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ERROR ESTIMATES FOR THE FINITE ELEMENT APPROXIMATION OF PROBLEMS IN UNBOUNDED DOMAINS*

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Abstract. In this paper we present error estimates for the finite element approximation of linear elliptic problems in unbounded domains that are outside an obstacle and a semi-infinite strip in the plane. The finite element approximation is formulated on a bounded domain using a nonlocal approximate artificial boundary condition. In fact there is a family of approximate boundary conditions with increasing accuracy (and computational cost) for a given artificial boundary. Our error estimates are based on the mesh size, the terms used in the approximate artificial boundary condition, and the location of the artificial boundary. Numerical examples for Poisson's problem outside a circle and in a semi-infinite strip are presented. Numerical results demonstrate the performance of our error estimates.

Key words. unbounded domain, finite element approximation, artificial boundary, nonlocal artificial boundary condition

AMS subject classification. 65N30

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1. Introduction.

In this paper we consider two types of unbounded domain problems that are exterior problems and problems in semi-infinite strips. Exterior problems are associated with the infinite space outside an obstacle (Figure 1.1). Problems in semi-infinite strips are encountered in applications involving waveguides (Figure 1.2) or flow around an obstacle in a channel (Figure 1.3). The corresponding unbounded domain is denoted by Ω.

We consider a linear elliptic second-order boundary value problem in two dimensions, with each of the three setups shown in Figures 1.1–1.3. The boundary Γ_i is decomposed into two disjoint parts, Γ_D where a Dirichlet boundary condition is given, and Γ_N where a Neumann boundary condition is given. The statement of the problem is

$$-\nabla \cdot \kappa(x)\nabla u(x) + \beta(x)u(x) = f(x) \qquad \text{in } \Omega, \tag{1.1}$$

$$u = g \qquad \text{on } \Gamma_D, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = k \qquad \text{on } \Gamma_N, \tag{1.3}$$

$$\text{some conditions at infinity}, \tag{1.4}$$

$$\text{if } \Omega \text{ is a strip}: \quad \text{some boundary conditions on } \Gamma_U \text{ and } \Gamma_L; \tag{1.5}$$

where $u(x)$ is the unknown function, $f(x)$, $\kappa(x) \geq 1$, $\beta(x) \geq \beta_0 \geq 0$, g and k are

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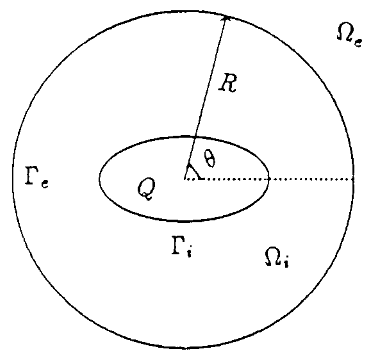

FIG. 1.1.

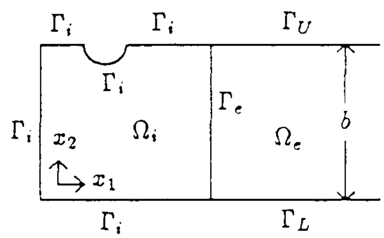

FIG. 1.2.

given functions satisfying supp(f), supp$(\kappa - 1)$ and supp$(\beta - \beta_0)$ are compact and $\kappa(x)$, $\beta(x) \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma_D)$, and $k \in H^{-1/2}(\Gamma_N)$.

In order to bound the physical domain of the problem, we introduce an artificial boundary Γ_e which divides the unbounded domain Ω into two parts, Ω_i and Ω_e, such that supp$(f)\cup$ supp$(\kappa-1)\cup$ supp$(\beta-\beta_0) \cup \Gamma_i \subset \bar{\Omega}_i$ (see Figures 1.1–1.3). In an exterior problem it is a circle and in a problem in semi-infinite strip it is a segment.

If Ω_e is a strip (Figures 1.2–1.3), then the boundary condition (1.5) on Γ_U and Γ_L is either the Dirichlet condition $u = 0$ or the Neumann condition $\frac{\partial u}{\partial x_2} = 0$. Let b be the width of the channel Ω_e. We introduce a Cartesian coordinate system (x_1, x_2), such that the ray Γ_L coincides with the x_1 axis and Γ_e is the segment $\{(d, x_2) \mid 0 \le x_2 \le b\}$. In the case of an exterior problem (Figure 1.1), we use a polar coordinate system (r, θ), such that Γ_e is the circle $\{(R, \theta) \mid 0 \le \theta \le 2\pi\}$. Furthermore let $R_0 = \inf\{r \mid (r, \theta) \in \Gamma_i \cup \text{supp}(f) \cup \text{supp}(\kappa - 1) \cup \text{supp}(\beta - \beta_0)\}$ and $\Gamma_0 = \{(R_0, \theta) \mid 0 \le \theta \le 2\pi\}$ in the case of an exterior problem and $d_0 = \inf\{x_1 \mid (x_1, x_2) \in \Gamma_i \cup \text{supp}(f) \cup \text{supp}(\kappa - 1) \cup \text{supp}(\beta - \beta_0)\}$ and $\Gamma_0 = \{(d_0, x_2) \mid 0 \le x_2 \le b\}$ in the case of a problem in a semi-infinite strip.

Let $|\Gamma_D|$ denote the length of the curve Γ_D. The condition at infinity (1.4) is as follows: In an exterior problem or problems in semi-infinite strips with Neumann boundary condition at Γ_U and Γ_L when $\beta_0 > 0$ or $|\Gamma_D| = 0$ and $\beta(x) \equiv 0$ or problems in semi-infinite strips with Dirichlet boundary condition at Γ_U and Γ_L, the solution u is required to varnish at infinity; in an exterior problem or problems in semi-infinite strips with the Neumann boundary condition at Γ_U and Γ_L when $\beta_0 = 0$ and $|\Gamma_D| \neq 0$ or $\beta(x) \not\equiv \beta_0 = 0$ and $|\Gamma_D| = 0$, the solution u is required to be bounded at infinity.

There are several methods to solve boundary value problems in unbounded domains [7]. One of the most popular methods is to introduce an artificial boundary Γ_e

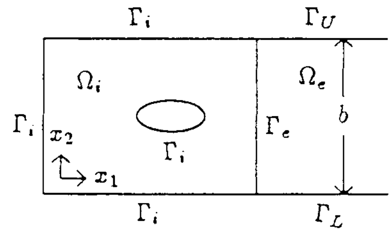

FIG. 1.3.

and set up artificial boundary conditions on it. Then the original problem is reduced to a boundary value problem in a bounded computational domain. Then a numerical approximation of the original problem in the bounded domain can be obtained by solving the reduced problem. In the last two decades, many authors have worked on this subject for various problems by different techniques. For instance, Engquist and Majda [4] designed absorbing boundary conditions for a wave equation. Goldstein [9] presented the exact boundary condition and a sequence of its approximations at an artificial boundary for a Helmholtz-type equation in waveguides. Feng [5] proposed the asymptotic radiation conditions for the reduced wave equation by using the asymptotic approximation of Hankel functions. Han and Wu [17, 18] obtained the exact boundary conditions and a series of their approximations at an artificial boundary for the Laplace equation and the linear elastic system. The exact boundary condition at an artificial boundary for partial differential equations in an infinite cylinder was proposed by Hagstrom and Keller [10, 11]. Shortly after, they used this technique to solve nonlinear problems. A family of artificial boundary conditions for unsteady Oseen equations in the velocity pressure formulation with small viscosity was developed by Halpern and Schatzman [12], which was then applied to unsteady Navier–Stokes equations. Han, Lu, and Bao [15] designed discrete artificial boundary conditions for incompressible viscous flows in an infinite channel by using a fast iterative method. Han and Bao [14, 13] proposed discrete artificial boundary conditions for incompressible viscous flows in a channel by using the method of lines. Han, Bao, and Wang [16] developed artificial boundary conditions for the problem of infinite elastic foundation. Recently Givoli, Patlashenko, and Keller [8] also analyze the boundary value problem (1.1)–(1.4).

In the above works, several authors gave error estimates for the numerical solution. Han and Wu analyzed the error estimates for the exterior problems of Poisson equation and the linear elastic equations. Givoli, Patlashenko, and Keller [8] discussed the error estimates for the exterior problem and a problem in a strip of second order elliptic equation. Masmoudi [21] and Harari and Hughues [19] considered the error estimates for the exterior problem of Helmholtz equation. But their error estimates only depend on the mesh size of a partition of Ω_i and the approximate artificial boundary condition. How does the error depend on the location of the artificial boundary Γ_e? This is a very interesting problem for engineers. In this paper, we will provide error estimates for the finite element approximation of the problem (1.1)–(1.4), which depends not only on the mesh size and the artificial boundary condition but also the location (R or d) of the artificial boundary Γ_e.

2. Nonlocal artificial boundary conditions at Γ_e.

In order to derive an exact boundary condition on Γ_e for the problem (1.1)–(1.4), we analyze the problem in Ω_e similarly to [17, 8]. The first case we considered is the exterior problem with $\beta_0 = 0$ and $|\Gamma_D| \neq 0$. By separation of variables we find that the general solution which satisfies (1.1) in the domain Ω_e and the condition at infinity mentioned in section 1 is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{R}{r}\right)^n (a_n \cos n\theta + b_n \sin n\theta)\,, \tag{2.1}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\theta)\cos n\theta\, d\theta \quad b_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\theta)\sin n\theta\, d\theta, \quad n = 0,1,2,\dots. \tag{2.2}$$

We differentiate (2.1) with respect to r and set $r = R$ to obtain

$$\begin{aligned} \frac{\partial u(R,\theta)}{\partial r} &= -\frac{1}{\pi R}\sum_{n=1}^{\infty} n \int_0^{2\pi} u(R,\phi)\cos n(\theta-\phi)\, d\phi \\ &= -\frac{1}{\pi R}\sum_{n=1}^{\infty} \int_0^{2\pi} \frac{\partial u(R,\phi)}{\partial \phi}\sin n(\theta-\phi)\, d\phi \triangleq Bu(R,\theta), \end{aligned} \tag{2.3}$$

where B is a bounded operator from the Sobolev space $H^{\frac{1}{2}}(\Gamma_e)$ to $H^{-\frac{1}{2}}(\Gamma_e)$ [6]. This is the desired exact boundary condition on Γ_e. Let

$$\begin{aligned} B_N u &= -\frac{1}{\pi R}\sum_{n=0}^{N} n \int_0^{2\pi} u(R,\phi)\cos n(\theta-\phi)\, d\phi \\ &= -\frac{1}{\pi R}\sum_{n=0}^{N} \int_0^{2\pi} \frac{\partial u(R,\phi)}{\partial \phi}\sin n(\theta-\phi)\, d\phi, \quad N = 0,1,2,\dots. \end{aligned} \tag{2.4}$$

Then we derive a series of approximate artificial boundary conditions on Γ_e:

(2.5)
$$\frac{\partial u(R,\theta)}{\partial r} = B_N u \equiv -\frac{1}{\pi R}\sum_{n=0}^{N} n \int_0^{2\pi} u(R,\phi)\cos n(\theta-\phi)\, d\phi, \quad N = 0,1,2,\dots,$$

where $N = 0$ is the Neumann boundary condition which is often used in engineering.

In a similar way, we get the exact boundary conditions and their approximations for all other cases considered. They are summarized in Tables 2.1 and 2.2 (see [17, 8]). There $K_n(r)$ is the modified Bessel function of the second kind. A prime after a summation sign indicates that a factor 1/2 multiplies the term with $n = 0$. Furthermore, in the case of $|\Gamma_D| = 0$ and $\beta(x) \equiv 0$ in an exterior problem or problems in semi-infinite strips with Neumann boundary condition at Γ_U and Γ_L, we need to add the following condition:

$$\int_{\Gamma_e} u\, ds = 0. \tag{2.6}$$

It is easy to see that the condition (2.6) is equivalent to

$$\int_{\Gamma_0} u\, ds = 0. \tag{2.7}$$

Thus we will use this condition (2.7) in the following finite element approximation in this case.

TABLE 2.1
Nonlocal exact boundary conditions for different problems.

Exterior problem	$\frac{\partial u}{\partial r}(R,\theta) = -\frac{1}{\pi R}\sum_{n=0}^{\infty}{}' Z_n \int_0^{2\pi} u(R,\theta')\cos n(\theta-\theta')\, d\theta'$
Semi-infinite strip $u = 0$ on Γ_U and Γ_L	$\frac{\partial u}{\partial x_1}(d,x_2) = -\frac{2\pi}{b^2}\sum_{n=1}^{\infty} Z_n \int_0^b u(d,x_2')\sin\frac{n\pi x_2}{b}\sin\frac{n\pi x_2'}{b}\, dx_2'$
Semi-infinite strip $\frac{\partial u}{\partial x_2} = 0$ on Γ_U and Γ_L	$\frac{\partial u}{\partial x_1}(d,x_2) = -\frac{2\pi}{b^2}\sum_{n=0}^{\infty}{}' Z_n \int_0^b u(d,x_2')\cos\frac{n\pi x_2}{b}\cos\frac{n\pi x_2'}{b}\, dx_2'$

Case	Exterior problem	Semi-infinite strip $u = 0$ on Γ_U and Γ_L	Semi-infinite strip $\frac{\partial u}{\partial x_2} = 0$ on Γ_U and Γ_L
$\beta_0 = 0$	$Z_n = n\ (n \ge 0)$	$Z_n = n\ (n \ge 0)$	$Z_n = n\ (n \ge 0)$
$\beta_0 > 0$	$Z_n = \frac{-R\beta_0^{\frac{1}{2}} K_n'(\beta_0^{\frac{1}{2}} R)}{K_n(\beta_0^{\frac{1}{2}} R)}$	$Z_n = \frac{b}{\pi}\sqrt{\beta_0 + \frac{n^2\pi^2}{b^2}}$	$Z_n = \frac{b}{\pi}\sqrt{\beta_0 + \frac{n^2\pi^2}{b^2}}$

TABLE 2.2
A series of approximate artificial boundary conditions ($N = 0, 1, 2, \ldots$) for different problems.

Exterior problem	$\frac{\partial u}{\partial r}(R,\theta) = -\frac{1}{\pi R}\sum_{n=0}^{N}{}' Z_n \int_0^{2\pi} u(R,\theta')\cos n(\theta-\theta')\, d\theta'$
Semi-infinite strip $u = 0$ on Γ_U and Γ_L	$\frac{\partial u}{\partial x_1}(d,x_2) = -\frac{2\pi}{b^2}\sum_{n=0}^{N} Z_n \int_0^b u(d,x_2')\sin\frac{n\pi x_2}{b}\sin\frac{n\pi x_2'}{b}\, dx_2'$
Semi-infinite strip $\frac{\partial u}{\partial x_2} = 0$ on Γ_U and Γ_L	$\frac{\partial u}{\partial x_1}(d,x_2) = -\frac{2\pi}{b^2}\sum_{n=0}^{N}{}' Z_n \int_0^b u(d,x_2')\cos\frac{n\pi x_2}{b}\cos\frac{n\pi x_2'}{b}\, dx_2'$

3. The finite element approximation. In this section we consider the finite element approximation of the problem (1.1)–(1.3) in Ω_i with nonlocal artificial boundary condition shown in Tables 2.1 and 2.2. First we consider the exterior problem. Problems in semi-infinite strips can be dealt with similarly. Let $H^m(\Omega_i)$ and $H^s(\Gamma_e)$ be usual Sobolev spaces on Ω_i and Γ_e with integer m and real number s [1]. Suppose

$$V_g = \begin{cases} \{v \in H^1(\Omega_i) \mid \int_{\Gamma_0} v\, ds = 0\} & \text{when } \beta(x) \equiv 0 \text{ and } |\Gamma_D| = 0, \\ \{v \in H^1(\Omega_i) \mid v = g \text{ on } \Gamma_D\} & \text{other cases.} \end{cases}$$

Then the weak form of the problem (1.1)–(1.3) with the exact boundary condition shown in Table 2.1 on Γ_e is

(P) find $u \in V_g$ such that

$$a(u,v) + b(u,v) = f(v) \quad \forall v \in V \equiv V_0, \tag{3.1}$$

where

$$a(u,v) = \int_{\Omega_i} \kappa(x)\nabla u(x)\cdot\nabla v(x)\,dx + \int_{\Omega_i} \beta(x)u(x)v(x)\,dx, \tag{3.2}$$

$$b(u,v) = {\sum_{m=0}^{\infty}}' \frac{Z_m}{\pi}\int_0^{2\pi}\int_0^{2\pi} \cos m(\theta-\theta')u(R,\theta)v(R,\theta')\,d\theta d\theta', \tag{3.3}$$

$$f(v) = \int_{\Omega_i} f(x)v(x)\,dx + \int_{\Gamma_N} \kappa k v\,ds. \tag{3.4}$$

Let

$$b_N(u,v) = {\sum_{m=0}^{N}}' \frac{Z_m}{\pi}\int_0^{2\pi}\int_0^{2\pi} \cos m(\theta-\theta')u(R,\theta)v(R,\theta')\,d\theta d\theta'. \tag{3.5}$$

Then the weak form of the problem (1.1)–(1.3) with the approximate artificial boundary condition shown in Table 2.2 on Γ_e is
(P_N) find $u_N \in V_g$ such that

$$a(u_N, v) + b_N(u_N, v) = f(v) \quad \forall v \in V. \tag{3.6}$$

Now we replace $V = V_0$ and V_g by two finite dimensional subsets, $V^h = V_0^h \subset V$ and $V_g^h \subset V_g$ in which h is the mesh size [3]. Then the finite element approximation of the problem (P_N) is
(P_N^h) find $u_N^h \in V_g^h$ such that

$$a(u_N^h, v^h) + b_N(u_N^h, v^h) = f(v^h) \quad \forall v^h \in V^h. \tag{3.7}$$

We note that the symmetric bilinear form $a(\cdot,\cdot)$ is bounded and coercive on $V\times V$ in all cases; i.e., there exist positive constants M_1, M_2 such that

$$|a(w,v)| \le M_1\|w\|_V\cdot\|v\|_V \qquad \forall\, w,\, v \in V, \tag{3.8}$$

$$M_2\|v\|_V^2 \le a(v,v) \qquad \forall\, v\in V. \tag{3.9}$$

Thus we can define an equivalent norm on the space V:

$$\|v\|_* = [a(v,v)]^{1/2} \quad \forall v \in V. \tag{3.10}$$

Therefore we have that

$$|a(w,v)| \le \|w\|_*\cdot\|v\|_* \qquad \forall\, w,\, v\in V, \tag{3.11}$$

$$\|v\|_*^2 \le a(v,v) \qquad \forall\, v \in V. \tag{3.12}$$

Before we discuss the bilinear forms $b(\cdot,\cdot)$ and $b_N(\cdot,\cdot)$, we prove some results about the modified Bessel functions $I_n(r)$ and $K_n(r)$ [2].

LEMMA 3.1. *The following inequalities hold:*

$$0 < K_n(r) < K_{n+1}(r), \qquad 0 < I_{n+1}(r) < I_n(r) \quad \forall r>0, \qquad n\ge 0, \tag{3.13}$$

$$0 < \frac{-rK_n'(r)}{K_n(r)} \le 2n + \alpha_0\frac{rI_n'(r)}{I_n(r)} \quad \forall r \ge R_0\sqrt{\beta_0}\,, \qquad n \ge 0, \tag{3.14}$$

where $\alpha_0 \geq 1$ *is a constant independent of* n *and* r.

$$\frac{K_{n+1}(r_1)}{K_{n+1}(r_2)} < \frac{K_n(r_1)}{K_n(r_2)} \quad \forall r_1 > r_2 > 0, \qquad n \geq 0. \tag{3.15}$$

Proof. By the equalities of $I_n(r)$ and $K_n(r)$ (see [2, p. 237])

$$I_n(r) = \frac{(r/2)^n}{\sqrt{\pi}\Gamma(n+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}} e^{-rt}\, dt, \qquad r > 0, \qquad n = 0, 1, 2, \dots, \tag{3.16}$$

$$K_n(r) = \frac{\sqrt{\pi}(r/2)^n}{\Gamma(n+\frac{1}{2})} \int_{1}^{\infty} (t^2-1)^{n-\frac{1}{2}} e^{-rt}\, dt, \qquad r > 0, \qquad n \geq 0. \tag{3.17}$$

Thus integrating by parts we have that

$$\begin{aligned} I_{n+1}(r) &= -\frac{(r/2)^n}{(2n+1)\sqrt{\pi}\Gamma(n+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{n+\frac{1}{2}}\, de^{-rt} \\ &= \frac{(r/2)^n}{\sqrt{\pi}\Gamma(n+\frac{1}{2})} \int_{-1}^{1} (-t)(1-t^2)^{n-\frac{1}{2}} e^{-rt}\, dt < I_n(r) \quad \forall r > 0, \quad n \geq 0. \end{aligned} \tag{3.18}$$

$$\begin{aligned} K_{n+1}(r) &= -\frac{\sqrt{\pi}(r/2)^n}{(2n+1)\Gamma(n+\frac{1}{2})} \int_{1}^{\infty} (t^2-1)^{n+\frac{1}{2}}\, de^{-rt} \\ &= \frac{\sqrt{\pi}(r/2)^n}{\Gamma(n+\frac{1}{2})} \int_{1}^{\infty} t(t^2-1)^{n-\frac{1}{2}} e^{-rt}\, dt > K_n(r) \quad \forall r > 0, \quad n \geq 0. \end{aligned} \tag{3.19}$$

By the equalities $K_0'(r) = -K_1(r)$ and $I_0'(r) = I_1(r)$ (see [2, p. 236]), we have that

$$-\frac{rK_0'(r)}{K_0(r)} = \frac{K_1(r)}{K_0(r)} \cdot \frac{I_0(r)}{I_1(r)} \cdot \frac{rI_0'(r)}{I_0(r)} \equiv F(r)\frac{rI_0'(r)}{I_0(r)} \quad \forall r > 0, \tag{3.20}$$

where

$$F(r) = \frac{K_1(r)I_0(r)}{K_0(r)I_1(r)}. \tag{3.21}$$

Then we know that $F(r)$ is continuous on $[R_0\sqrt{\beta_0}, +\infty)$ and $\lim_{r\to+\infty} F(r) = 1$ by the asymptotic formulas for large arguments about $I_n(r)$ and $K_n(r)$ (see [2, pp. 250–251]). Hence $F(r)$ is bounded on $[R_0\sqrt{\beta_0}, +\infty)$, i.e., there exists a constant $\alpha_0 \geq 1$ such that

$$F(r) \leq \alpha_0 \qquad \forall r \geq R_0\sqrt{\beta_0}. \tag{3.22}$$

By the equalities $-rK_n'(r) = nK_n(r) + rK_{n-1}(r)$ and $rI_n'(r) = -nI_n(r) + rI_{n-1}(r)$ ($n \geq 1$) (see [2, p. 236]), noting (3.13), we have that

$$\begin{aligned} 0 &< \frac{-rK_n'(r)}{K_n(r)} = n + \frac{rK_{n-1}(r)}{K_n(r)} \leq n + r \leq n + \frac{rI_{n-1}(r)}{I_n(r)} = 2n - n + \frac{rI_{n-1}(r)}{I_n(r)} \\ &= 2n + \frac{rI_n'(r)}{I_n(r)} \quad \forall r > 0, \qquad n \geq 1. \end{aligned} \tag{3.23}$$

Combining (3.22), (3.20), and (3.23), we get the desired inequality (3.14). For any $r_1 > r_2 > 0$ and $n = 0, 1, 2, \ldots$, noting (3.17) and (3.19), we have that

$$\begin{aligned} &K_{n+1}(r_2)K_n(r_1) - K_{n+1}(r_1)K_n(r_2) \\ &= \frac{\pi(r_1r_2/4)^n}{\Gamma(n+\frac{1}{2})^2}\int_1^\infty\int_1^\infty [(t_1^2-1)(t_2^2-1)]^{n-\frac{1}{2}} e^{-(r_1t_1+r_2t_2)}(t_2-t_1)\, dt_1dt_2 \\ &= \frac{\pi(r_1r_2/4)^n}{\Gamma(n+\frac{1}{2})^2}\int_1^\infty\int_1^{t_1} [(t_1^2-1)(t_2^2-1)]^{n-\frac{1}{2}} e^{-(r_1t_2+r_2t_1)}\left[1-e^{-(t_1-t_2)(r_1-r_2)}\right], \end{aligned}$$

$$(t_1-t_2)\, dt_2dt_1 > 0. \tag{3.24}$$

Thus the desired inequality (3.15) follows from (3.24) immediately.

LEMMA 3.2. *There exists a generic constant C_0 independent of R, N and h such that the following inequalities hold:*

$$0 \le b_{_N}(v,v) \le b(v,v) \le C_0\|v\|_*^2 \qquad \forall v \in V, \tag{3.25}$$

$$|b(u,v)| + |b_{_N}(u,v)| \le C_0\|u\|_* \cdot \|v\|_* \qquad \forall\ u,\, v \in V. \tag{3.26}$$

Proof. For any given $u, v \in V$, assume

$$u(R,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n \sin n\theta), \tag{3.27}$$

$$v(R,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n\cos n\theta + d_n \sin n\theta), \tag{3.28}$$

where $a_n,\ b_n$ are defined in (2.2) and

$$c_n = \frac{1}{\pi}\int_0^{2\pi} v(R,\theta)\cos n\theta\, d\theta \quad d_n = \frac{1}{\pi}\int_0^{2\pi} v(R,\theta)\sin n\theta\, d\theta, \qquad n \ge 0. \tag{3.29}$$

Inserting (3.27) and (3.28) into (3.3) and (3.5), we get

$$b(u,v) = \frac{Z_0\pi}{2}a_0c_0 + \pi\sum_{n=1}^{\infty} Z_n(a_nc_n + b_nd_n) \quad \forall u,v \in V, \tag{3.30}$$

$$b_{_N}(u,v) = \frac{Z_0\pi}{2}a_0c_0 + \pi\sum_{n=1}^{N} Z_n(a_nc_n + b_nd_n) \quad \forall u,v \in V. \tag{3.31}$$

We denote Q the domain enclosed by Γ_i, and $\hat{\Omega}$ the disk bounded by Γ_e (i.e., $\hat{\Omega} = Q \cup \Omega_i \cup \Gamma_i$); see Figure 1.1. For any $v \in V$, we define the function $v_0 \in H^1(\hat{\Omega})$ which satisfies

$$-\triangle v_0 + \beta_0 v_0 = 0 \qquad \text{in} \quad \Omega_i \cup Q, \tag{3.32}$$

$$v_0 = v \qquad \text{on} \quad \Gamma_e \cup \Gamma_i. \tag{3.33}$$

We also define the function $v_1 \in H^1(\hat{\Omega})$ which satisfies

$$-\triangle v_1 + \beta_0 v_1 = 0 \qquad \text{in} \quad \hat{\Omega}, \tag{3.34}$$

$$v_1 = v \qquad \text{on} \quad \Gamma_e. \tag{3.35}$$

Then $v_0|_{\Omega_i}$ minimizes the functional $\int_{\Omega_i}\left[|\nabla w|^2+\beta_0 w^2\right]\,dx$ among all functions $w \in V$ which are equal to v on $\Gamma_e \cup \Gamma_i$ and $v_0|_Q$ minimizes the functional $\int_Q \left[|\nabla w|^2 +\beta_0 w^2\right]\,dx$ among all functions $w \in H^1(Q)$ which are equal to v on Γ_i. Similarly, v_1 minimizes the functional $\int_{\hat\Omega}\left[|\nabla w|^2+\beta_0 w^2\right]\,dx$ among all functions $w \in H^1(\hat\Omega)$ which are equal to v on Γ_e. Therefore by the trace theorem and the Poincaré inequality, noting $v_0|_{\Gamma_i}=v|_{\Gamma_i}$, we have that

$$\begin{aligned}
&\int_{\hat\Omega}\left[|\nabla v_1|^2+\beta_0 v_1^2\right]\,dx\\
&\le \int_{\hat\Omega}\left[|\nabla v_0|^2+\beta_0 v_0^2\right]\,dx=\int_{\Omega_i}\left[|\nabla v_0|^2+\beta_0 v_0^2\right]\,dx+\int_Q\left[|\nabla v_0|^2+\beta_0 v_0^2\right]\,dx\\
&\le \int_{\Omega_i}\left[|\nabla v|^2+\beta_0 v^2\right]\,dx+C_0\|v\|^2_{\frac12,\Gamma_i}\\
&\le \int_{\Omega_i}\left[\kappa(x)|\nabla v|^2+\beta(x)v^2\right]\,dx+C_0\int_{\Omega_{i_0}}\left[\kappa(x)|\nabla v|^2+\beta(x)v^2\right]\,dx\\
&\le C_0\int_{\Omega_i}\left[\kappa(x)|\nabla v|^2+\beta(x)v^2\right]\,dx=C_0\|v\|_*^2,
\end{aligned} \tag{3.36}$$

where $\Omega_{i_0}=\{x\in\Omega_i\ :\ |x|<R_0\}\subset\Omega_i$. Recalling v_1 is a solution of the problem (3.34)–(3.35), using separation of variables, noting $v_1|_{\Gamma_e}=v_1(R,\theta)=v(R,\theta)$, we obtain

$$v_1(r,\theta)=\begin{cases}\dfrac{c_0}{2}+\displaystyle\sum_{m=1}^{\infty}\left(\frac{r}{R}\right)^m(c_m\cos m\theta+d_m\sin m\theta), & \beta_0=0,\\[2ex] \dfrac{c_0}{2}I_0\left(\sqrt{\beta_0}r\right)+\displaystyle\sum_{m=1}^{\infty}I_n\left(\sqrt{\beta_0}r\right)(c_m\cos m\theta+d_m\sin m\theta), & \beta_0>0,\end{cases} \tag{3.37}$$

where

$$c_m=\frac{1}{\pi}\int_0^{2\pi}v_1(R,\theta)\cos m\theta\,d\theta=\frac{1}{\pi}\int_0^{2\pi}v(R,\theta)\cos m\theta\,d\theta, \tag{3.38}$$

$$d_m=\frac{1}{\pi}\int_0^{2\pi}v_1(R,\theta)\sin m\theta\,d\theta=\frac{1}{\pi}\int_0^{2\pi}v(R,\theta)\sin m\theta\,d\theta\quad m=0,1,2,\dots. \tag{3.39}$$

From (3.36), (3.37), (3.38), (3.39), (3.34), and the divergence theorem, we obtain

$$\begin{aligned}
C_0\|v\|_*^2&\ge\int_{\hat\Omega}\left[\nabla v_1\cdot\nabla v_1+\beta_0 v_1^2\right]\,dx=\int_{\Gamma_e}v_1\frac{\partial v_1}{\partial n}\,ds\\
&=\int_0^{2\pi}v_1(R,\theta)\frac{\partial v_1}{\partial r}(R,\theta)R\,d\theta=\pi{\sum_{n=0}^{\infty}}'Z_n^*(c_n^2+d_n^2),
\end{aligned} \tag{3.40}$$

where

$$Z_n^* = \begin{cases} n, & \beta_0 = 0 \\ \dfrac{R\sqrt{\beta_0} I_n'(R\sqrt{\beta_0})}{I_n(R\sqrt{\beta_0})}, & \beta_0 > 0 \end{cases} \qquad \forall n \geq 0. \tag{3.41}$$

Combining (3.41), (3.40), (3.30), and (3.31) with $u = v$, we get the inequality (3.25) for the case of $\beta_0 = 0$ immediately; i.e.,

$$0 \leq \pi \sum_{n=1}^{N} n(c_n^2 + d_n^2) \leq \pi \sum_{n=1}^{\infty} n(c_n^2 + d_n^2) \leq C_0 \|v\|_*^2 \qquad \forall N \geq 0. \tag{3.42}$$

Combining (3.41), (3.40), (3.14), (3.42), (3.30), and (3.31) with $u = v$, we get the inequality (3.25) for the case of $\beta_0 > 0$ immediately. The inequality (3.26) follows from (3.25) and the Schwarz inequality immediately.

It follows immediately from (3.11), (3.12), (3.26), and (3.25) that the variational problems (P), (P_N), and (P_N^h) are well posed; that is, for $f \in V'$, the dual of V, there exists a unique $u \in V_g$ solving (P), a unique $u_N \in V_g$ solving (P_N), a unique $u_N^h \in V_g^h$ solving (P_N^h), and

$$\|u\|_* + \|u_N\|_* + \|u_N^h\|_* \leq M \left[\|f\|_{V'} + \|g\|_{\frac{1}{2},\Gamma_D} \right]. \tag{3.43}$$

Note that the well-posedness of (P) implies immediately the well-posedness of the original problem (1.1)–(1.4).

4. Error estimates. In this section we will establish error estimates for the finite element approximation of the problem (1.1)–(1.3) in Ω_i using nonlocal approximate artificial boundary conditions shown in Table 2.2. The first case we considered again is the exterior problem. For the sake of simplicity, we assume $g = 0$ on Γ_D.

We recall an equivalent definition of Sobolev space $H^s(\Gamma_0)$ for any real number s [20]:

$$v \in H^s(\Gamma_0) \Longleftrightarrow v(R_0, \theta) = \frac{c_0}{2} + \sum_{m=1}^{\infty} (c_m \cos m\theta + d_m \sin m\theta) \quad \text{and}$$

$$\frac{\pi c_0^2}{2} + \sum_{m=1}^{\infty} \pi(1+m^2)^s (c_m^2 + d_m^2) < \infty.$$

Thus we use

$$|v|_{s,\Gamma_0} = \left[\sum_{m=1}^{\infty} \pi(1+m^2)^s (c_m^2 + d_m^2) \right]^{1/2} \tag{4.1}$$

as a seminorm of the space $H^s(\Gamma_0)$. Then we have the following lemma.

Lemma 4.1. *Suppose $u \in H^1(\Omega_i)$ is a solution of the exterior problem (1.1)–(1.4) and there exists an integer $k \geq 1$ such that $u|_{\Gamma_0} \in H^{k-\frac{1}{2}}(\Gamma_0)$. Then for any $v \in V$ we* *have that*

(4.2)

$$|b(u,v) - b_{_N}(u,v)| \leq \begin{cases} \dfrac{C_0}{(N+1)^{k-1}} \left(\dfrac{R_0}{R}\right)^{N+1} |u|_{\Gamma_0}|_{k-\frac{1}{2},\Gamma_0} \cdot \|v\|_*, & \beta_0 = 0, \\ \dfrac{C_0(1+\sqrt{\beta_0}R)K_{N+1}(\sqrt{\beta_0}R)}{(N+1)^{k-1}K_{N+1}(\sqrt{\beta_0}R_0)} |u|_{\Gamma_0}|_{k-\frac{1}{2},\Gamma_0} \cdot \|v\|_*, & \beta_0 > 0. \end{cases}$$

Proof. Assume that

$$u(R_0,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta), \tag{4.3}$$

$$v(R,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n \cos n\theta + d_n \sin n\theta), \tag{4.4}$$

where $c_m,\ d_m$ are defined in (3.29) and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} u(R_0,\theta)\cos n\theta \; d\theta \quad b_n = \frac{1}{\pi}\int_0^{2\pi} u(R_0,\theta)\sin n\theta \; d\theta \quad n = 0,1,2,\ldots.$$

First we prove the case $\beta_0 = 0$, noting that u satisfies the Laplace equation in the domain $\{x \mid |x| > R_0\}$, we get

$$u(r,\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(\frac{R_0}{r}\right)^m (a_m \cos m\theta + b_m \sin m\theta), \qquad R_0 < r, \quad 0 \leq \theta \leq 2\pi. \tag{4.5}$$

Setting $r = R$ in (4.5), we have

$$u(R,\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(\frac{R_0}{R}\right)^m (a_m \cos m\theta + b_m \sin m\theta), \qquad 0 \leq \theta \leq 2\pi. \tag{4.6}$$

Inserting (4.6) and (4.4) into the forms of $b(u,v)$ (see (3.3)) and $b_{_N}(u,v)$ (see (3.5)), noting (4.1), (3.42), and the Schwarz inequality, we obtain

(4.7)

$$\begin{aligned} &|b(u,v) - b_{_N}(u,v)| \\ &= \left| \sum_{m=N+1}^{\infty} m\pi \left(\frac{R_0}{R}\right)^m (a_m c_m + b_m d_m) \right| \\ &\leq \frac{1}{(N+1)^{k-1}}\left(\frac{R_0}{R}\right)^{N+1} \sum_{m=N+1}^{\infty} \pi m^k (|a_m c_m| + |b_m d_m|) \\ &\leq \frac{1}{(N+1)^{k-1}}\left(\frac{R_0}{R}\right)^{N+1} \sqrt{\sum_{m=N+1}^{\infty} \pi m^{2k-1}(a_m^2 + b_m^2)} \cdot \sqrt{\sum_{m=N+1}^{\infty} \pi m (c_m^2 + d_m^2)} \\ &\leq \frac{C_0}{(N+1)^{k-1}}\left(\frac{R_0}{R}\right)^{N+1} |u|_{\Gamma_0}|_{k-\frac{1}{2},\Gamma_0} \cdot \|v\|_*. \end{aligned}$$

In a similar way, we can prove the case of $\beta_0 > 0$, just noting (3.23), (3.15).

THEOREM 4.2. *Let u be the solution of the problem (P) and u_{N}^h be the solution of the problem (P_{N}^h) with $\beta_0 = 0$. Suppose $f \in L^2(\Omega_i)$ and $u|_{\Gamma_0} \in H^{k-\frac{1}{2}}(\Gamma_0)$ ($k \geq 1$). Then we have the following error estimate:*

$$\|u - u_N^h\|_* \leq C_0 \left[\inf_{v^h \in V_0^h} \|u - v_h\|_* + \frac{1}{(N+1)^{k-1}} \left(\frac{R_0}{R}\right)^{N+1} |u|_{\Gamma_0}|_{k-\frac{1}{2},\Gamma_0} \right]. \tag{4.8}$$

Proof. Let $e := u - u_{N}^h$, $e^v := v^h - u$ and $e^h := v^h - u_{N}^h$. From (P) and (P_N^h) we have that

$$a(e, v^h) + b(u, v^h) - b_{N}(u_{N}^h, v^h) = 0 \qquad \forall\ v^h \in V_0^h. \tag{4.9}$$

From (3.2), (4.9) with $v^h = e^h$, (3.25), (3.26), and (4.2), we have that

$$\begin{aligned} \|e^h\|_*^2 &= a(e^h, e^h) \leq a(e^h, e^h) + b_{N}(e^h, e^h) \\ &= a(e^v, e^h) + b_{N}(e^v, e^h) + a(e, e^h) + b_{N}(e, e^h) \\ &= a(e^v, e^h) + b_{N}(e^v, e^h) + b_{N}(u, e^h) - b(u, e^h) \\ &\leq C_0 \left[\|e^v\|_* \cdot \|e^h\|_* + \frac{1}{(N+1)^{k-1}} \left(\frac{R_0}{R}\right)^{N+1} |u|_{\Gamma_0}|_{k-\frac{1}{2},\Gamma_0} \cdot \|e^h\|_* \right]. \end{aligned} \tag{4.10}$$

Thus

$$\|e^h\|_* \leq C_0 \left[\|e^v\|_* + \frac{1}{(N+1)^{k-1}} \left(\frac{R_0}{R}\right)^{N+1} |u|_{\Gamma_0}|_{k-\frac{1}{2},\Gamma_0} \right] \qquad \forall v^h \in V^h. \tag{4.11}$$

Then the desired result (4.8) follows from (4.11) and the triangle inequality.

If we suppose $u \in H^{p+1}(\Omega_i)$, $u|_{\Gamma_0} \in H^{p+\frac{1}{2}}(\Gamma_0)$ and the interpolation error of V_0^h approximate to V_0 is [3]

$$\inf_{v^h \in V_0^h} \|u - v^h\|_{1,\Omega_i} \leq C_0 h^p |u|_{p+1,\Omega_i}. \tag{4.12}$$

Then combining (4.12), (4.8), and the Poincaré inequality we get

(1) in the case of $\beta_0 = 0$

$$\begin{aligned} |u - u_{N}^h|_{1,\Omega_i} &\leq C_0 \|u - u_{N}^h\|_* \\ &\leq C_0 \left[h^p |u|_{p+1,\Omega_i} + \frac{1}{(N+1)^p} \left(\frac{R_0}{R}\right)^{N+1} |u|_{\Gamma_0}|_{p+\frac{1}{2},\Gamma_0} \right]; \end{aligned} \tag{4.13}$$

(2) in the case of $\beta_0 > 0$

$$\begin{aligned} \|u - u_{N}^h\|_{1,\Omega_i} &\leq C_0 \|u - u_{N}^h\|_* \\ &\leq C_0 \left[h^p |u|_{p+1,\Omega_i} + \frac{(1+\sqrt{\beta_0}R) K_{N+1}(\sqrt{\beta_0}R)}{(N+1)^{k-1} K_{N+1}(\sqrt{\beta_0}R_0)} |u|_{\Gamma_0}|_{p+\frac{1}{2},\Gamma_0} \right]. \end{aligned} \tag{4.14}$$

We now consider the other cases. For example we consider the problem (1.1)–(1.5) with Ω as a semi-infinite strip (Figure 1.3), $\beta_0 = 0$, $u = 0$ on Γ_U and Γ_L, $|\Gamma_N| = 0$, and $u \to 0$ when $x_2 \to \infty$. Let

$$V_g = \{v \in H^1(\Omega_i), v|_{\Gamma_i} = g\}.$$

Suppose that u is the unique solution of the above problem; then the restriction of u on the bounded domain Ω_i satisfies the following variational problem:

$$\text{Find } u \in V_g, \quad \text{such that}$$
$$a(u,v) + b'(u,v) = f(v) \quad \forall v \in V_0, \tag{4.15}$$

where

$$b'(u,v) = \frac{2\pi}{b^2}\sum_{m=1}^{\infty}\int_0^b\int_0^b \sin\frac{n\pi x_2}{b}\sin\frac{n\pi x_2'}{b}u(d,x_2)v(d,x_2')dx_2'dx_2.$$

Let

$$b_N'(u,v) = \frac{2\pi}{b^2}\sum_{m=1}^{N}\int_0^b\int_0^b \sin\frac{n\pi x_2}{b}\sin\frac{n\pi x_2'}{b}u(d,x_2)v(d,x_2')dx_2'dx_2.$$

Then the weak form of the problem with the artificial boundary condition on Γ_e shown in Table 2.2 is

$$\text{find } u_N \in V_g, \quad \text{such that}$$
$$a(u_N,v) + b_N'(u_N,v) = f(v) \quad \forall v \in V_0. \tag{4.16}$$

For the bilinear forms $b'(u,v)$ and $b_N'(u,v)$ we have the following estimates.

Suppose that u is the solution of the problem (1.1)–(1.5) and there exists an integer $k \geq 1$, such that $u|_{\Gamma_0} \in H^{k-1/2}(\Gamma_0)$. Then for any $v \in V_0$, we have

$$|b'(u,v) - b_N'(u,v)| \leq C\frac{1}{(N+1)^{k-1}}e^{-\pi(N+1)(d-d_0)/b}|u|_{k-1/2,\Gamma_0},$$

where C is a constant independent of h, N and d.

Assume that

$$u(d_0,x_2) = \sum_{m=1}^{\infty} a_m \sin\frac{m\pi x_2}{b}$$

with

$$a_m = \frac{2}{b}\int_0^b u(d_0,x_2')\sin\frac{m\pi x_2'}{b}dx_2'.$$

Then on the domain $d_0 \leq x_1 < +\infty, 0 \leq x_2 \leq b$,

$$u(x_1,x_2) = \sum_{m=1}^{\infty} a_m e^{-m\pi(x_1-d_0)/b}\sin\frac{m\pi x_2}{b}, \tag{4.17}$$

$$v(d,x_2) = \sum_{m=1}^{\infty} c_m \sin\frac{m\pi x_2}{b}. \tag{4.18}$$

Inserting (4.17) and (4.18) into the bilinear forms $b'(u,v)$ and $b_N'(u,v)$ we have

$$|b'(u,v) - b_N'(u,v)| = \frac{\pi}{b}\left|\sum_{m=N+1}^{\infty} m a_m c_m e^{-m\pi(d-d_0)/b}\right|$$

TABLE 4.1
Error estimate for different problems.

EP $\beta_0 = 0$	$\lvert u - u_N^h \rvert_{1,\Omega_i} \le C_0 \left[h^p \lvert u\rvert_{p+1,\Omega_i} + \frac{1}{(N+1)^{k-1}} \left(\frac{R_0}{R}\right)^{N+1} \left\lvert u\rvert_{\Gamma_0} \right\rvert_{k-\frac{1}{2},\Gamma_0} \right]$
EP $\beta_0 > 0$	$\lVert u - u_N^h \rVert_{1,\Omega_i} \le C_0 \left[h^p \lvert u\rvert_{p+1,\Omega_i} + \frac{(1+\sqrt{\beta_0}R) K_{N+1}(\sqrt{\beta_0}R)}{(N+1)^{k-1} K_{N+1}(\sqrt{\beta_0}R_0)} \left\lvert u\rvert_{\Gamma_0} \right\rvert_{k-\frac{1}{2},\Gamma_0} \right]$
PSS $\beta_0 = 0$	$\lvert u - u_N^h \rvert_{1,\Omega_i} \le C_0 \left[h^p \lvert u\rvert_{p+1,\Omega_i} + \frac{1}{(N+1)^{k-1}} e^{-(N+1)\pi(d-d_0)/b} \left\lvert u\rvert_{\Gamma_0} \right\rvert_{k-\frac{1}{2},\Gamma_0} \right]$
PSS $\beta_0 > 0$	$\lVert u - u_N^h \rVert_{1,\Omega_i} \le C_0 \left[h^p \lvert u\rvert_{p+1,\Omega_i} + \frac{1}{(N+1)^{k-1}} e^{-(d-d_0)\pi(N+1)/b} \left\lvert u\rvert_{\Gamma_0} \right\rvert_{k-\frac{1}{2},\Gamma_0} \right]$

$$
\begin{aligned}
&\le \frac{\pi}{b}\frac{1}{(N+1)^{k-1}} e^{-(N+1)\pi(d-d_0)/b} \sum_{m=N+1}^{\infty} m^{k-1/2} a_m m^{1/2} c_m \\
&\le C \frac{1}{(N+1)^{k-1}} e^{-(N+1)\pi(d-d_0)/b} \|u\|_{k-1/2,\Gamma_0} \|v\|_{1/2,\Gamma_e} \\
&\le C \frac{1}{(N+1)^{k-1}} e^{-(N+1)\pi(d-d_0)/b} \|u\|_{k-1/2,\Gamma_0} \|v\|_{*}
\end{aligned}
$$

The last inequality is from trace theorem. In a similar way, we can get the error estimate for this case and all other cases. They are summarized in Table 4.1. There EP and PSS denote "Exterior Problems" and "Problems in Semi-infinite Strip," respectively.

The similar results can be extended to the exterior problem of linear elastic equations, which will be reported in a separate paper.

5. Numerical implementation and results. In this section we present the numerical results which demonstrate the performance of the error estimate (4.13). In our computation continuous piecewise linear elements were used. That is to say, $p = 1$ in the interpolation error (4.12) [3].

Example 1. An exterior problem for Poisson equation.

We consider Poisson equation in the planar domain outside a circular obstacle of radius $a = 0.5$ (see Figure 1.1). The problem is governed by the following boundary value problem:

$$-\triangle u = f \qquad \text{in } \Omega = \{(r,\theta) \mid 0.5 < r, \qquad 0 \le \theta \le 2\pi\}, \tag{5.1}$$

$$u(0.5,\theta) = 0.4925 + 0.5 \ln \frac{20 + 4\sin\theta}{5 - \sin\theta}, \qquad 0 \le \theta \le 2\pi, \tag{5.2}$$

$$u \quad \text{is bounded} \qquad r \to +\infty, \tag{5.3}$$

where

$$
f(r,\theta) = \begin{cases} 8 - 16r^2, & 0.5 < r < 1.0 \quad 0 \le \theta \le 2\pi, \\ 0, & 1.0 \le r \quad 0 \le \theta \le 2\pi. \end{cases}
$$

This problem has an exact solution:

TABLE 5.1
The effect of the mesh size h for an exterior problem.

Mesh	$h = 0.31416$	$h = 0.15708$	$h = 0.07854$	$h = 0.03927$
$\max \vert u - u_N^h\vert$	1.3062E-2	3.2406E-3	8.0971E-4	2.0246E-4
$\Vert u - u_N^h\Vert_{0,\Omega_i}$	4.1749E-2	1.0810E-2	2.7280E-3	6.8367E-4
$\vert u - u_N^h\vert_{1,\Omega_i}$	0.5172	0.2752	0.13977	0.07016

(5.4)

$$u(r,\theta) = \begin{cases} 0.5 \ln \dfrac{4r^2 + 2r\sin\theta + 0.25}{r^2 - 0.5r\sin\theta + 0.0625} + (r^2-1)^2, & 0.5 \le r \le 1.0, \quad 0 \le \theta \le 2\pi, \\ 0.5 \ln \dfrac{4r^2 + 2r\sin\theta + 0.25}{r^2 - 0.5r\sin\theta + 0.0625}, & 1.0 \le r, \quad 0 \le \theta \le 2\pi. \end{cases}$$

First we test the effect of the mesh size h in the error estimate (4.13); we introduce a circular artificial boundary Γ_e of radius $R = R_0 = 1.0$. On Γ_e we apply the nonlocal artificial boundary condition (2.5) with $N = 0, 1, 2, \dots$. In the annular computational domain Ω_i, we use four meshes, respectively. The first mesh consists of 2 radial layers of elements, with 20 quadrilateral elements (in polar coordinates) in each layer. We denote it as 2×20. The other three meshes are 4×40, 8×80, and 16×160. Let g_h denote the interpolation of g on Γ_D, the boundary condition of u_h, the finite element approximation, is given by $u_h = g_h$ on Γ_D. Table 5.1 shows the maximum errors of $u - u_N^h$ over the mesh points, $\Vert u - u_N^h\Vert_{0,\Omega_i}$ and $|u - u_N^h|_{1,\Omega_i}$ for large N (say $N = 101$).

The results show that the convergent rates of $|u - u_N^h|_{1,\Omega_i}$ and $\Vert u - u_N^h\Vert_{0,\Omega_i}$ with respect to h is 1 and 2, respectively. Second we test the effect of N. Let u_∞^h denote the finite element approximation of the problem on the domain Ω_i with the mesh size h when N is very large (say $N = 101$). In this case $R_0 = R = 1.0$, so the effect of R in the error estimate (4.13) disappears. Figure 5.1 shows the errors $E_N := |u_\infty^h - u_N^h|_{k,\Omega_i}$ $(k = 0, 1)$ on the mesh 16×160 for different N. Third we test the effect of the location of the artificial boundary Γ_e. Let $\Omega_R = \{(r,\theta) \mid 0.5 < r < R \quad 0 \le \theta \le 2\pi\}$ denote the bounded computational domain with the artificial boundary Γ_R. We choose $R = 1.0,\ 1.5,\ 2.0,\ 2.5,\ 3.0$, respectively. The corresponding meshes we used were 8×40, 16×40, 24×40, 32×40, and 40×40. That is to say, each computational domain has a mesh with the fixed mesh size $h = 0.07854$. Let u_N^R denote the finite element approximation of the problem on the domain Ω_R with the corresponding mesh by using the nonlocal artificial boundary condition (2.5) on the artificial boundary Γ_R and u_∞^R corresponds to the solution when N is very large (say $N = 101$). Figure 5.2 shows the errors $E_R := |u_\infty^R - u_N^R|_{1,\Omega_R}$ for different R. The similar numerical results for a exterior problem of Laplace equation can also be found in [8].

Example 2. A problem for Poisson equation in semi-infinite strip.

The problem is governed by the following boundary value problem:

$$-\triangle u = f \qquad \text{in } \Omega = \{(x_1, x_2) \mid 0.0 < x_1 \quad 0 < x_2 < b\}, \tag{5.5}$$

$$u(0.0, x_2) = b + 0.25 + \sum_{m=1}^{50} \frac{1}{m^2} \cos\frac{m\pi x_2}{b}, \qquad 0 \le x_2 \le b, \tag{5.6}$$

$$u \quad \text{is bounded}, \qquad x_1 \to +\infty, \tag{5.7}$$

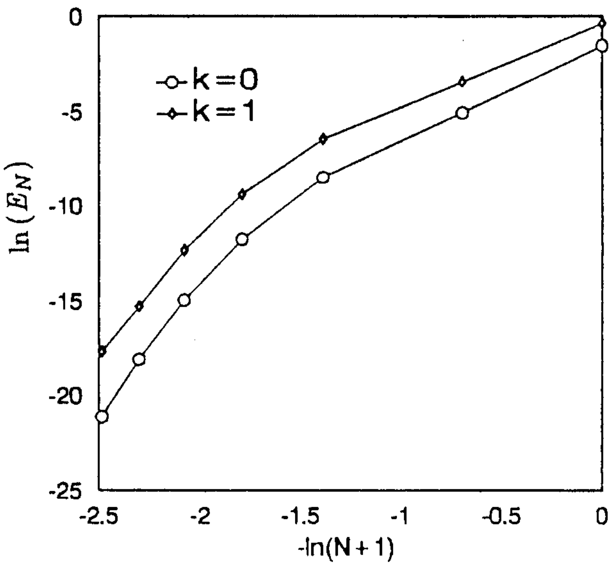

FIG. 5.1. *The effect of N for an exterior problem.*

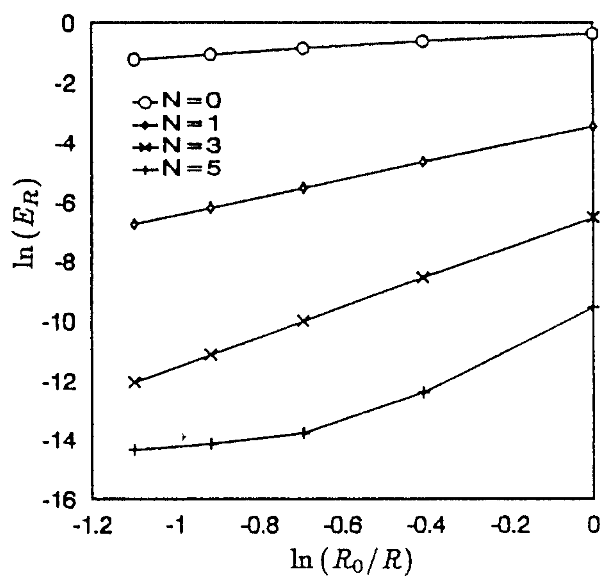

FIG. 5.2. *The effect of R for an exterior problem.*

where

$$f(x) = \begin{cases} -2, & 0.0 < x_1 < 0.5 \quad 0 < x_2 < b, \\ 0, & 0.5 \le x_1 < +\infty \quad 0 < x_2 < b. \end{cases}$$

This problem has an exact solution:

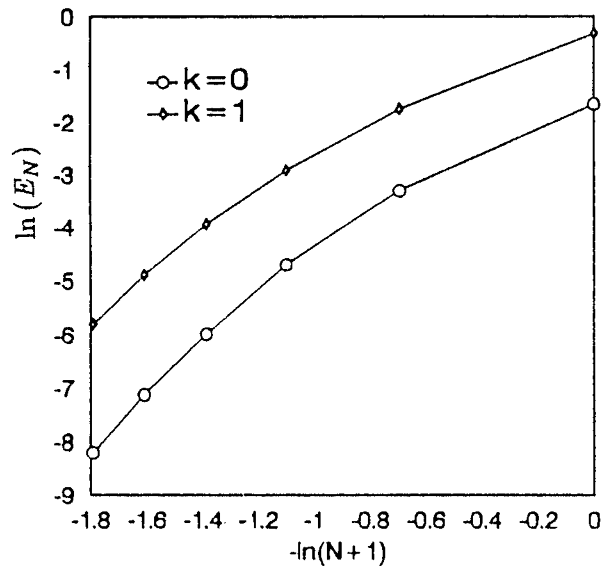

FIG. 5.3. *The effect of N for a problem in a semi-infinite strip.*

TABLE 5.2
The effect of the mesh size h for problem in a semi-infinite strip.

Mesh	$h = 0.25$	$h = 0.125$	$h = 0.0625$	$h = 0.03125$
$\max \lvert u - u_N^h \rvert$	1.5667E-2	7.3774E-3	6.9111E-3	2.1594E-3
$\lVert u - u_N^h \rVert_{0,\Omega_i}$	1.5903E-2	3.7692E-3	8.2931E-4	2.1206E-4
$\lvert u - u_N^h \rvert_{1,\Omega_i}$	0.2298	0.1246	6.8167E-2	3.5111E-2

(5.8)

$$u(x) = \begin{cases} b + \sum_{m=1}^{50} \frac{1}{m^2} e^{-\frac{m\pi x_1}{b}} \cos \frac{m\pi x_2}{b} + (x_1 - 0.5)^2, & 0 \le x_1, \quad 0 \le x_2 \le b, \\ b + \sum_{m=1}^{50} \frac{1}{m^2} e^{-\frac{m\pi x_1}{b}} \cos \frac{m\pi x_2}{b}, & 0 \le x_1, \quad 0 \le x_2 \le b. \end{cases}$$

We take $b = 2.5$ and introduce a segment $\Gamma_d = \{(d, x_2) \mid 0 \le x_2 \le b\}$ with $d = d_0 = 0.5$ as an artificial boundary to test the effect of mesh size h. On Γ_d we apply the nonlocal artificial boundary condition shown in Table 2.1 with $N = 0, 1, 2, \ldots$. Four uniform meshes 2×10, 4×20, 8×40 and 16×80 were used corresponding to the mesh size $h = 0.25,\ 0.125,\ 0.0625,\ 0.03125$ of the bounded computational domain $\Omega_d = \{x \in \mathcal{R}^2 \mid 0 < x_1 < d \quad 0 < x_2 < b\}$, respectively. Table 5.2 shows the maximum errors of $u - u_N^h$ over the mesh points, $\lVert u - u_N^h \rVert_{0,\Omega_i}$ and $\lvert u - u_N^h \rvert_{1,\Omega_i}$ for large N (say $N = 101$).

Furthermore, Figure 5.3 shows the errors $E_N := \lvert u_\infty^h - u_N^h \rvert_{k,\Omega_i}$ $(k = 1, 2)$ on the mesh 16×80 for different N. To test the effect of the location of the artificial boundary Γ_d, we choose $d = 0.5,\ 1.0,\ 1.5,\ 2.0,\ 2.5$, respectively. The uniform meshes we used for these bounded computational domains with the fixed mesh size $h = 0.0625$. That

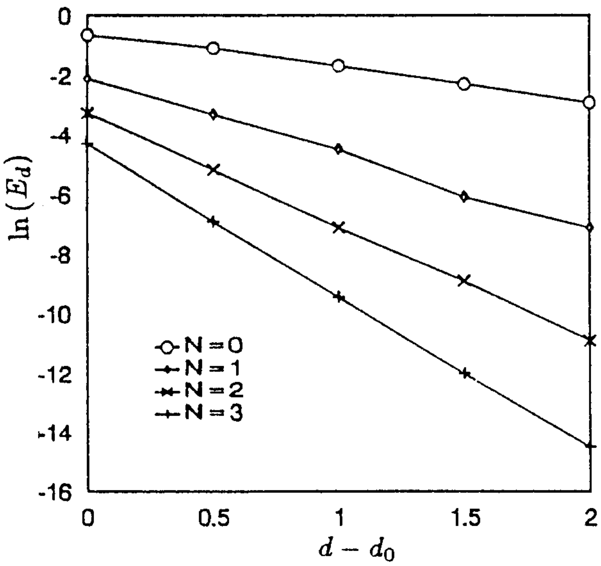

FIG. 5.4. *The effect of d for a problem in a semi-infinite strip.*

is to say, the corresponding meshes are 8×40, 16×40, 24×40, 32×40, and 40×40, respectively. Let u^d_N and u^d_∞ denote the similar meaning corresponding to Example 1. Figure 5.4 shows the errors $E_d := |u^d_\infty - u^d_N|_{1,\Omega_d}$ for different d.

Table 5.2 and Figures 5.3–5.4 demonstrate the error estimate in Table 4.1 for the problem in a semi-infinite strip.

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