \phi} - \sin^2\alpha_i \frac{\partial}{\partial \rho} + 2\sin\alpha_i \sin\phi \sin(\phi-\alpha_i) \frac{\partial^2}{\partial \rho \partial \phi} \\
& + \sin^2\phi \sin^2(\phi-\alpha_i) \frac{\partial^2}{\partial \phi^2} \Big],
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial x_2^2} = \rho_i^{-2} e^{-2\rho} \Big[ & \cos^2\alpha_i \frac{\partial^2}{\partial \rho^2} - \sin 2\phi \sin^2(\phi-\alpha_i) \frac{\partial}{\partial \phi} - \cos^2\alpha_i \frac{\partial}{\partial \rho} + 2\cos\alpha_i \cos\phi \sin(\phi-\alpha_i) \frac{\partial^2}{\partial \rho \partial \phi} \\
& + \cos^2\phi \sin^2(\phi-\alpha_i) \frac{\partial^2}{\partial \phi^2} \Big],
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
\Delta &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \\
&= \rho_i^{-2} e^{-2\rho} \left[ \frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\partial \rho} + \sin 2(\phi-\alpha_i) \frac{\partial^2}{\partial \rho \partial \phi} + \sin^2(\phi-\alpha_i) \frac{\partial^2}{\partial \phi^2} \right],
\end{aligned}
\tag{2.21}
$$

$$
dx = \frac{\rho_i^2 e^{2\rho}}{\sin^2(\phi-\alpha_i)} \, d\rho \, d\phi.
\tag{2.22}
$$

Furthermore we have that

$$
\left.\frac{\partial u}{\partial n}\right|_{\Gamma_e} \equiv \left.\left( \sin\alpha_i \frac{\partial u}{\partial x_1} - \cos\alpha_i \frac{\partial u}{\partial x_2} \right)\right|_{\rho=0} = -\rho_i^{-1} \left[ \frac{\partial u}{\partial \rho} + \frac{1}{2} \sin 2(\phi-\alpha_i) \frac{\partial u}{\partial \phi} \right]_{\rho=0} \quad \theta_i \leqslant \phi \leqslant \theta_{i+1}.
\tag{2.23}
$$

$$
\begin{aligned}
\left.\frac{\partial u}{\partial n}\right|_{\phi=\theta_i^+} &\equiv \left.\left( \sin\theta_i \frac{\partial u}{\partial x_1} - \cos\theta_i \frac{\partial u}{\partial x_2} \right)\right|_{\phi=\theta_i^+} = -\rho_i^{-1} e^{-\rho} \left[ \cos(\theta_i-\alpha_i) \frac{\partial u}{\partial \rho} + \sin(\theta_i-\alpha_i) \frac{\partial u}{\partial \phi} \right]_{\phi=\theta_i^+} \\
&= -R_i^{-1} e^{-\rho} \left[ \mathrm{ctg}(\theta_i-\alpha_i) \frac{\partial u}{\partial \rho} + \frac{\partial u}{\partial \phi} \right]_{\phi=\theta_i^+}.
\end{aligned}
\tag{2.24}
$$

$$
\begin{aligned}
\left.\frac{\partial u}{\partial n}\right|_{\phi=\theta_i^-} &\equiv \left.\left( \sin\theta_i \frac{\partial u}{\partial x_1} - \cos\theta_i \frac{\partial u}{\partial x_2} \right)\right|_{\phi=\theta_i^-} \\
&= -R_i^{-1} e^{-\rho} \left.\left[ \mathrm{ctg}(\theta_i-\alpha_{i-1}) \frac{\partial u}{\partial \rho} + \frac{\partial u}{\partial \phi} \right]\right|_{\phi=\theta_i^-}.
\end{aligned}
\tag{2.25}
$$

In the new coordinate $(\rho, \phi)$, the problem (2.5)–(2.7) is reduced to the following discontinuous coefficient problem on the semi-infinite strip $\tilde{\Omega}_e$.

$$
\frac{1}{\sin^2(\phi-\alpha_i)} \frac{\partial^2 u}{\partial \rho^2} + \mathrm{ctg}(\phi-\alpha_i) \frac{\partial^2 u}{\partial \rho \partial \phi} + \frac{\partial}{\partial \phi} \left[ \frac{\partial u}{\partial \phi} + \mathrm{ctg}(\phi-\alpha_i) \frac{\partial u}{\partial \rho} \right] = 0
$$

$$\theta_i < \phi < \theta_{i+1} \quad 1 \leqslant i \leqslant n, \tag{2.26}$$

$$u(\rho, \theta_i^-) = u(\rho, \theta_i^+) \quad 0 \leqslant \rho < +\infty \quad 1 \leqslant i \leqslant n-1, \tag{2.27}$$

$$\left(\mathrm{ctg}(\theta_i - \alpha_{i-1})\frac{\partial u}{\partial \rho} + \frac{\partial u}{\partial \phi}\right)\Big|_{(\rho,\theta_i^-)} = \left(\mathrm{ctg}(\theta_i - \alpha_i)\frac{\partial u}{\partial \rho} + \frac{\partial u}{\partial \phi}\right)\Big|_{(\rho,\theta_i^+)} \quad 0 \leqslant \rho < +\infty, 1 \leqslant i \leqslant n-1, \tag{2.28}$$

$$u|_{\rho=0} = u_0(\phi), \tag{2.29}$$

$$u \text{ is bounded} \quad \text{when } \rho \to +\infty, \tag{2.30}$$

where $\theta_1^- = \theta_n^-$, $\alpha_0 = \alpha_{n-1}$.

We notice that the coefficients of Eq. (2.26) are independent on the variable $\rho$. This fact will play an important role in considering the numerical solution of the problem (2.26)–(2.30). Let $H^1((0, 2\pi))$ denote the usual Sobolov space on the interval $(0, 2\pi)$ [19]. Furthermore we suppose that

$$V = \{v(\phi) \mid v(\phi) \in H^1((0, 2\pi)),\ v(0) = v(2\pi)\},$$

$$U = \left\{u(\rho, \phi) \mid \text{for fixed } \rho \in [0, +\infty),\ u,\ \frac{\partial u}{\partial \rho},\ \frac{\partial^2 u}{\partial \rho^2} \in V\right\}.$$

Then the discontinuous coefficient problem (2.26)–(2.30) is equivalent to the following variational–differential problem:

Find $u(\rho, \phi) \in U$ such that

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}A_2(u, v) + \frac{\mathrm{d}}{\mathrm{d}\rho}A_1(u, v) + A_0(u, v) = 0 \quad \forall v \in V, \tag{2.31}$$

$$u|_{\rho=0} = u_0(\phi), \tag{2.32}$$

$$u \text{ is bounded} \quad \text{when } \rho \to +\infty, \tag{2.33}$$

where

$$A_2(u, v) = \sum_{i=1}^{n-1} \int_{\theta_i}^{\theta_{i+1}} \frac{1}{\sin^2(\phi - \alpha_i)} u(\rho, \phi) v(\phi)\, \mathrm{d}\phi, \tag{2.34}$$

$$A_1(u, v) = \sum_{i=1}^{n-1} \int_{\theta_i}^{\theta_{i+1}} \mathrm{ctg}(\phi - \alpha_i)\left[\frac{\partial u}{\partial \phi}(\rho, \phi) v(\phi) - u(\rho, \phi)\frac{\mathrm{d}v(\phi)}{\mathrm{d}\phi}\right] \mathrm{d}\phi, \tag{2.35}$$

$$A_0(u, v) = -\int_0^{2\pi} \frac{\partial u}{\partial \phi}(\rho, \phi)\frac{\mathrm{d}v(\phi)}{\mathrm{d}\phi}\, \mathrm{d}\phi. \tag{2.36}$$

We now consider a semi-discrete approximation of the problem (2.31)–(2.33). Suppose that

$$0 = \phi_1 < \phi_2 < \cdots < \phi_{M+1} = 2\pi$$

is a partition of the interval $[0, 2\pi]$, such that each of $\{\theta_i,\ i = 1, 2, \ldots, n-1\}$ is a node of this partition, namely for every $\theta_i$ $(i = 1, 2, \ldots, n-1)$ there is a $\phi_j = \theta_i$. Let $h = \max_{1 \leqslant j \leqslant M}(\phi_{j+1} - \phi_j)$ and

$$V_h = \left\{v_h(\phi) \mid v_h(\phi) \in V,\ v_h|_{[\phi_j, \phi_{j+1}]} \in P_1([\phi_j, \phi_{j+1}]),\ j = 1, 2, \ldots, M\right\},$$

$$U_h = \left\{ u_h(\rho, \phi) \mid \text{for fixed } \rho \geqslant 0, \quad u_h, \ \frac{\partial u_h}{\partial \rho}, \ \frac{\partial^2 u_h}{\partial \rho^2} \in V_h \right\},$$

where $P_m([\phi_j, \phi_{j+1}])$ denotes the space of polynomals of degree not grsater that $m$. Then we get the numerical approximation of the problem (2.31)–(2.33):

Find $u_h(\rho, \phi) \in U_h$ such that

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} A_2(u_h, v_h) + \frac{\mathrm{d}}{\mathrm{d}\rho} A_1(u_h, v_h) + A_0(u_h, v_h) = 0 \quad \forall v_h \in V_h, \tag{2.37}$$

$$u_h|_{\rho=0} = u_{0,h}(\phi), \quad u_h \text{ is bounded when } \quad \rho \to +\infty, \tag{2.38}$$

where $u_{0,h}(\phi) \in V_h$ such that $u_{0,h}(\phi_j) = u_0(\phi_j)$ for $j = 1, 2, \ldots, M+1$. Assume that $\{N_j(\phi), \ j = 1, 2, \ldots, M\}$ is a basis of the finite dimensional space $V_h$ such that $N_j(\phi_i) = \delta_{ij}, \quad 1 \leqslant i, j \leqslant M$. Let $N(\phi) = [N_1(\phi), N_2(\phi), \ldots, N_M(\phi)]^{\mathrm{T}}$ and $U(\rho) = [u_h(\rho, \phi_1), u_h(\rho, \phi_2), \ldots, u_h(\rho, \phi_M)]^{\mathrm{T}}$ for $u_h(\rho, \phi) \in U_h$. Then we have that

$$u_h(\rho, \phi) = N(\phi)^{\mathrm{T}} U(\rho), \tag{2.39}$$

$$u_{0,h}(\phi) = N(\phi)^{\mathrm{T}} U_0 \quad \text{with } U_0 = [u_0(\phi_1), u_0(\phi_2), \ldots, u_0(\phi_M)]^{\mathrm{T}}. \tag{2.40}$$

Thus the discrete variational–differential problem (2.37) and (2.38) is equivalent to the following boundary value problem of a system of ordinary differential equations:

$$B_2 U''(\rho) + B_1 U'(\rho) + B_0 U(\rho) = 0 \quad 0 < \rho < +\infty, \tag{2.41}$$

$$U(0) = U_0, \quad U(\rho) \text{ is bounded when } \rho \to +\infty, \tag{2.42}$$

where

$$B_2 = \sum_{i=1}^{n-1} \int_{\theta_i}^{\theta_{i+1}} \frac{1}{\sin^2(\phi - \alpha_i)} N(\phi) N(\phi)^{\mathrm{T}} \, \mathrm{d}\phi, \tag{2.43}$$

$$B_1 = \sum_{i=1}^{n-1} \int_{\theta_i}^{\theta_{i+1}} \operatorname{ctg}(\phi - \alpha_i) [N(\phi) N'(\phi)^{\mathrm{T}} - N'(\phi) N(\phi)^{\mathrm{T}}] \, \mathrm{d}\phi, \tag{2.44}$$

$$B_0 = -\int_0^{2\pi} N'(\phi) N'(\phi)^{\mathrm{T}} \, \mathrm{d}\phi. \tag{2.45}$$

It is straightforward to check that $B_2$ is a positive definite symmetric matrix, $B_1$ is an antisymmetric matrix and $B_0$ is a semi-negative definite symmetric matrix. We use a direct method for solving the problem (2.41)–(2.42). Let

$$U(\rho) = \mathrm{e}^{\lambda \rho} \xi, \tag{2.46}$$

where $\lambda$ is a constant, $\xi \in \mathbb{C}^M$ to be determined. Substituting Eq. (2.46) into the Eqs. (2.41) and (2.42), we get the following generalized eigenvalue problem

$$[\lambda^2 B_2 + \lambda B_1 + B_0] \, \xi = 0. \tag{2.47}$$

Let $\eta = \lambda \xi$, then the eigenvalue problem (2.47) is equivalent to the following standard eigenvalue problem:

$$\begin{pmatrix} 0 & I_M \\ -B_0 & -B_1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} I_M & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{2.48}$$

where $I_M$ denotes the $M \times M$ unit matrix. After solving the eigenvalue problem (2.48), we obtain the eigenvalues $\lambda_j$ $(j = 1, 2, \ldots, M)$ with non-positive real part corresponding to the eigenvalues

$$\begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix} \quad j = 1, 2, \cdots, M$$

and $\lambda_1 = 0,\ \xi_1 = (1, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^M,\ \eta_1 = 0 \in \mathbb{R}^M$. Particularly we assume that $\lambda_j$ $(1 \leqslant j \leqslant r)$ are real eigenvalues and $\lambda_j$ $(r+1 \leqslant j \leqslant M)$ are complex eigenvalues with nonzero imaginary parts satisfying $\lambda_{2l} = \bar{\lambda}_{2l-1}$ $\left(\frac{r}{2} + 1 \leqslant l \leqslant M/2\right)$. Without losing the generality, we suppose that $M$ is an even number. Thus we obtain

$$U(\rho) = \sum_{j=1}^{r} b_j\, \mathrm{e}^{\rho\lambda_j} \xi_j + \sum_{j=r/2+1}^{M/2} \left[ b_{2j-1}\, \Re(\mathrm{e}^{\rho\lambda_{2j}} \xi_{2j}) + b_{2j}\, \Im(\mathrm{e}^{\rho\lambda_{2j}} \xi_{2j}) \right], \tag{2.49}$$

where $\Re(\lambda)$ and $\Im(\lambda)$ denote the real and imaginary parts of the complex number $\lambda$, respectively. Thus $U(\rho)$ satisfies the ordinary equations in Eq. (2.41) and the boundary condition: $U(\rho)$ is bounded, when $\rho \to +\infty$. By using the condition $U(0) = U_0$, we have that

$$U_0 = \sum_{j=1}^{r} b_j \xi_j + \sum_{j=r/2+1}^{M/2} [b_{2j-1}\, \Re(\xi_{2j}) + b_{2j}\, \Im(\xi_{2j})]. \tag{2.50}$$

Introducing matrices

$$D(\rho) = \left[\mathrm{e}^{\rho\lambda_1}\xi_1, \ldots, \mathrm{e}^{\rho\lambda_r}\xi_r, \Re(\mathrm{e}^{\rho\lambda_{r+2}}\xi_{r+2}), \Im(\mathrm{e}^{\rho\lambda_{r+2}}\xi_{r+2}), \ldots, \Re(\mathrm{e}^{\rho\lambda_M}\xi_M), \Im(\mathrm{e}^{\rho\lambda_M}\xi_M)\right],$$

$$D_0 \equiv D(0) = [\xi_1, \ldots, \xi_r, \Re(\xi_{r+2}), \Im(\xi_{r+2}), \ldots, \Re(\xi_M), \Im(\xi_M)],$$

$$B = [b_1, b_2, \ldots, b_M]^{\mathrm{T}}.$$

From Eq. (2.50) we obtain

$$B = D_0^{-1} U_0. \tag{2.51}$$

Substituting Eq. (2.51) into Eq. (2.49), we have that

$$U(\rho) = D(\rho) D_0^{-1} U_0. \tag{2.52}$$

Finally we get the semi-discrete approximate solution of problem (2.5)–(2.7) for given $u_0(\theta)$:

$$u_h(\rho, \phi) = N(\phi)^{\mathrm{T}} D(\rho) D_0^{-1} U_0. \tag{2.53}$$

Noting Eqs. (2.53) and (2.23), we have that

$$\left.\frac{\partial u_h}{\partial n}\right|_{\Gamma_e} = -\rho_i^{-1} \left[ N(\phi)^{\mathrm{T}} D'(0) D_0^{-1} + \frac{1}{2} \sin 2(\phi - \alpha_i) N'(\phi)^{\mathrm{T}} \right] U_0 \qquad \theta_i < \phi < \theta_{i+1}, 1 \leqslant i \leqslant n-1. \tag{2.54}$$

In fact the equality (2.54) is a discrete artificial boundary condition for the exterior problem of Poisson equation on the given polygonal boundary $\Gamma_e$, which is an approximation of the exact boundary condition (2.8).

## 3. The numerical solution of the problem (2.1)–(2.3)

On the bounded computational domain $\Omega_i$, we now consider the numerical solution of the problem (2.1)–(2.3). As we know that the restriction of $u$, the solution of the problem (2.1)–(2.3), satisfies the boundary value problem (2.9)–(2.11). Let $H^1(\Omega_i)$ denote the usual Sobolov space on $\Omega_i$ [19] and assume that

$$T_g = \{v \in H^1(\Omega_i) \ \mid v\,|_{\Gamma_i} = g\},$$
$$T_0 = \{v \in H^1(\Omega_i) \ \mid v\,|_{\Gamma_i} = 0\}.$$

Then the boundary value problem (2.9)–(2.11) is equivalent to the following variational problem:

Find $u \in T_g$ such that

$$a(u,v) + b(u,v) = f(v) \quad \forall v \in T_0, \tag{3.1}$$

with

$$a(u,v) = \int_{\Omega_i} \nabla u \ \nabla v \, \mathrm{d}x, \tag{3.2}$$

$$b(u,v) = -\int_{\Gamma_e} (Ku)v \, \mathrm{d}s, \tag{3.3}$$

$$f(v) = \int_{\Omega_i} fv \, \mathrm{d}x. \tag{3.4}$$

For the simplicity, we suppose that $\Gamma_i$ is a polygonal line in $\mathbb{R}^2$. Let $\mathscr{T}^h$ be a regular triangulation such that the nodes on the boundary $\Gamma_e$ are mapped into the points $(0, \phi_j)$, for $j = 1, 2, \ldots, M$ by the mapping (2.13). Furthermore let

$$T^h = \{v_h \in C^{(0)}(\Omega_i) \ |v_h|_K \in P_1(K), \ \forall K \in \mathscr{T}^h\},$$
$$T_g^h = \{v_h \in T^h \ |v_h(d_j) = g(d_j), \ \text{for the node } d_j \in \Gamma_i\},$$
$$T_0^h = \{v_h \in T^h \ |v_h|_{\Gamma_i} = 0\}.$$

Thus we obtain the discrete form of the problem (3.1):

Find $u_h \in T_g^h$ such that

$$a(u_h, v_h) + b(u_h, v_h) = f(v_h) \quad \forall v_h \in T_0^h. \tag{3.5}$$

Since the bounded operator $K$ is unknown, we cannot solve the problem (3.5) directly. But we have had the discrete artificial boundary condition (2.54). Thus for $u_h, v_h \in T^h$, let

$$\begin{aligned} b_h(u_h, v_h) = & -\sum_{i=1}^{n-1} \rho_i^{-1} \int_{\theta_i}^{\theta_{i+1}} v_h \left[ N(\phi)^{\mathrm{T}} D'(0) D_0^{-1} + \frac{1}{2} \sin 2(\phi - \alpha_i) N'(\phi)^{\mathrm{T}} \right] U_0 \, \mathrm{d}s \\ = & -V_0^{\mathrm{T}} \sum_{i=1}^{n-1} \int_{\theta_i}^{\theta_{i+1}} \frac{1}{\sin^2(\phi - \alpha_i)} \cdot \left[ N(\phi) N(\phi)^{\mathrm{T}} D'(0) D_0^{-1} + \frac{1}{2} \sin 2(\phi - \alpha_i) N(\phi) N'(\phi)^{\mathrm{T}} \right] U_0 \, \mathrm{d}\phi \end{aligned} \tag{3.6}$$

with $U_0 = [u_h(0, \phi_1), \ldots, u_h(0, \phi_M)]^{\mathrm{T}}$ and $V_0 = [v_h(0, \phi_1), \ldots, v_h(0, \phi_M)]^{\mathrm{T}}$. Using the bilinear form $b_h(u_h, v_h)$ instead of $b(u_h, v_h)$ in the problem (3.5) we obtain the approximation of the original problem (3.1) (say (2.1)–(2.3)):

Find $u_h \in T_g^h$ such that

$$a(u_h, v_h) + b_h(u_h, v_h) = f(v_h) \quad \forall v_h \in T_0^h. \tag{3.7}$$

After solving the problem (3.7), the solution $u_h \in T_g^h$ is an approximation of the original problem (2.1)–(2.3) in the bounded computational domain $\Omega_i$.

For the bilinear form $b_h(u_h, v_h)$, we have that

**Lemma 3.1.** *The bilinear form $b_h(u_h, v_h)$ is bounded and symmetric on $T^h \times T^h$. Furthermore $b_h(v_h, v_h) \geqslant 0$ for all $v_h \in T^h$.*

**Proof.** For any given $u_h, v_h \in T^h$, noting Eqs. (2.39) and (3. 6), we have that

$$u_h|_{\tilde{\Gamma}_e} = N(\phi)^{\mathrm{T}} U_0, \tag{3.8}$$

$$v_h|_{\tilde{\Gamma}_e} = N(\phi)^{\mathrm{T}} V_0. \tag{3.9}$$

On the domain $\tilde{\Omega}_e$, let

$$u_h = N(\phi)^{\mathrm{T}} D(\rho) D_0^{-1} U_0, \tag{3.10}$$

$$v_h = N(\phi)^{\mathrm{T}} D(\rho) D_0^{-1} V_0, \tag{3.11}$$

Thus we get the continuously extensions of $u_h$ and $v_h$ on the domain $\Omega_e$. A computation shows

$$\begin{aligned} \int_{\Omega_e} \nabla u_h \ \nabla v_h \,\mathrm{d}x &= \sum_{i=1}^{n-1} \int_0^{+\infty} \int_{\theta_i}^{\theta_{i+1}} \frac{1}{\sin^2(\phi-\alpha_i)} \cdot \left\{ \left[ \sin\alpha_i \frac{\partial u_h}{\partial \rho} + \sin\phi \sin(\phi-\alpha_i) \frac{\partial u_h}{\partial \phi} \right] \right. \\ &\quad \times \left[ \sin\alpha_i \frac{\partial v_h}{\partial \rho} + \sin\phi \sin(\phi-\alpha_i) \frac{\partial v_h}{\partial \phi} \right] + \left[ \cos\alpha_i \frac{\partial u_h}{\partial \rho} + \cos\phi \sin(\phi-\alpha_i) \frac{\partial u_h}{\partial \phi} \right] \\ &\quad \left. \times \left[ \cos\alpha_i \frac{\partial v_h}{\partial \phi} + \cos\phi \sin(\phi-\alpha_i) \frac{\partial v_h}{\partial \phi} \right] \right\} \mathrm{d}\phi \,\mathrm{d}\rho \\ &= b_h(u_h, v_h) - \sum_{i=1}^{n-1} \int_0^{+\infty} \int_{\theta_i}^{\theta_{i+1}} \left[ A_2\left( \frac{\partial^2 u_h}{\partial \rho^2}, v_h \right) + A_1\left( \frac{\partial u_h}{\partial \rho}, v_h \right) + A_0(u_h, v_h) \right] \mathrm{d}\phi \,\mathrm{d}\rho \\ &= b_h(u_h, v_h). \end{aligned} \tag{3.12}$$

Hence

$$b_h(u_h, v_h) = \int_{\Omega_e} \nabla u_h \ \nabla v_h \,\mathrm{d}x = b_h(v_h, u_h) \quad \forall u_h, v_h \in T^h, \tag{3.13}$$

$$b_h(v_h, v_h) = \int_{\Omega_e} |\nabla v_h|^2 \,\mathrm{d}x \geqslant 0 \quad \forall v_h \in T^h. \tag{3.14}$$

From the Lemma 3.1 it is straight forward to check that the problem (3.7) is a well-posed problem.

## 4. Numerical implementation and examples

Let $\Omega$ denote the exterior domain of the unit square, namely

$$\Omega = \{x = (x_1, x_2) \mid |x_1| > 1 \quad \text{or} \quad |x_2| > 1\}.$$

We consider the numerical solution of the original problem (2.1)–(2.3) with given $f$ and $g$. We take the artificial boundary $\Gamma_e = \{(x_1, x_2) \mid x_1 = \pm 2, \ -2 \leqslant x_2 \leqslant 2\} \cup \{(x_1, x_2) \mid x_2 = \pm 2, \ -2 \leqslant x_1 \leqslant 2\}$. Hence $\Omega_e = \{(x_1, x_2) \mid |x_1| > 2 \quad \text{or} \quad |x_2| > 2\}$ and $\Omega_i = \Omega \setminus \bar{\Omega}_e$. Since the solution of each example, $u(x_1, x_2)$, is symmetric about $x_2$ axes and antisymmetric about $x_1$ axes, respectively, the domain of computation is taken to be the part of $\Omega_i$ lying in the first quadrant. The symmetric and antisymmetric boundary conditions are used along $x_1 = 0$ and $x_2 = 0$, respectively

$$\frac{\partial u(0, x_2)}{\partial x_1} = 0 \quad 1 \leqslant x_2 \leqslant 2, \tag{4.1}$$

$$u(x_1,0) = 0 \quad 1 \leqslant x_1 \leqslant 2. \tag{4.2}$$

Three meshes were used in the computation. Fig. 3 shows the triangulation for mesh A. On each triangle in mesh A, we connected the midpoints of every two sides, thus this triangle was divided into four small triangles. Then we obtained the refined mesh B. Mesh C was similarly generated from mesh B. Linear finite element was used in our computation.

**Example 1.** An unbounded membrane with a square hole. Let

$$f = 0 \quad \text{in } \Omega, \tag{4.3}$$

$$g(x_1,x_2) = \frac{1}{2}\ln\frac{x_1^2 + (x_2+0.5)^2}{x_1^2 + (x_2-0.5)^2} \quad \text{on } \Gamma_i. \tag{4.4}$$

Then the original problem (2.1)–(2.3) with the given $f$ in (4.3) and $g$ in (4.4) has a unique solution $u(x_1,x_2)$:

$$u(x_1,x_2) = \frac{1}{2}\ln\frac{x_1^2 + (x_2+0.5)^2}{x_1^2 + (x_2-0.5)^2} \quad \text{in } \Omega. \tag{4.5}$$

Let $u_h$ denote the finite element approximation by using the discrete artificial boundary condition (2.54). Table 1 shows the maximum of the errors $u - u_h$ over the mesh points for meshes A, B and C. Furthermore Table 2 shows the errors of $\|u-u_h\|_{0,2,\Omega_i}$, $|u-u_h|_{1,2,\Omega_i}$ and $\|u-u_h\|_{1,2,\Omega_i}$ for meshes A, B and C. For comparison we also compute the finite element approximation $u_h^N$ with Neumann boundary condition on the artificial boundary $\Gamma_e$ for meshes A, B and C. The maximum of the error $u - u_h^N$ over the mesh points is given in Table 3 and errors of $\|u-u_h^N\|_{0,2,\Omega_i}$, $|u-u_h^N|_{1,2,\Omega_i}$ and $\|u-u_h^N\|_{1,2,\Omega_i}$ for meshes A, B and C are given in Table 4. Furthermore Fig. 4 shows the related errors $|u-u_h|/|u| \times 100$ on the artificial boundary $\Gamma_e$. For comparison Fig. 5 shows the related errors $|u-u_h^N|/|u| \times 100$ on the artificial boundary $\Gamma_e$. Fig. 6 shows the contour plot of the approximate solution $u_h$ on mesh C.

**Example 2.** A two-dimensional ideal (potential) flow around a square obstacle (see Fig. 7). In this example, the unknown $u$ denotes the streamfunction of the fluid. It is a solution of the following boundary value problem:

$$-\Delta u = 0 \quad \text{in } \Omega, \tag{4.6}$$

$$u|_{\Gamma_i} = 0, \tag{4.7}$$

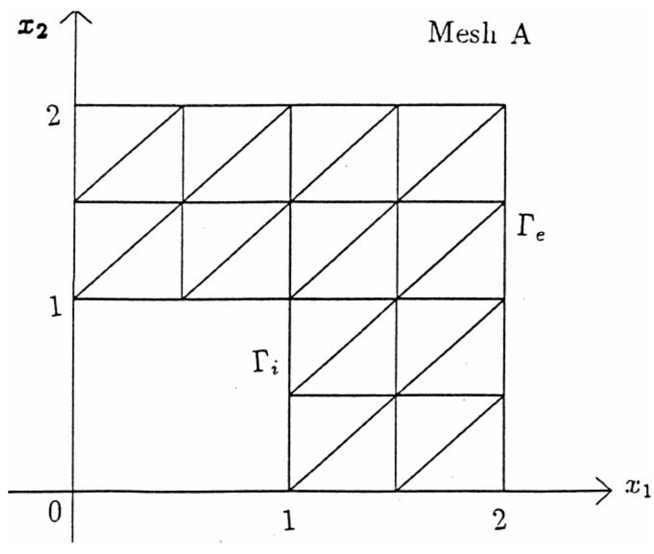


Fig. 3. Mesh A.

Table 1
Maximum error of $u - u_h$ over mesh points

| Mesh | A | B | C |
|---|---|---|---|
| $\max\|u - u_h\|$ | $2.1229 \times 10^{-2}$ | $5.6912 \times 10^{-3}$ | $1.4788 \times 10^{-3}$ |

Table 2
Error of $u - u_h$

| Mesh | A | B | C |
|---|---|---|---|
| $\\|u - u_h\\|_{0,2,\Omega_i}$ | $1.0665 \times 10^{-1}$ | $2.7974 \times 10^{-2}$ | $7.0909 \times 10^{-3}$ |
| $\|u - u_h\|_{1,2,\Omega_i}$ | 1.0311 | $5.4020 \times 10^{-1}$ | $2.7408 \times 10^{-1}$ |
| $\\|u - u_h\\|_{1,2,\Omega_i}$ | 1.0366 | $5.4093 \times 10^{-1}$ | $2.7417 \times 10^{-1}$ |

Table 3
Maximum error of $u - u_h^N$ over mesh points

| Mesh | A | B | C |
|---|---|---|---|
| $\max\|u - u_h^N\|$ | 0.2511 | 0.2386 | 0.2364 |

Table 4
Error of $u - u_h^N$

| Mesh | A | B | C |
|---|---|---|---|
| $\\|u - u_h^N\\|_{0,2,\Omega_i}$ | $8.4904 \times 10^{-1}$ | $8.5449 \times 10^{-1}$ | $8.6269 \times 10^{-1}$ |
| $\|u - u_h^N\|_{1,2,\Omega_i}$ | 1.5801 | 1.3387 | 1.2645 |
| $\\|u - u_h^N\\|_{1,2,\Omega_i}$ | 1.7937 | 1.5882 | 1.5308 |

$$\nabla u \to (0, V)^{\mathrm{T}} \quad \text{when } r \to +\infty. \tag{4.8}$$

we choose $V = 1.0$. Fig. 8 shows the contour plot of the approximate streamfunction $u_h$ on the mesh C.

**Example 3.** A two-dimensional incompressible flow around a square obstacle. In this example, the unknown $u$ also denotes the streamfunction of the fluid. It is a solution of the following boundary value problem:

$$-\Delta u = f \quad \text{in } \Omega, \tag{4.9}$$

$$u|_{\Gamma_i} = 0, \tag{4.10}$$

$$\nabla u \to (0, V)^{\mathrm{T}} \quad \text{when } r \to +\infty, \tag{4.11}$$

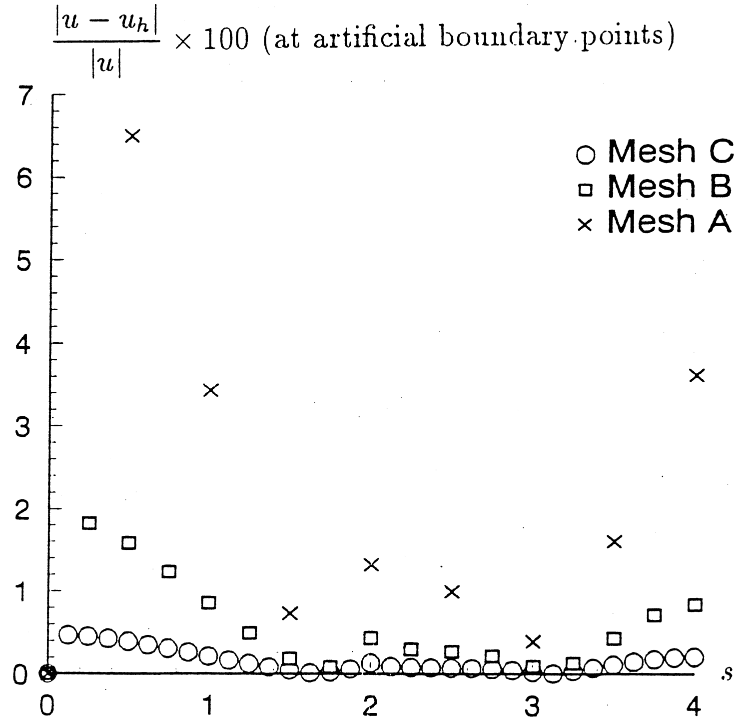
$\frac{|u-u_h|}{|u|} \times 100$ (at artificial boundary points)
7
6
5
4
3
2
1
0
Mesh C
Mesh B
Mesh A
0
1
2
3
4
s

Fig. 4.

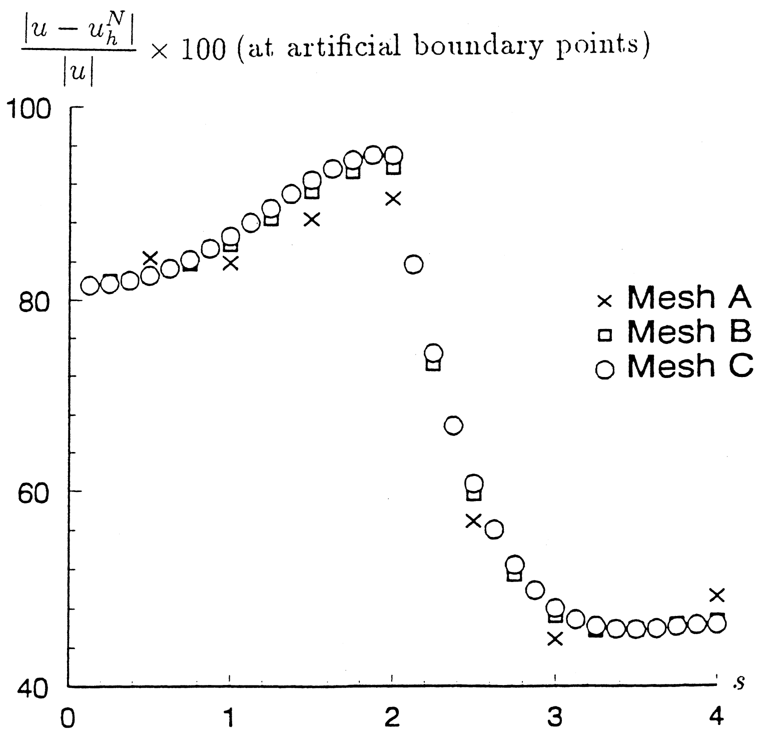
$\frac{|u-u_h^N|}{|u|} \times 100$ (at artificial boundary points)
100
80
60
40
Mesh A
Mesh B
Mesh C
0
1
2
3
4
s

Fig. 5.

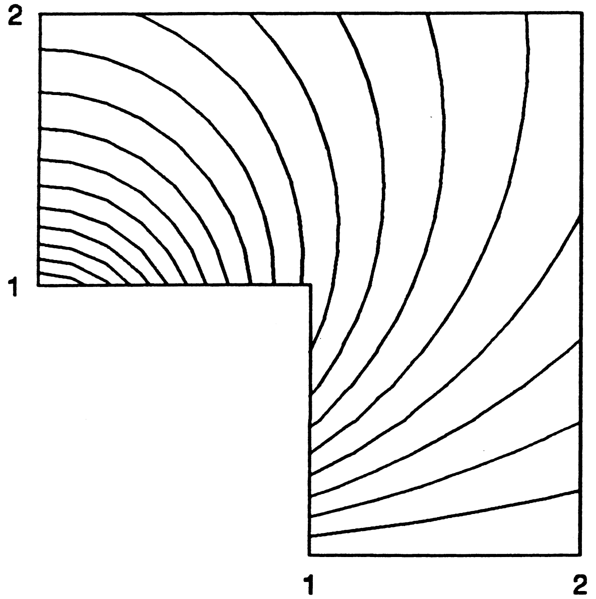


Fig. 6.

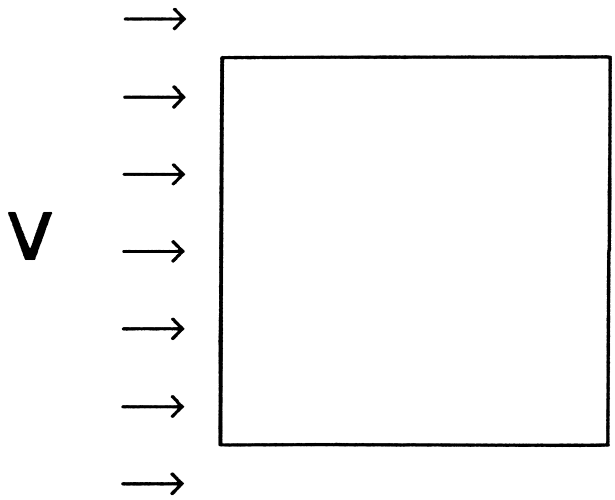


Fig. 7.

where $f(x)$ is the given vorticity of the fluid:

$$f(x_1,x_2) = \begin{cases} -4x_2\Big(\big(3x_1^2-4\big)\big(x_2^2-4\big)^2 + \big(x_1^2-4\big)^2\big(5x_2^2-12\big)\Big) & x \in \Omega_i, \\ 0 & x \in \bar{\Omega}_e. \end{cases} \tag{4.12}$$

we choose $V = 1.0$. Fig. 9 shows the contour plot of the approximate streamfunction $u_h$ on the mesh C.

The examples show that the discrete artificial boundary condition presented in this paper is very effective for Poisson equation in exterior domain and more accurate that Neumann boundary condition which is often used in engineering literatures. Furthermore this approach can be applied to solve some realistic physics problems, such as computation of a membrane, a potential flow, an incompressible flow or a static electromagetic field in an unbounded domain.

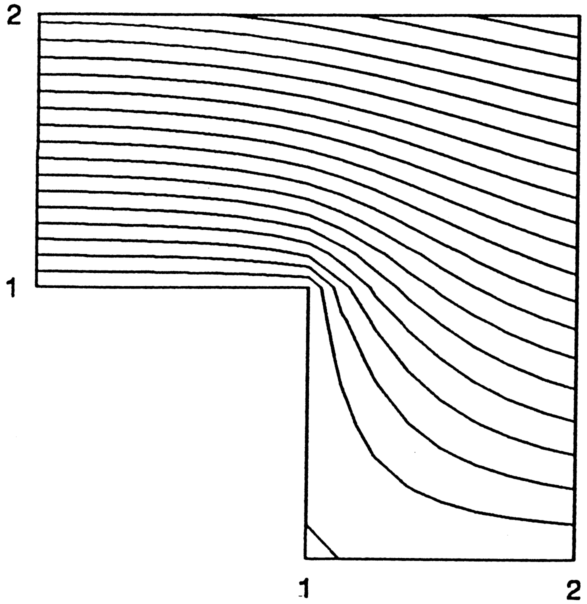


Fig. 8.

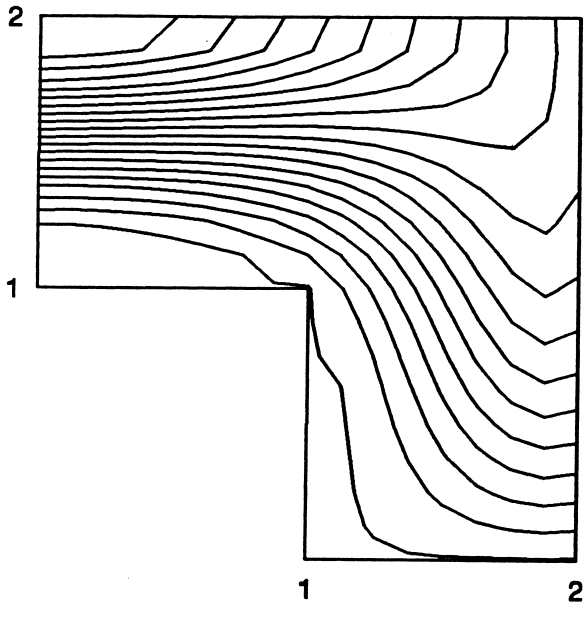


Fig. 9.

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