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Numerical simulation for the problem of infinite elastic foundation

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Abstract

In this paper we consider the numerical simulation for the problem of infinite elastic foundation. A half circle artificial boundary is introduced and discrete artificial boundary condition on it is designed by using the direct method of lines. Then the original problem is reduced to a boundary value problem on a bounded computational domain. Furthermore, finite element approximation of this boundary value problem defined in the bounded domain is considered. The numerical results show that the artificial boundary condition given in this paper is very effective and more accurate than the Neumann boundary condition which is often used in engineering literatures.

1. Introduction

Let Ω be an unbounded domain with boundaries Γ_0 and Γ_i (see Fig. 1). Consider the following problem of infinite elastic foundation:

$$-\mu\,\Delta u-(\lambda+\mu)\text{ grad div } u=f \quad \text{in } \Omega, \tag{1.1}$$

$$u|_{\Gamma_i}=g, \tag{1.2}$$

$$\sigma_{12}|_{\Gamma_0}=\sigma_{22}|_{\Gamma_0}=0, \tag{1.3}$$

$$u \text{ is bounded when } r=\sqrt{x_1^2+x_2^2}\to+\infty\,, \tag{1.4}$$

where $u=(u_1,u_2)^t$ denotes the displacement, λ,μ are Lamé constants, $g=(g_1,g_2)^t$ is a given function on Γ_i, $f=(f_1,f_2)^t$ is the applied body force and its support is compact. Let $\sigma=(\sigma_{ij})_{2\times2}$ be the stress tensor with entries

$$\sigma_{ij}=\lambda\delta_{ij}\text{ div } u+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \quad 1\leqslant i,j\leqslant 2, \tag{1.5}$$

where δ_{ij} is the Kronecker delta.

The problem (1.1)–(1.4) is a boundary value problem of Navier equations defined in an unbounded domain. In engineering computation, the stress analysis of a dam in plane with infinite elastic foundation is usually reduced to a similar problem. In numerical simulation of this kind problem, the unboundedness

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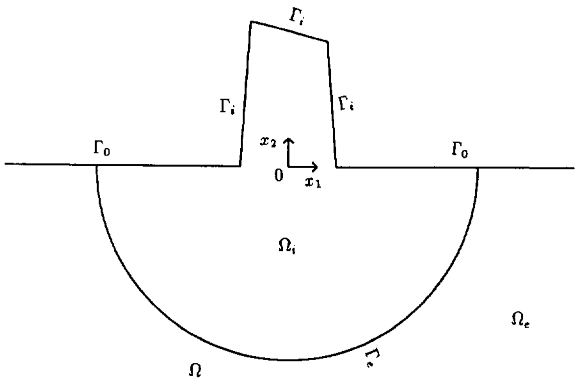


Fig. 1.

of the domain Ω is a common difficulty. In practical computation, it is a usual method to introduce an artificial boundary and design appropriate artificial boundary condition on it. Then the original problem is reduced to a boundary value problem defined in a bounded domain. Thus, one can use the traditional finite element or finite difference method to solve the new problem and derive a numerical solution of the original problem in the bounded domain. It is a common problem that how to introduce an appropriate artificial boundary and design artificial boundary condition with high accuracy on it. This question attracts many engineers and mathematicians working on it in the last decade. And many works on the related problems are appeared. For instance, Goldstein [7] presented the exact boundary condition at an artificial boundary for Helmholtz-type equation in waveguides; moreover, a sequence of approximations to the exact boundary condition at the artificial boundary was given. Feng [5] designed the asymptotic radiation conditions for the reduced wave equation by using the asymptotic approximation of Hankel functions. Bayliss et al. [2] proposed a series of radiation conditions for Laplace and Helmholtz equations. Han and Wu [16,17] presented the exact boundary conditions at an artificial boundary for the Laplace equation and the linear elastic system; moreover, a sequence of approximations to the exact boundary condition at the artificial boundary was given. The exact boundary condition at an artificial boundary for partial differential equations in an infinite cylinder was obtained by Hagstrom and Keller [10,11]. Shortly thereafter, they used this technique to solve nonlinear problems. A family of artificial boundary conditions for unsteady Oseen equations in the velocity pressure formulation with small viscosity was developed by Halpern and Schatzman [12], which was then applied to unsteady Navier–Stokes (N–S) equations. Nataf [19] designed an open boundary condition for steady Oseen equation in streamfunction vorticity formulation, which is applied to viscous incompressible fluid flow around a body in a flat channel with slip boundary conditions on the wall. Hagstrom [8,9] proposed asymptotic boundary conditions at artificial boundary for the simulation of time-dependent fluid flows. Han et al. [13] designed discrete artificial boundary conditions for N–S equations in an infinite channel by using a fast iterative method. Han and Bao [14,15] proposed discrete artificial boundary conditions for incompressible viscous flows in a channel by using the method of lines. One can find more references in [1,6].

Another approach in solving the problem defined in unbounded domain is to use infinite element, i.e. use traditional finite element in a bounded domain and infinite element in the outer domain. For instance, see Zienkiewicz et al. [20], Moriya [18], Beer and Meek [3], Bettess [4] and the references therein.

In this paper we introduce a half circle as artificial boundary for the problem of infinite elastic foundation and proposed artificial boundary condition on the half circle by using the direct method of lines to overcome the difficulty of the unboundedness of the domain. Then, the original problem is reduced to a problem defined in a bounded computational domain. We use a finite element method to solve this new problem and obtain an approximate solution of the original problem in the bounded computational domain. Furthermore, numerical example shows that the method given in this paper is very effective.

Let $\Gamma_e = \{x = (x_1, x_2) \mid x_1 = R\cos\theta,\ x_2 = R\sin\theta,\ -\pi \leqslant \theta \leqslant 0\}$, with $R > 0$ is a constant. In fact Γ_e is a half circle in Ω (see Fig. 1). Then the domain Ω is divided into a bounded part Ω_i and an unbounded part Ω_e. We can choose R such that supp $f \subset \Omega_i$. We expect to solve the original problem in the bounded domain Ω_i. The key point is that we must present 'appropriate' artificial boundary conditions on the artificial boundary Γ_e in order to derive a good approximation of the original problem. In the following section we will present a method to design a discrete artificial boundary condition on Γ_e.

2. The discrete artificial boundary condition on Γ_e

The solution of the boundary value problem (1.1)–(1.4), $u(x)$, in the domain Ω_e satisfies

$$-\mu\,\Delta u - (\lambda + \mu)\ \text{grad div}\ u = 0 \quad \text{in } \Omega_e, \tag{2.1}$$

$$\sigma_{12} = \sigma_{22} = 0, \qquad r \geqslant R, \qquad \theta = 0 \text{ or } -\pi, \tag{2.2}$$

$$u \text{ is bounded when } r \to +\infty. \tag{2.3}$$

Since the boundary condition of $u(x)$ on the artificial boundary Γ_e is unknown, the problem (2.1)–(2.3) is an incompletely posed problem. If we suppose that the value of $u(x)$ on Γ_e, $u(R, \theta)$, is known, i.e.

$$u|_{\Gamma_e} = u(R, \theta), \tag{2.4}$$

then the problem (2.1)–(2.4) has a unique solution $u(x)$.

Let

$$\sigma_n = \sigma\, n|_{\Gamma_e} = \begin{pmatrix} \sigma_{11}\cos\theta + \sigma_{12}\sin\theta \\ \sigma_{21}\cos\theta + \sigma_{22}\sin\theta \end{pmatrix}\Bigg|_{\Gamma_e}, \tag{2.5}$$

where $n = (\cos\theta, \sin\theta)^t$ is the unit outward normal on Γ_e. If $u|_{\Gamma_e} \in [H^{1/2}(\Gamma_e)]^2$ then we have that $\sigma_n \in [H^{-1/2}(\Gamma_e)]^2$, where $H^\alpha(\Gamma_e)$ denotes the usual Sobolov space on Γ_e with real number α. Hence, we obtain a bounded operator $K:\ [H^{1/2}(\Gamma_e)]^2 \to [H^{-1/2}(\Gamma_e)]^2$, namely

$$\sigma_n = K(u|_{\Gamma_e}). \tag{2.6}$$

In fact the condition (2.6) is an exact boundary condition satisfied by the solution of the original problem (1.1)–(1.4). Hence, the restriction of the solution of the problem (1.1)–(1.4) in Ω_i satisfies

$$-\mu\,\Delta u - (\lambda + \mu)\ \text{grad div}\ u = f \quad \text{in } \Omega_i, \tag{2.7}$$

$$u|_{\Gamma_i} = g, \tag{2.8}$$

$$\sigma_{12} = \sigma_{22} = 0 \quad \text{on } \Gamma_0 \cap \bar{\Omega}_i, \tag{2.9}$$

$$\sigma_n = K(u|_{\Gamma_e}). \tag{2.10}$$

Unfortunately, the bounded operator K is unknown, the problem (2.7)–(2.10) cannot be solved directly. In the following we solve the boundary value problem (2.1)–(2.4) by the method of lines with semi-discretization. Then we can find an approximate relation between the value $u(R, \theta)$ and the boundary stress σ_n by using the numerical solution of the problem (2.1)–(2.4). Thus, we derive a discrete artificial boundary condition on Γ_e.

Introducing variable transformation:

$$\begin{cases} x_1 = e^\rho \cos\theta, \\ x_2 = e^\rho \sin\theta. \end{cases} \tag{2.11}$$

Then, the domain Ω_e maps to a strip $\tilde{\Omega}_e = \{(\rho, \theta) \mid \ln R < \rho,\ -\pi < \theta < 0\}$ and Γ_e maps to a segment $\tilde{\Gamma}_e = \{(\ln R, \theta) \mid -\pi \leqslant \theta \leqslant 0\}$. Furthermore, we have that

$$\begin{cases} \dfrac{\partial}{\partial\rho} = e^{\rho}\left(\cos\theta\dfrac{\partial}{\partial x_1} + \sin\theta\dfrac{\partial}{\partial x_2}\right), \\ \dfrac{\partial}{\partial\theta} = e^{\rho}\left(-\sin\theta\dfrac{\partial}{\partial x_1} + \cos\theta\dfrac{\partial}{\partial x_2}\right); \end{cases} \tag{2.12}$$

$$\begin{cases} \dfrac{\partial}{\partial x_1} = e^{-\rho}\left(\cos\theta\dfrac{\partial}{\partial\rho} - \sin\theta\dfrac{\partial}{\partial\theta}\right), \\ \dfrac{\partial}{\partial x_2} = e^{-\rho}\left(\sin\theta\dfrac{\partial}{\partial\rho} + \cos\theta\dfrac{\partial}{\partial\theta}\right); \end{cases} \tag{2.13}$$

and

$$\frac{\partial^2}{\partial x_1^2} = e^{-2\rho}\left[\cos^2\theta\frac{\partial^2}{\partial\rho^2} - \sin 2\theta\frac{\partial^2}{\partial\rho\,\partial\theta} + \sin^2\theta\frac{\partial^2}{\partial\theta^2} - \cos 2\theta\frac{\partial}{\partial\rho} + \sin 2\theta\frac{\partial}{\partial\theta}\right], \tag{2.14}$$

$$\frac{\partial^2}{\partial x_2^2} = e^{-2\rho}\left[\sin^2\theta\frac{\partial^2}{\partial\rho^2} + \sin 2\theta\frac{\partial^2}{\partial\rho\,\partial\theta} + \cos^2\theta\frac{\partial^2}{\partial\theta^2} + \cos 2\theta\frac{\partial}{\partial\rho} - \sin 2\theta\frac{\partial}{\partial\theta}\right], \tag{2.15}$$

$$\frac{\partial^2}{\partial x_1\,\partial x_2} = e^{-2\rho}\left[\frac{1}{2}\sin 2\theta\frac{\partial^2}{\partial\rho^2} + \cos 2\theta\frac{\partial^2}{\partial\rho\,\partial\theta} - \frac{1}{2}\sin 2\theta\frac{\partial^2}{\partial\theta^2} - \sin 2\theta\frac{\partial}{\partial\rho} - \cos 2\theta\frac{\partial}{\partial\theta}\right], \tag{2.16}$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = e^{-2\rho}\left[\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\theta^2}\right]. \tag{2.17}$$

Then, the boundary value problem (2.1)–(2.4) is equivalent to the following boundary value problem defined in the domain $\tilde{\Omega}_e$:

$$\begin{aligned} &-\left[\mu + (\lambda+\mu)\cos^2\theta\right]\frac{\partial^2\tilde{u}_1}{\partial\rho^2} + (\lambda+\mu)\left[\cos 2\theta\frac{\partial\tilde{u}_1}{\partial\rho} + \sin 2\theta\frac{\partial^2\tilde{u}_1}{\partial\rho\,\partial\theta} - \sin 2\theta\frac{\partial\tilde{u}_1}{\partial\theta}\right] \\ &-\left[\mu + (\lambda+\mu)\sin^2\theta\right]\frac{\partial^2\tilde{u}_1}{\partial\theta^2} - (\lambda+\mu)\left[\frac{1}{2}\sin 2\theta\frac{\partial^2\tilde{u}_2}{\partial\rho^2} + \cos 2\theta\frac{\partial^2\tilde{u}_2}{\partial\rho\,\partial\theta} - \frac{1}{2}\sin 2\theta\frac{\partial^2\tilde{u}_2}{\partial\theta^2}\right. \\ &\left. - \sin 2\theta\frac{\partial\tilde{u}_2}{\partial\rho} - \cos 2\theta\frac{\partial\tilde{u}_2}{\partial\theta}\right] = 0 \quad \text{in } \tilde{\Omega}_e, \end{aligned} \tag{2.18}$$

$$\begin{aligned} &-(\lambda+\mu)\left[\frac{1}{2}\sin 2\theta\frac{\partial^2\tilde{u}_1}{\partial\rho^2} + \cos 2\theta\frac{\partial^2\tilde{u}_1}{\partial\rho\,\partial\theta} - \frac{1}{2}\sin 2\theta\frac{\partial^2\tilde{u}_1}{\partial\theta^2} - \sin 2\theta\frac{\partial\tilde{u}_1}{\partial\rho} - \cos 2\theta\frac{\partial\tilde{u}_1}{\partial\theta}\right] \\ &-\left[\mu + (\lambda+\mu)\sin^2\theta\right]\frac{\partial^2\tilde{u}_2}{\partial\rho^2} - (\lambda+\mu)\left[\cos 2\theta\frac{\partial\tilde{u}_2}{\partial\rho} + \sin 2\theta\frac{\partial^2\tilde{u}_2}{\partial\rho\,\partial\theta} - \sin 2\theta\frac{\partial\tilde{u}_2}{\partial\theta}\right] \\ &-\left[\mu + (\lambda+\mu)\cos^2\theta\right]\frac{\partial^2\tilde{u}_2}{\partial\theta^2} = 0 \quad \text{in } \tilde{\Omega}_e, \end{aligned} \tag{2.19}$$

$$\left(\frac{\partial\tilde{u}_1}{\partial\theta} + \frac{\partial\tilde{u}_2}{\partial\rho}\right)\bigg|_{\theta=-\pi,0} = \left(\lambda\frac{\partial\tilde{u}_1}{\partial\rho} + (\lambda+2\mu)\frac{\partial\tilde{u}_2}{\partial\theta}\right)\bigg|_{\theta=-\pi,0} = 0 \quad \rho \geqslant \rho_0, \tag{2.20}$$

$$\tilde{u} \text{ is bounded when } \rho \to +\infty, \tag{2.21}$$

$$\tilde{u}|_{\rho=\rho_0} = \tilde{u}(\rho_0, \theta); \tag{2.22}$$

where $\rho_0 = \ln R$, $\tilde{u}(\rho,\theta) = u(e^{\rho},\theta)$.

In order to reduce the boundary value problem (2.18)–(2.22) to a variational ordinary differential equation, we rewrite (2.18), (2.19) in the following form:

$$\begin{aligned}&\frac{\partial}{\partial\theta}\left\{\left[\mu+(\lambda+\mu)\sin^2\theta\right]\frac{\partial\tilde{u}_1}{\partial\theta}-\frac{\lambda+\mu}{2}\sin 2\theta\left(\frac{\partial\tilde{u}_1}{\partial\rho}+\frac{\partial\tilde{u}_2}{\partial\theta}\right)+\left(\mu\cos^2\theta-\lambda\sin^2\theta\right)\frac{\partial\tilde{u}_2}{\partial\rho}\right\}\\&\quad+\left[\mu+(\lambda+\mu)\cos^2\theta\right]\frac{\partial^2\tilde{u}_1}{\partial\rho^2}+\frac{(\lambda+\mu)}{2}\sin 2\theta\left[\frac{\partial^2\tilde{u}_2}{\partial\rho^2}-\frac{\partial^2\tilde{u}_1}{\partial\rho\,\partial\theta}\right]\\&\quad+\left[\lambda\cos^2\theta-\mu\sin^2\theta\right]\frac{\partial^2\tilde{u}_2}{\partial\rho\,\partial\theta}=0\quad\text{in }\tilde{\Omega}_e;\end{aligned}\tag{2.23}$$

$$\begin{aligned}&\frac{\partial}{\partial\theta}\left\{\left[\mu+(\lambda+\mu)\cos^2\theta\right]\frac{\partial\tilde{u}_2}{\partial\theta}+\frac{\lambda+\mu}{2}\sin 2\theta\left(\frac{\partial\tilde{u}_2}{\partial\rho}-\frac{\partial\tilde{u}_1}{\partial\theta}\right)+\left(\lambda\cos^2\theta-\mu\sin^2\theta\right)\frac{\partial\tilde{u}_1}{\partial\rho}\right\}\\&\quad+\frac{\lambda+\mu}{2}\sin 2\theta\left[\frac{\partial^2\tilde{u}_1}{\partial\rho^2}+\frac{\partial^2\tilde{u}_2}{\partial\rho\,\partial\theta}\right]+\left[\mu\cos^2\theta-\lambda\sin^2\theta\right]\frac{\partial^2\tilde{u}_1}{\partial\rho\,\partial\theta}\\&\quad+\left[\mu+(\lambda+\mu)\sin^2\theta\right]\frac{\partial^2\tilde{u}_2}{\partial\rho^2}=0\quad\text{in }\tilde{\Omega}_e.\end{aligned}\tag{2.24}$$

Let $H^1((-\pi,0))$ denote the usual Sobolov space on the interval $(-\pi,0)$, i.e.

$$H^1((-\pi,0))=\{v(\theta)\mid v(\theta),\ v'(\theta)\in L^2((-\pi,0))\}.$$

We also let

$$W=H^1((-\pi,0))\times H^1((-\pi,0)),$$

and

$$V=\left\{v(\rho,\theta)=(v_1(\rho,\theta),v_2(\rho,\theta))^t\mid\text{for fixed }\rho\geqslant\rho_0,\ v(\rho,\theta),\ \frac{\partial v(\rho,\theta)}{\partial\rho}\ \text{ and }\ \frac{\partial^2 v(\rho,\theta)}{\partial\rho^2}\in W\right\}.$$

Then the boundary value problem (2.18)–(2.22) is equivalent to the following variational ordinary differential equations:

Find $\tilde{u}(\rho,\theta)\in V$ such that

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}A_2(\tilde{u},v)+\frac{\mathrm{d}}{\mathrm{d}\rho}A_1(\tilde{u},v)+A_0(\tilde{u},v)=0,\quad\forall v\in W,\tag{2.25}$$

$$\tilde{u}\text{ is bounded when }\rho\to+\infty,\tag{2.26}$$

$$\tilde{u}|_{\rho=\rho_0}=\tilde{u}(\rho_0,\theta);\tag{2.27}$$

where

$$\begin{aligned}A_2(\tilde{u},v)=\int_{-\pi}^{0}\Big\{&\left[\mu+(\lambda+\mu)\cos^2\theta\right]\tilde{u}_1(\rho,\theta)v_1(\theta)+\left[\mu+(\lambda+\mu)\sin^2\theta\right]\tilde{u}_2(\rho,\theta)v_2(\theta)\\&+\frac{\lambda+\mu}{2}\sin 2\theta\left[\tilde{u}_1(\rho,\theta)v_2(\theta)+\tilde{u}_2(\rho,\theta)v_1(\theta)\right]\Big\}\,\mathrm{d}\theta,\end{aligned}\tag{2.28}$$

$$\begin{aligned}A_1(\tilde{u},v)=\int_{-\pi}^{0}\Big\{&\frac{\lambda+\mu}{2}\sin 2\theta\left[\tilde{u}_1(\rho,\theta)v_1'(\theta)-\frac{\partial\tilde{u}_1(\rho,\theta)}{\partial\theta}v_1(\theta)+\frac{\partial\tilde{u}_2(\rho,\theta)}{\partial\theta}v_2(\theta)\right.\\&\left.-\tilde{u}_2(\rho,\theta)v_2'(\theta)\right]+(\lambda\cos^2\theta-\mu\sin^2\theta)\left[\frac{\partial\tilde{u}_2(\rho,\theta)}{\partial\theta}v_1(\theta)-\tilde{u}_1(\rho,\theta)v_2'(\theta)\right]\end{aligned}$$

$$+(\mu\cos^2\theta-\lambda\sin^2\theta)\left[\frac{\partial\tilde{u}_1(\rho,\theta)}{\partial\theta}v_2(\theta)-\tilde{u}_2(\rho,\theta)v_1'(\theta)\right]\Bigg\}\,\mathrm{d}\theta, \tag{2.29}$$

$$A_0(\tilde{u},v)=-\int_{-\pi}^{0}\Bigg\{\left[\mu+(\lambda+\mu)\sin^2\theta\right]\frac{\partial\tilde{u}_1(\rho,\theta)}{\partial\theta}v_1'(\theta)+\left[\mu+(\lambda+\mu)\cos^2\theta\right]\frac{\partial\tilde{u}_2(\rho,\theta)}{\partial\theta}v_2'(\theta)$$
$$-\frac{\lambda+\mu}{2}\sin 2\theta\left[\frac{\partial\tilde{u}_1(\rho,\theta)}{\partial\theta}v_2'(\theta)+\frac{\partial\tilde{u}_2(\rho,\theta)}{\partial\theta}v_1'(\theta)\right]\Bigg\}\,\mathrm{d}\theta. \tag{2.30}$$

$A_j(\tilde{u},v)$ $(j=0,1,2)$ are bilinear forms on $V\times W$. They can also be considered bilinear forms on $W\times W$ by replacing $\tilde{u}(\rho,\theta)$ by $u(\theta)\in W$ and $\frac{\partial\tilde{u}(\rho,\theta)}{\partial\theta}$ by $u'(\theta)$. Then, it is straightforward to check that they have the following properties on $W\times W$.

LEMMA 2.1. (a) $A_j(u,v)$ $(j=0,1,2)$ are bounded bilinear forms on $W\times W$.
(b) $A_0(u,v)$ and $A_2(u,v)$ are symmetric forms, $A_1(u,v)$ is antisymmetric form.
(c) The following inequalities hold:

$$-A_0(v,v)\geqslant\mu|v|^2_{1,(-\pi,0)},\quad\forall v\in W, \tag{2.31}$$

$$A_2(v,v)\geqslant\mu\|v\|^2_{0,(-\pi,0)},\quad\forall v\in W. \tag{2.32}$$

In the following we will approximate the variational ordinary differential equation (2.25) by semi-discretization. Let

$$-\pi=\theta_1<\theta_2<\cdots<\theta_M=0$$

be a partition of interval $[-\pi,0]$. Furthermore, let $h_\theta=\max_{1\leqslant j\leqslant M-1}(\theta_{j+1}-\theta_j)$ and

$$S^h=\left\{v^h(\theta)\mid v^h(\theta)\in C^0([-\pi,0]),\ v^h(\theta)|_{[\theta_j,\theta_{j+1}]}\in P_1([\theta_j,\theta_{j+1}]),\ 1\leqslant j\leqslant M-1\right\}.$$

Then, S^h is a finite element subspace of $H^1((-\pi,0))$. In addition let

$$W^h=S^h\times S^h,$$

$$V^h=\left\{v^h(\rho,\theta)\mid \text{for fixed }\rho\geqslant\rho_0\quad v^h(\rho,\theta),\ \frac{\partial v^h(\rho,\theta)}{\partial\rho},\ \frac{\partial^2v^h(\rho,\theta)}{\partial\rho^2}\in W^h\right\}.$$

Then, we derive an approximation of problem (2.25)–(2.27) with semi-discretization by replacing V by V^h and W by W^h:

Find $\tilde{u}^h(\rho,\theta)\in V^h$ such that

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}A_2(\tilde{u}^h,v^h)+\frac{\mathrm{d}}{\mathrm{d}\rho}A_1(\tilde{u}^h,v^h)+A_0(\tilde{u}^h,v^h)=0\quad\forall v^h\in W^h, \tag{2.33}$$

$$\tilde{u}^h \text{ is bounded when } \rho\to+\infty, \tag{2.34}$$

$$\tilde{u}^h|_{\rho=\rho_0}=\tilde{u}^h(\rho_0,\theta); \tag{2.35}$$

where $\tilde{u}^h(\rho_0,\theta)\in W^h$ such that $\tilde{u}^h(\rho_0,\theta_j)=\tilde{u}(\rho_0,\theta_j)$ for $j=1,2,\ldots,M$.

Suppose that $\{\phi_j(\theta),\ j=1,2,\ldots,M\}$ is a basis of the finite element space S^h such that $\phi_j(\theta_i)=\delta_{ij}$, $1\leqslant i,j\leqslant M$. Then, $\{(\phi_j(\theta),0)^t,\ (0,\phi_j(\theta))^t,\ 1\leqslant j\leqslant M\}$ is a basis of the finite element space W^h. Thus, for $\tilde{u}^h(\rho,\theta)=(\tilde{u}^h_1(\rho,\theta),\tilde{u}^h_2(\rho,\theta))^t\in V^h$, we have that

$$\tilde{u}_i^h(\rho,\theta) = \sum_{j=1}^{M} u_i^{j,h}(\rho)\phi_j(\theta), \quad i = 1,2. \tag{2.36}$$

Setting

$$U(\rho) = \left[u_1^{1,h}(\rho), u_1^{2,h}(\rho), \ldots, u_1^{M,h}(\rho), u_2^{1,h}(\rho), u_2^{2,h}(\rho), \ldots, u_2^{M,h}(\rho)\right]^t, \tag{2.37}$$

$$\begin{aligned} U(\rho_0) &= [\tilde{u}_1(\rho_0,\theta_1), \tilde{u}_1(\rho_0,\theta_2), \ldots, \tilde{u}_1(\rho_0,\theta_M), \tilde{u}_2(\rho_0,\theta_1), \tilde{u}_2(\rho_0,\theta_2), \ldots, \tilde{u}_2(\rho_0,\theta_M)]^t \\ &\equiv [u_1(R,\theta_1), u_1(R,\theta_2), \ldots, u_1(R,\theta_M), u_2(R,\theta_1), u_2(R,\theta_2), \ldots, u_2(R,\theta_M)]^t, \end{aligned} \tag{2.38}$$

$$N(\theta) = \begin{pmatrix} \phi_1(\theta) & \phi_2(\theta) & \cdots & \phi_M(\theta) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \phi_1(\theta) & \phi_2(\theta) & \cdots & \phi_M(\theta) \end{pmatrix}^t. \tag{2.39}$$

Substituting (2.36), (2.37), (2.38) and (2.39) into (2.33)–(2.35), a computation shows that the problem (2.33)–(2.35) is equivalent to the following ordinary differential equations:

$$B_2 U''(\rho) + B_1 U'(\rho) + B_0 U(\rho) = 0, \quad \rho > \rho_0, \tag{2.40}$$

$$U(\rho) \text{ is bounded when } \rho \to +\infty, \tag{2.41}$$

$$U(\rho)|_{\rho=\rho_0} = U(\rho_0), \tag{2.42}$$

where

$$B_2 = \int_{-\pi}^{0} N(\theta) \begin{pmatrix} \mu + (\lambda+\mu)\cos^2\theta & \frac{\lambda+\mu}{2}\sin 2\theta \\ \frac{\lambda+\mu}{2}\sin 2\theta & \mu + (\lambda+\mu)\sin^2\theta \end{pmatrix} N(\theta)^t \, \mathrm{d}\theta, \tag{2.43}$$

$$\begin{aligned} B_1 = \int_{-\pi}^{0} \Bigg[& N(\theta) \begin{pmatrix} -\frac{\lambda+\mu}{2}\sin 2\theta & \lambda\cos^2\theta - \mu\sin^2\theta \\ \mu\cos^2\theta - \lambda\sin^2\theta & \frac{\lambda+\mu}{2}\sin 2\theta \end{pmatrix} N'(\theta)^t \\ & + N'(\theta) \begin{pmatrix} \frac{\lambda+\mu}{2}\sin 2\theta & \lambda\sin^2\theta - \mu\cos^2\theta \\ \mu\sin^2\theta - \lambda\cos^2\theta & -\frac{\lambda+\mu}{2}\sin 2\theta \end{pmatrix} N(\theta)^t \Bigg] \mathrm{d}\theta, \end{aligned} \tag{2.44}$$

$$B_0 = -\int_{-\pi}^{0} N'(\theta) \begin{pmatrix} \mu + (\lambda+\mu)\sin^2\theta & -\frac{\lambda+\mu}{2}\sin 2\theta \\ -\frac{\lambda+\mu}{2}\sin 2\theta & \mu + (\lambda+\mu)\cos^2\theta \end{pmatrix} N'(\theta)^t \, \mathrm{d}\theta. \tag{2.45}$$

B_0, B_1 and B_2 are $2M \times 2M$ matrices. From Lemma 2.1 we have that

LEMMA 2.2. B_2 is a symmetric positive-definite matrix, B_1 is an antisymmetric matrix and B_0 is a symmetric negative semi-definite matrix.

Since the elements in the matrices B_j, $j = 0,1,2$ are real constants, the boundary value problem of ordinary differential equations (2.40)–(2.42) can be solved by a direct method. Suppose

$$U(\rho) = \mathrm{e}^{\gamma\rho}\xi, \tag{2.46}$$

where γ is a constant, $\xi \in \mathbb{C}^{2M}$ is to be determined. Substituting (2.46) into the ordinary differential equations (2.40), we derive the following generalised eigenvalue problem:

$$\left[\gamma^2 B_2 + \gamma B_1 + B_0\right] \xi = 0. \tag{2.47}$$

Let $\eta = \gamma\xi$, then we have the following standard eigenvalue problem:

$$\begin{pmatrix} 0 & I_{2M} \\ -B_0 & -B_1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \gamma \begin{pmatrix} I_{2M} & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{2.48}$$

where I_{2M} denotes $2M \times 2M$ unit matrix.

LEMMA 2.3. If $\gamma \in \mathbb{C}$ is an eigenvalue of problem (2.47) ($\equiv$ (2.48)), then $-\gamma$ is also an eigenvalue of problem (2.47).

PROOF. Let $\gamma \in \mathbb{C}$ be an eigenvalue of problem (2.47) and $\xi \in \mathbb{C}^{2M}$ be the corresponding eigenvector. Then, by Lemma 2.2, we have that

$$\begin{aligned} 0 &= \det[\gamma^2 B_2 + \gamma B_1 + B_0] \\ &= \det[(\gamma^2 B_2 + \gamma B_1 + B_0)^t] \\ &= \det[(-\gamma)^2 B_2 - \gamma B_1 + B_0]. \end{aligned} \tag{2.49}$$

Thus, we know that there exists a vector $\eta \in \mathbb{C}^{2M}$ such that

$$[(-\gamma)^2 B_2 - \gamma B_1 + B_0]\eta = 0. \tag{2.50}$$

This implies that $-\gamma$ is also an eigenvalue of problem (2.47) and the vector η is the corresponding eigenvector.

We can solve the standard eigenvalue problem (2.48) by the numerical method. From Lemma 2.3 we know that we can derive $2M$ eigenvalues with non-positive real part, say $\gamma_1, \gamma_2, \dots, \gamma_{2M}$ and the corresponding eigenvectors:

$$\begin{pmatrix} \xi_1 \\ \gamma_1\xi_1 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \gamma_2\xi_2 \end{pmatrix} \cdots \begin{pmatrix} \xi_{2M} \\ \gamma_{2M}\xi_{2M} \end{pmatrix},$$

with $\gamma_1 = \gamma_2 = 0$, $\xi_1 = (1,1,\dots,1,0,0,\dots,0)^t \in \mathbb{R}^{2M}$, $\xi_2 = (0,0,\dots,0,1,1,\dots,1)^t \in \mathbb{R}^{2M}$. Particularly, we assume γ_j $(1 \leqslant j \leqslant 2r)$ are real eigenvalues and γ_j $(2r+1 \leqslant j \leqslant 2M)$ are complex eigenvalues with nonzero imaginary parts such that $\gamma_{2l} = \bar{\gamma}_{2l-1}$ $(r+1 \leqslant l \leqslant M)$. Thus, we have that

$$U(\rho) = \sum_{j=1}^{2r} b_j\, e^{\gamma_j(\rho-\rho_0)}\xi_j + \sum_{j=r+1}^{M} \left[b_{2j-1}\, \mathrm{Re}(e^{\gamma_{2j}(\rho-\rho_0)}\xi_{2j}) + b_{2j}\, \mathrm{Im}(e^{\gamma_{2j}(\rho-\rho_0)}\xi_{2j})\right], \tag{2.51}$$

where $\mathrm{Re}(\gamma)$ and $\mathrm{Im}(\gamma)$ denote the real part and the imaginary part of the complex number γ and b_j $(1 \leqslant j \leqslant 2M)$ are constants. Thus, $U(\rho)$ satisfies the ordinary equations (2.40) and the boundary condition, $U(\rho)$ is bounded when $\rho \to +\infty$.

Introduce matrices

$$\begin{aligned} W(\rho) = \Big[&e^{\gamma_1(\rho-\rho_0)}\xi_1, \dots, e^{\gamma_{2r}(\rho-\rho_0)}\xi_{2r}, \mathrm{Re}(e^{\gamma_{2r+2}(\rho-\rho_0)}\xi_{2r+2}), \mathrm{Im}(e^{\gamma_{2r+2}(\rho-\rho_0)}\xi_{2r+2}), \dots, \\ &\mathrm{Re}(e^{\gamma_{2M}(\rho-\rho_0)}\xi_{2M}), \mathrm{Im}(e^{\gamma_{2M}(\rho-\rho_0)}\xi_{2M})\Big], \end{aligned}$$

$$W(\rho_0) \equiv W_0 = [\xi_1, \dots, \xi_{2r}, \mathrm{Re}(\xi_{2r+2}), \mathrm{Im}(\xi_{2r+2}), \cdots, \mathrm{Re}(\xi_{2M}), \mathrm{Im}(\xi_{2M})],$$

$$W'(\rho_0) \equiv W_1 = \Big[\gamma_1\xi_1, \ \dots, \ \gamma_{2r}\xi_{2r}, \ \mathrm{Re}(\gamma_{2r+2}\xi_{2r+2}), \ \mathrm{Im}(\gamma_{2r+2}\xi_{2r+2}), \dots, \ \mathrm{Re}(\gamma_{2M}\xi_{2M}), \ \mathrm{Im}(\gamma_{2M}\xi_{2M})\Big],$$

$$b = [b_1, \ b_2, \ \dots, \ b_{2M}]^t.$$

Thus

$$U(\rho) = W(\rho)b. \tag{2.52}$$

From (2.52) and (2.42) we obtain that

$$b = W_0^{-1}U(\rho_0). \tag{2.53}$$

Substituting (2.53) into (2.52), we have that

$$U(\rho) = W(\rho)W_0^{-1}U(\rho_0). \tag{2.54}$$

Thus, we get the semi-discrete approximate solution of problem (2.18)–(2.22) for given $u(\rho_0, \theta)$:

$$\tilde{u}^h(\rho,\theta) = \begin{pmatrix} \tilde{u}_1^h(\rho,\theta) \\ \tilde{u}_2^h(\rho,\theta) \end{pmatrix} = N(\theta)^t U(\rho) = N(\theta)^t W(\rho) W_0^{-1} U(\rho_0). \tag{2.55}$$

Therefore, the approximate solution of problem (2.1)–(2.4) in polar coordinate (r, θ) is:

$$u^h(r,\theta) = \tilde{u}^h(\ln r, \theta) = N(\theta)^t W(\ln r) W_0^{-1} U(\rho_0). \tag{2.56}$$

In the polar coordinate (r, θ), we have that

$$\sigma_n = \frac{1}{R}\begin{pmatrix} \mu + (\lambda+\mu)\cos^2\theta & \frac{\lambda+\mu}{2}\sin 2\theta \\ \frac{\lambda+\mu}{2}\sin 2\theta & \mu + (\lambda+\mu)\sin^2\theta \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{u}_1}{\partial \rho} \\ \frac{\partial \tilde{u}_2}{\partial \rho} \end{pmatrix}\Bigg|_{\rho=\rho_0} + \frac{1}{R}\begin{pmatrix} -\frac{\lambda+\mu}{2}\sin 2\theta & \lambda\cos^2\theta - \mu\sin^2\theta \\ \mu\cos^2\theta - \lambda\sin^2\theta & \frac{\lambda+\mu}{2}\sin 2\theta \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{u}_1}{\partial \theta} \\ \frac{\partial \tilde{u}_2}{\partial \theta} \end{pmatrix}\Bigg|_{\rho=\rho_0}. \tag{2.57}$$

Substituting (2.55) into (2.57), we have that:

$$\sigma_n \approx \frac{1}{R}\begin{pmatrix} \mu + (\lambda+\mu)\cos^2\theta & \frac{\lambda+\mu}{2}\sin 2\theta \\ \frac{\lambda+\mu}{2}\sin 2\theta & \mu + (\lambda+\mu)\sin^2\theta \end{pmatrix} N(\theta)^t W_1 W_0^{-1} U(\rho_0) + \frac{1}{R}\begin{pmatrix} -\frac{\lambda+\mu}{2}\sin 2\theta & \lambda\cos^2\theta - \mu\sin^2\theta \\ \mu\cos^2\theta - \lambda\sin^2\theta & \frac{\lambda+\mu}{2}\sin 2\theta \end{pmatrix} N'(\theta)^t U(\rho_0). \tag{2.58}$$

The formula (2.58) is a relation between the normal stress and displacement on Γ_e. This is a discrete artificial boundary condition.

3. The numerical solution of the problem (1.1)–(1.4)

We now consider the numerical solution of the problem (1.1)–(1.4) on the bounded computational domain Ω_i. As we know the restriction on Ω_i of the solution of the problem (1.1)–(1.4), $u(x)$, satisfies the boundary value problem (2.7)–(2.10). Let $H^1(\Omega_i)$ denote the usual Sobolov space on Ω_i and assume that

$$T_g = \{v \in H^1(\Omega_i) \times H^1(\Omega_i) \mid v|_{\Gamma_i} = g\},$$

$$T_0 = \{v \in H^1(\Omega_i) \times H^1(\Omega_i) \mid v|_{\Gamma_i} = 0\}.$$

Then, the boundary value problem (2.7)–(2.10) is equivalent to the following variational problem:

Find $u \in T_g$ such that

$$a(u,v) + b(u,v) = f(v), \quad \forall v \in T_0, \tag{3.1}$$

where

$$a(u,v) = \int_{\Omega_i} \left[\lambda \operatorname{div} u \operatorname{div} v + 2\mu \left(\frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2} \right) + \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) \right] \mathrm{d}x, \tag{3.2}$$

$$b(u,v) = -\int_{\Gamma_e} K(u) \cdot v \, \mathrm{d}s, \tag{3.3}$$

$$f(v) = \int_{\Omega_i} f \cdot v \, \mathrm{d}x. \tag{3.4}$$

For the simplicity, we suppose that Γ_i is a polygonal line in $\mathbb{R}^2$. Let $\mathcal{T}^h$ be a regular triangulation of Ω_i, i.e. $\bar{\Omega}_i \equiv \cup_{T \in \mathcal{T}^h} \bar{T}$ with those T such that $\bar{T} \cap \Gamma_e \neq \emptyset$, and only these, having a curved side (on Γ_e). Furthermore, let

$$T^h = \{v^h = (v_1^h, v_2^h)^t \mid v_j^h|_T \in P_1(T), \quad \forall T \in \mathcal{T}^h, \quad j = 1,2\},$$

$$T_g^h = \{v^h \in T^h \mid v^h(d_j) = g(d_j) \text{ for the node } d_j \in \Gamma_i\},$$

$$T_0^h = \{v^h \in T^h \mid v^h|_{\Gamma_i} = 0\}.$$

Thus, we obtain the discrete form of the problem (3.1):

Find $u^h \in T_g^h$ such that

$$a(u^h, v^h) + b(u^h, v^h) = f(v^h), \quad \forall v^h \in T_0^h. \tag{3.5}$$

Since the bounded operator K is unknown, we cannot solve the problem (3.5) directly. But we can use the procedure in Section 2 in the case that the nodes on the boundary Γ_e are mapped into the points $(0, \phi_j)$ for $j = 1, 2, \ldots, M$ by the mapping (2.11). Then we have the discrete artificial boundary condition (2.58). Thus, for $u^h, v^h \in T^h$, let

$$b_h(u^h, v^h) = -\int_{\Gamma_e} \sigma_n \cdot v^h \, \mathrm{d}s$$

$$
\begin{aligned}
&-\int_{-\pi}^{0} \sigma_n \cdot v^h\, R\, \mathrm{d}\theta \\
&= -V_{h,0}^t \int_{-\pi}^{0} \Bigg[N(\theta) \begin{pmatrix} \mu + (\lambda+\mu)\cos^2\theta & \frac{\lambda+\mu}{2}\sin 2\theta \\ \frac{\lambda+\mu}{2}\sin 2\theta & \mu + (\lambda+\mu)\sin^2\theta \end{pmatrix} N(\theta)^t W_1 W_0^{-1} \\
&+ N(\theta) \begin{pmatrix} -\frac{\lambda+\mu}{2}\sin 2\theta & \lambda\cos^2\theta - \mu\sin^2\theta \\ \mu\cos^2\theta - \lambda\sin^2\theta & \frac{\lambda+\mu}{2}\sin 2\theta \end{pmatrix} N'(\theta)^t \Bigg] \mathrm{d}\theta\, U_{h,0},
\end{aligned} \tag{3.6}
$$

with

$$U_{h,0} = [u_1^h(R,\theta_1), \ldots, u_1^h(R,\theta_M), u_2^h(R,\theta_1), \ldots, u_2^h(R,\theta_M)]^t, \tag{3.7}$$

and

$$V_{h,0} = [v_1^h(R,\theta_1), \ldots, v_1^h(R,\theta_M), v_2^h(R,\theta_1), \ldots, v_2^h(R,\theta_M)]^t\,. \tag{3.8}$$

Using bilinear form $b_h(u^h, v^h)$ instead of $b(u^h, v^h)$ in the problem (3.5) we obtain:

Find $u^h \in T_g^h$ such that

$$a(u^h, v^h) + b_h(u^h, v^h) = f(v^h), \quad \forall v^h \in T_0^h. \tag{3.9}$$

After solving the problem (3.9), the solution $u^h \in T_g^h$ is an approximation of the original problem (1.1)–(1.4) on the computational domain Ω_i.

For the bilinear form $b_h(u^h, v^h)$, we have that

LEMMA 3.1. The bilinear form $b_h(u^h, v^h)$ is bounded and symmetric on $T^h \times T^h$. Furthermore, $b_h(v^h, v^h) \geqslant 0$ for all $v^h \in T^h$.

PROOF. For given $u^h, v^h \in T^h$, noting (3.6), (3.7), (3.8) and (2.36), we have that

$$u^h|_{\tilde{\Gamma}_e} = N(\theta)^t U_{h,0}, \tag{3.10}$$

$$v^h|_{\tilde{\Gamma}_e} = N(\theta)^t V_{h,0}. \tag{3.11}$$

On the domain $\tilde{\Omega}_e$, let

$$u^h = N(\theta)^t W(\rho) W_0^{-1} U_{h,0}, \tag{3.12}$$

$$v^h = N(\theta)^t W(\rho) W_0^{-1} V_{h,0}. \tag{3.13}$$

Thus, we have the continuous extension of u^h and v^h on $\tilde{\Omega}_e$ (say Ω_e). A computation shows that

$$
\begin{aligned}
&\int_{\Omega_e} \left[\lambda\, \mathrm{div}\, u^h\, \mathrm{div}\, v^h + 2\mu \left(\frac{\partial u_1^h}{\partial x_1}\frac{\partial v_1^h}{\partial x_1} + \frac{\partial u_2^h}{\partial x_2}\frac{\partial v_2^h}{\partial x_2} \right) + \mu \left(\frac{\partial u_1^h}{\partial x_2} + \frac{\partial u_2^h}{\partial x_1} \right)\left(\frac{\partial v_1^h}{\partial x_2} + \frac{\partial v_2^h}{\partial x_1} \right) \right] \mathrm{d}x \\
&= \int_{\ln R}^{+\infty} \int_{-\pi}^{0} \Bigg[(\lambda+2\mu) \left(\cos\theta \frac{\partial u_1^h}{\partial \rho} - \sin\theta \frac{\partial u_1^h}{\partial \theta} \right)\left(\cos\theta \frac{\partial v_1^h}{\partial \rho} - \sin\theta \frac{\partial v_1^h}{\partial \theta} \right) \\
&+ (\lambda+2\mu) \left(\sin\theta \frac{\partial u_2^h}{\partial \rho} + \cos\theta \frac{\partial u_2^h}{\partial \theta} \right)\left(\sin\theta \frac{\partial v_2^h}{\partial \rho} + \cos\theta \frac{\partial v_2^h}{\partial \theta} \right) \\
&+ \lambda \left(\cos\theta \frac{\partial u_1^h}{\partial \rho} - \sin\theta \frac{\partial u_1^h}{\partial \theta} \right)\left(\sin\theta \frac{\partial v_2^h}{\partial \rho} + \cos\theta \frac{\partial v_2^h}{\partial \theta} \right)
\end{aligned}
$$

$$+\lambda\left(\sin\theta\frac{\partial u_2^h}{\partial\rho}+\cos\theta\frac{\partial u_2^h}{\partial\theta}\right)\left(\cos\theta\frac{\partial v_1^h}{\partial\rho}-\sin\theta\frac{\partial v_1^h}{\partial\theta}\right)$$
$$+\mu\left(\sin\theta\frac{\partial u_1^h}{\partial\rho}+\cos\theta\frac{\partial u_1^h}{\partial\theta}\right)\left(\sin\theta\frac{\partial v_1^h}{\partial\rho}+\cos\theta\frac{\partial v_1^h}{\partial\theta}\right)$$
$$+\mu\left(\sin\theta\frac{\partial u_1^h}{\partial\rho}+\cos\theta\frac{\partial u_1^h}{\partial\theta}\right)\left(\cos\theta\frac{\partial v_2^h}{\partial\rho}-\sin\theta\frac{\partial v_2^h}{\partial\theta}\right)$$
$$+\mu\left(\cos\theta\frac{\partial u_2^h}{\partial\rho}-\sin\theta\frac{\partial u_2^h}{\partial\theta}\right)\left(\sin\theta\frac{\partial v_1^h}{\partial\rho}+\cos\theta\frac{\partial v_1^h}{\partial\theta}\right)$$
$$+\mu\left(\cos\theta\frac{\partial u_2^h}{\partial\rho}-\sin\theta\frac{\partial u_2^h}{\partial\theta}\right)\left(\cos\theta\frac{\partial v_2^h}{\partial\rho}-\sin\theta\frac{\partial v_1^h}{\partial\theta}\right)\Bigg]\,\mathrm{d}\rho\,\mathrm{d}\theta$$
$$=b_h(u^h,v^h)-\int_{\ln R}^{+\infty}\int_{-\pi}^{0}\left[A_2\left(\frac{\partial^2u^h}{\partial\rho^2},v^h\right)+A_1\left(\frac{\partial u^h}{\partial\rho},v^h\right)+A_0(u^h,v^h)\right]\mathrm{d}\rho\,\mathrm{d}\theta$$
$$=b_h(u^h,v^h). \tag{3.14}$$

Hence

$$b_h(u^h,v^h)=\int_{\Omega_e}\left[\lambda\,\mathrm{div}\,u^h\,\mathrm{div}\,v^h+2\mu\left(\frac{\partial u_1^h}{\partial x_1}\frac{\partial v_1^h}{\partial x_1}+\frac{\partial u_2^h}{\partial x_2}\frac{\partial v_2^h}{\partial x_2}\right)+\mu\left(\frac{\partial u_1^h}{\partial x_2}+\frac{\partial u_2^h}{\partial x_1}\right)\left(\frac{\partial v_1^h}{\partial x_2}+\frac{\partial v_2^h}{\partial x_1}\right)\right]\mathrm{d}x$$
$$=b_h(v^h,u^h),\quad\forall u^h,v^h\in T^h. \tag{3.15}$$

$$b_h(v^h,v^h)=\int_{\Omega_e}\left[\lambda|\,\mathrm{div}\,v^h|^2+2\mu\left(\frac{\partial v_1^h}{\partial x_1}\right)^2+2\mu\left(\frac{\partial v_2^h}{\partial x_2}\right)^2+\mu\left(\frac{\partial v_1^h}{\partial x_2}+\frac{\partial v_2^h}{\partial x_1}\right)^2\right]\mathrm{d}x\geqslant 0,\quad\forall v^h\in T^h. \tag{3.16}$$

From the Lemma 3.1, it is straightforward to check that the problem (3.9) is a well posed problem.

4. Numerical example

Let $\omega=\lambda/(\lambda+2\mu)$ and

$$h_1(x,t)=(1-\omega)\,\mathrm{arctg}\,\frac{x_1-t}{x_2}-(1+\omega)\frac{(x_1-t)x_2}{(x_1-t)^2+x_2^2}, \tag{4.1}$$

$$h_2(x,t)=\ln[(x_1-t)^2+x_2^2]+(1+\omega)\frac{(x_1-t)^2}{(x_1-t)^2+x_2^2}, \tag{4.2}$$

$$h(x,t)=(h_1(x,t),h_2(,x,t))'; \tag{4.3}$$

$$u(x)=(u_1(x),u_2(x))'=h(x,0)-\frac{1}{2}[h(x,0.5)+h(x,-0.5)]. \tag{4.4}$$

It is straightforward to check that $u(x)$ is the unique solution of the following boundary value problem:

$$-\mu\,\Delta v-(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,v=0\quad\text{in }\Omega, \tag{4.5}$$

$$v|_{x_1=\pm1}=u(\pm1,x_2),\quad -1\leqslant x_2\leqslant 0, \tag{4.6}$$

$$v|_{x_2=-1} = u(x_1, -1), \quad -1 \leqslant x_1 \leqslant 1, \tag{4.7}$$

$$\lambda \frac{\partial v_1}{\partial x_1} + (\lambda + 2\mu)\frac{\partial v_2}{\partial x_2} = \mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) = 0, \quad x_2 = 0, \quad |x_1| \geqslant 1, \tag{4.8}$$

$$v \text{ is bounded when } r \to +\infty; \tag{4.9}$$

where Ω is the domain of the lower half plane $\mathbb{R}^2_- = \{x = (x_1, x_2) \mid x_2 < 0\}$ subtracted a rectangle $\bar{\Omega}_0 = \{x = (x_1, x_2) \mid -1 \leqslant x_1 \leqslant 1, \ -1 \leqslant x_2 \leqslant 0\}$, i.e. $\Omega = \mathbb{R}^2_- \setminus \bar{\Omega}_0$. We take $\Gamma_e = \{x = (2\cos\theta, 2\sin\theta) \mid -\pi \leqslant \theta \leqslant 0\}$ as artificial boundary. Then the domain Ω is divided into a bounded part Ω_i and an unbounded part Ω_e with

$$\Omega_i = \{x = (x_1, x_2) \mid x \in \Omega, \ x_1^2 + x_2^2 < 4\},$$

$$\Omega_e = \{x = (x_1, x_2) \mid x \in \Omega, \ x_1^2 + x_2^2 > 4\}.$$

Since the first and second parts $u_1(x)$ and $u_2(x)$ of $u(x)$ are antisymmetric and symmetric about x_2 axes, respectively. The domain of computation is taken to be the part of Ω_i lying in the fourth quadrant (say Ω_i^+). The following boundary condition is posed along $x_2 = 0$:

$$v_1(0, x_2) = \frac{\partial v_2(0, x_2)}{\partial x_1} = 0, \quad -2 \leqslant x_2 \leqslant -1. \tag{4.10}$$

The boundary condition (4.10) is equivalent to the following condition:

$$V_1(0, x_2) = \sigma_{12}(0, x_2) = 0, \quad -2 \leqslant x_2 \leqslant -1. \tag{4.11}$$

We use the method proposed in Section 3 to solve the problem (4.5)–(4.9) on the domain Ω_i^+.

Three meshes are used in the computation. Fig. 2 shows the triangulation for mesh A. On each triangle in mesh A, we connect the midpoints of every two sides, thus this triangle is divided into four small triangles. Then we obtained the refined mesh B. Mesh C is similarly generated from mesh B. Linear finite element is used in our computation. We take $\lambda = 1.0$ and $\mu = 2.0$. Let $u^h = (u_1^h, u_2^h)^t$ denote the finite element approximation in the domain Ω_i^+ by using the discrete artificial boundary condition (2.58). For comparison we also compute the finite element approximation $u^{h,N} = (u_1^{h,N}, u_2^{h,N})^t$ of problem (4.5)–(4.9) in the domain Ω_i^+ by using the following Neumann artificial boundary condition on Γ_e:

$$\sigma_n|_{\Gamma_e} = 0. \tag{4.12}$$

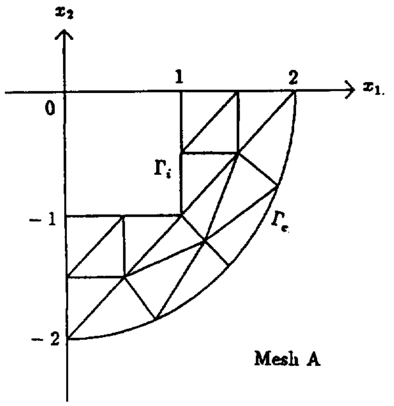


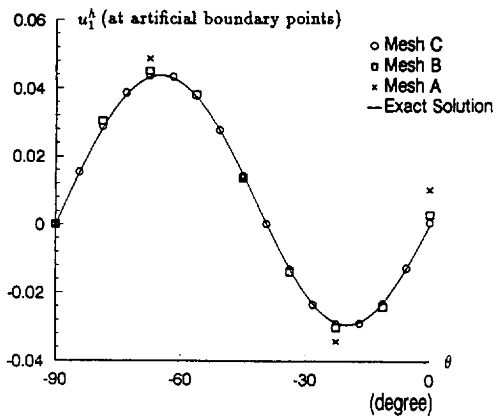
Fig. 2.

The Neumann artificial boundary condition (4.12) is often used in engineering literatures for simulating the problem of infinite elastic foundation.

Table 1 shows the maximum of the errors $u - u^h$ over the mesh points for mesh A, B and C. Furthermore, Table 2 gives the errors $\|u - u^h\|_{0,2,\Omega_i}$, $|u - u^h|_{1,2,\Omega_i}$ and $\|u - u^h\|_{1,2,\Omega_i}$ for mesh A, B and C. For comparison Table 3 shows the maximum of the errors $u - u^{h,N}$ over the mesh points for mesh A, B and C and Table 4 gives the errors $\|u - u^{h,N}\|_{0,2,\Omega_i}$, $|u - u^{h,N}|_{1,2,\Omega_i}$ and $\|u - u^{h,N}\|_{1,2,\Omega_i}$ for mesh A, B and C.

Furthermore, Figs. 3 and 4 show the values of numerical solution u_1^h and u_2^h on the mesh points of the artificial boundary Γ_e. Figs. 5 and 6 show the related errors $|u_1 - u_1^h|/|u_1| \times 100$ and $|u_2 - u_2^h|/|u_2| \times 100$ on the artificial boundary Γ_e. Figs. 7 and 8 show $u_1^{h,N}$ and $u_2^{h,N}$ on the artificial boundary Γ_e.

From Tables 1–4 and Figs. 3–8, we can see that our discrete artificial boundary condition (2.58) is very effective for the problem of infinite elastic foundation and more accurate than the Neumann boundary condition (4.12) which is often used in engineering literatures. We can derive a good numerical approximation u^h of the solution u of the original problem in a small domain Ω_i by using our discrete artificial



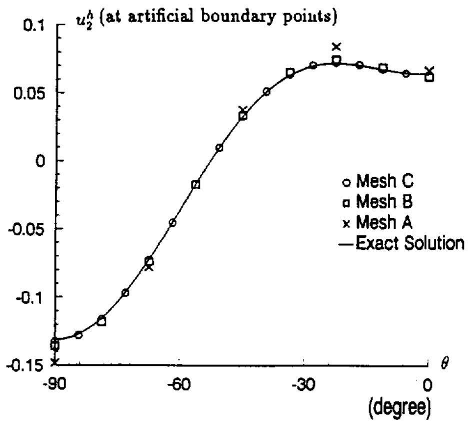
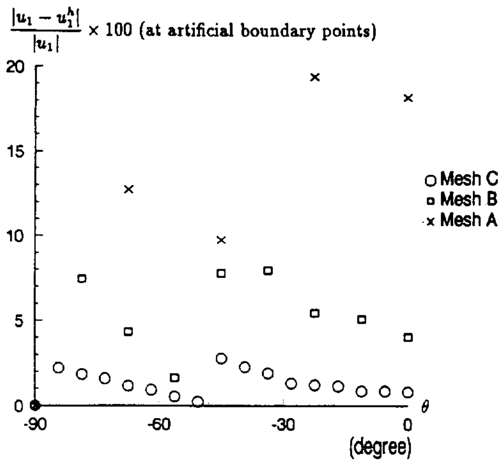


Fig. 3.

Fig. 4.



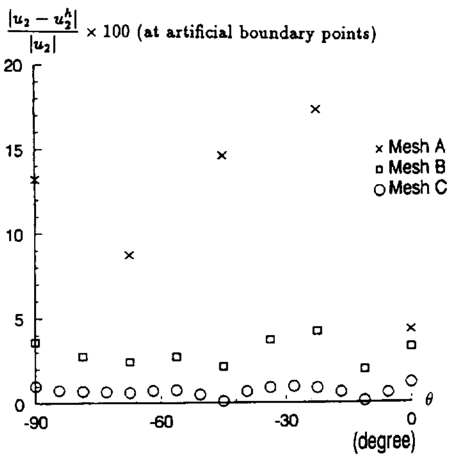
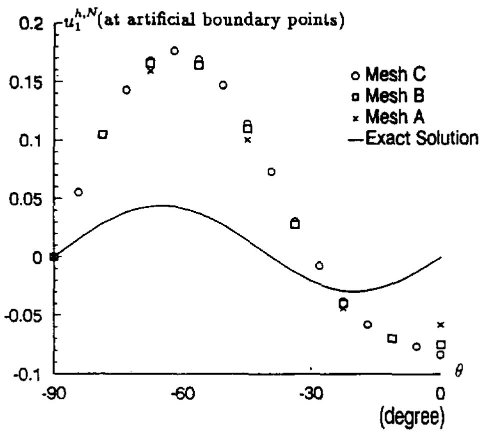


Fig. 5.

Fig. 6.



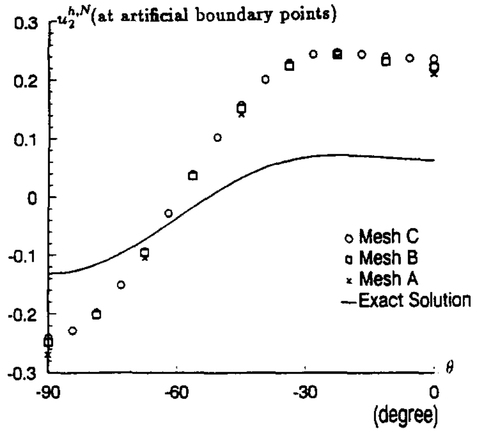


Fig. 7.

Fig. 8.

Table 1
Maximum error of $u - u^h$ over mesh points

Mesh	A	B	C
$\max \lvert u_1 - u_1^h \rvert$	1.1961E-2	6.7392E-3	2.6952E-3
$\max \lvert u_2 - u_2^h \rvert$	1.9399E-2	6.6557E-3	2.1782E-3

Table 2
Errors of $u - u^h$

Mesh	A	B	C
$\lVert u_1 - u_1^h \rVert_{0,2,\Omega_i}$	4.2899E-2	1.1832E-2	3.2777E-3
$\lvert u_1 - u_1^h \rvert_{1,2,\Omega_i}$	2.9494E-1	1.7532E-1	9.1848E-2
$\lVert u_1 - u_1^h \rVert_{1,2,\Omega_i}$	2.9804E-1	1.7572E-1	9.1907E-2
$\lVert u_2 - u_2^h \rVert_{0,2,\Omega_i}$	4.8181E-2	1.5243E-2	4.1713E-3
$\lvert u_2 - u_2^h \rvert_{1,2,\Omega_i}$	4.6169E-1	2.5481E-1	1.3099E-1
$\lVert u_2 - u_2^h \rVert_{1,2,\Omega_i}$	4.6420E-1	2.5527E-1	1.3105E-1

Table 3
Maximum error of $u - u^{h,N}$ over mesh points

Mesh	A	B	C
$\max \lvert u_1 - u_1^{h,N} \rvert$	1.1554E-1	1.2648E-1	1.3283E-1
$\max \lvert u_2 - u_2^{h,N} \rvert$	1.6878E-1	1.7154E-1	1.7611E-1

Table 4
Errors of $u - u^{h,N}$

Mesh	A	B	C
$\lVert u_1 - u_1^{h,N} \rVert_{0,2,\Omega_i}$	9.6826E-2	1.2118E-1	1.3004E-1
$\lvert u_1 - u_1^{h,N} \rvert_{1,2,\Omega_i}$	4.4996E-1	4.2557E-1	4.1974E-1
$\lVert u_1 - u_1^{h,N} \rVert_{1,2,\Omega_i}$	4.6026E-1	4.4249E-1	4.3943E-1
$\lVert u_2 - u_2^{h,N} \rVert_{0,2,\Omega_i}$	2.0704E-1	2.2585E-1	2.3780E-1
$\lvert u_2 - u_2^{h,N} \rvert_{1,2,\Omega_i}$	6.4550E-1	5.5591E-1	5.3290E-1
$\lVert u_2 - u_2^{h,N} \rVert_{1,2,\Omega_i}$	6.7790E-1	6.0004E-1	5.8355E-1

boundary condition at the artificial boundary. On the other hand, the Neumann artificial boundary condition is a very crude artificial boundary condition. We can also see that the finite element approximation u^h converges to the solution u of the original problem in the bounded computational domain Ω_i when the finite element mesh size h of Ω_i goes to 0 and $u^{h,N}$ converges, but not to u, in Ω_i when h goes to 0.

5. Conclusions

A discrete artificial boundary condition on a half-circle artificial boundary is designed for the problem of infinite elastic foundation by using the direct method of lines. The discrete artificial boundary condition is compared to the Neumann boundary condition which is often used in engineering literatures. Numerical results show that our discrete artificial boundary condition is more accurate than the Neumann boundary condition. Therefore it is time saving, since for a given accuracy it is possible to use a smaller computational domain. Furthermore, when the bounded computational domain is fixed, the finite element approximation u^h converges to the solution u of the original problem by using our discrete artificial boundary condition at the artificial boundary as the finite element mesh size h of the bounded computational domain tends to 0. On the other hand, if we use Neumann boundary condition at the artificial boundary, the finite element approximation $u^{h,N}$ converges, but not to the exact solution u, when the mesh size h tends to 0.

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