



The direct method of lines for the problem of infinite elastic foundation

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Abstract

We consider the numerical simulation for the problem of infinite elastic foundation. A polygonal artificial boundary is introduced and a discrete artificial boundary condition on it is presented by using the direct method of lines. Then, the original problem is reduced to a boundary value problem on a bounded computational domain, which is solved by the finite element method. In addition, we prove an optimal a priori error bound for the displacement in the bounded computational domain. Finally, numerical example shows that the discrete artificial boundary condition given in this paper is very effective and more accurate than Neumann boundary condition which is often used in engineering literatures. © 1999 Elsevier Science S.A. All rights reserved.

1. Introduction

Let Ω be an unbounded domain with boundaries Γ_0 and Γ_i (see Fig. 1). Consider the following problem of infinite elastic foundation:

$$-\mu \Delta u - (\lambda + \mu) \operatorname{grad} \operatorname{div} u = f \quad \text{in } \Omega, \quad (1.1)$$

$$u = g \quad \text{on } \Gamma_i, \quad (1.2)$$

$$\sigma_{12} \equiv \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \sigma_{22} \equiv \lambda \frac{\partial u_1}{\partial x_1} + (\lambda + 2\mu) \frac{\partial u_2}{\partial x_2} = 0 \quad \text{on } \Gamma_0, \quad (1.3)$$

$$u \text{ is bounded when } r = \sqrt{x_1^2 + x_2^2} \rightarrow +\infty; \quad (1.4)$$

where $u = (u_1, u_2)^t$ denotes the displacement, λ, μ are Lamé constants, $g = (g_1, g_2)^t$ is given function on Γ_i , $f = (f_1, f_2)^t$ be the applied body force and its support is compact. Let $\sigma = (\sigma_{ij})_{2 \times 2}$ be the stress tensor with entries:

$$\sigma_{ij} = \lambda \delta_{ij} \operatorname{div} u + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad 1 \leq i, j \leq 2, \quad (1.5)$$

where δ_{ij} is Kronecker Delta.

The problem (1.1)–(1.4) is a boundary value problem of Navier equations defined in an unbounded domain. In engineering computation, the stress analysis of a dam in plane with infinite elastic foundation is usually reduced to a similar problem. In numerical simulation of this kind of problem, the unboundedness of the domain Ω is a common difficulty. In practical computations, it is usual to introduce an artificial boundary and design

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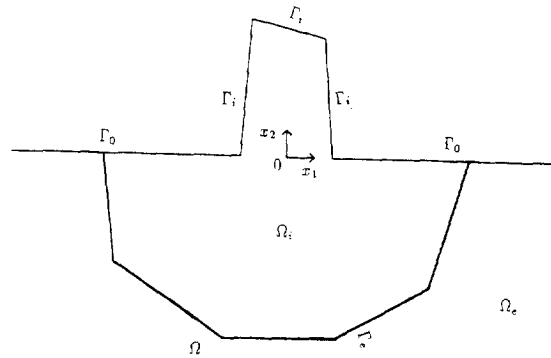


Fig. 1.

appropriate artificial boundary conditions on it. Then, the original problem is reduced to a boundary value problem defined in a bounded computational domain. Thus, one can use the traditional finite element or finite difference method to solve the new problem and derive a numerical solution of the original problem in the bounded computational domain. Therefore, it is a common problem to introduce an appropriate artificial boundary and design artificial boundary condition with high accuracy on it, which attracts many mathematicians and engineers. During recent years, many authors have worked on this subject for various problems by different techniques. For details, refer to the works by Goldstein [7], Feng [5], Han and Wu [15], Han et al. [14], Hagstrom and Keller [8,9], Halpern and Schatzman [16], Nataf [18], Han and Bao [10,11], Givoli [6], Han et al. [13] and references therein. Since the restriction of the methods they used, they mainly consider the regular artificial boundaries, such as circumferences, straight lines or segments of straight lines in two-dimensional problems. But in engineering literature, the polygonal artificial boundaries are often used. Thus, for a given polygonal artificial boundary, how to design an artificial boundary condition with high accuracy is an important problem. In [12], we proposed the direct method of lines to answer this question for an exterior problem of Poisson equation. In this paper, we extend the method to the problem of infinite elastic foundation. For a given polygonal artificial boundary, we set up a discrete artificial boundary condition by the direct method of lines. Then, the original problem is reduced to a boundary value problem on a bounded computational domain, which is solved by finite element method. Moreover, the error bound of the displacement in the bounded computational domain is given, and numerical example shows that the discrete artificial boundary condition given in this paper is very effective and more accurate than the Neumann boundary condition which is often used in engineering literatures.

Another approach in solving the problem defined in unbounded domain is to use infinite element, i.e. use traditional finite element in a bounded domain and infinite element in the outer domain. For instance, see Zienkiewicz et al. [19], Moriya [17], Beer and Meek [2], Bettess [3] and references therein.

The layout of this paper is as follows. In the next section we construct the discrete artificial boundary condition on a given polygonal artificial boundary by using the direct method of lines. In Section 3 we approximate the original problem in the bounded computational domain by finite element method. In Section 4 we prove optimal a priori error bound for the displacement in the bounded computational domain. Finally, in Section 5 we report on a numerical example, which confirms our a priori error bound. Throughout, C denotes a positive generic constant independent of the mesh size h .

2. The discrete artificial boundary condition on a given polygonal artificial boundary

We introduce a polygonal artificial boundary Γ_e in Ω , then the domain Ω is divided into two parts, the bounded part Ω_i and the unbounded part $\Omega_e = \Omega \setminus \Omega_i$ (see Fig. 1). Γ_e is given by

$$r = e(\theta) \quad -\pi \leq \theta \leq 0, \quad (2.6)$$

where (r, θ) is pole coordinate. Suppose that the support of f belongs to Ω_i .

If a suitable boundary condition at Γ_c is given, then we can consider the boundary value problem on the bounded domain Ω_c . The goal of this section is to construct the artificial boundary condition at the given polygonal artificial boundary Γ_c by the direct method of lines. We consider the restriction of u , the solution of problem (1.1)–(1.4), in Ω_c , then we obtain

$$-\mu \Delta u - (\lambda + \mu) \text{grad div } u = 0 \quad \text{in } \Omega_c, \tag{2.7}$$

$$u|_{\Gamma_c} = u(e(\theta), \theta) \equiv u^0(\theta) \quad -\pi \leq \theta \leq 0, \tag{2.8}$$

$$\sigma_{12} = \sigma_{22} = 0 \quad \text{on } \Gamma_0 \cap \bar{\Omega}_c, \tag{2.9}$$

$$u \text{ is bounded when } r \rightarrow +\infty; \tag{2.10}$$

where $u^0(\theta) = (u_1^0(\theta), u_2^0(\theta))^t$. Since the value $u|_{\Gamma_c}$ is unknown, problem (2.7)–(2.10) is an incompletely posed problem, it cannot be solved independently. Let $\dot{H}^\alpha(\Gamma_c)$ denote the usual Sobolev space on Γ_c with real number α [1]. If $u_1|_{\Gamma_c}, u_2|_{\Gamma_c} \in H^{1/2}(\Gamma_c)$ are given, then the problem (2.7)–(2.10) has a unique solution $u = (u_1, u_2)^t$. From (1.5), we obtain the vector components of stress acting on the boundary Γ_c :

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix}_{\Gamma_c} = \begin{pmatrix} n_1 \sigma_{11} + n_2 \sigma_{12} \\ n_1 \sigma_{21} + n_2 \sigma_{22} \end{pmatrix}_{\Gamma_c}, \tag{2.11}$$

where $(n_1(x), n_2(x))^t$ denotes the unit outward normal on the boundary Γ_c of the domain Ω_c . Hence, for given $(u_1|_{\Gamma_c}, u_2|_{\Gamma_c})^t \in [H^{1/2}(\Gamma_c)]^2$, we obtained a bounded operator $K : [H^{1/2}(\Gamma_c)]^2 \rightarrow [H^{-1/2}(\Gamma_c)]^2$, namely

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix}_{\Gamma_c} = K(u|_{\Gamma_c}). \tag{2.12}$$

The boundary condition (2.12) is the exact boundary condition satisfying by the solution of the original problem (1.1)–(1.4). Thus, the restriction of the solution of the problem (1.1)–(1.4) on Ω_c satisfies:

$$-\mu \Delta u - (\lambda + \mu) \text{grad div } u = f \quad \text{in } \Omega_c, \tag{2.13}$$

$$u = g \quad \text{on } \Gamma_c, \tag{2.14}$$

$$\sigma_{12} = \sigma_{22} = 0 \quad \text{on } \Gamma_0 \cap \bar{\Omega}_c, \tag{2.15}$$

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix}_{\Gamma_c} = K(u|_{\Gamma_c}). \tag{2.16}$$

But the bounded operator K is unknown, the problem (2.13)–(2.16) cannot be solved independently as well. We now return to the problem (2.7)–(2.10) under the assumption, $u|_{\Gamma_c}$ is given. We will obtain a discrete approximation of the bounded operator K . We suppose that the polygonal artificial boundary Γ_c has $n + 1$ vertexes $\{a_i = (x_1^i, x_2^i), i = 1, 2, \dots, n + 1\}$ with

$$x_1^i = R_i \cos \theta_i \quad x_2^i = R_i \sin \theta_i \quad 1 \leq i \leq n + 1, \tag{2.17}$$

as shown in Fig. 1. For the ease of exposition, we assume that $\theta_1 = -\pi$ and $\theta_{n+1} = 0$. The rays $\{\theta = \theta_i, 1 \leq i \leq n + 1\}$ divide Ω_c into n parts

$$\Omega_c^i = \{x = (x_1, x_2) : x \in \Omega_c, \theta_i < \theta < \theta_{i+1}\} \quad 1 \leq i \leq n.$$

On each subdomain Ω_c^i ($1 \leq i \leq n$), we introduce the mapping

$$\begin{cases} x_1 = \frac{\rho_i e^\rho \cos \phi}{\sin(\phi - \alpha_i)} \\ x_2 = \frac{\rho_i e^\rho \sin \phi}{\sin(\phi - \alpha_i)} \end{cases} \quad \theta_i \leq \phi \leq \theta_{i+1} \quad 0 \leq \rho < +\infty; \tag{2.18}$$

with

$$\rho_i = \frac{x_1^{i+1}x_2^i - x_1^i x_2^{i+1}}{|a_i a_{i+1}|},$$

$$\sin \alpha_i = \frac{x_2^{i+1} - x_2^i}{|a_i a_{i+1}|} \quad \cos \alpha_i = \frac{x_1^{i+1} - x_1^i}{|a_i a_{i+1}|} \quad 1 \leq i \leq n,$$

$$|a_i a_{i+1}| = \sqrt{(x_1^{i+1} - x_1^i)^2 + (x_2^{i+1} - x_2^i)^2}.$$

The mapping (2.18) maps Ω_e^i onto a semi-infinite strip:

$$\widetilde{\Omega}_e^i = \{(\rho, \phi) : \theta_i < \phi < \theta_{i+1}, 0 < \rho < +\infty\} \quad i = 1, 2, \dots, n.$$

Then, Ω_e is mapped onto $\widetilde{\Omega}_e = \{(\rho, \phi) : -\pi \leq \phi \leq 0, 0 < \rho < +\infty\}$ and Γ_e is mapped onto $\widetilde{\Gamma}_e = \{(0, \phi) : -\pi \leq \phi \leq 0\}$. In addition, on Ω_e^i , we have that

$$\begin{cases} \frac{\partial}{\partial \rho} = \frac{\rho_i e^\rho \cos \phi}{\sin(\phi - \alpha_i)} \frac{\partial}{\partial x_1} + \frac{\rho_i e^\rho \sin \phi}{\sin(\phi - \alpha_i)} \frac{\partial}{\partial x_2}, \\ \frac{\partial}{\partial \phi} = -\frac{\rho_i e^\rho \cos \alpha_i}{\sin^2(\phi - \alpha_i)} \frac{\partial}{\partial x_1} - \frac{\rho_i e^\rho \sin \alpha_i}{\sin^2(\phi - \alpha_i)} \frac{\partial}{\partial x_2}; \end{cases} \quad (2.19)$$

$$\begin{cases} \frac{\partial}{\partial x_1} = -\rho_i^{-1} e^{-\rho} \left[\sin \alpha_i \frac{\partial}{\partial \rho} + \sin \phi \sin(\phi - \alpha_i) \frac{\partial}{\partial \phi} \right], \\ \frac{\partial}{\partial x_2} = \rho_i^{-1} e^{-\rho} \left[\cos \alpha_i \frac{\partial}{\partial \rho} + \cos \phi \sin(\phi - \alpha_i) \frac{\partial}{\partial \phi} \right]; \end{cases} \quad (2.20)$$

$$\begin{aligned} \frac{\partial^2}{\partial x_1^2} = & \rho_i^{-2} e^{-2\rho} \left[\sin^2 \alpha_i \frac{\partial^2}{\partial \rho^2} + \sin 2\phi \sin^2(\phi - \alpha_i) \frac{\partial}{\partial \phi} - \sin^2 \alpha_i \frac{\partial}{\partial \rho} \right. \\ & \left. + 2 \sin \alpha_i \sin \phi \sin(\phi - \alpha_i) \frac{\partial^2}{\partial \rho \partial \phi} + \sin^2 \phi \sin^2(\phi - \alpha_i) \frac{\partial^2}{\partial \phi^2} \right]; \end{aligned} \quad (2.21)$$

$$\begin{aligned} \frac{\partial^2}{\partial x_2^2} = & \rho_i^{-2} e^{-2\rho} \left[\cos^2 \alpha_i \frac{\partial^2}{\partial \rho^2} - \sin 2\phi \sin^2(\phi - \alpha_i) \frac{\partial}{\partial \phi} - \cos^2 \alpha_i \frac{\partial}{\partial \rho} \right. \\ & \left. + 2 \cos \alpha_i \cos \phi \sin(\phi - \alpha_i) \frac{\partial^2}{\partial \rho \partial \phi} + \cos^2 \phi \sin^2(\phi - \alpha_i) \frac{\partial^2}{\partial \phi^2} \right]; \end{aligned} \quad (2.22)$$

$$\begin{aligned} \frac{\partial^2}{\partial x_1 \partial x_2} = & \rho_i^{-2} e^{-2\rho} \left[-\frac{1}{2} \sin 2\alpha_i \frac{\partial^2}{\partial \rho^2} - \cos 2\phi \sin^2(\phi - \alpha_i) \frac{\partial}{\partial \phi} + \frac{1}{2} \sin 2\alpha_i \frac{\partial}{\partial \rho} \right. \\ & \left. - \sin(\phi + \alpha_i) \sin(\phi - \alpha_i) \frac{\partial^2}{\partial \rho \partial \phi} - \frac{1}{2} \sin 2\phi \sin^2(\phi - \alpha_i) \frac{\partial^2}{\partial \phi^2} \right]; \end{aligned} \quad (2.23)$$

$$\begin{aligned} \Delta = & \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \\ = & \rho_i^{-2} e^{-2\rho} \left[\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\partial \rho} + \sin 2(\phi - \alpha_i) \frac{\partial^2}{\partial \rho \partial \phi} + \sin^2(\phi - \alpha_i) \frac{\partial^2}{\partial \phi^2} \right]. \end{aligned} \quad (2.24)$$

Let

$$\begin{cases} X_{\theta_i^+} = (\sin \theta_i \sigma_{11} - \cos \theta_i \sigma_{12})|_{\phi=\theta_i^+} \\ Y_{\theta_i^-} = (\sin \theta_i \sigma_{21} - \cos \theta_i \sigma_{22})|_{\phi=\theta_i^-} \end{cases} \quad 1 \leq i \leq n; \quad (2.25)$$

$$\begin{cases} X_{\theta_i^-} = (\sin \theta_i \sigma_{11} - \cos \theta_i \sigma_{12})|_{\phi=\theta_i^-} \\ Y_{\theta_i^-} = (\sin \theta_i \sigma_{21} - \cos \theta_i \sigma_{22})|_{\phi=\theta_i^-} \end{cases} \quad 2 \leq i \leq n + 1; \tag{2.26}$$

$$\begin{cases} X_n = (\sin \alpha_i \sigma_{11} - \cos \alpha_i \sigma_{12})|_{\rho=0} \\ Y_n = (\sin \alpha_i \sigma_{21} - \cos \alpha_i \sigma_{22})|_{\rho=0} \end{cases} \quad \theta_i \leq \phi \leq \theta_{i+1}. \tag{2.27}$$

Then, from (1.5) and (2.25) we have

$$\begin{aligned} X_{\theta_i^+} &= (\sin \theta_i \sigma_{11} - \cos \theta_i \sigma_{12})|_{\phi=\theta_i^+} \\ &= \sin \theta_i \left[(\lambda + 2\mu) \frac{\partial u_1}{\partial x_1} + \lambda \frac{\partial u_2}{\partial x_2} \right]_{\phi=\theta_i^+} - \mu \cos \theta_i \left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right]_{\phi=\theta_i^+}. \end{aligned} \tag{2.28}$$

By the equalities (2.20), (2.28) and $R_i = \rho_i / (\sin(\theta_i - \alpha_i))$, we get

$$\begin{aligned} X_{\theta_i^+} &= R_i^{-1} e^{-\rho} \left[-\frac{(\lambda + 2\mu) \sin \theta_i \sin \alpha_i + \mu \cos \theta_i \cos \alpha_i}{\sin(\theta_i - \alpha_i)} \frac{\partial u_1}{\partial \rho} - [\mu + (\lambda + \mu) \sin^2 \theta_i] \frac{\partial u_1}{\partial \phi} \right. \\ &\quad \left. + \frac{\mu \cos \theta_i \sin \alpha_i + \lambda \sin \theta_i \cos \alpha_i}{\sin(\theta_i - \alpha_i)} \frac{\partial u_2}{\partial \rho} + \frac{\lambda + \mu}{2} \sin 2\theta_i \frac{\partial u_2}{\partial \phi} \right]_{\phi=\theta_i^+}. \end{aligned} \tag{2.29}$$

Similarly, we obtain

$$\begin{aligned} Y_{\theta_i^+} &= R_i^{-1} e^{-\rho} \left[-\frac{\mu \sin \theta_i \sin \alpha_i + (\lambda + 2\mu) \cos \theta_i \cos \alpha_i}{\sin(\theta_i - \alpha_i)} \frac{\partial u_2}{\partial \rho} - [\mu + (\lambda + \mu) \cos^2 \theta_i] \frac{\partial u_2}{\partial \phi} \right. \\ &\quad \left. + \frac{\lambda \cos \theta_i \sin \alpha_i + \mu \sin \theta_i \cos \alpha_i}{\sin(\theta_i - \alpha_i)} \frac{\partial u_1}{\partial \rho} + \frac{\lambda + \mu}{2} \sin 2\theta_i \frac{\partial u_1}{\partial \phi} \right]_{\phi=\theta_i^+}; \end{aligned} \tag{2.30}$$

$$\begin{aligned} X_{\theta_i^-} &= R_i^{-1} e^{-\rho} \left[-\frac{(\lambda + 2\mu) \sin \theta_i \sin \alpha_{i-1} + \mu \cos \theta_i \cos \alpha_{i-1}}{\sin(\theta_i - \alpha_{i-1})} \frac{\partial u_1}{\partial \rho} - [\mu + (\lambda + \mu) \sin^2 \theta_i] \frac{\partial u_1}{\partial \phi} \right. \\ &\quad \left. + \frac{\mu \cos \theta_i \sin \alpha_{i-1} + \lambda \sin \theta_i \cos \alpha_{i-1}}{\sin(\theta_i - \alpha_{i-1})} \frac{\partial u_2}{\partial \rho} + \frac{\lambda + \mu}{2} \sin 2\theta_i \frac{\partial u_2}{\partial \phi} \right]_{\phi=\theta_i^-}; \end{aligned} \tag{2.31}$$

$$\begin{aligned} Y_{\theta_i^-} &= R_i^{-1} e^{-\rho} \left[-\frac{\mu \sin \theta_i \sin \alpha_{i-1} + (\lambda + 2\mu) \cos \theta_i \cos \alpha_{i-1}}{\sin(\theta_i - \alpha_{i-1})} \frac{\partial u_2}{\partial \rho} - [\mu + (\lambda + \mu) \cos^2 \theta_i] \frac{\partial u_2}{\partial \phi} \right. \\ &\quad \left. + \frac{\lambda \cos \theta_i \sin \alpha_{i-1} + \mu \sin \theta_i \cos \alpha_{i-1}}{\sin(\theta_i - \alpha_{i-1})} \frac{\partial u_1}{\partial \rho} + \frac{\lambda + \mu}{2} \sin 2\theta_i \frac{\partial u_1}{\partial \phi} \right]_{\phi=\theta_i^-}. \end{aligned} \tag{2.32}$$

Furthermore, from (1.5), (2.20) and (2.27) we have that

$$\begin{aligned} X_n &= (\sin \alpha_i \sigma_{11} - \cos \alpha_i \sigma_{12})|_{\rho=0} \\ &= \frac{1}{\rho_i} \left[-[\mu + (\lambda + \mu) \sin^2 \alpha_i] \frac{\partial u_1}{\partial \rho} - \sin(\phi - \alpha_i) [\mu \cos \phi \cos \alpha_i + (\lambda + 2\mu) \sin \phi \sin \alpha_i] \frac{\partial u_1}{\partial \phi} \right. \\ &\quad \left. + \frac{\lambda + \mu}{2} \sin 2\alpha_i \frac{\partial u_2}{\partial \rho} + \sin(\phi - \alpha_i) (\lambda \sin \alpha_i \cos \phi + \mu \cos \alpha_i \sin \phi) \frac{\partial u_2}{\partial \phi} \right]_{\rho=0} \\ &\quad \theta_i \leq \phi \leq \theta_{i+1}; \end{aligned} \tag{2.33}$$

$$\begin{aligned} Y_n &= (\sin \alpha_i \sigma_{21} - \cos \alpha_i \sigma_{22})|_{\rho=0} \\ &= \frac{1}{\rho_i} \left[\frac{\lambda + \mu}{2} \sin 2\alpha_i \frac{\partial u_1}{\partial \rho} + \sin(\phi - \alpha_i) (\mu \sin \alpha_i \cos \phi + \lambda \cos \alpha_i \sin \phi) \frac{\partial u_1}{\partial \phi} \right. \\ &\quad \left. - [\mu + (\lambda + \mu) \cos^2 \alpha_i] \frac{\partial u_2}{\partial \rho} - \sin(\phi - \alpha_i) [\mu \sin \phi \sin \alpha_i + (\lambda + 2\mu) \cos \phi \cos \alpha_i] \frac{\partial u_2}{\partial \phi} \right]_{\rho=0} \\ &\quad \theta_i \leq \phi \leq \theta_{i+1}. \end{aligned} \tag{2.34}$$

In the new coordinate (ρ, ϕ) , the problem (2.7)–(2.10) is reduced to the following discontinuous coefficient problem on the semi-infinite strip Ω_c :

$$\begin{aligned} & \frac{\mu + (\lambda + \mu) \sin^2 \alpha_i}{\sin^2(\phi - \alpha_i)} \frac{\partial^2 u_1}{\partial \rho^2} + \frac{\mu \cos \phi \cos \alpha_i + (\lambda + 2\mu) \sin \phi \sin \alpha_i}{\sin(\phi - \alpha_i)} \frac{\partial^2 u_1}{\partial \rho \partial \phi} - \frac{(\lambda + \mu) \sin 2\alpha_i}{2 \sin^2(\phi - \alpha_i)} \frac{\partial^2 u_2}{\partial \rho^2} \\ & - \frac{\mu \sin \phi \cos \alpha_i + \lambda \cos \phi \sin \alpha_i}{\sin(\phi - \alpha_i)} \frac{\partial^2 u_2}{\partial \rho \partial \phi} + \frac{\partial}{\partial \phi} \left\{ \frac{(\lambda + 2\mu) \sin \phi \sin \alpha_i + \mu \cos \phi \cos \alpha_i}{\sin(\phi - \alpha_i)} \frac{\partial u_1}{\partial \rho} \right. \\ & \left. + [\mu + (\lambda + \mu) \sin^2 \phi] \frac{\partial u_1}{\partial \phi} - \frac{\mu \cos \phi \sin \alpha_i + \lambda \sin \phi \cos \alpha_i}{\sin(\phi - \alpha_i)} \frac{\partial u_2}{\partial \rho} - \frac{\lambda + \mu}{2} \sin 2\phi \frac{\partial u_2}{\partial \phi} \right\} \\ & = 0 \quad \theta_i < \phi < \theta_{i+1} \quad 0 < \rho < +\infty, \end{aligned} \tag{2.35}$$

$$\begin{aligned} & - \frac{(\lambda + \mu) \sin 2\alpha_i}{2 \sin^2(\phi - \alpha_i)} \frac{\partial^2 u_1}{\partial \rho^2} - \frac{\lambda \sin \phi \cos \alpha_i + \mu \cos \phi \sin \alpha_i}{\sin(\phi - \alpha_i)} \frac{\partial^2 u_1}{\partial \rho \partial \phi} + \frac{\mu + (\lambda + \mu) \cos^2 \alpha_i}{\sin^2(\phi - \alpha_i)} \frac{\partial^2 u_2}{\partial \rho^2} \\ & + \frac{\mu \sin \phi \sin \alpha_i + (\lambda + 2\mu) \cos \phi \cos \alpha_i}{\sin(\phi - \alpha_i)} \frac{\partial^2 u_2}{\partial \rho \partial \phi} + \frac{\partial}{\partial \phi} \left\{ - \frac{\lambda \cos \phi \sin \alpha_i + \mu \sin \phi \cos \alpha_i}{\sin(\phi - \alpha_i)} \frac{\partial u_1}{\partial \rho} \right. \\ & \left. - \frac{\lambda + \mu}{2} \sin 2\phi \frac{\partial u_1}{\partial \phi} + \frac{\mu \sin \phi \sin \alpha_i + (\lambda + 2\mu) \cos \phi \cos \alpha_i}{\sin(\phi - \alpha_i)} \frac{\partial u_2}{\partial \rho} + [\mu + (\lambda + \mu) \cos^2 \phi] \frac{\partial u_2}{\partial \phi} \right\} \\ & = 0 \quad \theta_i < \phi < \theta_{i+1} \quad 0 < \rho < +\infty, \end{aligned} \tag{2.36}$$

$$u(\rho, \theta_i^+) = u(\rho, \theta_i^-) \quad 0 \leq \rho < +\infty \quad 1 < i \leq n, \tag{2.37}$$

$$X_{\theta_i^+} = X_{\theta_i^-} \quad Y_{\theta_i^+} = Y_{\theta_i^-} \quad 0 \leq \rho < +\infty \quad 1 < i \leq n, \tag{2.38}$$

$$X_{\theta_1^+} = Y_{\theta_1^+} = X_{\theta_{n-1}^-} = Y_{\theta_{n-1}^-} = 0 \quad 0 \leq \rho < +\infty, \tag{2.39}$$

$$u|_{\rho=0} = u^0(\phi) \quad -\pi \leq \phi \leq 0, \tag{2.40}$$

$$u \text{ is bounded when } \rho \rightarrow +\infty. \tag{2.41}$$

Let $H^1((-\pi, 0))$ denote the usual Sobolev space on the interval $(-\pi, 0)$ [1]. Furthermore, we introduce

$$W_1 = H^1((-\pi, 0)),$$

$$W = W_1 \times W_1,$$

$$V_1 = \left\{ v_1(\rho, \phi) \mid \text{for fixed } \rho \in [0, +\infty), v_1, \frac{\partial v_1}{\partial \rho}, \frac{\partial^2 v_1}{\partial \rho^2} \in W_1 \right\},$$

$$V = V_1 \times V_1.$$

Then, the boundary value problem (2.35)–(2.41) is equivalent to the following variational-differential problem:

Find $u(\rho, \phi) \in V$ such that

$$\frac{d^2}{d\rho^2} A_2(u, v) + \frac{d}{d\rho} A_1(u, v) + A_0(u, v) = 0 \quad \forall v \in W \quad 0 < \rho < +\infty, \tag{2.42}$$

$$u|_{\rho=0} = u^0(\phi) \quad -\pi \leq \phi \leq 0, \tag{2.43}$$

$$u \text{ is bounded when } \rho \rightarrow +\infty; \tag{2.44}$$

where

$$A_2(u, v) = \sum_{i=1}^n \int_{\theta_i}^{\theta_{i+1}} \frac{v(\phi) \mathbb{K}_1(\alpha_i) u(\rho, \phi)}{\sin^2(\phi - \alpha_i)} d\phi, \tag{2.45}$$

$$A_1(u, v) = \sum_{i=1}^n \int_{\theta_i}^{\theta_{i+1}} \frac{1}{\sin(\phi - \alpha_i)} \left[v(\phi)' \mathbb{K}_2 \frac{\partial u(\rho, \phi)}{\partial \phi} - v'(\phi)' \mathbb{K}_2' u(\rho, \phi) \right] d\phi, \tag{2.46}$$

$$A_0(u, v) = - \int_{-\pi}^0 v'(\phi)' \mathbb{K}_1(\phi) \frac{\partial u(\rho, \phi)}{\partial \phi} d\phi; \tag{2.47}$$

with

$$\mathbb{K}_1(\psi) = \begin{pmatrix} \mu + (\lambda + \mu) \sin^2 \psi & -\frac{\lambda + \mu}{2} \sin 2\psi \\ -\frac{\lambda + \mu}{2} \sin 2\psi & \mu + (\lambda + \mu) \cos^2 \psi \end{pmatrix},$$

$$\mathbb{K}_2 = \begin{pmatrix} \mu \cos \phi \cos \alpha_i + (\lambda + 2\mu) \sin \phi \sin \alpha_i & -\mu \sin \phi \cos \alpha_i - \lambda \cos \phi \sin \alpha_i \\ -\mu \cos \phi \sin \alpha_i - \lambda \sin \phi \cos \alpha_i & \mu \sin \phi \sin \alpha_i + (\lambda + 2\mu) \cos \phi \cos \alpha_i \end{pmatrix},$$

We consider the semi-discrete approximation of the problem (2.42)–(2.44). Assume that

$$-\pi = \phi_1 < \phi_2 < \dots < \phi_M = 0$$

is a partition of the interval $[-\pi, 0]$ and for every θ_i , ($i = 1, 2, \dots, n$) there is ϕ_j such that $\phi_j = \theta_i$. Let $n = \max_{1 \leq j \leq M-1} (\phi_{j+1} - \phi_j)$ and

$$W_1^h = \{v_1^h(\phi) \in W_1 : v_1^h(\phi)|_{[\phi_j, \phi_{j+1}]} \in P_1([\phi_j, \phi_{j+1}]), 1 \leq j \leq M-1\},$$

$$W_h = W_1^h \times W_1^h,$$

$$V_1^h = \left\{ v_1^h(\rho, \phi) \in V_1 : \text{for fixed } \rho \in [0, +\infty), v_1^h, \frac{\partial v_1^h}{\partial \rho}, \frac{\partial^2 v_1^h}{\partial \rho^2} \in W_1^h \right\},$$

$$V_h = V_1^h \times V_1^h.$$

Then, we obtain the semi-discrete formulation of problem (2.42)–(2.44):

Find $u_h(\rho, \phi) \in V_h$ such that

$$\frac{d^2}{d\rho^2} A_2(u_h, v_h) + \frac{d}{d\rho} A_1(u_h, v_h) + A_0(u_h, v_h) = 0 \quad \forall v_h \in W_h, \tag{2.48}$$

$$u_h|_{\rho=0} = u_h^0(\phi), \tag{2.49}$$

$$u_h \text{ is bounded when } \rho \rightarrow +\infty; \tag{2.50}$$

where $u_h^0(\phi) \in W_h$ and $u_h^0(\phi_j) = u^0(\phi_j)$ for $j = 1, 2, \dots, M$. Suppose that $\{N_j(\phi), j = 1, 2, \dots, M\}$ is a basis of the finite dimensional space W_1^h such that $N_j(\phi_i) = \delta_{ij}$, $1 \leq i, j \leq M$. Let

$$N(\phi) = \begin{bmatrix} N_1(\phi) & N_2(\phi) & \dots & N_M(\phi) & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & N_1(\phi) & N_2(\phi) & \dots & N_M(\phi) \end{bmatrix}'. \tag{2.51}$$

For $u_h(\rho, \phi) \in V_h$, we have that

$$u_h(\rho, \phi) = \begin{pmatrix} u_1^h(\rho, \phi) \\ u_2^h(\rho, \phi) \end{pmatrix} = N'(\phi)U(\rho), \tag{2.52}$$

where

$$U(\rho) = [u_1^h(\rho, \phi_1), \dots, u_1^h(\rho, \phi_M), u_2^h(\rho, \phi_1), \dots, u_2^h(\rho, \phi_M)]'. \tag{2.53}$$

Thus, the semi-discrete problem (2.48)–(2.50) is equivalent to the following boundary value problem of a system of ordinary differential equations:

$$B_2 U''(\rho) + B_1 U'(\rho) + B_0 U(\rho) = 0 \quad 0 < \rho < +\infty, \tag{2.54}$$

$$U|_{\rho=0} = U_0, \tag{2.55}$$

$$U \text{ is bounded when } \rho \rightarrow +\infty; \tag{2.56}$$

where

$$U_0 = [u_1^0(\phi_1), \dots, u_1^0(\phi_M), u_2^0(\phi_1), \dots, u_2^0(\phi_M)]^t, \tag{2.57}$$

$$B_2 = \sum_{i=1}^n \int_{\theta_i}^{\theta_{i+1}} \frac{N(\phi) \mathbb{K}_1(\alpha_i) N(\phi)^t}{\sin^2(\phi - \alpha_i)} d\phi, \tag{2.58}$$

$$B_1 = \sum_{i=1}^n \int_{\theta_i}^{\theta_{i+1}} \frac{N(\phi) \mathbb{K}_2 N'(\phi)^t - N'(\phi) \mathbb{K}_2^t N(\phi)^t}{\sin(\phi - \alpha_i)} d\phi, \tag{2.59}$$

$$B_0 = - \int_{-\pi}^0 N'(\phi) \mathbb{K}_1(\phi) N'(\phi)^t d\phi. \tag{2.60}$$

For the $2M \times 2M$ matrices B_2 , B_1 and B_0 , we know that B_2 is a positive definite symmetric matrix, B_1 is an antisymmetric matrix and B_0 is a semi-negative definite symmetric matrix. We now solve the boundary value problem (2.54)–(2.56) by a direct method. Let

$$U(\rho) = e^{\gamma\rho} \xi, \tag{2.61}$$

where γ is a constant, $\xi \in \mathbb{C}^{2M}$ to be determined. Substituting (2.61) into the equations in (2.54), we obtain the following generalised eigenvalue problem for determining γ and ξ

$$[\gamma^2 B_2 + \gamma B_1 + B_0] \xi = 0. \tag{2.62}$$

Let $\eta = \gamma\xi$, then the eigenvalue problem (2.62) is reduced to the following standard eigenvalue problem:

$$\begin{pmatrix} 0 & I_{2M} \\ -B_0 & -B_1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \gamma \begin{pmatrix} I_{2M} & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{2.63}$$

where I_{2M} denotes the $2M \times 2M$ unit matrix. After solving the eigenvalue problem (2.63), we get the eigenvalues γ_j ($j = 1, 2, \dots, 2M$) with non-positive real part corresponding to the eigenvectors

$$\begin{pmatrix} \xi_j \\ \xi_j \end{pmatrix} \quad j = 1, 2, \dots, 2M$$

and $\lambda_1 = \lambda_2 = 0$, $\xi_1 = (1, \dots, 1, 0, \dots, 0)^t \in \mathbb{R}^{2M}$, $\xi_2 = (0, \dots, 0, 1, \dots, 1)^t \in \mathbb{R}^{2M}$. In particular, we suppose that γ_j ($1 \leq j \leq 2r$) are real eigenvalue and γ_j ($2r + 1 \leq j \leq 2M$) are complex eigenvalue with nonzero imaginary parts such that $\gamma_{2l} = \bar{\gamma}_{2l-1}$ ($r + 1 \leq l \leq M$). Thus, we have that

$$U(\rho) = \sum_{j=1}^{2r} b_j e^{\rho\gamma_j} \xi_j + \sum_{j=r+1}^M [b_{2j-1} \text{Re}(e^{\rho\gamma_{2j}} \xi_{2j}) + b_{2j} \text{Im}(e^{\rho\gamma_{2j}} \xi_{2j})], \tag{2.64}$$

where $\text{Re}(\gamma)$ and $\text{Im}(\gamma)$ denote the real part and the imaginary part of the complex number γ . Then, we know that $U(\rho)$ satisfies the ordinary equation (2.54) and the boundary condition (2.56). By the condition $U(0) = U_0$, we have that

$$U_0 = \sum_{j=1}^{2r} b_j \xi_j + \sum_{j=r+1}^M [b_{2j-1} \text{Re}(\xi_{2j}) + b_{2j} \text{Im}(\xi_{2j})]. \tag{2.65}$$

Introduce matrices

$$G(\rho) = [e^{\rho\gamma_1} \xi_1, \dots, e^{\rho\gamma_{2r}} \xi_{2r}, \text{Re}(e^{\rho\gamma_{2r+2}} \xi_{2r+2}), \text{Im}(e^{\rho\gamma_{2r+2}} \xi_{2r+2}), \dots, \text{Re}(e^{\rho\gamma_{2M}} \xi_{2M}), \text{Im}(e^{\rho\gamma_{2M}} \xi_{2M})],$$

$$G_0 \equiv G(0) = [\xi_1, \dots, \xi_{2r}, \text{Re}(\xi_{2r+2}), \text{Im}(\xi_{2r+2}), \dots, \text{Re}(\xi_{2M}), \text{Im}(\xi_{2M})],$$

$$B = [b_1, b_2, \dots, b_{2M}]^t.$$

From (2.65), we obtain

$$B = G_0^{-1}U_0. \tag{2.66}$$

Inserting (2.66) into (2.64) we get

$$U(\rho) = G(\rho)G_0^{-1}U_0. \tag{2.67}$$

Finally, we get a semi-discrete approximate solution of problem (2.48)–(2.50):

$$u_h(\rho, \phi) = N(\phi)'G(\rho)G_0^{-1}U_0. \tag{2.68}$$

Substituting (2.68) into (2.33) and (2.34) we have

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix}_{\Gamma_c} = -\frac{1}{\rho_i} \mathbb{K}_1(\alpha_i)N(\phi)'G'(0)G_0^{-1}U_0 - \frac{\sin(\phi - \alpha_i)}{\rho_i} \mathbb{K}_2N'(\phi)'U_0 \quad \theta_i \leq \phi \leq \theta_{i+1}. \tag{2.69}$$

The equality (2.69) is an approximation of the condition (2.12), which is a discrete artificial boundary condition on the artificial boundary Γ_c .

3. The finite element approximation of problem (1.1)–(1.4)

On the bounded computational domain Ω_i , we consider the numerical solution of the problem (1.1)–(1.4). As we have shown, the restriction of u , the solution of the problem (1.1)–(1.4), on the bounded domain Ω_i satisfies the boundary value problem (2.13)–(2.16). Let $H^1(\Omega_i)$ denote the usual Sobolev space on Ω_i [1] and suppose that

$$T_g = \{v = (v_1, v_2)' \in [H^1(\Omega_i)]^2 : v|_{\Gamma_i} = g\},$$

$$T_0 = \{v = (v_1, v_2)' \in [H^1(\Omega_i)]^2 : v|_{\Gamma_i} = 0\}.$$

Then, the boundary value problem (2.13)–(2.16) is equivalent to the following variational problem:

Find $u \in T_g$ such that

$$a(u, v) + b(u, v) = f(v) \quad \forall v \in T_0, \tag{3.1}$$

where

$$a(u, v) = \int_{\Omega_i} \left[\lambda \operatorname{div} u \operatorname{div} v + 2\mu \left(\frac{\partial u_1}{\partial x_1} \frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \frac{\partial v_2}{\partial x_2} \right) + \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) \right] dx, \tag{3.2}$$

$$b(u, v) = - \int_{\Gamma_c} v'K(u) ds, \tag{3.3}$$

$$f(v) = \int_{\Omega_i} v'f dx. \tag{3.4}$$

For the ease of exposition, we assume that Γ_i is a polygonal line in \mathbb{R}^2 . Let \mathcal{T}^h be a regular triangulation of Ω_i such that the nodes on the boundary Γ_c are mapped onto the points $(0, \phi_j)$, $j = 1, 2, \dots, M$ by the mapping (2.18). Furthermore, we introduce the finite element space T^h :

$$T^h = \{v_h = (v_1^h, v_2^h)' : v_j^h \in C^{(0)}(\bar{\Omega}_i) \text{ and } v_j^h|_T \in P_1(T) \quad \forall T \in \mathcal{T}^h \quad j = 1, 2\},$$

$$T_g^h = \{v_h \in T^h : v_h(d_j) = g(d_j) \text{ for the node } d_j \in \Gamma_i\},$$

$$T_0^h = \{v_h \in T^h : v_h|_{\Gamma_i} = 0\}.$$

Hence, we obtain the finite element approximation of the problem (3.1):

Find $u_h \in T_g^h$ such that

$$a(u_h, v_h) + b(u_h, v_h) = f(v_h) \quad \forall v_h \in T_0^h. \tag{3.5}$$

Since the bounded operator K is unknown, we cannot solve the problem (3.5) directly. By the discrete artificial boundary condition (2.69) we introduce an approximation of the bilinear form $b(u_h, v_h)$: For $u_h, v_h \in T^h$, let

$$\begin{aligned} b_h(u_h, v_h) &= - \int_{\Gamma_e} [X_n(u_h)v_1^h + Y_n(u_h)v_2^h] ds \\ &= - \sum_{i=1}^n \int_{\theta_i}^{\theta_{i+1}} \frac{-\rho_i}{\sin^2(\phi - \alpha_i)} [X_n(u_h)v_1^h + Y_n(u_h)v_2^h] d\phi. \end{aligned} \tag{3.6}$$

Since (2.69) and we use

$$u_h|_{\tilde{\Gamma}_e} = N(\phi)^t u_e^h \quad v_h|_{\tilde{\Gamma}_e} = N(\phi)^t v_e^h, \tag{3.7}$$

with

$$u_e^h = [u_1^h(0, \phi_1), \dots, u_1^h(0, \phi_M), u_2^h(0, \phi_1), \dots, u_2^h(0, \phi_M)]^t; \tag{3.8}$$

$$v_e^h = [v_1^h(0, \phi_1), \dots, v_1^h(0, \phi_M), v_2^h(0, \phi_1), \dots, v_2^h(0, \phi_M)]^t. \tag{3.9}$$

From (2.69), (3.6) and (3.7) we have

$$b_h(u_h, v_h) = -(v_e^h)^t \sum_{i=1}^n \int_{\theta_i}^{\theta_{i+1}} \left[\frac{N(\phi)K_1(\alpha_i)N(\phi)^t G'(0)G_0^{-1}}{\sin^2(\phi - \alpha_i)} + \frac{N(\phi)K_2 N'(\phi)^t}{\sin(\phi - \alpha_i)} \right] d\phi u_e^h. \tag{3.10}$$

Using the bilinear form $b_h(u_h, v_h)$ instead of $b(u_h, v_h)$ in the problem (3.5) we obtain

Find $u_h \in T_g^h$ such that

$$a(u_h, v_h) + b_h(u_h, v_h) = f(v_h) \quad \forall v_h \in T_0^h. \tag{3.11}$$

For the bilinear form $b_h(u_h, v_h)$, we have that

LEMMA 3.1. *The bilinear form $b_h(u_h, v_h)$ is bounded and symmetric on $T^h \times T^h$. Furthermore, $b_h(v_h, v_h) \geq 0$ for all $v_h \in T^h$.*

PROOF. From the definition of $b_h(u_h, v_h)$, we know that $b_h(u_h, v_h)$ is a bounded bilinear form on $T^h \times T^h$. For given $u_h, v_h \in T^h$, noting (3.7), we have that

$$u_h|_{\tilde{\Gamma}_e} = N(\phi)^t u_e^h \quad v_h|_{\tilde{\Gamma}_e} = N(\phi)^t v_e^h. \tag{3.12}$$

On the domain $\tilde{\Omega}_e$, let

$$u_h = N(\phi)^t G(\rho)G_0^{-1} u_e^h \quad v_h = N(\phi)^t G(\rho)G_0^{-1} v_e^h. \tag{3.13}$$

Thus, we have the continuous extensions of u_h and v_h on $\tilde{\Omega}_e$ (say Ω_e). Let

$$\begin{aligned} D(u_h, v_h) &= \int_{\Omega_e} \left[\lambda \operatorname{div} u_h \operatorname{div} v_h + 2\mu \left(\frac{\partial u_1^h}{\partial x_1} \frac{\partial v_1^h}{\partial x_1} + \frac{\partial u_2^h}{\partial x_2} \frac{\partial v_2^h}{\partial x_2} \right) \right. \\ &\quad \left. + \mu \left(\frac{\partial u_1^h}{\partial x_2} + \frac{\partial u_2^h}{\partial x_1} \right) \left(\frac{\partial v_1^h}{\partial x_2} + \frac{\partial v_2^h}{\partial x_1} \right) \right] dx. \end{aligned} \tag{3.14}$$

Then, for any $u_h, v_h \in T^h$, we have on recalling (3.14), integrating by parts, noting (2.18), (2.20), (2.48), (2.54) and (3.13) that

$$\begin{aligned}
 D(u_h, v_h) &= b_h(u_h, v_h) + \int_{\Omega_r} \left[A_2 \left(\frac{d^2 u_h}{d\rho^2}, v_h \right) + A_1 \left(\frac{du_h}{d\rho}, v_h \right) + A_0(u_h, v_h) \right] d\rho d\phi \\
 &= b_h(u_h, v_h).
 \end{aligned}
 \tag{3.15}$$

Hence

$$b_h(u_h, v_h) = D(u_h, v_h) = D(v_h, u_h) = b_h(v_h, u_h) \quad \forall u_h, v_h \in T^h,
 \tag{3.16}$$

$$b_h(v_h, v_h) = D(v_h, v_h) \geq 0 \quad \forall v_h \in T^h.
 \tag{3.17}$$

It is straightforward to check that the problem (3.11) is a well posed problem. After solving the problem (3.11) we obtain u_h , the approximate solution of the original problem of (1.1)–(1.4) on the bounded computational domain Ω_i .

4. The error bound for the displacement in Ω_i

For ease of exposition, in this section we assume that $g = 0$. Let u denote the solution of the original problem (1.1)–(1.4) with $g = 0$ and u_h denote the solution of approximate problem (3.11) with $g = 0$. Then, we have that

$$\begin{aligned}
 &\text{Find } u \in T_0 \quad \text{such that} \\
 &a(u, v) + b(u, v) = f(v) \quad \forall v \in T_0
 \end{aligned}
 \tag{4.1}$$

and

$$\begin{aligned}
 &\text{Find } u_h \in T_0^h \quad \text{such that} \\
 &a(u_h, v_h) + b_h(u_h, v_h) = f(v_h) \quad \forall v_h \in T_0^h.
 \end{aligned}
 \tag{4.2}$$

We have the following result:

THEOREM 4.1. *Suppose that $u|_{\Omega_i} \in [H^2(\Omega_i)]^2$, then the following abstract error estimate holds*

$$\|u - u_h\|_{1, \Omega_i} \leq C \left[\|u - \Pi_h u\|_{1, \Omega_i} + \sup_{w_h \in T_0^h} \frac{|b(u, w_h) - b_h(\Pi_h u, w_h)|}{\|w_h\|_{1, \Omega_i}} \right],
 \tag{4.3}$$

where $\Pi_h u$ is the interpolation of u on Ω_i , namely $\Pi_h u \in T_0^h$, $\Pi_h u(d_j) = u(d_j)$, $\{d_j\}$ are the nodes of the triangulation \mathcal{T}^h .

PROOF. Let

$$e := u - u_h, \quad e_a := u - \Pi_h u, \quad e_h := \Pi_h u - u_h.$$

Then, from (4.1), (4.2), korn inequality and Lemma 3.1, we know that there exist a positive constant β_0 such that

$$\begin{aligned}
 \beta_0 \|e_h\|_{1, \Omega_i}^2 &\leq a(e_h, e_h) + b_h(e_h, e_h) \\
 &= -a(e_a, e_h) + a(e, e_h) + b_h(e_h, e_h) \\
 &= -a(e_a, e_h) + b_h(e_h, e_h) - b(u, e_h) + b_h(u_h, e_h) \\
 &= -a(e_a, e_h) - (b(u, e_h) - b_h(\Pi_h u, e_h)) \\
 &\leq \|a\| \|e_a\|_{1, \Omega_i} \|e_h\|_{1, \Omega_i} + |b(u, e_h) - b_h(\Pi_h u, e_h)|.
 \end{aligned}
 \tag{4.4}$$

Thus, from (4.4) we have that

$$\|e_h\|_{1, \Omega_i} \leq \frac{1}{\beta_0} \left[\|a\| \|e_a\|_{1, \Omega_i} + \sup_{w_h \in T_0^h} \frac{|b(u, w_h) - b_h(\Pi_h u, w_h)|}{\|w_h\|_{1, \Omega_i}} \right].
 \tag{4.5}$$

By the triangle inequality, noting (4.5), we obtain

$$\begin{aligned} \|e\|_{1,\Omega_i} &\leq \|e_h\|_{1,\Omega_i} + \|e_a\|_{1,\Omega_i} \\ &\leq C \left[\|e_a\|_{1,\Omega_i} + \sup_{w_h \in T_0^h} \frac{|b(u, w_h) - b_h(I_h u, w_h)|}{\|w_h\|_{1,\Omega_i}} \right], \end{aligned} \tag{4.6}$$

with $C = \max\{1 + \|a\|/\beta_0, 1/\beta_0\}$. The proof is completed. \square

For the first term in (4.3), we have [4]

$$\|e_a\|_{1,\Omega} = \|u - I_h u\|_{1,\Omega} \leq Ch|u|_{2,\Omega}. \tag{4.7}$$

Thus, we only need to estimate the second term in (4.3). On the domain $\widetilde{\Omega}_e$, for $w_h \in T_0^h$, let

$$\widetilde{w}_e^h = N(\phi)^t G(\rho) G_0^{-1} w_e^h \quad \widetilde{I}_h u = N(\phi)^t G(\rho) G_0^{-1} (I_h u)_e^h, \tag{4.8}$$

where

$$\begin{aligned} w_e^h &= (w_1^h(0, \phi_1), \dots, w_1^h(0, \phi_M), w_2^h(0, \phi_1), \dots, w_2^h(0, \phi_M))^t, \\ (I_h u)_e^h &= (I_h u_1(0, \phi_1), \dots, I_h u_1(0, \phi_M), I_h u_2(0, \phi_1), \dots, I_h u_2(0, \phi_M))^t \\ &\equiv (u_1(0, \phi_1), \dots, u_1(0, \phi_M), u_2(0, \phi_1), \dots, u_2(0, \phi_M))^t. \end{aligned}$$

Then, from we have that

$$\begin{aligned} b(u, w_h) &= b(u, w_h - \widetilde{w}_h) + b(u, \widetilde{w}_h) \\ &= b(u, w_h - \widetilde{w}_h) + D(u, \widetilde{w}_h), \end{aligned} \tag{4.9}$$

$$b_h(I_h u, w_h) = D(\widetilde{I}_h u, \widetilde{w}_h). \tag{4.10}$$

$$|b(u, w_h) - b_h(I_h u, w_h)| \leq |b(u, w_h - \widetilde{w}_h)| + |D(u - \widetilde{I}_h u, \widetilde{w}_h)|. \tag{4.11}$$

$$\begin{aligned} |b(u, w_h - \widetilde{w}_h)| &= \left| \int_{\Gamma} (w_h - \widetilde{w}_h)^t K(u) \, ds \right| \\ &\leq \|w_h - \widetilde{w}_h\|_{1/2,\Gamma} \|K(u)\|_{1/2,\Gamma} \\ &\leq Ch \|w_h\|_{1/2,\Gamma} \|K(u)\|_{1/2,\Gamma} \\ &\leq Ch \|w_h\|_{1,\Omega_i} \|u\|_{2,\Omega_i}. \end{aligned} \tag{4.12}$$

$$|D(u - \widetilde{I}_h u, \widetilde{w}_h)| \leq |u - \widetilde{I}_h u|_{*,\Omega_e} |\widetilde{w}_h|_{*,\Omega_e}, \tag{4.13}$$

where $|\widetilde{w}_h|_{*,\Omega_e}^2 = D(\widetilde{w}_h, \widetilde{w}_h)$ and

$$|\widetilde{w}_h|_{*,\Omega_e} \leq C |\widetilde{w}_h|_{1,\Omega_e} \leq C \|w_h\|_{1,\Omega_i}. \tag{4.14}$$

Let $I_h u$ denote the interpolating of u on the domain Ω_e , namely $I_h u \in V_h$ and $I_h u(\rho, \phi_j) = u(\rho, \phi_j)$, $j = 1, \dots, M$. For the interpolating error we have that

$$|u - I_h u|_{*,\Omega_e} \leq C(u)h, \tag{4.15}$$

with

$$C(u) = C_0 \sqrt{\sum_{i=1}^n \int_0^{+\infty} \int_{\theta_i}^{\theta_{i+1}} \left[\left(\frac{\partial^2 u_1}{\partial \rho \partial \phi} \right)^2 + \left(\frac{\partial^2 u_1}{\partial \phi^2} \right)^2 + \left(\frac{\partial^2 u_2}{\partial \rho \partial \phi} \right)^2 + \left(\frac{\partial^2 u_2}{\partial \phi^2} \right)^2 \right] d\rho \, d\phi}. \tag{4.16}$$

On the other hand, we have

$$D(u - \widetilde{I}_h u, u - \widetilde{I}_h u) = D(u - \widetilde{I}_h u, u - I_h u), \tag{4.17}$$

$$|u - \widetilde{I}_h u|_{*, \Omega_\epsilon} \leq |u - I_h u|_{*, \Omega_\epsilon} \leq C(u)h. \tag{4.18}$$

Therefore, from (4.11), (4.12), (4.13), (4.14), (4.18) we get

$$|b(u, w_h) - b_h(I_h u, w_h)| \leq Ch \|w_h\|_{1, \Omega_i} (\|u\|_{2, \Omega_i} + C(u)). \tag{4.19}$$

Finally, combining (4.3), (4.7) and (4.19) we obtain the following error bound:

$$\|u - u_h\|_{1, \Omega_i} \leq Ch[\|u\|_{2, \Omega_i} + C(u)]. \tag{4.20}$$

5. Numerical example

Let $\omega = \lambda / (\lambda + 2\mu)$ and

$$h_1(x, t) = (1 - \omega) \operatorname{arctg} \frac{x_1 - t}{x_2} - (1 + \omega) \frac{(x_1 - t)x_2}{(x_1 - t)^2 + x_2^2}, \tag{5.1}$$

$$h_2(x, t) = \ln[(x_1 - t)^2 + x_2^2] + (1 + \omega) \frac{(x_1 - t)^2}{(x_1 - t)^2 + x_2^2}, \tag{5.2}$$

$$h(x, t) = (h_1(x, t), h_2(x, t))'; \tag{5.3}$$

$$u^0(x) = (u_1^0(x), u_2^0(x))' = h(x, 0) - \frac{1}{2} [h(x, 0.5) + h(x, -0.5)]. \tag{5.4}$$

It is straightforward to check that $u^0(x)$ is the unique solution of the following boundary value problem:

$$-\mu \Delta u - (\lambda + \mu) \operatorname{grad} \operatorname{div} u = 0 \quad \text{in } \Omega, \tag{5.5}$$

$$u|_{x_1 = \pm 1} = u^0(\pm 1, x_2) \quad -1 \leq x_2 \leq 0, \tag{5.6}$$

$$u|_{x_2 = -1} = u^0(x_1, -1) \quad -1 \leq x_1 \leq 1, \tag{5.7}$$

$$\lambda \frac{\partial u_1}{\partial x_1} + (\lambda + 2\mu) \frac{\partial u_2}{\partial x_2} = \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = 0 \quad x_2 = 0 \quad |x_1| \geq 1, \tag{5.8}$$

$$u \text{ is bounded when } r \rightarrow +\infty; \tag{5.9}$$

where Ω is the domain of the lower half plane $\mathbb{R}_-^2 = \{x = (x_1, x_2) : x_2 < 0\}$ subtracted a rectangle $\bar{\Omega}_0 = \{x = (x_1, x_2) : -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 0\}$, i.e. $\Omega = \mathbb{R}_-^2 \setminus \bar{\Omega}_0$. We take $\Gamma_\epsilon = \{x : x_1 = \pm 2, -2 \leq x_2 \leq 0 \text{ and } x_2 = -2, -2 \leq x_1 \leq 2\}$ as artificial boundary. Then, the domain Ω is divided into a bounded part Ω_i and an unbounded part Ω_e with

$$\Omega_i = \{x : 1 < |x_1| < 2, -2 < x_2 < 0 \text{ and } -2 < x_2 < -1, -1 \leq x_1 \leq 1\},$$

$$\Omega_e = \Omega \setminus \bar{\Omega}_i.$$

Since the first component $u_1^0(x)$ and the second component $u_2^0(x)$ of $u^0(x)$ are antisymmetric and symmetric about x_2 axes, respectively. The domain of computation is taken to be the part of Ω_i lying in the fourth quadrant (say Ω_i^+). The following boundary condition is posed along $x_1 = 0$:

$$u_1(0, x_2) = \frac{\partial u_2(0, x_2)}{\partial x_1} = 0 \quad -2 \leq x_2 \leq -1. \tag{5.10}$$

The boundary condition (5.10) is equivalent to the following condition:

$$u_1(0, x_2) = \sigma_{12}(0, x_2) = 0 \quad -2 \leq x_2 \leq -1. \tag{5.11}$$

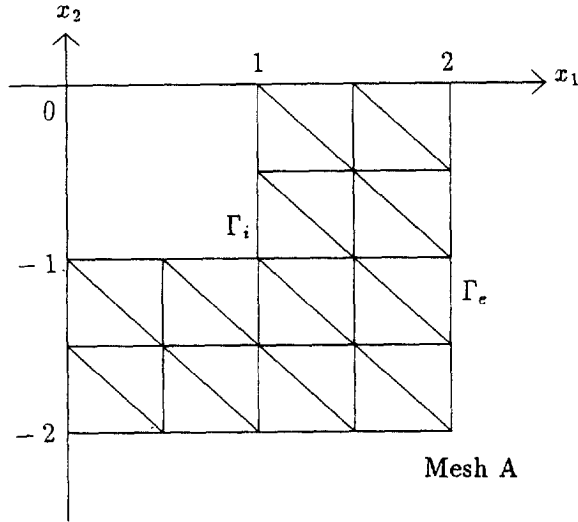


Fig. 2.

Table 1
Maximum error of $u - u_h$ over mesh points

Mesh	A	B	C
$\max u_1 - u_1^h $	3.1726E-2	1.1186E-2	3.6913E-3
$\max u_2 - u_2^h $	2.8594E-2	8.1036E-3	2.3139E-3

Three meshes are used in the computation. Fig. 2 shows the triangulation for mesh A. On each triangle in mesh A, we connect the midpoints of every two sides, thus this triangle is divided into four small triangles. Then, we obtained the refined mesh B. Mesh C is similarly generated from mesh B. Linear finite element is used in our computation. We take $\lambda = 1.0$ and $\mu = 2.0$. Let $u_h = (u_1^h, u_2^h)^t$ denote the finite element approximation in the domain Ω_i^+ by using the discrete artificial boundary condition (2.69). For comparison we also compute the finite element approximation $u_h^N = (u_1^{h,N}, u_2^{h,N})^t$ of problem (5.5)–(5.9) in the domain Ω_i^+ by using the following Neumann artificial boundary condition on Γ_e :

$$\sigma_n|_{\Gamma_e} = 0. \tag{5.12}$$

The Neumann artificial boundary condition (5.12) is often used in engineering literatures for simulating the problem of infinite elastic foundation.

Table 1 shows the maximum of the errors $u - u_h$ over the mesh points for meshes A, B and C. Furthermore, Table 2 gives the errors $\|u - u_h\|_{0,2,\Omega_i}$, $|u - u_h|_{1,2,\Omega_i}$ and $\|u - u_h\|_{1,2,\Omega_i}$ for meshes A, B and C. For comparison, Table 3 shows the maximum of the errors $u - u_h^N$ over the mesh points for meshes A, B and C and Table 4 gives the errors $\|u - u_h^N\|_{0,2,\Omega_i}$, $|u - u_h^N|_{1,2,\Omega_i}$ and $\|u - u_h^N\|_{1,2,\Omega_i}$ for meshes A, B and C.

Furthermore, Figs. 3 and 4 show the values of numerical solution u_1^h and u_2^h on the mesh points of the artificial

Table 2
Errors of $u - u_h$

Mesh	A	B	C
$\ u_1 - u_1^h\ _{0,2,\Omega_i}$	5.4189E-2	1.3505E-2	3.3784E-3
$ u_1 - u_1^h _{1,2,\Omega_i}$	2.8736E-1	1.5918E-1	8.2317E-2
$\ u_1 - u_1^h\ _{1,2,\Omega_i}$	2.9242E-1	1.5975E-1	8.2387E-2
$\ u_2 - u_2^h\ _{0,2,\Omega_i}$	4.4933E-2	1.2433E-2	3.2787E-3
$ u_2 - u_2^h _{1,2,\Omega_i}$	4.4296E-1	2.3448E-1	1.1906E-1
$\ u_2 - u_2^h\ _{1,2,\Omega_i}$	4.4524E-1	2.3481E-1	1.1910E-1

Table 3
Maximum error of $u - u_h^N$ over mesh points

Mesh	A	B	C
$\max u_1 - u_1^{h,N} $	1.8181E-1	2.0972E-1	2.1950E-1
$\max u_2 - u_2^{h,N} $	1.9999E-1	2.4008E-1	2.5482E-1

Table 4
Errors of $u - u_h^N$

Mesh	A	B	C
$\ u_1 - u_1^{h,N}\ _{0,2,\Omega_1}$	2.2429E-1	2.4891E-1	2.6189E-1
$\ u_1 - u_1^{h,N}\ _{1,2,\Omega_1}$	5.3353E-1	5.5275E-1	5.7002E-1
$\ u_1 - u_1^{h,N}\ _{1,2,\Omega_2}$	5.7876E-1	6.0621E-1	6.2730E-1
$\ u_2 - u_2^{h,N}\ _{0,2,\Omega_1}$	2.9147E-1	3.3671E-1	3.6038E-1
$\ u_2 - u_2^{h,N}\ _{1,2,\Omega_1}$	6.5077E-1	6.1770E-1	6.2359E-1
$\ u_2 - u_2^{h,N}\ _{1,2,\Omega_2}$	7.1306E-1	7.0351E-1	7.2023E-1

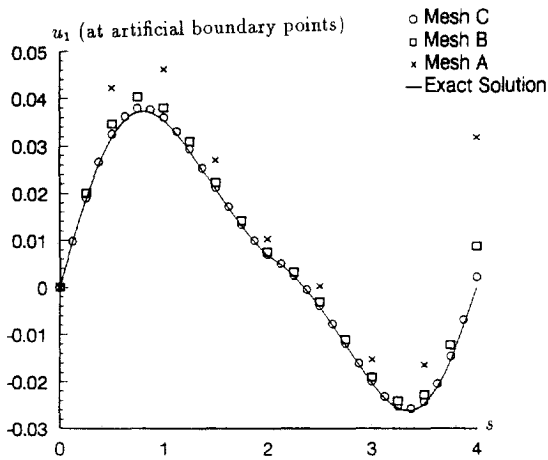


Fig. 3.

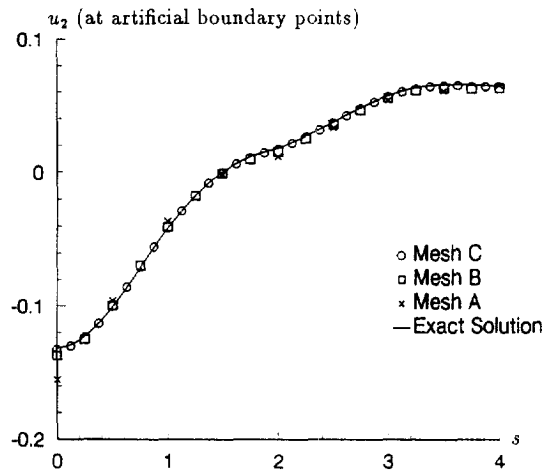


Fig. 4.

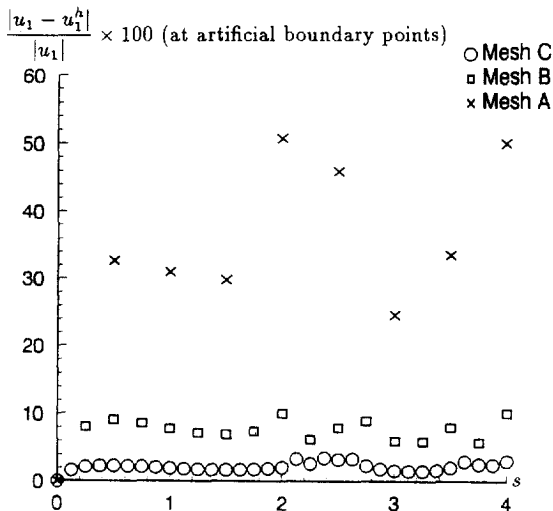


Fig. 5.

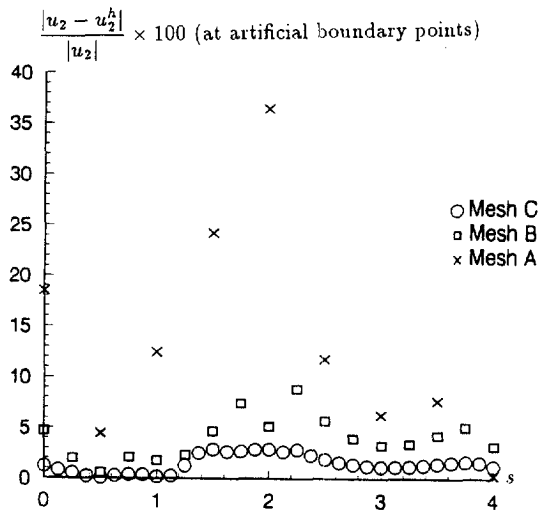


Fig. 6.

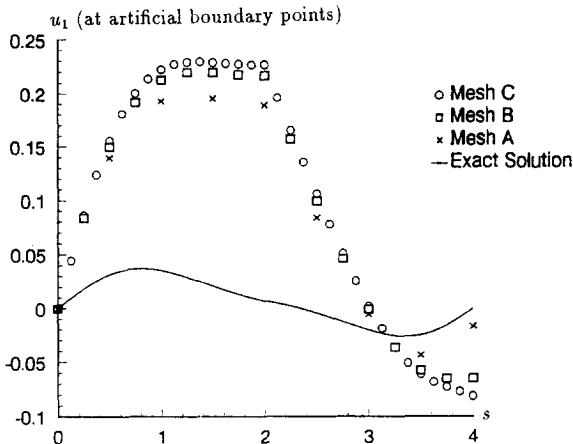


Fig. 7.

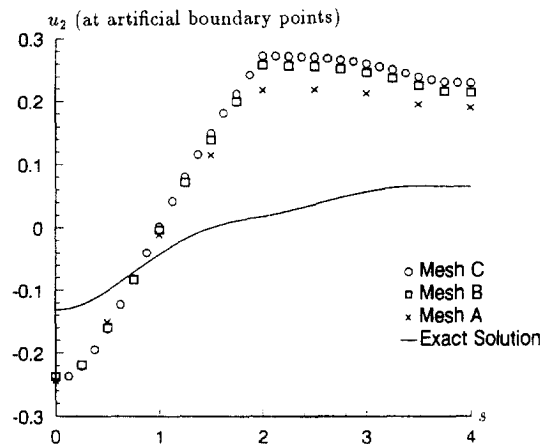


Fig. 8.

boundary Γ_c . Figs. 5 and 6 show the related errors $(|u_1 - u_1^h|)/|u_1| \times 100$ and $(|u_2 - u_2^h|)/|u_2| \times 100$ on the artificial boundary Γ_c . Figs. 7 and 8 show $u_1^{h,N}$ and $u_2^{h,N}$ on the artificial boundary Γ_c .

From Tables 1–4 and Figs. 3–8, we can see that our discrete artificial boundary condition (2.69) is very effective for the problem of infinite elastic foundation and more accurate than the Neumann boundary condition (5.12) which is often used in engineering literatures. We can derive a good numerical approximation u_h of the solution u of the original problem in a small domain, say Ω_l , by using our discrete artificial boundary condition at the artificial boundary.

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