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The artificial boundary conditions for incompressible materials on an unbounded domain*,**

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Summary. In this paper we consider the numerical simulations of the incompressible materials on an unbounded domain in $\Re^2$. A series of artificial boundary conditions at a circular artificial boundary for solving incompressible materials on an unbounded domain is given. Then the original problem is reduced to a problem on a bounded domain, which be solved numerically by a mixed finite element method. The numerical example shows that our artificial boundary conditions are very effective.

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1. Introduction

We now turn our attention to the incompressible materials on an unbounded domain in $\Re^2$. For infinite small deformations, the incompressibility is manifested in the requirement that the displacement field be solenoidal, i.e. $\operatorname{div} u = 0$, and the problems are reduced to the boundary value problems of Stokes equations on an unbounded domain in $\Re^2$ [1, p. 29]. Finite element approximations of incompressible materials on the bounded domain were studied using various penalty and mixed methods by Zienkiewicz [2, p. 284], Fried [3], Kikuchi and Oden [4, Chapter 7], Brezzi and Fortin [5, Chapter VI] and many others. The numerical approximation of the boundary value problem of partial differential equations on unbounded domains have been studied during recent ten years. One of the effective approaches is to introduce artificial boundaries and set up artificial boundary conditions on them. Then the original problem is reduced to a

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problem on a bounded computational domain. There are many authors, who have worked on this subject for various problems by different techniques. For more information, refer to the works by Goldstein [6], Feng [7], Han and Wu [8,9], Han, Lu and Bao [10], Hagstrom and Keller [11, 12], Halpern [13], Nataf [14] and the references quoted by them. The purpose of this paper is to show that the technique developed in [8, 9] applies to the linear incompressible materials on an unbounded domain in $\Re^2$.

Let Γ be a bounded simple closed curve in $\Re^2$, and let Ω be the unbounded domain with boundary Γ. We shall try to determine the displacement $u = (u_1, u_2)^{\mathrm{T}}$ and the hydrostatic pressure p of an incompressible elastic material on the unbounded domain Ω under the action of some external forces. Suppose that the displacement to be small and the material to be isotropic and homogenous. To set our problem, we introduce some notation from continuum mechanic. Let $\varepsilon(u) = (\varepsilon_{ij}(u))$ be the strain tensor corresponding to the displacement u, which is given by:

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad 1 \le i, j \le 2. \tag{1.1}$$

Let $\sigma(u,p) = (\sigma_{ij}(u,p))$ be the stress tensor corresponding to the displacement u and the hydrostatic pressure p, $\sigma(u,p)$ is given by [1, p. 69, 79]:

$$\sigma_{ij}(u,p) = 2\mu\varepsilon_{ij}(u) - \delta_{ij}\ p, \quad 1 \le i, j \le 2, \tag{1.2}$$

where $\mu > 0$ is the Lame constant, δ_{ij} is the Kronecker delta. Consider the following boundary value problem on unbounded domain Ω

$$-2\mu \text{ div } \varepsilon(u) + \text{grad } p = f, \quad \text{in } \Omega, \tag{1.3}$$

$$\text{div } u = 0, \quad \text{in } \Omega, \tag{1.4}$$

$$u|_\Gamma = 0, \tag{1.5}$$

$$u \text{ is bounded}, \quad p \to 0, \quad \text{when } |x| \to +\infty. \tag{1.6}$$

Here $f = (f_1, f_2)^{\mathrm{T}}$ is the body force, the support of which is compact. div $\varepsilon(u)$ is given by:

$$\text{div } \varepsilon(u) = \begin{pmatrix} \frac{\partial}{\partial x_1}\varepsilon_{11}(u) + \frac{\partial}{\partial x_2}\varepsilon_{12}(u) \\ \\ \frac{\partial}{\partial x_1}\varepsilon_{21}(u) + \frac{\partial}{\partial x_2}\varepsilon_{22}(u) \end{pmatrix}. \tag{1.7}$$

Since the problem (1.3)-(1.6) is defined on the unbounded domain Ω, in finding the numerical solutions of this problem, it is often difficult to use the classical finite element method or finite difference method. We now introduce an artificial boundary by drawing a circumference Γ_R with radius R in the domain Ω. Then Ω is divided into two parts; the bounded part Ω_i and the unbounded part Ω_R (See Fig. 1). Furthermore assume that the support of the body force f is in Ω_i. If a suitable boundary condition on the artificial boundary Γ_R is given, then we could solve the problem (1.3)-(1.6) on the bounded domain Ω_i. In the following section, we shall derive the exact and approximate boundary conditions on Γ_R for the solution of problem (1.3)-(1.6).

2. The exact and approximate boundary conditions at the artificial boundary Γ_R

We now concern the restriction of the solution of problem (1.3)-(1.6) on the regular unbounded domain $\Omega_R = \left\{ x = (x_1, x_2) \mid r = \sqrt{x_1^2 + x_2^2} > R \right\}$, which satisfies:

$$-2\mu \operatorname{div} \varepsilon(u) + \operatorname{grad} p = 0, \quad \text{in } \Omega_R, \tag{2.1}$$

$$\operatorname{div} u = 0, \quad \text{in } \Omega_R, \tag{2.2}$$

$$u|_{\Gamma_R} = u(R, \theta), \tag{2.3}$$

$$u \text{ is bounded}, \quad p \to 0, \quad \text{when} \quad r \to +\infty, \tag{2.4}$$

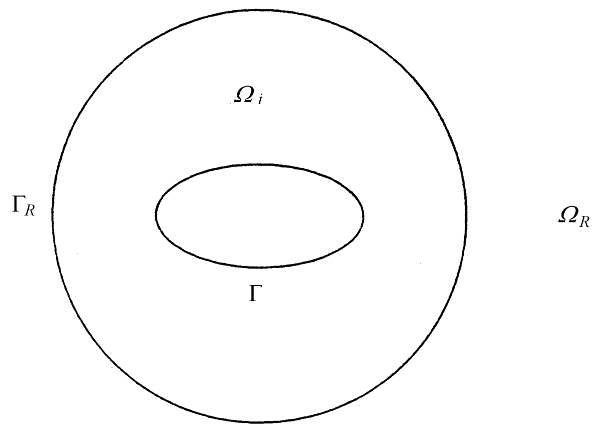

Fig. 1.

where (r, θ) denote the polar coordinates in the plane. Unfortunately, the value $u(R,\theta) = (u_1(R,\theta), u_2(R,\theta))^{\mathrm{T}}$ is unknown, the problem (2.1)-(2.4) cannot be solved independently. If $u(R,\theta)$ is given, then we know that the problem (2.1)-(2.4) has a unique solution (u, p), which can be found analytically. For our application, suppose that the displacement $u = (u_1, u_2)^{\mathrm{T}}$ is given in the following form:

$$u_j = G_j + (r^2 - R^2)\frac{\partial W}{\partial x_j}, \qquad j = 1, 2, \tag{2.5}$$

where G_1, G_2 and W are three harmonic functions to be determined. From the boundary conditions (2.3) and (2.4), one is led to the Dirichlet problems of Laplace equation in order to determine G_1 and G_2. We obtain:

$$\begin{aligned} \triangle G_j &= 0, \quad \text{in } \Omega_R, \\ G_j|_{\Gamma_R} &= u_j(R,\theta), \\ G_j \text{ is bounded}, & \quad \text{when} \quad r \to +\infty, \end{aligned}$$

where $j = 1, 2$. Furthermore we have:

$$G_1 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)r^{-n}, \tag{2.6}$$

$$G_2 = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n \cos n\theta + d_n \sin n\theta)r^{-n}, \tag{2.7}$$

with

$$\begin{aligned} a_n &= \frac{R^n}{\pi}\int_0^{2\pi} u_1(R,\theta)\cos n\theta d\theta, \\ c_n &= \frac{R^n}{\pi}\int_0^{2\pi} u_2(R,\theta)\cos n\theta d\theta, \quad n = 0,1,2,\cdots, \end{aligned} \tag{2.8}$$

$$\begin{aligned} b_n &= \frac{R^n}{\pi}\int_0^{2\pi} u_1(R,\theta)\sin n\theta d\theta, \\ d_n &= \frac{R^n}{\pi}\int_0^{2\pi} u_2(R,\theta)\sin n\theta d\theta, \quad n = 1,2,\cdots \end{aligned} \tag{2.9}$$

Furthermore in order to determine harmonic function W, substitute (2.5) into (2.2) and obtain:

$$2x_1\frac{\partial W}{\partial x_1} + 2x_2\frac{\partial W}{\partial x_2} = -\left(\frac{\partial G_1}{\partial x_1} + \frac{\partial G_2}{\partial x_2}\right),$$

namely

$$r\frac{\partial W}{\partial r} = -\frac{1}{2}\left(\frac{\partial G_1}{\partial x_1} + \frac{\partial G_2}{\partial x_2}\right). \tag{2.10}$$

A computation shows

$$\begin{aligned} -\frac{1}{2}\left(\frac{\partial G_1}{\partial x_1} + \frac{\partial G_2}{\partial x_2}\right) &= \sum_{n=1}^{\infty}\frac{n}{2}\,[(a_n - d_n)\cos(n+1)\theta \\ &\quad +(b_n + c_n)\sin(n+1)\theta]\,r^{-n-1}. \end{aligned}$$

Hence from (2.10) we have

$$\begin{aligned} W &= -\sum_{n=1}^{\infty}\frac{n}{2(n+1)}\,[(a_n - d_n)\cos(n+1)\theta \\ &\quad +(b_n + c_n)\sin(n+1)\theta]\,r^{-n-1}. \end{aligned} \tag{2.11}$$

Inserting (2.5) into (2.1), we get

$$\frac{\partial p}{\partial x_1} = \frac{\partial}{\partial x_1}\left[4\mu(x_1\frac{\partial W}{\partial x_1} + x_2\frac{\partial W}{\partial x_2})\right],$$

$$\frac{\partial p}{\partial x_2} = \frac{\partial}{\partial x_2}\left[4\mu(x_1\frac{\partial W}{\partial x_1} + x_2\frac{\partial W}{\partial x_2})\right].$$

Combining the boundary condition, $p \to 0$, when $r \to +\infty$, we get

$$\begin{aligned} p &= 4\mu\left(x_1\frac{\partial W}{\partial x_1} + x_2\frac{\partial W}{\partial x_2}\right) \\ &= 4\mu r\frac{\partial W}{\partial r} \\ &= 2\mu\sum_{n=1}^{\infty} n\,[(a_n - d_n)\cos(n+1)\theta + (b_n + c_n)\sin(n+1)\theta]\,r^{-n-1}. \qquad (2.12) \end{aligned}$$

Hence (u, p) given by (2.5) and (2.12) is the unique solution of the problem (2.1)- (2.4) for a given $u(R,\theta)$. We now discuss the stress on the boundary Γ_R. The vector components of stress acting on the boundary Γ_R is given by:

$$\sigma_{n_1} = (\sigma_{11}\cos\theta + \sigma_{12}\sin\theta)|_{\Gamma_R},$$

$$\sigma_{n_2} = (\sigma_{21}\cos\theta + \sigma_{22}\sin\theta)|_{\Gamma_R}.$$

Since the relationship of stress and strain (1.2), we get

$$(2.13) \qquad \sigma_{n_1} = \left[\left(2\mu\frac{\partial u_1}{\partial x_1} - p\right)\cos\theta + \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\sin\theta\right]_{\Gamma_R},$$

$$(2.14) \qquad \sigma_{n_2} = \left[\mu\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)\cos\theta + \left(2\mu\frac{\partial u_2}{\partial x_2} - p\right)\sin\theta\right]_{\Gamma_R}.$$

Substituting (2.5) and (2.12) into (2.13) and (2.14), the equations (2.13) and (2.14) are reduced to

$$(2.15) \qquad \sigma_{n_1} = \left[\mu\frac{\partial G_1}{\partial r} - \frac{\mu}{r}\frac{\partial G_2}{\partial\theta} - 2\mu\left(\frac{x_2}{r}\frac{\partial W}{\partial\theta} + x_1\frac{\partial W}{\partial r}\right)\right]_{\Gamma_R},$$

$$(2.16) \qquad \sigma_{n_2} = \left[\mu\frac{\partial G_2}{\partial r} + \frac{\mu}{r}\frac{\partial G_1}{\partial\theta} - 2\mu\left(-\frac{x_1}{r}\frac{\partial W}{\partial\theta} + x_2\frac{\partial W}{\partial r}\right)\right]_{\Gamma_R}.$$

A computation shows that

$$(2.17) \qquad -2\mu\left(\frac{x_2}{r}\frac{\partial W}{\partial\theta} + x_1\frac{\partial W}{\partial r}\right)\Big|_{\Gamma_R} = \mu\left[\frac{\partial G_1}{\partial r} + \frac{1}{r}\frac{\partial G_2}{\partial\theta}\right]_{\Gamma_R},$$

$$(2.18) \qquad -2\mu\left(-\frac{x_1}{r}\frac{\partial W}{\partial\theta} + x_2\frac{\partial W}{\partial r}\right)\Big|_{\Gamma_R} = \mu\left[\frac{\partial G_2}{\partial r} - \frac{1}{r}\frac{\partial G_1}{\partial\theta}\right]_{\Gamma_R}.$$

Inserting (2.17) and (2.18) into (2.15) and (2.16), and combining (2.5)-(2.10), we obtain:

$$\sigma_{n_1} = 2\mu \frac{\partial G_1}{\partial r}\Big|_{\Gamma_R} = \frac{2\mu}{\pi R}\sum_{n=1}^{\infty}\frac{\partial}{\partial\theta}\int_0^{2\pi}\frac{\cos n(\varphi-\theta)}{n}\frac{\partial u_1(R,\varphi)}{\partial\varphi}d\varphi \equiv T_1(u), \tag{2.19}$$

$$\sigma_{n_2} = 2\mu \frac{\partial G_2}{\partial r}\Big|_{\Gamma_R} = \frac{2\mu}{\pi R}\sum_{n=1}^{\infty}\frac{\partial}{\partial\theta}\int_0^{2\pi}\frac{\cos n(\varphi-\theta)}{n}\frac{\partial u_2(R,\varphi)}{\partial\varphi}d\varphi \equiv T_2(u). \tag{2.20}$$

The boundary condition (2.19)-(2.20) is the exact boundary condition at artificial boundary Γ_R satisfied by the solution (u,p) of problem (1.3)-(1.6). Then the restriction of the solution (u,p) of problem (1.3)-(1.6) on the bounded domain Ω_i is a solution of the following problem:

$$-2\mu \text{ div } \varepsilon(u) + \text{grad } p = f, \quad \text{in} \quad \Omega_i, \tag{2.21}$$

$$\text{div } u = 0, \quad \text{in} \quad \Omega_i, \tag{2.22}$$

$$u = 0, \quad \text{on} \quad \Gamma, \tag{2.23}$$

$$\sigma_{n_1} = T_1(u), \quad \sigma_{n_2} = T_2(u), \quad \text{on} \quad \Gamma_R. \tag{2.24}$$

We introduce

$$T(u) = (T_1(u), T_2(u))^{\mathrm{T}},$$

$$T^N(u) = (T_1^N(u), T_2^N(u))^{\mathrm{T}},$$

with

$$T_j^N(u) = \begin{cases} 0, & N = 0, \quad j = 1,2, \\ \dfrac{2\mu}{\pi R}\displaystyle\sum_{n=1}^{N}\frac{\partial}{\partial\theta}\int_0^{2\pi}\frac{\cos n(\varphi-\theta)}{n}\frac{\partial u_j(R,\varphi)}{\partial\varphi}d\varphi, & N = 1,2,\cdots, \quad j = 1,2. \end{cases} \tag{2.25}$$

Then we obtain a sequence of the approximate boundary conditions on the artificial boundary Γ_R:

$$\sigma_{n_1} = T_1^N(u), \quad \sigma_{n_2} = T_2^N(u), \quad N = 0,1,2,\cdots, \quad \text{on} \quad \Gamma_R. \tag{2.26}$$

Using the approximate boundary conditions (2.26), the original problem (1.3)-(1.6) is reduced to the following problems on the bounded domain Ω_i approximately for $N = 0,1,2,\cdots$

(2.27) $$-2\mu \operatorname{div} \varepsilon(u) + \operatorname{grad} p = f, \quad \text{in } \Omega_i,$$

(2.28) $$\operatorname{div} u = 0, \quad \text{in } \Omega_i$$

(2.29) $$u = 0, \quad \text{on } \Gamma,$$

(2.30) $$\sigma_{n_1} = T_1^N(u), \quad \sigma_{n_2} = T_2^N(u), \quad \text{on } \Gamma_R.$$

In the following section, we reduce problem (2.21)-(2.24) and problem (2.27)-(2.30) to the equivalent variational problems, then we will show that the equivalent variational problems are well posed.

3. The equivalent variational problems of problem (2.21)-(2.24) and problem (2.27)-(2.30)

Let $H^m(\Omega_i)$ and $H^s(\Gamma_R)$ denote the usual Sobolev spaces on the domain Ω_i and the boundary Γ_R with integer m and real number s. Furthermore we introduce the following spaces :

$$\begin{aligned} H_*^1(\Omega_i) &= \{w \in H^1(\Omega_i), \quad w|_\Gamma = 0\}; \\ X &= H_*^1(\Omega_i) \times H_*^1(\Omega_i), \quad \text{with norm } \|u\|_X = \sqrt{\|u_1\|_{1,\Omega_i}^2 + \|u_2\|_{1,\Omega_i}^2}; \\ M &= L^2(\Omega_i). \end{aligned}$$

Then the problem (2.21)-(2.24) is reduced to the following equivalent variational problem :

$$\text{Find } (u,p) \in X \times M, \text{ such that}$$

(3.1) $$A(u,v) + A_0(u,v) + B(p,v) = F(v), \quad \forall v \in X,$$

(3.2) $$B(q,u) = 0, \quad \forall q \in M,$$

where

$$A(u,v) = 2\mu \int_{\Omega_i} \varepsilon(u) : \varepsilon(v) dx$$

(3.3) $$\equiv 2\mu \int_{\Omega_i} \sum_{k,l=1}^{2} \varepsilon_{kl}(u)\varepsilon_{kl}(v) dx,$$

$$A_0(u,v) = -\int_{\Gamma_R} T(u) \cdot v ds$$

(3.4) $$= \frac{2\mu}{\pi} \sum_{j=1}^{2} \sum_{n=1}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos n(\varphi - \theta)}{n} \frac{\partial u_j(R,\varphi)}{\partial \varphi} \frac{\partial v_j(R,\theta)}{\partial \theta} d\varphi d\theta,$$

(3.5) $$B(q,v) = -\int_{\Omega_i} q \operatorname{div} v dx,$$

$$(3.6) \quad F(v) = \int_{\Omega_i} f \cdot v dx.$$

Similarly the problem (2.27)-(2.30) is reduced to the following problem:

Find $(u, p) \in X \times M$, such that

$$(3.7) \quad A(u, v) + A_0^N(u, v) + B(p, v) = F(v), \quad \forall v \in X,$$

$$(3.8) \quad B(q, u) = 0, \quad \forall q \in M,$$

with

$$A_0^N(u, v) = -\int_{\Gamma_R} T^N(u) \cdot v ds$$

$$(3.9) \quad = \frac{2\mu}{\pi} \sum_{j=1}^{2} \sum_{n=1}^{N} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos n(\varphi - \theta)}{n} \frac{\partial u_j(R, \varphi)}{\partial \varphi} \frac{\partial v_j(R, \theta)}{\partial \theta} d\varphi d\theta.$$

From the definitions of $A(u, v)$ and $B(q, v)$ we know that $A(u, v)$ is a bounded bilinear form on $X \times X$ and $B(q, v)$ is a bounded bilinear form on $M \times X$. Furthermore, from Korn inequality [15], we have known that $A(u, v)$ is coercive on $X \times X$. Namely, there are three positive constants M_1, M_2 and α_1, such that

$$(3.10) \quad |A(u, v)| \leq M_1 \|u\|_X \|v\|_X, \quad \forall u, v \in X,$$

$$(3.11) \quad |B(q, v)| \leq M_2 \|q\|_M \|v\|_X, \quad \forall q \in M, v \in X,$$

$$(3.12) \quad A(u, u) \geq \alpha_1 \|u\|_X^2, \quad \forall u \in X.$$

For the bilinear forms $A_0(u, v)$ and $A_0^N(u, v)$, we have

Lemma 3.1 *The bilinear forms $A_0(u, v)$ and $A_0^N(u, v)$ are symmetrical and bounded on $X \times X$, i.e. there exists a constant $M_3 > 0$ independent of integer N, such that*

$$(3.13) \quad |A_0(u, v)| \leq M_3 \|u\|_X \|v\|_X, \quad \forall u, v \in X,$$

$$(3.14) \quad |A_0^N(u, v)| \leq M_3 \|u\|_X \|v\|_X, \quad \forall u, v \in X.$$

Moreover

$$(3.15) \quad A_0(u, u) \geq 0, \quad A_0^N(u, u) \geq 0, \quad \forall u \in X.$$

Proof. For any given $u = (u_1, u_2)^{\mathrm{T}} \in X$, $v = (v_1, v_2)^{\mathrm{T}} \in X$, we know that $u_j \in H_*^1(\Omega_i)$ and $v_j \in H_*^1(\Omega_i)$ $(j = 1, 2)$. By the trace theorem, we get $u_j|_{\Gamma_R} = u_j(R, \theta) \in H^{\frac{1}{2}}(\Gamma_R)$ and $v_j|_{\Gamma_R} = v_j(R, \theta) \in H^{\frac{1}{2}}(\Gamma_R)$ $(j = 1, 2)$. Suppose

$$(3.16) \quad u_j(R, \theta) = \frac{a_0^j}{2} + \sum_{n=1}^{\infty} (a_n^j \cos n\theta + b_n^j \sin n\theta), \quad j = 1, 2,$$

$$(3.17) \quad v_j(R, \theta) = \frac{c_0^j}{2} + \sum_{n=1}^{\infty} (c_n^j \cos n\theta + d_n^j \sin n\theta), \quad j = 1, 2,$$

and

$$\|u_j(R,\theta)\|_* = \sqrt{\frac{(a_0^j)^2}{2} + \sum_{n=1}^{\infty}(1+n^2)^{\frac{1}{2}}[(a_n^j)^2+(b_n^j)^2]}, \tag{3.18}$$

$$\|v_j(R,\theta)\|_* = \sqrt{\frac{(b_0^j)^2}{2} + \sum_{n=1}^{\infty}(1+n^2)^{\frac{1}{2}}[(c_n^j)^2+(d_n^j)^2]}. \tag{3.19}$$

Then $\|\cdot\|_*$ is an equivalent norm of the space $H^{\frac{1}{2}}(\Gamma_R)$ [16], namely, there is a constant $C>0$, such that

$$\|u_j(R,\theta)\|_* \le C\|u_j\|_{\frac{1}{2},\Gamma_R}, \tag{3.20}$$

$$\|v_j(R,\theta)\|_* \le C\|v_j\|_{\frac{1}{2},\Gamma_R}. \tag{3.21}$$

Substituting (3.16), (3.17) into (3.4) and (3.9), we obtain

$$A_0(u,v) = 2\pi\mu\sum_{j=1}^{2}\sum_{n=1}^{\infty} n(a_n^j c_n^j + b_n^j d_n^j), \quad \forall u,v\in X, \tag{3.22}$$

$$A_0^N(u,v) = 2\pi\mu\sum_{j=1}^{2}\sum_{n=1}^{N} n(a_n^j c_n^j + b_n^j d_n^j), \quad \forall u,v\in X. \tag{3.23}$$

From (3.22)-(3.23), we obtain the inequality (3.15) immediately. Furthermore, using inequalities (3.18)-(3.21) and the trace theorem of $H^1(\Omega_i)$, the estimates (3.13) and (3.14) follow directly. □

For the bilinear from $B(q,v)$, we have

Lemma 3.2 *There is a constant $\beta>0$, such that*

$$\sup_{v\in X\setminus\{0\}}\frac{B(q,v)}{\|v\|_X} \ge \beta\|q\|_M, \quad \forall q\in M. \tag{3.24}$$

Proof. (i) There exist $p_0\in L^2(\Omega_i)$ and $v^0\in X$, such that

$$\text{div } v^0 = p_0, \quad \int_{\Omega_i} p_0 dx \neq 0.$$

(ii) For any $p\in L^2(\Omega_i)$, we have

$$p = p^* + \alpha p_0, \quad \text{with} \quad \alpha = \frac{\int_{\Omega_i} pdx}{\int_{\Omega_i} p_0 dx}.$$

Hence $p^*\in L_0^2(\Omega_i)\equiv\{q \mid q\in L^2(\Omega_i) \text{ and } \int_{\Omega_i} qdx = 0\}$ and

$$\|p^*\|_M \le c\|p\|_M .$$

By virtue of Corollary 2.4 ([16], p. 24) there exists an element $v^* \in X$, such that

$$\text{div } v^* = p^*, \quad \|v^*\|_X \le c\|p^*\|_M .$$

Let $v_p = v^* + \alpha v^0$, then

$$\text{div } v_p = \text{div } v^* + \alpha \text{ div } v^0 = p^* + \alpha p^0 = p,$$

and

$$\|v_p\|_X \le \|v^*\|_X + |\alpha| \cdot \|v^0\|_X \le c\|p\|_M .$$

Then inf-sup condition (3.24) follows immediately.

Combining Lemma 3.1-3.2 and Theorem 4.1 in Chapter I of [4], we obtain

Theorem 3.1 *Suppose $f \in X'$, then the variational problem (3.1)-(3.2) has a unique solution $(u,p) \in X \times M$ and the variational problem (3.7)-(3.8) has a unique solution $(u_N, P_N) \in X \times M$ for any integer $N \ge 0$.*

In order to estimate the error between (u,p) and (u_N, P_N), we prove the following result:

Lemma 3.3 *For $u, v \in X$, assume that $u|_{\Gamma_R} \in H^{m+\frac{1}{2}}(\Gamma_R) \times H^{m+\frac{1}{2}}(\Gamma_R)$ with positive integer m, then the following estimate holds*

$$|A_0(u,v) - A_0^N(u,v)| \le \frac{c}{(N+1)^m} |u|_{m+\frac{1}{2},\Gamma_R} \|v\|_X , \tag{3.25}$$

where $c > 0$ is a constant independent of N.

Proof. By virtue of (3.22) and (3.23), we have

$$\begin{aligned}
|A_0(u,v) - A_0^N(u,v)| &= 2\pi\mu \left| \sum_{j=1}^{2} \sum_{n=N+1}^{\infty} n(a_n^j c_n^j + b_n^j d_n^j) \right| \\
&\le 2\pi\mu \sum_{j=1}^{2} \sqrt{\sum_{n=N+1}^{\infty} n \left[(a_n^j)^2 + (b_n^j)^2 \right]} \\
&\quad \times \sqrt{\sum_{n=N+1}^{\infty} n \left[(c_n^j)^2 + (d_n^j)^2 \right]} \\
&\le \frac{2\pi\mu}{(N+1)^m} \sqrt{\sum_{n=N+1}^{\infty} n^{2m+1} \left[(a_n^j)^2 + (b_n^j)^2 \right]} \; \|v\|_{\frac{1}{2},\Gamma_R} \\
&\le \frac{c}{(N+1)^m} |u|_{m+\frac{1}{2},\Gamma_R} \|v\|_X .
\end{aligned}$$

The last inequality follows from the equivalent norm theorem in space $H^{m+\frac{1}{2}}(\Gamma_R)$ [16], then the estimate (3.25) is completely proved. □

Furthermore we have

Theorem 3.2 *Suppose that* $(u,p) \in X \times M$ *is the solution of problem (3.1)-(3.2) and* $(u_N, P_N) \in X \times M$ *is the solution of problem (3.7)-(3.8), Futhermore* $u|_{\Gamma_R} \in H^{m+\frac{1}{2}}(\Gamma_R) \times H^{m+\frac{1}{2}}(\Gamma_R)$ *with positive integer* m*, then the following error estimate holds*

$$\|u - u_N\|_X + \|p - p_N\|_M \le \frac{c}{(N+1)^m} |u|_{m+\frac{1}{2},\Gamma_R}, \tag{3.26}$$

where c *is a constant independent of* N *and* m*.*

Proof. Let $u_e = u - u_N$, $p_e = p - p_N$, then $(u_e, p_e) \in X \times M$ satisfies the following problem :

$$A(u_e, v) + A_0^N(u_e, v) + B(p_e, v) = A_0^N(u, v) - A_0(u, v), \quad \forall v \in X, \tag{3.27}$$

$$B(q, u_e) = 0, \quad \forall q \in M. \tag{3.28}$$

Taking $v = u_e$ in (3.27), we obtain

$$A(u_e, u_e) + A_0^N(u_e, u_e) = A_0^N(u, u_e) - A_0(u, u_e).$$

Combining the inequalities (3.12), (3.15) and (3.25), we get

$$\|u_e\|_X \le \frac{c}{(N+1)^m} |u|_{m+\frac{1}{2},\Gamma_R}. \tag{3.29}$$

On the other hand, from (3.27), we arrive at

$$\begin{aligned} |B(p_e, v)| &= |A_0^N(u, v) - A_0(u, v) - A(u_e, v) - A_0^N(u_e, v)| \\ &\le \frac{c}{(N+1)^m} |u|_{m+\frac{1}{2},\Gamma_R} \|v\|_X + (M_1 + M_3)\|u_e\|_X \|v\|_X \\ &\le \left[\frac{c}{(N+1)^m} |u|_{m+\frac{1}{2},\Gamma_R} + (M_1 + M_3)\|u_e\|_X\right] \|v\|_X. \end{aligned}$$

Combining the inequalities (3.24) and (3.29), the estimate (3.26) follows immediately. □

4. The finite element approximation of variational problem (3.7)-(3.8)

In this section we consider the finite element approximation of problem (3.7)-(3.8). Suppose that X_h and M_h are two finite-dimensional subspace of X and M satisfying the following condition:

There exists a positive constant β^* independent of h, such that

$$\sup_{v_h \in X_h \setminus \{0\}} \frac{B(q_h, v_h)}{\|v_h\|_X} \ge \beta^* \|q_h\|_M, \quad \forall q_h \in M_h. \tag{4.1}$$

We now consider the discrete problem of (3.7)-(3.8):

$$\text{Find } \left(u^{h,N}, p^{h,N}\right) \in X_h \times M_h, \quad \text{such that}$$

$$A(u^{h,N},v_h)+A_0^N(u^{h,N},v_h)+B(p^{h,N},v_h) = F(v_h), \quad \forall v_h\in X_h, \tag{4.2}$$

$$B(q_h,u^{h,N}) = 0, \quad \forall q_h\in M_h. \tag{4.3}$$

Applying Theorem 4.1 in Chapter I of [4] again, we know that variational problem (4.2)-(4.3) has a unique solution $(u^{h,N},p^{h,N})\in X_h\times M_h$ for any integer $N\geq 0$. Furthermore we have

Theorem 4.1 *Suppose that $(u,p)\in X\times M$ is the solution of problem (3.1)-(3.2) and $(u^{h,N},P^{h,N})\in X_h\times M_h$ is the solution of problem (4.2)-(4.3), then the following abstract error estimate holds:*

$$\begin{aligned}\|u-u^{h,N}\|_X+\|p-p^{h,N}\|_M \quad \leq \quad & C\{\inf_{v_h\in X_h}\|u-v_h\|_X+\inf_{q_h\in M_h}\|p-q_h\|_M \\ & +\frac{1}{(N+1)^m}|u|_{m+\frac{1}{2},\Gamma_R}\},\end{aligned} \tag{4.4}$$

where C is a constant independent of h and N.

Throughout this section C and C_0 denote positive generic constants independent of N and the mesh size h.

Proof. For the errors $\|u^N-u^{h,N}\|_X$ and $\|p^N-p^{h,N}\|_M$, by a standard technique of mixed finite element method [17] we have

$$\begin{aligned}\|u^N-u^{h,N}\|_X+\|p^N-p^{h,N}\|_M \quad \leq \quad & C_0\left\{\inf_{v_h\in X_h}\|u^N-v_h\|_X\right. \\ & \left.+\inf_{q_h\in M_h}\|p^N-q_h\|_M\right\}.\end{aligned} \tag{4.5}$$

Hence we obtain

$$\begin{aligned} & \|u-u^{h,N}\|_X+\|p-p^{h,N}\|_M \\ \leq \quad & \|u^N-u^{h,N}\|_X+\|p^N-p^{h,N}\|_M+\|u-u^N\|_X+\|p-p^N\|_M \\ \leq \quad & C_0\left\{\inf_{v_h\in X_h}\|u^N-v_h\|_X+\inf_{q_h\in M_h}\|p^N-q_h\|_M\right\}+\|u-u^N\|_X+\|p-p^N\|_M \\ \leq \quad & C_0\left\{\inf_{v_h\in X_h}\|u-v_h\|_X+\inf_{q_h\in M_h}\|p-q_h\|_M\right\} \\ & +(C_0+1)(\|u-u^N\|_X+\|p-p^N\|_M).\end{aligned}$$

Combining the estimate (3.26), the error estimate (4.4) follows immediately. □

Suppose the solution of (3.1)-(3.2), $(u,p)\in(X\cap(H^{k+1}(\Omega_i)\times H^{k+1}(\Omega_i))$, $M\cap H^k(\Omega_i))$ with positive integer k and the subspace X_h and M_h satisfy the approximate properties

$$\inf_{v_h\in X_h}\|u-v_h\|_X \quad \leq \quad Ch^k|u|_{k+1}, \tag{4.6}$$

$$\inf_{q_h\in M_h}\|p-q_h\|_M \quad \leq \quad Ch^k|p|_k. \tag{4.7}$$

Combining (4.6), (4.7) and (4.4) with $m=k$ we obtain the following error estimate

$$\|u-u^{h,N}\|_X+\|p-p^{h,N}\|_M \leq C\left\{h^k(|u|_{k+1}+|p|_k)+\frac{1}{(N+1)^k}|u|_{k+\frac{1}{2},\Gamma_R}\right\}. \tag{4.8}$$

Furthermore if we take $N=O(h^{-1})$, then it is evaluated as

$$\|u-u^{h,N}\|_X+\|p-p^{h,N}\|_M \leq Ch^k\left\{|u|_{k+1}+|u|_{k+\frac{1}{2},\Gamma_R}+|p|_k\right\}. \tag{4.9}$$

From the error estimate (4.8), we can see that the error consists of two parts, one of them is from finite element approximation, the other is from the approximate artificial boundary condition.

For example, for P_2/P_1 elements the error estimate (4.8) holds with $k=2$.

5. Numerical experiment

We consider the numerical solution of the original problem on a given bounded computational domain by a mixed finite element method. In our example, we take $\mu=1$, $x^+=(0,\frac{1}{2})$, $x^-=(0,-\frac{1}{2})$ and the unbounded domain $\Omega=\{x\in\Re^2,\ 1<|x_1| \text{ or } 1<|x_2|\}$ is the exterior domain of the square $[-1,1]\times[-1,1]$ with boundary Γ. Let

$$\begin{aligned}
u_1^0(x) &= \frac{1}{4\mu}\left[\frac{(x_1-x_1^+)^2}{|x-x^+|^2}-\frac{(x_1-x_1^-)^2}{|x-x^-|^2}-\ln\frac{|x-x^+|}{|x-x^-|}\right],\\
u_2^0(x) &= \frac{1}{4\mu}\left[\frac{(x_1-x_1^+)(x_2-x_2^+)}{|x-x^+|^2}-\frac{(x_1-x_1^-)(x_2-x_2^-)}{|x-x^-|^2}\right],\\
p_0(x) &= \frac{1}{2}\left[\frac{x_1-x_1^+}{|x-x^+|^2}-\frac{x_1-x_1^-}{|x-x^-|^2}\right].
\end{aligned}$$

Then $\{u^0(x)=\left(u_1^0(x),u_2^0(x)\right)^{\mathrm{T}},p_0(x)\}$ is the unique solution of the following boundary value problem on the unbounded domain Ω:

$$-2\mu\,\mathrm{div}\,\varepsilon(u)+\mathrm{grad}\,p = 0,\quad \text{in } \Omega, \tag{5.1}$$

$$\mathrm{div}\,u = 0,\quad \text{in } \Omega, \tag{5.2}$$

$$u = u^0,\quad \text{on } \Gamma, \tag{5.3}$$

$$u \text{ is bounded},\quad p\to 0,\qquad \text{when } |x|\to+\infty. \tag{5.4}$$

We take Γ_R as a circumference with radius 2, then we obtain a bounded computational domain $\Omega_i=\{x\mid x\in\Omega \text{ and } |x|<2\}$. Using the artificial boundary conditions (2.26), the problem (4.1)-(4.4) is reduced to the following problem on the bounded computational domain Ω_i:

$$-2\mu\,\mathrm{div}\,\varepsilon(u)+\mathrm{grad}\,p = 0,\quad \text{in } \Omega_i, \tag{5.5}$$

$$\mathrm{div}\,u = 0,\quad \text{in } \Omega_i, \tag{5.6}$$

$$u = u^0,\quad \text{on } \Gamma, \tag{5.7}$$

$$\sigma_{n_1}=T_1^N(u),\qquad \sigma_{n_2}=T_2^N(u),\qquad \text{on } \Gamma_R. \tag{5.8}$$

Introduce a set

$$X_* = \{u = (u_1, u_2)^{\mathrm{T}} \mid u_j \in H^1(\Omega_i) \text{ and } u_j|_\Gamma = u_j^0, \quad j = 1, 2\}.$$

Then the problem (4.5)-(4.8) is equivalent to the following variational problem:

$$\text{Find} \quad (u, p) \in X_* \times M, \quad \text{such that}$$

$$A(u, v) + A_0^N(u, v) + B(p, v) = 0, \quad \forall v \in X, \tag{5.9}$$

$$B(q, u) = 0, \quad \forall q \in M. \tag{5.10}$$

We now solve the variational problem (4.9)-(4.10) by a mixed finite element method. Suppose that J_h is a regular triangulation of Ω_i satisfying

$$\overline{\Omega_i} = \bigcup_{K \in J_h} K$$

where K is a triangle or a curved triangle with a curved side on Γ_R. h_K denotes the diameter of K and $h = \max_{K \in J_h} h_K$. We construct a finite dimensional approximate space X_h of X by quadratic conforming triangle elements and a finite dimensional approximate space M_h of M by linear triangle elements. Let $(u^{h,N}, p^{h,N})$ denote the finite element approximation of the problem (4.9)-(4.10) corresponding to the given subspaces X_h, M_h and the given positive integer N.

Three meshes were used in our computation. Figure 2 shows the triangulation for mesh A, Mesh B was generated by dividing the triangles in mesh A into four small triangles and mesh C was similarly generated from mesh B.

Table 1 shows the maximum of the errors $u - u^{h,N}$ and $p - p^{h,N}$ over the mesh points when $N = 7$ for mesh A, mesh B and mesh C. Table 2 shows the maximum of the errors $u - u^{h,N}$ and $p - p^{h,N}$ for mesh C when $N = 1, 3, 5$ and 7. By the error estimate (4.8), considering $h \sim \frac{1}{N+1}$, then we have Table 3 for the maximum of the errors $u - u^{h,N}$ and $p - p^{h,N}$ for mesh A and $N = 1$, mesh B and $N = 3$, mesh C and $N = 7$.

Table 1. Maximum error for $N = 7$

mesh	A	B	C
h	0.36	0.18	0.09
$\max \lvert u_1 - u_1^{h,N} \rvert$	2.4186E-2	4.4264E-3	9.3290E-4
$\max \lvert u_2 - u_2^{h,N} \rvert$	2.1980E-2	4.4328E-3	6.7933E-4
$\max \lvert p - p^{h,N} \rvert$	1.4190	0.6095	0.1591
$\lVert p - p^{h,N} \rVert_{0,2,\Omega_i}$	0.7472	0.1271	1.8370E-2

Figures 3–7 show the results u, p and $u^{h,N}, p^{h,N}$ for mesh C along some curves, where the interior points are the points along the curve ABCDE shown in Fig. 2, and the boundary points are the points along the boundary Γ_R. In the figures for $N = 5$ and $N = 7$, we can not find any difference. As shown in the figures, $N = 5$ gives good approximations and therefore in the computation very few terms in the bilinear from $A_0^N(u, v)$ are needed in order to get good accuracy.

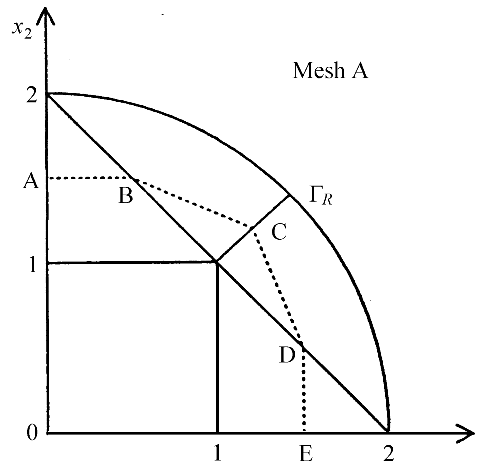

Fig. 2.

Table 2. Maximum error for mesh C

N	0	1	3	5	7
$\max \lvert u_1 - u_1^{h,N} \rvert$	2.039E-1	4.849E-2	3.681E-3	1.138E-3	9.329E-4
$\max \lvert u_2 - u_2^{h,N} \rvert$	2.228E-1	4.656E-2	2.983E-3	6.733E-4	6.793E-4
$\max \lvert p - p^{h,N} \rvert$	0.509	0.282	0.160	0.159	0.159
$\lVert p - p^{h,N} \rVert_{0,2,\Omega_i}$	0.257	0.237	2.848E-2	1.850E-2	1.837E-2

Table 3. Maximum error

	(mesh A, $N = 1$)	(mesh B, $N = 3$)	(mesh C, $N = 7$)
$\max \lvert u_1 - u_1^{h,N} \rvert$	8.188E-2	7.532E-3	9.329E-4
$\max \lvert u_2 - u_2^{h,N} \rvert$	7.936E-2	5.124E-3	6.793E-4
$\max \lvert p - p^{h,N} \rvert$	1.379	0.614	0.159
$\lVert p - p^{h,N} \rVert_{0,2,\Omega_i}$	0.737	0.128	1.837E-2

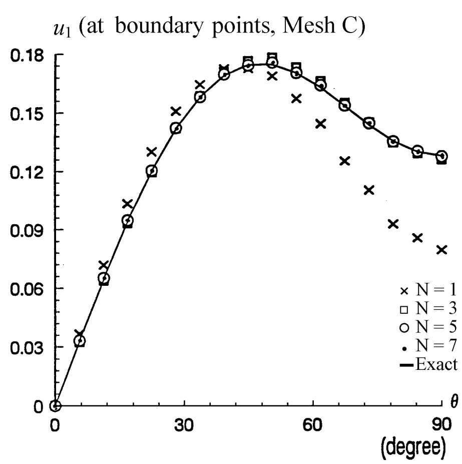

Fig. 3.

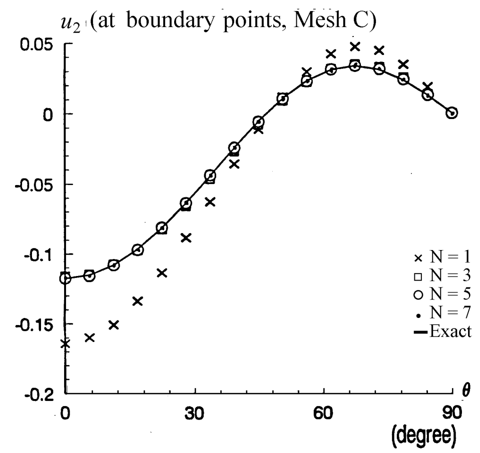

Fig. 4.

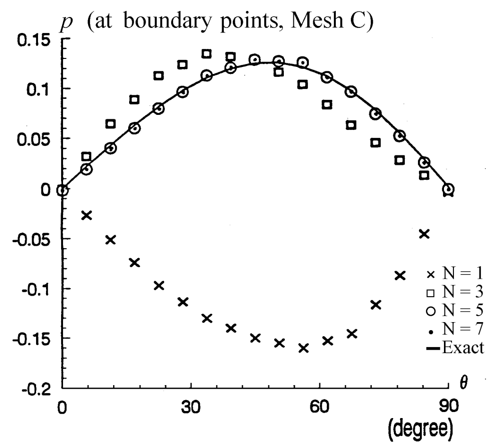

Fig. 5.

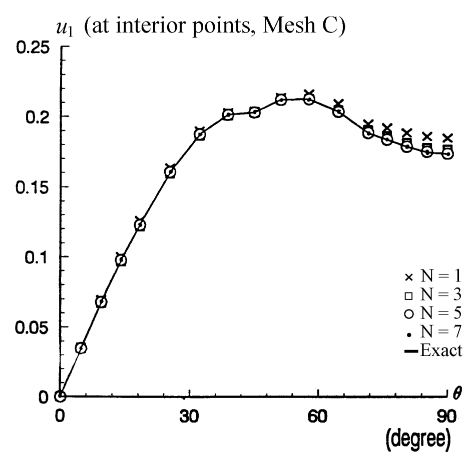

Fig. 6.

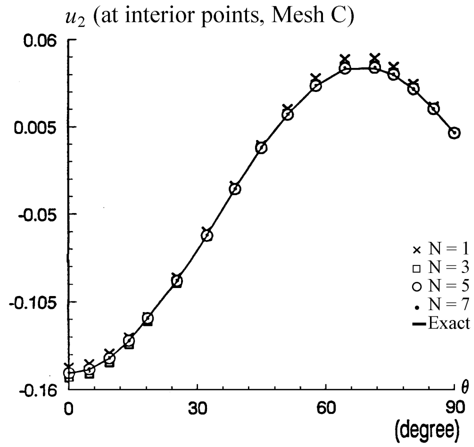

Fig. 7.

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