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ERROR ESTIMATES FOR THE FINITE ELEMENT APPROXIMATION OF LINEAR ELASTIC EQUATIONS IN AN UNBOUNDED DOMAIN

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ABSTRACT. In this paper we present error estimates for the finite element approximation of linear elastic equations in an unbounded domain. The finite element approximation is formulated on a bounded computational domain using a nonlocal approximate artificial boundary condition or a local one. In fact there are a family of nonlocal approximate boundary conditions with increasing accuracy (and computational cost) and a family of local ones for a given artificial boundary. Our error estimates show how the errors of the finite element approximations depend on the mesh size, the terms used in the approximate artificial boundary condition, and the location of the artificial boundary. A numerical example for Navier equations outside a circle in the plane is presented. Numerical results demonstrate the performance of our error estimates.

1. INTRODUCTION

Let Γ_i be a bounded simple closed curve in $\mathbb{R}^2$ and Ω be the unbounded domain with the boundary Γ_i (see Figure 1). We consider the following linear elastic problem:

$$-\mu \triangle u - (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\, u = f \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on } \Gamma_i, \tag{1.2}$$

$$u \text{ is bounded when } \quad r = \sqrt{x_1^2 + x_2^2} \to +\infty, \tag{1.3}$$

where $x = (x_1, x_2)$ is the Cartesian coordinate system and the corresponding polar coordinate system is (r, θ), $u = (u_1, u_2)^T$ is the displacement, $\lambda,\ \mu > 0$ are the Lamé constants and $f = (f_1, f_2)^T$ is the density of applied body force whose support is compact.

Since the problem (1.1)-(1.3) is defined in the unbounded domain Ω, in finding the numerical solution of this problem, it is often difficult to use the classical finite element method or finite difference method directly. In the last two decades, several methods were proposed to solve boundary value problems in unbounded domains

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[6]. One of the most popular methods is to introduce an artificial boundary and set up artificial boundary conditions on it. Then the original problem is reduced to a boundary value problem in a bounded computational domain. Thus a numerical approximation of the original problem can be obtained by solving the reduced problem. In recent years many authors have worked on this subject for various problems by different techniques, see Engquist and Majda [3], Goldstein [11], Feng [4], Han and Wu [20, 21], Hagstrom and Keller [12, 13], Halpern and Schatzman [14], Han et al. [18, 19], Han and Bao [15, 16, 17], Givoli et al. [7, 8, 9], and the references therein.

In the above works, several authors also gave error estimates for the numerical solution, see [20, 21, 10]. But their error estimates only depend on the mesh size and the approximate artificial boundary condition. How the error depends on the location of the artificial boundary is unknown. But this is a very interesting problem for engineers. In this paper, we will discuss high-order local artificial boundary conditions and provide error estimates for the finite element approximation of the exterior problem (1.1)-(1.3). Our error estimates depend on not only the mesh size and the approximate artificial boundary condition but also the location of the artificial boundary.

The layout of this paper is as follows. In Section 2 we derive high-order local artificial boundary conditions at a given artificial boundary for the problem (1.1)-(1.3). In Section 3 we introduce the finite element formulation of the problem (1.1)-(1.3) in a bounded computational domain using an approximate nonlocal artificial boundary condition and prove an error estimate for the finite element approximation. In Section 4 we propose the finite element formulation of the problem (1.1)-(1.3) in a bounded computational domain using a high-order local artificial boundary condition and establish an error estimate for the finite element approximation. Finally in Section 5 we report on some numerical experiments.

2. High-order local artificial boundary conditions

In order to derive high-order local artificial boundary conditions, we recall here the derivation of the exact boundary condition at a given artificial boundary for the linear elastic problem (1.1)-(1.3) as described in [7, 21].

Introducing a circle Γ_e with radius R such that $\operatorname{supp} f \subset B_R(0) := \{x \in \mathbb{R}^2 : |x| < R\}$, then Ω is divided into two parts: the unbounded part $\Omega_e := \Omega \setminus B_R(0)$

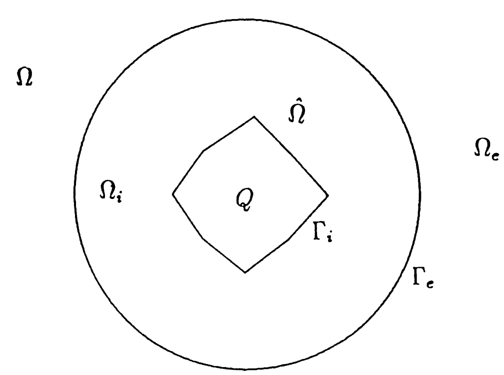

Figure 1.

and the bounded part $\Omega_i := \Omega \setminus \bar{\Omega}_e$ (see Figure 1). The restriction of a solution u of the problem (1.1)-(1.3) to the unbounded domain Ω_e is then a solution of the following problem:

$$-\mu \triangle u - (\lambda + \mu) \operatorname{grad} \operatorname{div} u = 0 \qquad \text{in } \Omega_e, \tag{2.1}$$

$$u|_{\Gamma_e} = u(R, \theta), \tag{2.2}$$

$$u \text{ is bounded when} \quad r = \sqrt{x_1^2 + x_2^2} \to +\infty. \tag{2.3}$$

We know that the general solution of (2.1)-(2.3) is (see [21] for detail)

$$u_i(r,\theta) = (r^2 - R^2) W_i(r,\theta) + G_i(r,\theta) \qquad R \le r < +\infty \quad 0 \le \theta \le 2\pi \qquad i = 1, 2, \tag{2.4}$$

where G_1, G_2, W_1 and W_2 are harmonic functions which satisfy

$$G_i(r,\theta) = \frac{a_0^i}{2} + \sum_{n=1}^{\infty} (a_n^i \cos n\theta + b_n^i \sin n\theta) \frac{R^n}{r^n} \qquad R \le r < +\infty \quad 0 \le \theta \le 2\pi \qquad i = 1, 2, \tag{2.5}$$

$$W_i(r,\theta) = \sum_{n=3}^{\infty} (p_n^i \cos n\theta + q_n^i \sin n\theta) \frac{R^{n-2}}{r^n} \qquad R \le r < +\infty \quad 0 \le \theta \le 2\pi \qquad i = 1, 2; \tag{2.6}$$

with $\kappa = \frac{\mu}{\lambda + \mu}$ and

$$a_n^i = \frac{1}{\pi} \int_0^{2\pi} G_i(R,\theta) \cos n\theta \, d\theta = \frac{1}{\pi} \int_0^{2\pi} u_i(R,\theta) \cos n\theta \, d\theta \qquad i = 1, 2 \quad n \ge 0, \tag{2.7}$$

$$b_n^i = \frac{1}{\pi} \int_0^{2\pi} G_i(R,\theta) \sin n\theta \, d\theta = \frac{1}{\pi} \int_0^{2\pi} u_i(R,\theta) \sin n\theta \, d\theta \qquad i = 1, 2 \quad n \ge 1; \tag{2.8}$$

$$p_n^1 = q_n^2 = \frac{n-2}{2+4\kappa} \left(a_{n-2}^1 - b_{n-2}^2 \right), \quad q_n^1 = -p_n^2 = \frac{n-2}{2+4\kappa} \left(b_{n-2}^1 + a_{n-2}^2 \right) \qquad n \ge 3. \tag{2.9}$$

Let $\varepsilon(u) = (\varepsilon_{ij}(u))_{2\times 2}$ and $\sigma(u) = (\sigma_{ij}(u))_{2\times 2}$ be the strain and stress tensors, respectively, which satisfy

$$\varepsilon_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \qquad \sigma_{ij}(u) = \lambda \operatorname{div} u \, \delta_{ij} + 2\mu \varepsilon_{ij}(u) \qquad 1 \le i, j \le 2. \tag{2.10}$$

Furthermore let $\sigma_n(u) = (\sigma_{n_1}(u), \sigma_{n_2}(u))^T$ be the normal stress corresponding to the displacement u at the artificial boundary Γ_e, say

$$\sigma_{n_i}(u) = \sigma_{i1}(u) \cos\theta + \sigma_{i2}(u) \sin\theta |_{\Gamma_e} \qquad i = 1, 2. \tag{2.11}$$

Combining (2.4) with $r = R$, (2.10) and (2.11), a computation shows (see details in [21])

(2.12)
$$\sigma_n(u) = \begin{pmatrix} \sigma_{n_1}(u) \\ \sigma_{n_2}(u) \end{pmatrix} = \begin{pmatrix} \dfrac{\mu(2+2\kappa)}{1+2\kappa} \left.\dfrac{\partial G_1}{\partial r}\right|_{r=R} - \dfrac{2\mu\kappa}{(1+2\kappa)R} \left.\dfrac{\partial G_2}{\partial \theta}\right|_{r=R} \\ \dfrac{\mu(2+2\kappa)}{1+2\kappa} \left.\dfrac{\partial G_2}{\partial r}\right|_{r=R} + \dfrac{2\mu\kappa}{(1+2\kappa)R} \left.\dfrac{\partial G_1}{\partial \theta}\right|_{r=R} \end{pmatrix}.$$

We differentiate (2.5) with respect to r on noting (2.4) and (2.12) and set $r = R$ to obtain

(2.13)
$$\begin{aligned} \sigma_{n_1}(u) &= -\frac{2+2\kappa}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{\infty} n\int_0^{2\pi} u_1(R,\phi)\cos n(\theta-\phi)\,d\phi - \frac{2\kappa}{1+2\kappa}\frac{\mu}{R}\frac{\partial u_2(R,\theta)}{\partial\theta} \\ &= \frac{-1}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{\infty} n\int_0^{2\pi} [(2+2\kappa)u_1(R,\phi)\cos n(\theta-\phi) \\ &\qquad\qquad - 2\kappa u_2(R,\theta)\sin n(\theta-\phi)]d\phi \\ &\equiv T_1(u), \end{aligned}$$

(2.14)
$$\begin{aligned} \sigma_{n_2}(u) &= \frac{2\kappa}{1+2\kappa}\frac{\mu}{R}\frac{\partial u_1(R,\theta)}{\partial\theta} - \frac{2+2\kappa}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{\infty} n\int_0^{2\pi} u_2(R,\phi)\cos n(\theta-\phi)\,d\phi \\ &= \frac{-1}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{N} n\int_0^{2\pi} [2\kappa u_1(R,\theta)\sin n(\theta-\phi) \\ &\qquad\qquad +(2+2\kappa)u_2(R,\phi)\cos n(\theta-\phi)]\,d\phi \\ &\equiv T_2(u). \end{aligned}$$

This is the desired exact boundary condition at Γ_e for the problem (1.1)-(1.3). Thus the restriction of the solution u of the problem (1.1)-(1.3) to the bounded domain Ω_i is a solution of the following problem.

(P) Find u such that

(2.15) $$-\mu \triangle u - (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\, u = f \qquad \text{in } \Omega_i,$$

(2.16) $$u = 0 \qquad \text{on } \Gamma_i,$$

(2.17) $$\sigma_n(u) = (T_1(u), T_2(u))^T \qquad \text{on } \Gamma_e.$$

Let

(2.18)
$$\begin{aligned} T_1^N(u) = \frac{-1}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{N} n\int_0^{2\pi} &[(2+2\kappa)u_1(R,\phi)\cos n(\theta-\phi) \\ &- 2\kappa u_2(R,\theta)\sin n(\theta-\phi)]d\phi, \end{aligned}$$

(2.19)
$$\begin{aligned} T_2^N(u) = \frac{-1}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{N} n\int_0^{2\pi} &[2\kappa u_1(R,\theta)\sin n(\theta-\phi) \\ &+ (2+2\kappa)u_2(R,\phi)\cos n(\theta-\phi)]d\phi. \end{aligned}$$

Then we obtain a series of approximate artificial boundary conditions at Γ_e

(2.20)
$$\sigma_n(u) = T^N(u) \equiv \left(T_1^N(u), T_2^N(u)\right)^T \qquad \text{on } \Gamma_e \qquad N = 0, 1, 2, \ldots,$$

where $T^0(u) = (0,0)^T$ is the stress free boundary condition which is often used in engineering literature. Then the original problem (1.1)-(1.3) can be reduced to the following problem defined on the bounded domain Ω_i approximately for $N = 0, 1, 2, \ldots$.

(P_N) Find $u_{_N}$ such that

$$-\mu \triangle u_{_N} - (\lambda + \mu) \operatorname{grad} \operatorname{div} u_{_N} = f \qquad \text{in } \Omega_i, \tag{2.21}$$

$$u_{_N} = 0 \qquad \text{on } \Gamma_i, \tag{2.22}$$

$$\sigma_n(u_{_N}) = T^N(u_{_N}) \qquad \text{on } \Gamma_e. \tag{2.23}$$

Now we discuss high-order local artificial boundary conditions at Γ_e for the problem (1.1)-(1.3). We consider a solution u of the problem (1.1)-(1.3), which consists of the first N harmonics at Γ_e. Thus we assume

$$u_i(R,\theta) = \frac{a_0^i}{2} + \sum_{n=1}^{N} \left(a_n^i \cos n\theta + b_n^i \sin n\theta\right) \qquad i = 1, 2, \tag{2.24}$$

where the a_n^1, b_n^1, a_n^2 and b_n^2 are constants (Fourier coefficients, see (2.7) and (2.8)). Substituting (2.24) into (2.13) and (2.14), we get

(2.25)
$$\begin{cases} \sigma_{n_1}(u) = -\dfrac{2+2\kappa}{1+2\kappa}\dfrac{\mu}{R}\displaystyle\sum_{n=1}^{N} n\left(a_n^1 \cos n\theta + b_n^1 \sin n\theta\right) - \dfrac{2\kappa}{1+2\kappa}\dfrac{\mu}{R}\dfrac{\partial u_2(R,\theta)}{\partial\theta}, \\[2ex] \sigma_{n_2}(u) = \dfrac{2\kappa}{1+2\kappa}\dfrac{\mu}{R}\dfrac{\partial u_1(R,\theta)}{\partial\theta} - \dfrac{2+2\kappa}{1+2\kappa}\dfrac{\mu}{R}\displaystyle\sum_{n=1}^{N} n\left(a_n^2 \cos n\theta + b_n^2 \sin n\theta\right). \end{cases}$$

It is desired to find a linear differential operator L_N which does not depend on n, such that

(2.26)
$$L_N[1] = 0 \quad L_N[\cos n\theta] = n\cos n\theta \quad L_N[\sin n\theta] = n \sin n\theta \qquad n = 1, 2, \ldots, N.$$

With such an operator at hand, noting (2.24), then (2.25) can be written

(2.27)
$$\begin{cases} \sigma_{n_1}(u) = -\dfrac{2+2\kappa}{1+2\kappa}\dfrac{\mu}{R} L_N[u_1(R,\theta)] - \dfrac{2\kappa}{1+2\kappa}\dfrac{\mu}{R}\dfrac{\partial u_2(R,\theta)}{\partial\theta}, \\[2ex] \sigma_{n_2}(u) = \dfrac{2\kappa}{1+2\kappa}\dfrac{\mu}{R}\dfrac{\partial u_1(R,\theta)}{\partial\theta} - \dfrac{2+2\kappa}{1+2\kappa}\dfrac{\mu}{R} L_N[u_2(R,\theta)]. \end{cases}$$

The equality (2.27) is a local boundary condition at Γ_e which is exact for all solutions consisting of at most the first N harmonics at Γ_e. Noting the fact that

(2.28)
$$\frac{d^{2m}}{d\theta^{2m}} \cos n\theta = (-1)^m n^{2m} \cos n\theta, \quad \frac{d^{2m}}{d\theta^{2m}} \sin n\theta = (-1)^m n^{2m} \sin n\theta$$
$$m \geq 0 \quad n \geq 0,$$

TABLE 1. The coefficients $\alpha_m^{(N)}$ in the first five local artificial boundary conditions

	$\alpha_1^{(N)}$	$\alpha_2^{(N)}$	$\alpha_3^{(N)}$	$\alpha_4^{(N)}$	$\alpha_5^{(N)}$
$N=1$	1				
$N=2$	7/6	−1/6			
$N=3$	74/60	−15/60	1/60		
$N=4$	533/420	−43/144	11/360	−1/1008	
$N=5$	3881/3780	−214/643	71/1728	−13/6048	1/25920

we can assume the operator L_N has the following form:

$$L_N[u(R,\theta)] = \sum_{m=1}^{N} (-1)^m \alpha_m^{(N)} \frac{\partial^{2m}}{\partial\theta^{2m}} u(R,\theta). \tag{2.29}$$

Inserting (2.29) into (2.27), noting (2.25) and (2.24), we obtain

$$\sum_{m=1}^{N} n^{2m} \alpha_m^{(N)} = n \qquad n = 1, 2, \ldots, N. \tag{2.30}$$

It is straightforward to check that the linear system (2.30) has a unique solution for any $n \in \mathbb{N}$. Table 1 shows the coefficients $\alpha_m^{(N)}$ in the first five local artificial boundary conditions. In the paper [25] by A. Sidi, a similar linear system was discussed. Combining (2.27) and (2.29), we get high-order local artificial boundary conditions at Γ_e for the problem (1.1)-(1.3):

$$\sigma_n(u) = \tilde{T}^N(u) \equiv \left(\tilde{T}_1^N(u), \tilde{T}_2^N(u)\right)^T \qquad N = 1, 2, \ldots, \tag{2.31}$$

where

$$\begin{cases} \tilde{T}_1^N(u) = -\dfrac{2+2\kappa}{1+2\kappa}\dfrac{\mu}{R}\displaystyle\sum_{m=1}^{N}(-1)^m\alpha_m^{(N)}\dfrac{\partial^{2m}u_1(R,\theta)}{\partial\theta^{2m}} - \dfrac{2\kappa}{1+2\kappa}\dfrac{\mu}{R}\dfrac{\partial u_2(R,\theta)}{\partial\theta}, \\ \tilde{T}_2^N(u) = \dfrac{2\kappa}{1+2\kappa}\dfrac{\mu}{R}\dfrac{\partial u_1(R,\theta)}{\partial\theta} - \dfrac{2+2\kappa}{1+2\kappa}\dfrac{\mu}{R}\displaystyle\sum_{m=1}^{N}(-1)^m\alpha_m^{(N)}\dfrac{\partial^{2m}u_2(R,\theta)}{\partial\theta^{2m}}. \end{cases} \tag{2.32}$$

Then the original problem (1.1)-(1.3) can be reduced to the following problem defined on the bounded domain Ω_i approximately for $N = 1, 2, \ldots$.
($\tilde{\mathrm{P}}_N$) Find $\tilde{u}_{N}$ such that

$$-\mu \triangle \tilde{u}_N - (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\tilde{u}_N = f \qquad \text{in } \Omega_i, \tag{2.33}$$

$$\tilde{u}_N = 0 \qquad \text{on } \Gamma_i, \tag{2.34}$$

$$\sigma_n(\tilde{u}_N) = \tilde{T}^N(\tilde{u}_N) \qquad \text{on } \Gamma_e. \tag{2.35}$$

3. The error estimates for the case of using nonlocal artificial boundary conditions

In the work [21], the authors have already given error estimates for the finite element approximation of the problem (P_N). But from their estimates, we don't know how the errors depend on the location of the artificial boundary. In this section, we will present new error estimates for the finite element approximation of problems (P_N). These error estimates depend not only the mesh size and the approximate artificial boundary condition but also the location of the artificial boundary. This kind of error estimate is very useful in engineering applications.

Let $H^m(\Omega_i)$ and $H^s(\Gamma_e)$ be the usual Sobolev spaces on the domain Ω_i and the boundary Γ_e with integer m and real number s. Suppose

$$V = \{v = (v_1, v_2)^T \in H^1(\Omega_i) \times H^1(\Omega_i) \mid \ v|_{\Gamma_i} = 0\}.$$

Then the boundary value problem (P) is equivalent to the following variational problem.

(VP) Find $u \in V$ such that

$$a(u, v) + b(u, v) = f(v) \qquad \forall v \in V, \tag{3.1}$$

where

(3.2)

$$\begin{aligned} a(u, v) &= \int_{\Omega_i} \left[\lambda \operatorname{div} u \operatorname{div} v + 2\mu \sum_{i,j=1}^{2} \varepsilon_{ij}(u)\varepsilon_{ij}(v) \right] dx \\ &\equiv \int_{\Omega_i} [\lambda \operatorname{div} u \operatorname{div} v + 2\mu\varepsilon(u) : \varepsilon(v)]\, dx \qquad \forall u, v \in V, \end{aligned}$$

(3.3)

$$\begin{aligned} b(u, v) = &\frac{2+2\kappa}{1+2\kappa} \frac{\mu}{\pi} \sum_{n=1}^{\infty} n \int_0^{2\pi} \int_0^{2\pi} [u_1(R, \phi)v_1(R, \theta) \\ &\qquad + u_2(R, \phi)v_2(R, \theta)] \cos n(\theta - \phi) d\theta\, d\phi \\ &+ \frac{2\kappa}{1+2\kappa} \frac{\mu}{\pi} \sum_{n=1}^{\infty} n \int_0^{2\pi} \int_0^{2\pi} [u_1(R, \phi)v_2(R, \theta) \\ &\qquad - u_2(R, \phi)v_1(R, \theta)] \sin n(\theta - \phi) d\theta\, d\phi \\ = &\frac{2+2\kappa}{1+2\kappa} \frac{\mu}{\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \left[\frac{\partial u_1(R, \phi)}{\partial \phi} \frac{\partial v_1(R, \theta)}{\partial \theta} + \frac{\partial u_2(R, \phi)}{\partial \phi} \frac{\partial v_2(R, \theta)}{\partial \theta} \right] \\ &\qquad \cdot \frac{\cos n(\theta - \phi)}{n} d\theta\, d\phi \\ &+ \frac{2\kappa}{1+2\kappa} \frac{\mu}{\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \left[\frac{\partial u_1(R, \phi)}{\partial \phi} \frac{\partial v_2(R, \theta)}{\partial \theta} - \frac{\partial u_2(R, \phi)}{\partial \phi} \frac{\partial v_1(R, \theta)}{\partial \theta} \right] \\ &\qquad \cdot \frac{\sin n(\theta - \phi)}{n} d\theta\, d\phi \qquad \forall u, v \in V, \end{aligned}$$

(3.4)

$$f(v) = \int_{\Omega_i} f \cdot v \, dx \qquad \forall v \in V.$$

Furthermore let

(3.5)

$$b_{N}(u,v) = \frac{2+2\kappa}{1+2\kappa}\frac{\mu}{\pi}\sum_{n=1}^{N} n\int_0^{2\pi}\int_0^{2\pi}[u_1(R,\phi)v_1(R,\theta) + u_2(R,\phi)v_2(R,\theta)]\cos n(\theta-\phi)d\theta\, d\phi + \frac{2\kappa}{1+2\kappa}\frac{\mu}{\pi}\sum_{n+1}^{N} n\int_0^{2\pi}\int_0^{2\pi}[u_1(R,\phi)v_2(R,\theta) - u_2(R,\phi)v_1(R,\theta)]\sin n(\theta-\phi)d\theta\, d\phi \quad \forall u,v\in V.$$

Then the boundary value problem (P_N) is equivalent to the following variational problem.

(VP_N) Find $u_N \in V$ such that

$$a(u_N,v) + b_N(u_N,v) = f(v) \qquad \forall v\in V. \tag{3.6}$$

If we replace V by its finite element subspace V^h in which h represents the mesh size [11], then the finite element approximation of the problem (VP_N) is as follows.

(VP_N^h) Find $u_N^h \in V^h$ such that

$$a(u_N^h,v^h) + b_N(u_N^h,v^h) = f(v^h) \qquad \forall v^h\in V^h. \tag{3.7}$$

We note that the symmetric bilinear form $a(\cdot,\cdot)$ is bounded and coercive on $V\times V$ from the Körn inequality [23] and Poincaré inequality [1], i.e., there exist positive constants M_1, M_2 such that

$$|a(u,v)| \le M_1\|u\|_V\cdot\|v\|_V \qquad \forall u,v\in V, \tag{3.8}$$

$$M_2\|v\|_V^2 \le a(v,v) \qquad \forall v\in V. \tag{3.9}$$

Thus we can define an equivalent norm on the space V:

$$\|v\|_* = [a(v,v)]^{1/2} \qquad \forall v\in V. \tag{3.10}$$

Therefore we have that

$$|a(u,v)| \le \|u\|_*\cdot\|v\|_* \qquad \forall u,v\in V, \tag{3.11}$$

$$\|v\|_*^2 \le a(v,v) \qquad \forall v\in V. \tag{3.12}$$

For the symmetric bilinear forms $b(\cdot,\cdot)$ and $b_N(\cdot,\cdot)$, we have that

Lemma 3.1. *The following inequality holds:*

$$0\le b_N(v,v)\le b(v,v)\le a(v,v)\equiv\|v\|_*^2 \qquad \forall v\in V \quad N\ge 0, \tag{3.13}$$

$$|b(u,v)|\le\|u\|_*\cdot\|v\|_*, \quad |b_N(u,v)|\le\|u\|_*\cdot\|v\|_* \qquad \forall u,v\in V \quad N\ge 0. \tag{3.14}$$

where $b(u,v)$ and $b_N(u,v)$ are defined in (3.3) and (3.5), respectively.

Proof. For any given $u,v\in V$, we formally expand $u|_{\Gamma_e} = u(R,\theta)$ and $v|_{\Gamma_e} = v(R,\theta)$ in Fourier series, i.e.,

$$u_i(R,\theta) = \tfrac{a_0^i}{2} + \sum_{n=1}^{\infty}(a_n^i\cos n\theta + b_n^i\sin n\theta) \qquad i=1,2, \tag{3.15}$$

$$v_i(R,\theta) = \tfrac{c_0^i}{2} + \sum_{n=1}^{\infty}(c_n^i\cos n\theta + d_n^i\sin n\theta) \qquad i=1,2, \tag{3.16}$$

where $a_n^i,\ b_n^i$ are defined in (2.7) and (2.8), and

(3.17)
$$c_n^i = \frac{1}{\pi}\int_0^{2\pi} v_i(R,\theta)\cos n\theta\, d\theta, \quad d_n^i = \frac{1}{\pi}\int_0^{2\pi} v_i(R,\theta)\sin n\theta\, d\theta, \qquad i=1,2 \quad n\geq 0.$$

Inserting (3.15) and (3.16) into (3.3) and (3.5), we get

(3.18)
$$b(u,v) = \frac{2\mu\pi}{1+2\kappa}\sum_{n=1}^{\infty} n\left[\sum_{i=1}^{2}(a_n^i c_n^i + b_n^i d_n^i) + \kappa(a_n^1 + b_n^2)(c_n^1 + d_n^2) + \kappa(b_n^1 - a_n^2)(d_n^1 - c_n^2)\right],$$

(3.19)
$$b_N(u,v) = \frac{2\mu\pi}{1+2\kappa}\sum_{n=1}^{N} n\left[\sum_{i=1}^{2}(a_n^i c_n^i + b_n^i d_n^i) + \kappa(a_n^1 + b_n^2)(c_n^1 + d_n^2) + \kappa(b_n^1 - a_n^2)(d_n^1 - c_n^2)\right].$$

We denote Q the domain enclosed by Γ_i, and $\hat{\Omega}$ the disk bounded by Γ_e (i.e., $\hat{\Omega} = Q\cup\Omega_i\cup\Gamma_i$) (see Figure 1). Then for any $v\in V$, we define the function $v^{(0)}$ which satisfies

$$-\mu\triangle v^{(0)} - (\lambda+\mu)\,\mathrm{grad\,div}\, v^{(0)} = 0 \qquad \text{in } \Omega_i, \tag{3.20}$$
$$v^{(0)} = v \qquad \text{on} \quad \Gamma_e\cup\Gamma_i. \tag{3.21}$$
$$v^{(0)} \equiv 0 \qquad \text{in} \quad Q. \tag{3.22}$$

Then we know $v^{(0)}\in\left[H^1(\hat{\Omega})\right]^2$. We also define the function $v^{(1)}\in\left[H^1(\hat{\Omega})\right]^2$ which satisfies

$$-\mu\triangle v^{(1)} - (\lambda+\mu)\,\mathrm{grad\,div}\, v^{(1)} = 0 \qquad \text{in } \hat{\Omega}, \tag{3.23}$$
$$v^{(1)} = v \qquad \text{on} \quad \Gamma_e. \tag{3.24}$$

Then $v^{(0)}|_{\Omega_i}$ minimizes the functional $\int_{\Omega_i}\left[\lambda(\mathrm{div}\, w)^2 + 2\mu|\varepsilon(w)|^2\right]dx$ among all functions $w\in[H^1(\Omega_i)]^2$ which are equal to v on $\Gamma_e\cup\Gamma_i$. Similarly, $v^{(1)}$ minimizes the functional $\int_{\hat{\Omega}}\left[\lambda(\mathrm{div}\, w)^2 + 2\mu|\varepsilon(w)|^2\right]dx$ among all functions $w\in\left[H^1(\hat{\Omega})\right]^2$ which are equal to v on Γ_e. Therefore we have that

$$\begin{aligned} a(v,v) &= \int_{\Omega_i}\left[\lambda(\mathrm{div}\, v)^2 + 2\mu|\varepsilon(v)|^2\right]dx \geq a(v^{(0)},v^{(0)}) \\ &= \int_{\hat{\Omega}}\left[\lambda(\mathrm{div}\, v^{(0)})^2 + 2\mu|\varepsilon(v^{(0)})|^2\right]dx \\ &\geq \int_{\hat{\Omega}}\left[\lambda(\mathrm{div}\, v^{(1)})^2 + 2\mu|\varepsilon(v^{(1)})|^2\right]dx. \end{aligned} \tag{3.25}$$

Recalling that $v^{(1)}$ is a solution of the problem (3.23)-(3.24), using separation of variables and noting $v^{(1)}|_{\Gamma_e} = v^{(1)}(R,\theta) = v(R,\theta)$, we obtain

(3.26)
$$\begin{aligned} v_1^{(1)}(r,\theta) = \frac{R^2-r^2}{2+4\kappa}\sum_{n=0}^{\infty}(n+2)[(c_{n+2}^1 - d_{n+2}^2)\cos n\theta \\ + (d_{n+2}^1 + c_{n+2}^2)\sin n\theta]\frac{r^n}{R^{n+2}} \\ + \frac{c_0^1}{2} + \sum_{n=1}^{\infty}\frac{r^n}{R^n}(c_n^1\cos n\theta + d_n^1 \sin n\theta) \qquad 0\le r\le R \quad 0\le\theta\le 2\pi, \end{aligned}$$

(3.27)
$$\begin{aligned} v_2^{(1)}(r,\theta) = \frac{R^2-r^2}{2+4\kappa}\sum_{n=0}^{\infty}(n+2)[-(d_{n+2}^1 + c_{n+2}^2)\cos n\theta \\ + (c_{n+2}^1 - d_{n+2}^2)\sin n\theta]\frac{r^n}{R^{n+2}} \\ + \frac{c_0^2}{2} + \sum_{n=1}^{\infty}\frac{r^n}{R^n}(c_n^2\cos n\theta + d_n^2 \sin n\theta) \qquad 0\le r\le R \quad 0\le\theta\le 2\pi, \end{aligned}$$

where

(3.28)
$$c_n^i = \frac{1}{\pi}\int_0^{2\pi} v_i^{(1)}(R,\theta)\cos n\theta\, d\theta = \frac{1}{\pi}\int_0^{2\pi} v_i(R,\theta)\cos n\theta\, d\theta \qquad i=1,2 \quad n\ge 0,$$

(3.29)
$$d_n^i = \frac{1}{\pi}\int_0^{2\pi} v_i^{(1)}(R,\theta)\sin n\theta\, d\theta = \frac{1}{\pi}\int_0^{2\pi} v_i(R,\theta)\sin n\theta\, d\theta \qquad i=1,2 \quad n\ge 1.$$

Combining (3.25), (3.26) and (3.27) with $r=R$, (3.28), (3.29), (2.10) and (2.11) with $u=v^{(1)}$, (3.18) and (3.19) with $u=v$, and integration by parts, we obtain

(3.30)
$$\begin{aligned} a(v,v) &\ge \int_{\hat\Omega}\left[\lambda(\operatorname{div} v^{(1)})^2 + 2\mu|\varepsilon(v^{(1)})|^2\right]dx = \int_{\Gamma_e}\sigma_n(v^{(1)})\cdot v^{(1)}\, ds \\ &= \frac{2+2\kappa}{1+2\kappa}\frac{\mu}{\pi}\sum_{n=1}^{\infty} n\int_0^{2\pi}\int_0^{2\pi}[v_1^{(1)}(R,\phi)v_1^{(1)}(R,\theta) \\ &\qquad + v_2^{(1)}(R,\phi)v_2^{(1)}(R,\theta)]\cos n(\theta-\phi)d\theta\, d\phi \\ &\quad + \frac{2\kappa}{1+2\kappa}\frac{\mu}{\pi}\sum_{n=1}^{\infty} n\int_0^{2\pi}\int_0^{2\pi}[v_1^{(1)}(R,\phi)v_2^{(1)}(R,\theta) \\ &\qquad - v_2^{(1)}(R,\phi)v_1^{(1)}(R,\theta)]\sin n(\theta-\phi)d\theta\, d\phi \\ &= b(v^{(1)},v^{(1)}) = b(v,v) \\ &= \frac{2\mu\pi}{1+2\kappa}\sum_{n=1}^{\infty} n\left[(c_n^1)^2 + (d_n^1)^2 + (c_n^2)^2 + (d_n^2)^2 + \kappa(c_n^1+d_n^2)^2 + \kappa(d_n^1 - c_n^2)^2\right] \\ &\ge \frac{2\mu\pi}{1+2\kappa}\sum_{n=1}^{N} n\left[(c_n^1)^2 + (d_n^1)^2 + (c_n^2)^2 + (d_n^2)^2 + \kappa(c_n^1+d_n^2)^2 + \kappa(d_n^1 - c_n^2)^2\right] \\ &= b_N(v,v) \ge 0. \end{aligned}$$

Then the desired inequality (3.13) is proved. Thus the inequality (3.14) follows from (3.13) and the Schwarz inequality immediately.

It follows immediately from (3.11), (3.12) (3.14) and (3.13) that the variational problems (VP), (VP_N) and (VP_N^h) are well posed; that is, for $f \in V'$, the dual of V, there exists a unique $u \in V$ solving (VP), a unique $u_{N} \in V$ solving (VP_N), a unique $u_{N}^h \in V^h$ solving (VP_N^h), and

$$\|u\|_* + \|u_{N}\|_* + \|u_{N}^h\|_* \leq 3\|f\|_{V'}. \tag{3.31}$$

Note that the well-posedness of (VP) implies immediately the well-posedness of the original problem (1.1)-(1.3).

Let $R_0 = \max\{|x| \ : \ x \in \ \operatorname{supp} f \cup \Gamma_i\}$, $\Gamma_0 = \{(R_0, \theta) \ : \ 0 \leq \theta \leq 2\pi\}$ and $\Omega_0 = \{x \in \Omega_i \ : \ |x| < R_0\}$ and $\Gamma_r = \{(r, \theta) \ : \ 0 \leq \theta \leq 2\pi\}$. We recall an equivalent definition of Sobolev space $H^s(\Gamma_r)$ for any real number s [22]:

$$w \in H^s(\Gamma_r) \Longleftrightarrow w(r,\theta) = \frac{p_0}{2} + \sum_{m=1}^{\infty} (p_m \cos m\theta + q_m \sin m\theta)$$

and

$$\frac{\pi p_0^2}{2} + \sum_{m=1}^{\infty} \pi (1+m^2)^s (p_m^2 + q_m^2) < \infty.$$

Thus we use

$$|w|_{s,\Gamma_r} = \left[\sum_{m=1}^{\infty} \pi m^{2s} (p_m^2 + q_m^2) \right]^{1/2} \tag{3.32}$$

as a semi-norm of the space $H^s(\Gamma_r)$. Then we have the following estimate.

Lemma 3.2. *Suppose $u \in V$ is a solution of the exterior problem* (1.1)-(1.3) *and there exists an integer $k \geq 1$ such that $u|_{\Gamma_0} \in [H^{k+\frac{1}{2}}(\Gamma_0)]^2$. Then we have that*

(3.33)
$$|b(u,v) - b_{N}(u,v)| \leq \frac{C_0}{(N+1)^{k-1}} \left(\frac{R_0}{R} \right)^{\max\{1,N-1\}} |u|_{k+\frac{1}{2},\Gamma_0} \cdot \|v\|_* \qquad \forall v \in V,$$

where C_0 is a generic constant independent of u, N, h and R.

Proof. Assume that

$$u_i(R_0,\theta) = \tfrac{p_0^i}{2} + \sum_{n=1}^{\infty} (p_n^i \cos n\theta + q_n^i \sin n\theta) \qquad i = 1,2, \tag{3.34}$$

$$v_i(R,\theta) = \tfrac{c_0^i}{2} + \sum_{n=1}^{\infty} (c_n^i \cos n\theta + d_n^i \sin n\theta) \qquad i = 1,2, \tag{3.35}$$

where c_n^i and d_n^i are defined in (3.28) and (3.29), respectively, and

(3.36)
$$p_n^i = \frac{1}{\pi} \int_0^{2\pi} u_i(R_0,\theta) \cos n\theta \, d\theta, \quad q_n^i = \frac{1}{\pi} \int_0^{2\pi} u_i(R_0,\theta) \sin n\theta \, d\theta, \qquad i = 1,2 \quad n \geq 0.$$

Note that u satisfies the homogeneous Navier equations (say (1.1) with $f = 0$) in the domain $\{x \,:\, |x| > R_0\}$. By separation of variables, we get

(3.37)
$$\begin{aligned} u_1(r,\theta) = \frac{r^2 - R_0^2}{2+4\kappa} \sum_{n=3}^{\infty} (n-2)[(p_{n-2}^1 - q_{n-2}^2)\cos n\theta \\ + (q_{n-2}^1 + p_{n-2}^2)\sin n\theta] \frac{R_0^{n-2}}{r^n} \\ + \frac{p_0^1}{2} + \sum_{n=1}^{\infty} (p_n^1 \cos n\theta + q_n^1 \sin n\theta) \frac{R_0^n}{r^n} \qquad R_0 \le r \quad 0 \le \theta \le 2\pi, \end{aligned}$$

(3.38)
$$\begin{aligned} u_2(r,\theta) = \frac{r^2 - R_0^2}{2+4\kappa} \sum_{n=3}^{\infty} (n-2)[-(q_{n-2}^1 + p_{n-2}^2)\cos n\theta \\ + (p_{n-2}^1 - q_{n-2}^2)\sin n\theta] \frac{R_0^{n-2}}{r^n} \\ + \frac{p_0^2}{2} + \sum_{n=1}^{\infty} (p_n^2 \cos n\theta + q_n^2 \sin n\theta) \frac{R_0^n}{r^n} \qquad R_0 \le r \quad 0 \le \theta \le 2\pi. \end{aligned}$$

Setting $r = R$ in (3.37) and (3.38), we obtain

$$u_i(R,\theta) = \frac{a_0^i}{2} + \sum_{n=1}^{\infty} (a_n^i \cos n\theta + b_n^i \sin n\theta), \qquad i = 1, 2, \tag{3.39}$$

where

(3.40)
$$a_n^1 = \left(\frac{R_0}{R}\right)^n \begin{cases} p_n^1 & n = 0,1,2, \\ p_n^1 + \frac{(n-2)(R^2-R_0^2)}{(2+4\kappa)R_0^2}(p_{n-2}^1 - q_{n-2}^2) & n \ge 3; \end{cases}$$

(3.41)
$$b_n^1 = \left(\frac{R_0}{R}\right)^n \begin{cases} q_n^1 & n = 1,2, \\ q_n^1 + \frac{(n-2)(R^2-R_0^2)}{(2+4\kappa)R_0^2}(q_{n-2}^1 + p_{n-2}^2) & n \ge 3; \end{cases}$$

(3.42)
$$a_n^2 = \left(\frac{R_0}{R}\right)^n \begin{cases} p_n^2 & n = 0,1,2, \\ p_n^2 - \frac{(n-2)(R^2-R_0^2)}{(2+4\kappa)R_0^2}(q_{n-2}^1 + p_{n-2}^2) & n \ge 3; \end{cases}$$

(3.43)
$$b_n^2 = \left(\frac{R_0}{R}\right)^n \begin{cases} q_n^2 & n = 1,2, \\ q_n^2 + \frac{(n-2)(R^2-R_0^2)}{(2+4\kappa)R_0^2}(p_{n-2}^1 - q_{n-2}^2) & n \ge 3. \end{cases}$$

From (3.39), (3.35), (3.18) and (3.19), noting (3.40)-(3.43), (3.36), (3.14) and (3.13), we obtain

(3.44)
$$
\begin{aligned}
&|b(u,v) - b_{_N}(u,v)| \\
&\quad = \frac{2\mu\pi}{1+2\kappa}\left|\sum_{n=N+1}^{\infty} n\left[\sum_{i=1}^{2}(a_n^i c_n^i + b_n^i d_n^i) + \kappa(a_n^1 + b_n^2)(c_n^1 + d_n^2)\right.\right. \\
&\qquad\qquad \left.\left. + \kappa(b_n^1 - a_n^2)(d_n^1 - c_n^2)\right]\right| \\
&\quad \le \frac{2\mu\pi}{1+2\kappa}\left[\sum_{n=N+1}^{\infty} n((a_n^1)^2 + (b_n^1)^2 + (a_n^2)^2 + (b_n^2)^2\right. \\
&\qquad\qquad \left. + \kappa(a_n^1 + b_n^2)^2 + \kappa(b_n^1 - a_n^2)^2)\right]^{1/2} \\
&\qquad \cdot \left[\sum_{n=N+1}^{\infty} n\left((c_n^1)^2 + (d_n^1)^2 + (c_n^2)^2 + (d_n^2)^2 + \kappa(c_n^1 + d_n^2)^2 + \kappa(c_n^1 - d_n^2)^2\right)\right]^{1/2} \\
&\quad \le C_0\left[\sum_{n=N+1}^{\infty} n\sum_{i=1}^{2}\left((p_n^i)^2 + (q_n^i)^2\right)\frac{R_0^{2n}}{R^{2n}}\right. \\
&\qquad\qquad \left. + \sum_{n=\max\{1,N-1\}}^{\infty} n^3\sum_{i=1}^{2}\left((p_n^i)^2 + (q_n^i)^2\right)\frac{R_0^{2n}}{R^{2n}}\right]^{1/2} \cdot \|v\|_* \\
&\quad \le \frac{C_0}{(N+1)^{k-1}}\left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{k+\frac{1}{2},\Gamma_0} \cdot \|v\|_*.
\end{aligned}
$$

Combining Lemmas 3.1 and 3.2, we get the following error estimate.

Theorem 3.1. *Suppose u is the solution of the problem* (1.1)-(1.3) *and $u_{_N}^h$ is the solution of the problem (VP_N^h). Suppose $f \in [L^2(\Omega_i)]^2$ and $u|_{\Gamma_0} \in [H^{k+\frac{1}{2}}(\Gamma_0)]^2$ ($k \ge 1$). Then we have the following error estimate:*

(3.45)
$$
\|u - u_{_N}^h\|_* \le C_0\left[\inf_{v^h \in V^h}\|u - v_h\|_* + \frac{1}{(N+1)^{k-1}}\left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{k+\frac{1}{2},\Gamma_0}\right].
$$

Proof. Let $e := u - u_{_N}^h$, $e^v := v^h - u$ and $e^h := v^h - u_{_N}^h$. Subtracting (3.7) from (3.1), we have that

(3.46)
$$
a(e, v^h) + b_{_N}(e, v^h) = b(u, v^h) - b_{_N}(u, v^h) \qquad \forall\ v^h \in V^h.
$$

From (3.12), (3.11), (3.14), (3.13), (3.46) with $v^h = e^h$ and (3.33), we have that

(3.47)
$$\begin{aligned}\|e^h\|_*^2 &= a(e^h, e^h) \le a(e^h, e^h) + b_{_N}(e^h, e^h)\\ &= a(e^v, e^h) + b_{_N}(e^v, e^h) + a(e, e^h) + b_{_N}(e, e^h)\\ &= a(e^v, e^h) + b_{_N}(e^v, e^h) + b_{_N}(u, e^h) - b(u, e^h)\\ &\le 2\|e^v\|_* \cdot \|e^h\|_* + \frac{C_0}{(N+1)^{k-1}}\left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{k+\frac{1}{2},\Gamma_0} \cdot \|e^h\|_*.\end{aligned}$$

Thus

(3.48)
$$\|e^h\|_* \le C_0\left[\|e^v\|_* + \frac{1}{(N+1)^{k-1}}\left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{k+\frac{1}{2},\Gamma_0}\right] \qquad \forall v^h \in V^h.$$

Then the desired result (3.45) follows from (3.48) and the triangle inequality.

If we suppose $u \in [H^{p+1}(\Omega_i)]^2$, $u|_{\Gamma_0} \in [H^{p+\frac{1}{2}}(\Gamma_0)]^2$, and the interpolation error of V^h approximate to V is [2]

$$\inf_{v^h \in V^h} \|u - v^h\|_V \le C_0 h^p |u|_{p+1,\Omega_i}, \tag{3.49}$$

then combining (3.49) and (3.45), noting the Körn inequality and Poincaré inequality, we get

(3.50)
$$\begin{aligned}\|u - u_{_N}^h\|_{1,\Omega_0} \le C_0|u - u_{_N}^h|_{1,\Omega_0} &\le C_0 \int_{\Omega_0}\left[\lambda|\mathrm{div}(u - u_{_N}^h)|^2 + 2\mu|\varepsilon(u - u_{_N}^h)|^2\right] dx\\ &\le C_0\, a(u - u_{_N}^h, u - u_{_N}^h)\\ &\le C_0\left[h^p|u|_{p+1,\Omega_i} + \frac{1}{(N+1)^{p-1}}\left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{p+\frac{1}{2},\Gamma_0}\right].\end{aligned}$$

4. The error estimates for the case of using high-order local artificial boundary conditions

In this section, we will present the finite element formulation of the problem $(\tilde{\mathrm{P}}_N)$ and provide an error estimate for the finite element approximation. To cope with the high-order local artificial boundary condition (2.31), we define

$$\tilde{V} = \{v \in [H^1(\Omega_i)]^2 \mid v|_{\Gamma_e} \in [H^N(\Gamma_e)]^2 \quad v|_{\Gamma_i} = 0\}.$$

Let

(4.1)
$$\begin{aligned}\tilde{b}_{_N}(u, v) = &-\int_{\Gamma_e} v \cdot \tilde{T}^N(u)\, ds\\ = &\frac{\mu(2+2\kappa)}{1+2\kappa}\int_0^{2\pi} \sum_{m=1}^{N} \alpha_m^{(N)}\left[\frac{\partial^m u_1(R,\theta)}{\partial\theta^m}\frac{\partial^m v_1(R,\theta)}{\partial\theta^m}\right.\\ &\left.+\frac{\partial^m u_2(R,\theta)}{\partial\theta^m}\frac{\partial^m v_2(R,\theta)}{\partial\theta^m}\right] d\theta\\ &+ \frac{2\mu\kappa}{1+2\kappa}\int_0^{2\pi}\left[\frac{\partial u_2(R,\theta)}{\partial\theta} v_1(R,\theta) - \frac{\partial u_1(R,\theta)}{\partial\theta} v_2(R,\theta)\right] d\theta \quad \forall u, v \in \tilde{V}.\end{aligned}$$

Then the weak form of the problem $(\tilde{\mathrm{P}}_N)$ is as follows.
$(\tilde{\mathrm{V}}\mathrm{P}_N)$ Find $\tilde{u}_{N} \in \tilde{V}$ such that

$$a(\tilde{u}_{N}, v) + \tilde{b}_{N}(\tilde{u}_{N}, v) = f(v) \qquad \forall v \in \tilde{V}. \tag{4.2}$$

If we replace $\tilde{V}$ by its finite dimensional subspace, $\tilde{V}^h \subset \tilde{V}$ in which h is the mesh size [2] (a family of such subspaces were introduced by Givoli et al. [10]), then the finite element approximation of the problem $(\tilde{\mathrm{V}}\mathrm{P}_N)$ is as follows.
$(\tilde{\mathrm{V}}\mathrm{P}_N^h)$ Find $\tilde{u}_{N}^h \in \tilde{V}^h$ such that

$$a(\tilde{u}_{N}^h, v^h) + \tilde{b}_{N}(\tilde{u}_{N}^h, v^h) = f(v^h) \qquad \forall v^h \in \tilde{V}^h. \tag{4.3}$$

From (3.11) and (3.12), the well-posedness of the problems $(\tilde{\mathrm{V}}\mathrm{P}_N)$ and $(\tilde{\mathrm{V}}\mathrm{P}_N^h)$ depend on the property of the symmetric bilinear form $\tilde{b}_{N}(u, v)$. For any $u, v \in \tilde{V}$, we can also expand $u|_{\Gamma_e} = u(R, \theta)$ and $v|_{\Gamma_e} = v(R, \theta)$ in Fourier series (see (3.15) and (3.16)). Substituting (3.15) and (3.16) into (4.1) and using the orthogonality of the cosines and sines, we obtain

(4.4)
$$\begin{aligned}\tilde{b}_{N}(u, v) = \frac{2\mu}{1+2\kappa}\sum_{n=1}^{\infty}\big[\gamma_n^{(N)}\left(a_n^1 c_n^1 + b_n^1 d_n^1 + a_n^2 c_n^2 + b_n^2 d_n^2\right)\\ + \kappa n(a_n^1 + b_n^2)(c_n^1 + d_n^2) + \kappa n(b_n^1 - a_n^2)(d_n^1 - c_n^2)\big] \qquad \forall u, v \in \tilde{V},\end{aligned}$$

(4.5)
$$\begin{aligned}\tilde{b}_{N}(v, v) = \frac{2\mu}{1+2\kappa}\sum_{n=1}^{\infty}\Bigg[\gamma_n^{(N)}\sum_{i=1}^{2}((c_n^i)^2 + (d_n^i)^2)\\ + \kappa n(c_n^1 + d_n^2)^2 + \kappa n(d_n^1 - c_n^2)^2\Bigg] \quad \forall v \in \tilde{V},\end{aligned}$$

where

(4.6)
$$\gamma_n^{(N)} = (1+\kappa)\sum_{m=1}^{N} n^{2m}\alpha_m^{(N)} - \kappa n \equiv (1+\kappa)\beta_n^{(N)} - \kappa n \qquad \forall n \in \mathbb{N}.$$

Thus the property of $\tilde{b}_{N}(u, v)$ depends on the property of $\gamma_n^{(N)}$ (or $\beta_n^{(N)}$).

Table 3 shows $\alpha_m^{(N)}$ is positive for odd m, and is negative for even $m > 0$ for $1 \le N \le 5$. This property can be demonstrated numerically for $1 \le N \le 20$ (in fact this property can be proved for any positive integer N using the method given by Sidi in [25]). Thus we have that

(4.7)
$$\alpha_1^{(N)} > 0, \qquad \alpha_N^{(N)} = \begin{cases} > 0 & N \text{ is odd}, \\ < 0 & N \text{ is even}, \end{cases} \qquad 1 \le N \le 20.$$

From the engineering application point of view, the parameter N in (2.31) is always less than 10. Therefore in this paper we assume $N \le 20$ in (2.31).

Then for the $\beta_n^{(N)}$ we have

Lemma 4.1. *If $1 \le N \le 20$ is odd, then*

$$\beta_n^{(N)} \ge n \quad \forall n \ge 1 \qquad\qquad \lim_{n\to+\infty}\frac{\beta_n^{(N)}}{n^{2N}} = \alpha_N^{(N)} > 0. \tag{4.8}$$

If $1 \le N \le 20$ *is even, then*

(4.9)

$$\beta_n^{(N)} < 0 \qquad \text{when } n \text{ is sufficiently large} \qquad \lim_{n\to+\infty} \frac{\beta_n^{(N)}}{n^{2N}} = \alpha_N^{(N)} < 0.$$

Proof. We set a polynomial function whose degree is $2N$, say

$$\eta_N(t) = \sum_{m=1}^{N} \alpha_m^{(N)} t^{2m} - t. \tag{4.10}$$

Since $\eta_N''(t)$ is an even polynomial function whose degree is $2N-2$ and $\eta_N''(0) = 2\alpha_1^{(N)} > 0$ for $1 \le N \le 20$ by noting (4.7), we know that $\eta_N''(t) = 0$ has at most $N-1$ nonnegative roots. Thus $\eta_N(t) = 0$ has at most $N+1$ nonnegative roots. From (4.10) and (2.30), we know that $t = 0, 1, 2, \ldots, N$ are roots of $\eta_N(t) = 0$. Thus for $1 \le N \le 20$, we have that

$$\eta_N(t) \ne 0 \quad \forall t > N \qquad \lim_{t\to+\infty} \frac{\eta_N(t)}{t^{2N}} = \alpha_N^{(N)}. \tag{4.11}$$

Then the desired inequalities (4.8) and (4.9) follow immediately from (4.11) and (4.7).

From the above discussion we have

Lemma 4.2. *For odd* $1 \le N \le 20$*, there exist two generic positive constants* $C_N^{(1)}$ *and* $C_N^{(2)}$ *depending only on* N *such that*

$$|\tilde{b}_N(u,v)| \le C_N^{(2)} |u|_{N,\Gamma_R} \cdot |v|_{N,\Gamma_R} \qquad \forall u, v \in \tilde{V}, \tag{4.12}$$

$$C_N^{(1)} |v|_{N,\Gamma_R}^2 \le \tilde{b}_N(v,v) \qquad \forall v \in \tilde{V}. \tag{4.13}$$

Proof. For any odd $1 \le N \le 20$, noting (4.6) and (4.8), we know that there exist positive constants $C_N^{(1)}$ and $C_N^{(2)}$ such that

$$C_N^{(1)} n^{2N} \le \gamma_n^{(N)} \le C_N^{(2)} n^{2N} \qquad n = 1, 2, 3, \ldots. \tag{4.14}$$

From (4.14) and (4.4), noting (3.32) with $r = R$, (3.15) and (3.16), we have that

(4.15)

$$\begin{aligned}
\left|\tilde{b}_N(u,v)\right| &\le C_N^{(2)} \sum_{n=1}^{\infty} n^{2N} \left[\left| a_n^1 c_n^1 + b_n^1 d_n^1 + a_n^2 c_n^2 + b_n^2 d_n^2 \right| \right. \\
&\qquad\qquad \left. + \left| a_n^1 d_n^2 - b_n^1 c_n^2 - a_n^2 d_n^1 + b_n^2 c_n^1 \right| \right] \\
&\le C_N^{(2)} \left[\sum_{n=1}^{\infty} n^{2N} \sum_{i=1}^{2} \left[(a_n^i)^2 + (b_n^i)^2 \right] \right]^{1/2} \\
&\quad \cdot \left[\sum_{n=1}^{\infty} n^{2N} \sum_{i=1}^{2} \left[(d_n^i)^2 + (c_n^i)^2 \right] \right]^{1/2} \\
&\le C_N^{(2)} |u|_{N,\Gamma_e} \cdot |v|_{N,\Gamma_e} \qquad \forall u, v \in \tilde{V}.
\end{aligned}$$

Furthermore from (4.14) and (4.5), noting (3.32) with $r = R$ and (3.16), we obtain

$$
\begin{aligned}
\tilde{b}_{N}(v,v) &\geq \frac{2\mu}{1+2\kappa}\sum_{n=1}^{\infty}\gamma_n^{(N)}\left((c_n^1)^2+(d_n^1)^2+(c_n^2)^2+(d_n^2)^2\right)\\
&\geq C_N^{(1)}\sum_{n=1}^{\infty}n^{2N}\left((c_n^1)^2+(d_n^1)^2+(c_n^2)^2+(d_n^2)^2\right)\\
&= C_N^{(1)}|v|^2_{N,\Gamma_e} \qquad \forall v\in\tilde{V}.
\end{aligned}
\tag{4.16}
$$

Thus the desired inequalities (4.12) and (4.13) are proved.

From the discussion above, noting the Körn inequality and Poincaré inequality, we assign the following norm on $\tilde{\mathrm{V}}$:

$$
\|v\|_{\triangle} := \left[a(v,v)+|v|^2_{N,\Gamma_e}\right]^{1/2} \equiv \left[\|v\|_*^2+|v|^2_{N,\Gamma_e}\right]^{1/2} \qquad \forall v\in\tilde{V}.
\tag{4.17}
$$

It follows immediately from (3.11), (3.12), (4.12), (4.13), (4.6), (4.8) and (4.9) that the variational problems ($\tilde{\mathrm{V}}\mathrm{P}_N$) and ($\tilde{\mathrm{V}}\mathrm{P}_N^h$) are well posed in the case of odd $1 \leq N \leq 20$ or $N = 0$ and they are not well posed in the case of even $0 < N \leq 20$; that is, for $f \in \tilde{V}'$, the dual of $\tilde{V}$, there exists a unique $\tilde{u}_N \in \tilde{V}$ solving ($\tilde{\mathrm{V}}\mathrm{P}_N$), a unique $\tilde{u}_N^h \in \tilde{V}^h$ solving ($\tilde{\mathrm{V}}\mathrm{P}_N^h$), and

$$
\|\tilde{u}_N\|_{\triangle} + \|\tilde{u}_N^h\|_{\triangle} \leq M_N\|f\|_{\tilde{V}'} \qquad \forall \text{ odd } 1\leq N\leq 20,
\tag{4.18}
$$

where M_N is a constant.

Then we have the following estimate.

Lemma 4.3. *Suppose $u \in \tilde{V}$ is a solution of the exterior problem* (1.1)-(1.3) *and $u|_{\Gamma_0} \in [H^{N+1}(\Gamma_0)]^2$. Then we have the following estimate for odd $1 \leq N \leq 20$:*

$$
|b(u,v)-\tilde{b}_N(u,v)| \leq C_{(N)}\left(\frac{R_0}{R}\right)^{\max\{1,N-1\}}|u|_{N+1,\Gamma_0}\cdot|v|_{N,\Gamma_e} \qquad \forall v\in\tilde{V},
\tag{4.19}
$$

where $C_{(N)}$ is a constant independent of u, R and h.

Proof. Subtracting (3.18) from (4.4), noting (4.6), (4.8) (3.40)-(3.43) and (3.32), we obtain for odd $1 \le N \le 20$

(4.20)
$$\begin{aligned}
&\left| b(u,v) - \tilde{b}_{N}(u,v) \right| \\
&\quad = \frac{2\mu\pi}{1+2\kappa} \left| \sum_{n=N+1}^{\infty} (1+\kappa) \left(\sum_{m=1}^{N} n^{2m} \alpha_m^{(N)} - n \right) \left(a_n^1 c_n^1 + b_n^1 d_n^1 + a_n^2 c_n^2 + b_n^2 d_n^2 \right) \right| \\
&\quad \le C_{(N)} \sum_{n=N+1}^{\infty} n^{2N} \left| a_n^1 c_n^1 + b_n^1 d_n^1 + a_n^2 c_n^2 + b_n^2 d_n^2 \right| \\
&\quad \le C_{(N)} \left[\sum_{n=N+1}^{\infty} n^{2N} \sum_{i=1}^{2} \left((a_n^i)^2 + (b_n^i)^2 \right) \right]^{1/2} \\
&\qquad \cdot \left[\sum_{n=N+1}^{\infty} n^{2N} \sum_{i=1}^{2} \left((c_n^i)^2 + (d_n^i)^2 \right) \right]^{1/2} \\
&\quad \le C_{(N)} \left[\sum_{n=\max\{1,N-1\}}^{\infty} n^{2N+2} \left((p_n^1)^2 + (q_n^1)^2 + (p_n^2)^2 + (q_n^2)^2 \right) \frac{R_0^{2n}}{R^{2n}} \right]^{1/2} \cdot |v|_{N,\Gamma_e} \\
&\quad \le C_{(N)} \left(\frac{R_0}{R} \right)^{\max\{1,N-1\}} |u|_{N+1,\Gamma_0} \cdot |v|_{N,\Gamma_e} \qquad \forall v \in \tilde{V}.
\end{aligned}$$

Combining Lemmas 4.2 and 4.3, we get the following error estimate.

Theorem 4.1. *Let u be the solution of the problem (1.1)-(1.3) and $\tilde{u}_N^h$ the solution of the problem $(\tilde{V}P_N^h)$. Suppose $f \in [L^2(\Omega_i)]^2$ and $u|_{\Gamma_0} \in [H^{N+1}(\Gamma_0)]^2$. Then we have the following error estimate for odd $1 \le N \le 20$:*

(4.21)
$$\|u - \tilde{u}_N^h\|_\triangle \le C_{(N)} \left[\inf_{v^h \in \tilde{V}^h} \|u - v_h\|_\triangle + \left(\frac{R_0}{R} \right)^{\max\{1,N-1\}} |u|_{N+1,\Gamma_0} \right]$$

with constant $C_{(N)}$ independent of u, R and h.

Proof. The proof of this theorem is similarly to the proof of Theorem 3.1. It is omitted here.

If we suppose $u \in [H^{p+1}(\Omega_i)]^2$, $u|_{\Gamma_e} \in [H^{p+N}(\Gamma_e)]^2$ and the interpolation error of $\tilde{V}^h$ approximate to $\tilde{V}$ is

$$\inf_{v^h \in \tilde{V}^h} \|u - v^h\|_\triangle \le C_0 h^p \left[|u|_{p+1,\Omega_i} + |u|_{p+N,\Gamma_e} \right] \tag{4.22}$$

(a family of this kind of finite element subspaces was proposed in [10]), then combining (4.22) and (4.21), noting the Körn inequality and Poincaré inequality, we get for odd $1 \le N \le 20$

(4.23)
$$\begin{aligned}
\|u - \tilde{u}_N^h\|_{1,\Omega_0} &\le C_0 |u - \tilde{u}_N^h|_{1,\Omega_0} \le C_0 \|u - \tilde{u}_N^h\|_\triangle \\
&\le C_{(N)} \left[h^p \left(|u|_{p+1,\Omega_i} + |u|_{p+N,\Gamma_e} \right) + \left(\frac{R_0}{R} \right)^{\max\{1,N-1\}} |u|_{N+1,\Gamma_0} \right].
\end{aligned}$$

5. Numerical implementation and results

In this section we present the numerical results which demonstrate the performance of the error estimates (3.50) and (4.23). In our example, we take $\mu = 1$, $x^+ = (0, 0.25)$, and $x^- = (0, -0.25)$, and the unbounded domain $\Omega = \{x \in \mathbb{R}^2 \ : \ 0.5 < |x|\}$ is the exterior domain outside a circle $\Gamma_i = \{x \in \mathbb{R}^2 \ : \ |x| = 0.5\}$. In our computation, when dealing with nonlocal approximate artificial boundary conditions (2.20), continuous piecewise linear elements were used throughout the domain Ω_i. When dealing with high-order local artificial boundary conditions (2.31), continuous piecewise bilinear elements were used throughout the domain Ω_i, except in the single layer elements adjacent to the artificial boundary Γ_e. There, special finite elements, $C^{2,1}_{\Gamma_e}$ (which were introduced by Givoli et al., [10] and has $C^2(\Gamma_e)$ regularity at Γ_e), were used. That is to say, $p = 1$ in the interpolation errors (3.49) and (4.22) [2, 10].

Example. An exterior problem for Navier equations.

We consider the Navier equations in the planar domain outside a circular obstacle of radius $a = 0.5$ (see Figure 1). The problem is governed by the following boundary value problem:

$$-\mu \triangle u - (\lambda + \mu) \operatorname{grad} \operatorname{div} u = f = (f_1, f_2)^T \qquad \text{in } \Omega = \{x \ : \ 0.5 < |x|\}, \tag{5.1}$$

$$u(0.5, \theta) = g(\theta) = (g_1, g_2)^T \qquad \text{on } \Gamma_i = \partial\Omega, \tag{5.2}$$

$$u \text{ is bounded when } \quad r = \sqrt{x_1^2 + x_2^2} \to +\infty, \tag{5.3}$$

where

$$f_1(x) = \begin{cases} -8x_2[(3\lambda + 6\mu)x_1^2 + (\lambda + 4\mu)x_2^2 - (\lambda + 3\mu)] & 0.5 \le |x| < 1.0, \\ 0 & 1.0 \le |x|, \end{cases}$$

$$f_2(x) = \begin{cases} -8x_1[(\lambda + 4\mu)x_1^2 + (3\lambda + 6\mu)x_2^2 - (\lambda + 3\mu)] & 0.5 \le |x| < 1.0, \\ 0 & 1.0 \le |x|; \end{cases}$$

$$g_1(\theta) = \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)} \ln \frac{1.25 + \sin\theta}{1.25 - \sin\theta} + \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \frac{2\cos^3\theta \sin\theta}{1.5625 - \sin^2\theta} + 0.28125 \sin\theta \quad 0 \le \theta < 2\pi,$$

$$g_2(\theta) = \frac{(\lambda + \mu)\cos\theta}{\mu(\lambda + 2\mu)} \left[\frac{\sin\theta - 0.5)}{1.25 - \sin\theta} - \frac{\sin\theta + 0.5}{1.25 + \sin\theta} \right] + 0.28125 \cos\theta \quad 0 \le \theta < 2\pi.$$

This problem has an exact solution:

$$u_1(x) = \begin{cases} \frac{\lambda+3\mu}{\mu(\lambda+2\mu)} \ln \frac{|x-x^-|}{|x-x^+|} + \frac{\lambda+\mu}{\mu(\lambda+\mu)} \left[\frac{x_1^2}{|x-x^+|^2} - \frac{x_1^2}{|x-x^-|^2} \right] \\ \qquad + x_2(|x|^2 - 1)^2 & 0.5 \le |x| < 1.0, \\ \frac{\lambda+3\mu}{\mu(\lambda+2\mu)} \ln \frac{|x-x^-|}{|x-x^+|} + \frac{\lambda+\mu}{\mu(\lambda+\mu)} \left[\frac{x_1^2}{|x-x^+|^2} - \frac{x_1^2}{|x-x^-|^2} \right] & 1.0 \le |x|; \end{cases}$$

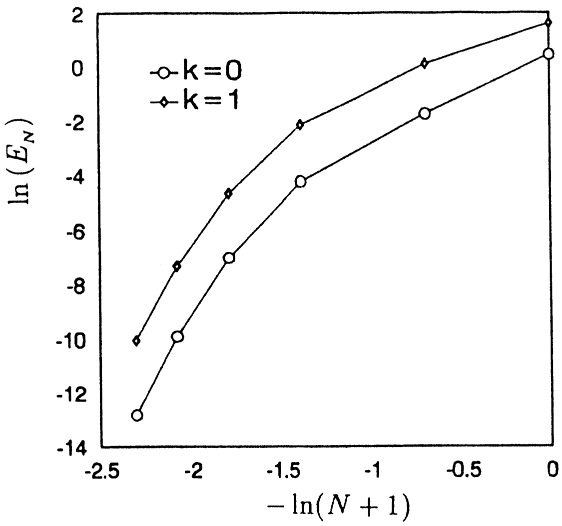

FIGURE 2. The effect of N using nonlocal artificial boundary conditions

TABLE 2. The effect of the mesh size h using nonlocal artificial boundary conditions

Mesh	$h = 0.31416$	$h = 0.15708$	$h = 0.07854$	$h = 0.03927$
$\max\lvert u - u_N^h\rvert$	1.5631E-2	4.1784E-3	1.0707E-3	2.6869E-4
$\lVert u - u_N^h\rVert_{0,\Omega_0}$	4.4819E-2	1.2438E-2	3.1956E-3	8.0471E-4
$\lvert u - u_N^h\rvert_{1,\Omega_0}$	0.7745	0.4067	0.2060	0.1034

$$u_2(x) = \begin{cases} \frac{\lambda+\mu}{\mu(\lambda+2\mu)}\left[\frac{x_1(x_2-x_2^+)}{|x-x^+|^2} - \frac{x_1(x_2-x_2^-)}{|x-x^-|^2}\right] & \\ \qquad + x_1(|x|^2-1)^2 & 0.5 \le |x| < 1.0, \\ \frac{\lambda+\mu}{\mu(\lambda+2\mu)}\left[\frac{x_1(x_2-x_2^+)}{|x-x^+|^2} - \frac{x_1(x_2-x_2^-)}{|x-x^-|^2}\right] & 1.0 \le |x|. \end{cases}$$

First we test the effect of the mesh size h in the error estimates (3.50). We introduce a circular artificial boundary $\Gamma_e = \Gamma_0$ of radius $R = R_0 = 1.0$. On Γ_0 we apply the nonlocal artificial boundary condition (2.20) with $N = 0, 1, 2, \ldots$ or high-order local artificial boundary condition (2.31). In the annular computational domain Ω_0, we use four meshes. The first mesh consists of 2 radial layers of elements, with 20 quadrilateral elements in each layer. We denote it as 2×20. The other three meshes are 4×40, 8×80 and 16×160. Table 2 shows the maximum errors of $u - u_N^h$ over the mesh points, $\|u - u_N^h\|_{0,\Omega_0}$ and $|u - u_N^h|_{1,\Omega_0}$ for large N (say $N = 51$).

The results show that the convergent rates of $|u - u_N^h|_{1,\Omega_0}$ and $\|u - u_N^h\|_{0,\Omega_0}$ with respect to h are 1 and 2 when using nonlocal artificial boundary conditions, respectively. Second we test the effect of N in the error estimate (3.50). Let u_∞^h denote the finite element approximation of the problem on the domain Ω_0 with

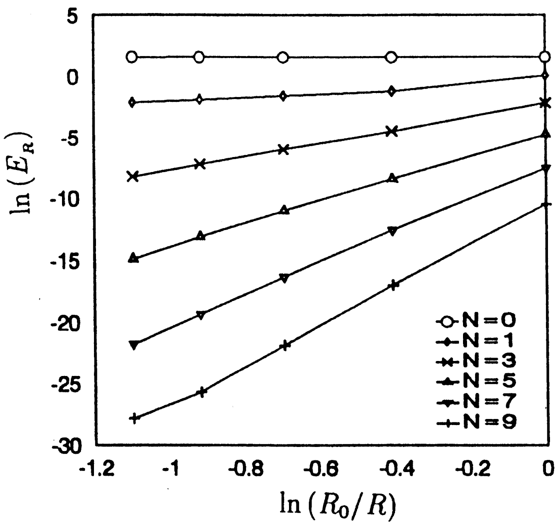

FIGURE 3. The effect of R using nonlocal artificial boundary conditions

TABLE 3. The effect of the location of Γ_R using a local artificial boundary condition ($N = 1$)

Location	$R = 1.0$	$R = 1.5$	$R = 2.0$	$R = 2.5$	$R = 3.0$
$\max \lvert u - \tilde{u}_N^h \rvert$	9.5675E-2	6.0485E-2	4.3960E-2	3.4578E-2	2.8531E-2
$\lVert u - \tilde{u}_N^h \rVert_{0,\Omega_0}$	5.9929E-2	2.2710E-2	1.5862E-2	1.0996E-2	7.8054E-3
$\lvert u - \tilde{u}_N^h \rvert_{1,\Omega_0}$	0.6290	0.3570	0.3096	0.2740	0.2483

the mesh size h when N is very large (say $N = 51$). In this case $R_0 = R$, so the effect of R in the error estimate (3.50) disappears. Figure 2 shows the errors $E_N := \|u_\infty^h - u_N^h\|_{k,\Omega_0}$ ($k = 0, 1$) on the mesh 16×160 for different N. Third we test the effect of the location of the artificial boundary Γ_e. Let $\Omega_R = \{x \ : \ 0.5 < |x| < R\}$ denotes the bounded computational domain with the artificial boundary Γ_R. We choose $R = 1.0,\ 1.5,\ 2.0,\ 2.5,\ 3.0$. The corresponding meshes we used were 8×40, 16×40, 24×40, 32×40 and 40×40, respectively. That is to say, each computational domain has a mesh with the fixed mesh size $h = 0.07854$. Let u_N^R denote the finite element approximation of the problem on the domain Ω_R with the corresponding mesh by using the nonlocal artificial boundary condition (2.20) on the artificial boundary Γ_R. u_∞^R corresponds to the solution when N is very large (say $N = 51$) and $\tilde{u}_N^R$ corresponds to the solution using the high-order local artificial boundary condition (2.31) at Γ_R. Tables 3 and 4 shows the maximum errors of $u - \tilde{u}_N^R$ over the mesh points, $\|u - \tilde{u}_N^R\|_{0,\Omega_0}$ and $|u - \tilde{u}_N^R|_{1,\Omega_0}$ for $N = 1, 3$. Further, Figure 3 shows the errors $E_R := \|u_\infty^R - u_N^R\|_{1,\Omega_0}$ for different R, and Figure 4 shows the errors $E_R := \|u_\infty^R - \tilde{u}_N^R\|_{1,\Omega_0}$ for different R.

Tables 2–4 and Figures 2–4 demonstrate the performance of the error estimate (3.50) and (4.23). In practice, if one wants to use a local artificial boundary condition, we advise using the one corresponding to $N = 1$. This condition is very simple

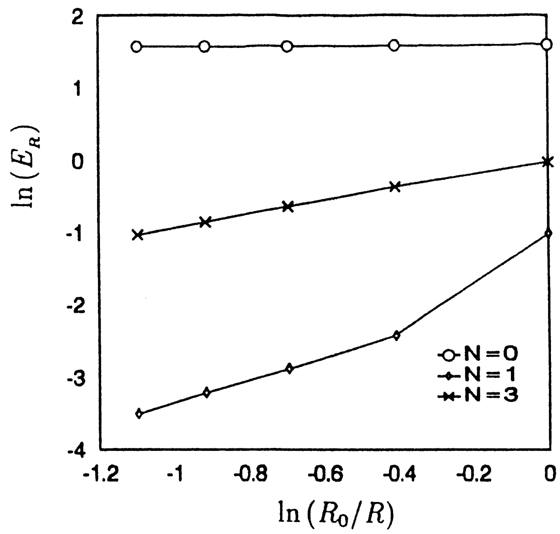

FIGURE 4. The effect of R using local artificial boundary conditions

TABLE 4. The effect of the location of Γ_R using a local artificial boundary condition ($N = 3$)

Location	$R = 1.0$	$R = 1.5$	$R = 2.0$	$R = 2.5$	$R = 3.0$
$\max \lvert u - \tilde{u}_N^h \rvert$	0.2552	0.1910	0.1489	0.1212	0.1019
$\lVert u - \tilde{u}_N^h \rVert_{0,\Omega_0}$	0.2020	0.1136	7.0255E-2	4.6411E-2	3.2470E-2
$\lvert u - \tilde{u}_N^h \rvert_{1,\Omega_0}$	1.3323	0.7880	0.6025	0.4889	0.4142

and easy to deal with by using the standard finite elements. From our numerical results, the local artificial boundary condition corresponding to $N = 3$ is not better than the one corresponding to $N = 1$. One reason is the constant $C_{(1)} \ll C_{(3)}$ in (4.13) and errors in the numerical integration for the part $\tilde{b}_N(u, v)$. Thus in pratical computation, the cases $N \geq 3$ need further study. The analogous numerical results for the Laplacian equation can be found in [10].

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