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GROSS–PITAEVSKII–POISSON EQUATIONS FOR DIPOLAR BOSE–EINSTEIN CONDENSATE WITH ANISOTROPIC CONFINEMENT*

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Abstract. Ground states and dynamical properties of a dipolar Bose–Einstein condensate are analyzed based on the Gross–Pitaevskii–Poisson system (GPPS) and its dimension reduction models under an anisotropic confining potential. We begin with the three-dimensional (3D) GPPS and review its quasi-two-dimensional (2D) approximate equations when the trap is strongly confined in the z-direction and quasi-one-dimensional (1D) approximate equations when the trap is strongly confined in the x- and y-directions. In fact, in the quasi-2D equations, a fractional Poisson equation with the operator $(-\Delta)^{1/2}$ is involved which brings significant difficulties into the analysis. Existence and uniqueness as well as nonexistence of the ground state under different parameter regimes are established for the quasi-2D and quasi-1D equations. Well-posedness of the Cauchy problem for both types of equations and finite time blow-up in two dimensions are analyzed. Finally, we rigorously prove the convergence with linear convergence rate for the solutions of the 3D GPPS and its quasi-2D and quasi-1D approximate equations in the weak interaction regime.

Key words. Gross–Pitaevskii–Poisson system, dipolar Bose–Einstein condensate, ground state, dimension reduction

AMS subject classifications. 35Q55, 35A01, 81Q99

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1. Introduction.

Quantum degenerate gases have received considerable interest, both theoretically and experimentally, since the first observation of a Bose–Einstein condensate (BEC) with dilute bosonic gases in 1995. The properties of these ultracold dilute quantum gases are determined by the short-range, isotropic contact interactions between the particles, which have been studied extensively. For those particles with large permanent magnetic or electric dipole moment, dipole-dipole interactions are nonnegligible, and the dipolar interactions are long-range and anisotropic, different from contact interactions. Due to these remarkable properties of dipolar interactions, there has been great interest in studying dipolar BECs in the last decade. In 2005, the first dipolar BEC with ^{52}Cr atoms was successfully realized in experiments at the University of Stuttgart [15]. Very recently in 2011, a dipolar BEC with ^{164}Dy atoms, whose dipole-dipole interaction is much stronger than that of ^{52}Cr, was achieved in experiments at Stanford University [21]. The success of these experiments has renewed interest in theoretically studying dipolar BECs.

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In this paper, we will consider the zero temperature mean-field model of dipolar BECs, the three-dimensional (3D) Gross–Pitaevskii equation (GPE) with dipolar interaction in dimensionless form [2, 12, 24, 30, 31, 32]:

$$i\partial_t\psi(\mathbf{r},t) = \left[-\frac{1}{2}\nabla^2 + V(\mathbf{r}) + \beta|\psi|^2 + \lambda\left(U_{\rm dip} * |\psi|^2\right)\right]\psi, \qquad \mathbf{r} = (\mathbf{x}, z) \in \mathbb{R}^3, \quad t > 0, \tag{1.1}$$

where t is time, $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and $\mathbf{r} = (\mathbf{x}, z) = (x, y, z) \in \mathbb{R}^3$ are the Cartesian coordinates, $\psi = \psi(\mathbf{r}, t)$ is the dimensionless complex-valued wave function, $V(\mathbf{r})$ is a given real-valued trapping potential in the experiments, β and λ are dimensionless constants representing the contact interaction and dipolar interaction, respectively, and $U_{\rm dip}(\mathbf{r})$ is given as

$$U_{\rm dip}(\mathbf{r}) = \frac{3}{4\pi}\,\frac{1 - 3(\mathbf{r}\cdot\mathbf{n})^2/|\mathbf{r}|^2}{|\mathbf{r}|^3} = \frac{3}{4\pi}\,\frac{1 - 3\cos^2(\theta)}{|\mathbf{r}|^3}, \qquad \mathbf{r} \in \mathbb{R}^3, \tag{1.2}$$

with the dipolar axis $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$ satisfying $|\mathbf{n}| = \sqrt{n_1^2 + n_2^2 + n_3^3} = 1$. Although the kernel $U_{\rm dip}$ is highly singular near the origin, the convolution is well defined for $\rho \in L^p(\mathbb{R}^3)$ with $U_{\rm dip} * \rho \in L^p(\mathbb{R}^3)$ $(p \in (1, \infty))$ [12]. In the context of a BEC, the initial data is usually normalized such that $\|\psi(\cdot, 0)\|_{L^2} = 1$.

Denote the differential operators $\partial_{\mathbf{n}} = \mathbf{n} \cdot \nabla$ and $\partial_{\mathbf{nn}} = \partial_{\mathbf{n}}\partial_{\mathbf{n}}$, and note the identity [2]

$$U_{\rm dip}(\mathbf{r}) = \frac{3}{4\pi|\mathbf{r}|^3}\left(1 - \frac{3(\mathbf{r}\cdot\mathbf{n})^2}{|\mathbf{r}|^2}\right) = -\delta(\mathbf{r}) - 3\partial_{\mathbf{nn}}\left(\frac{1}{4\pi|\mathbf{r}|}\right), \qquad \mathbf{r} \in \mathbb{R}^3, \tag{1.3}$$

with δ being the Dirac distribution. We can reformulate the GPE (1.1) as the following Gross–Pitaevskii–Poisson system (GPPS) [2, 11]:

$$i\partial_t\psi(\mathbf{r},t) = \left[-\frac{1}{2}\nabla^2 + V(\mathbf{r}) + (\beta - \lambda)|\psi|^2 - 3\lambda\partial_{\mathbf{nn}}\varphi\right]\psi, \quad \mathbf{r} \in \mathbb{R}^3, \quad t > 0, \tag{1.4}$$

$$\nabla^2\varphi(\mathbf{r}, t) = -|\psi(\mathbf{r}, t)|^2, \qquad \mathbf{r} \in \mathbb{R}^3, \qquad \lim_{|\mathbf{r}|\to\infty}\varphi(\mathbf{r}, t) = 0, \qquad t \geq 0. \tag{1.5}$$

The above GPPS in three dimensions (3D) conserves the *mass*, or the *normalization condition*,

$$\|\psi(\cdot, t)\|_2^2 = \int_{\mathbb{R}^3}|\psi(\mathbf{r}, t)|^2\, d\mathbf{r} \equiv \int_{\mathbb{R}^3}|\psi(\mathbf{r}, 0)|^2\, d\mathbf{r} = 1, \qquad t \geq 0, \tag{1.6}$$

and *energy* per particle,

$$E_{3D}(\psi) = \int_{\mathbb{R}^3}\left[\frac{1}{2}|\nabla\psi|^2 + V(\mathbf{r})|\psi|^2 + \frac{\beta - \lambda}{2}|\psi|^4 + \frac{3\lambda}{2}\,|\partial_{\mathbf{n}}\nabla\varphi|^2\right]d\mathbf{r}, \quad \varphi = \frac{1}{4\pi|\mathbf{r}|} * |\psi|^2. \tag{1.7}$$

It was proved [2] that when $\beta \geq 0$ and $-\frac{\beta}{2} \leq \lambda \leq \beta$, there exists a unique positive ground state Φ_g which is defined as the minimizer of the energy functional, i.e., $E_{3D}(\Phi_g) = \min_{\|\Phi\|_2=1} E_{3D}(\Phi)$, and the Cauchy problem of the GPPS (1.4)–(1.5) is globally well-posed; otherwise there exists no ground state, and the Cauchy problem is locally well-posed and finite time blow-up may happen under certain conditions [2].

In many physical experiments with dipolar BECs, the condensates are confined by a strong harmonic trap in one or two axis directions, resulting in a pancake- or cigar-shaped dipolar BEC, respectively. Mathematically speaking, this corresponds to the anisotropic potentials $V(\mathbf{r})$ of the following forms:

Case I *(pancake-shaped).* The potential is strongly confined in the vertical z-direction with

$$V(\mathbf{r}) = V_2(\mathbf{x}) + \frac{z^2}{2\varepsilon^4}, \qquad \mathbf{r} = (\mathbf{x}, z) \in \mathbb{R}^3. \tag{1.8}$$

Case II *(cigar-shaped).* The potential is strongly confined in the horizontal $\mathbf{x} = (x, y) \in \mathbb{R}^2$ plane with

$$V(\mathbf{r}) = V_1(z) + \frac{x^2 + y^2}{2\varepsilon^4}, \qquad \mathbf{r} = (\mathbf{x}, z) \in \mathbb{R}^3, \tag{1.9}$$

where $0 < \varepsilon \ll 1$ is a small parameter describing the strength of confinement. In such cases, the above GPPS in 3D can be formally reduced to two dimensions (2D) and one dimension (1D), respectively [11].

In Case I, when $\varepsilon \to 0^+$, evolution of the solution $\psi(\mathbf{r}, t)$ of GPPS (1.4)–(1.5) in the z-direction would essentially occur in the ground state mode of $L_z := -\frac{1}{2}\partial_{zz} + \frac{z^2}{2\varepsilon^4}$, which is spanned by $w_\varepsilon(z) = \varepsilon^{-1/2}\pi^{-1/4}e^{-\frac{z^2}{2\varepsilon^2}}$ [11, 4]. By taking the ansatz

$$\psi(\mathbf{x}, z, t) = e^{-it/2\varepsilon^2}\phi(\mathbf{x}, t)w_\varepsilon(z), \qquad (\mathbf{x}, z) \in \mathbb{R}^3, \quad t \geq 0, \tag{1.10}$$

the 3D GPPS (1.4)–(1.5) will be formally reduced to a *quasi-two-dimensional (2D) equation* I [11]:

$$i\partial_t\phi = \left[-\frac{1}{2}\Delta + V_2 + \frac{\beta - \lambda + 3\lambda n_3^2}{\sqrt{2\pi}\,\varepsilon}|\phi|^2 - \frac{3\lambda}{2}(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta)\varphi^{2D}\right]\phi, \quad \mathbf{x} \in \mathbb{R}^2,\ t > 0, \tag{1.11}$$

where $\mathbf{x} = (x, y)$, $\mathbf{n}_\perp = (n_1, n_2)$, $\partial_{\mathbf{n}_\perp} = \mathbf{n}_\perp \cdot \nabla$, $\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} = \partial_{\mathbf{n}_\perp}(\partial_{\mathbf{n}_\perp})$, $\Delta = \partial_{xx} + \partial_{yy}$, and

$$\varphi^{2D}(\mathbf{x}, t) = U_\varepsilon^{2D} * |\phi|^2, \quad U_\varepsilon^{2D}(\mathbf{x}) = \frac{1}{2\sqrt{2}\pi^{3/2}} \int_{\mathbb{R}} \frac{e^{-s^2/2}}{\sqrt{x^2 + y^2 + \varepsilon^2 s^2}}\, ds, \ \mathbf{x} \in \mathbb{R}^2,\ t \geq 0. \tag{1.12}$$

In addition, as $\varepsilon \to 0^+$, φ^{2D} can be approximated by φ_∞^{2D} [11] as

$$\varphi_\infty^{2D}(\mathbf{x}, t) = U_{\text{dip}}^{2D} * |\phi|^2 \quad \text{with} \quad U_{\text{dip}}^{2D}(\mathbf{x}) = \frac{1}{2\pi\sqrt{x^2 + y^2}}, \quad \mathbf{x} \in \mathbb{R}^2, \quad t \geq 0, \tag{1.13}$$

which can be rewritten as a fractional Poisson equation [11]

$$(-\Delta)^{1/2}\varphi_\infty^{2D}(\mathbf{x}, t) = |\phi(\mathbf{x}, t)|^2, \quad \mathbf{x} \in \mathbb{R}^2, \quad \lim_{|\mathbf{x}| \to \infty} \varphi_\infty^{2D}(\mathbf{x}, t) = 0, \qquad t \geq 0. \tag{1.14}$$

Thus an alternative *quasi-2D equation* II can be obtained as [11]

$$i\partial_t\phi = \left[-\frac{1}{2}\Delta + V_2 + \frac{\beta - \lambda + 3\lambda n_3^2}{\sqrt{2\pi}\,\varepsilon}|\phi|^2 - \frac{3\lambda}{2}(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta)(-\Delta)^{-1/2}(|\phi|^2)\right]\phi. \tag{1.15}$$

Similarly, in Case II, evolution of the solution $\psi(\mathbf{x},z,t)$ of GPPS (1.4)–(1.5) in $\mathbf{x}=(x,y)$-directions would essentially occur in the ground state mode of $L_{\mathbf{x}} := -\frac{1}{2}(\partial_{xx}+\partial_{yy})+\frac{x^2+y^2}{2\varepsilon^4}$, which is spanned by $w_\varepsilon(\mathbf{x})=\varepsilon^{-1}\pi^{-1/2}e^{-\frac{|\mathbf{x}|^2}{2\varepsilon^2}}$ [11, 4]. Again, by taking the ansatz

$$\psi(\mathbf{x},z,t)=e^{-it/\varepsilon^2}\phi(z,t)w_\varepsilon(\mathbf{x}), \qquad (\mathbf{x},z)\in\mathbb{R}^3, \quad t\ge 0, \tag{1.16}$$

the 3D GPPS (1.4)–(1.5) will be formally reduced to a *quasi-one-dimensional (1D) equation* [11]:

$$i\partial_t\phi=\left[-\frac{1}{2}\partial_{zz}+V_1+\frac{\beta+\frac{1}{2}\lambda(1-3n_3^2)}{2\pi\varepsilon^2}|\phi|^2-\frac{3\lambda(3n_3^2-1)}{8\sqrt{2\pi}\,\varepsilon}\partial_{zz}\varphi^{1D}\right]\phi, \quad z\in\mathbb{R},\ t>0, \tag{1.17}$$

where

$$\varphi^{1D}(z,t)=U_\varepsilon^{1D}*|\phi|^2, \qquad U_\varepsilon^{1D}(z)=\frac{\sqrt{2}e^{z^2/2\varepsilon^2}}{\sqrt{\pi}\,\varepsilon}\int_{|z|}^{\infty}e^{-s^2/2\varepsilon^2}\,ds, \qquad z\in\mathbb{R}, \quad t\ge 0. \tag{1.18}$$

The above effective lower dimensional models in 2D and 1D are very useful in the study of dipolar BECs since they are much easier and cheaper to simulate in practical computation. In fact, for the GPE without the dipolar term, i.e., $\lambda=0$, there have been extensive studies on this subject. For formal analysis and numerical simulation, the convergence rate of such dimension reduction was investigated numerically in [3, 5], and a nonlinear Schrödinger equation with polynomial nonlinearity in reduced dimensions was proposed in [23]. For rigorous analysis, convergence of the dimension reduction under anisotropic confinement has been proved in the weak interaction regime [7, 9], i.e., $\beta=O(\varepsilon)$ in 2D and $\beta=O(\varepsilon^2)$ in 1D. However, with the dipolar term, i.e., $\lambda\ne 0$, there have been few works addressing the mathematical analysis for this dimension reduction except for some preliminary results in [12], where different scalings and formulations were adapted. In fact, our quasi-2D models (1.11) and (1.15) and quasi-1D model (1.17) are much easier to use in mathematical analysis and practical numerical computation.

The main aims of this paper are to establish existence and uniqueness of the ground states and well-posedness of the Cauchy problems associated to the quasi-2D equations I and II and quasi-1D equation, and to analyze the convergence and convergence rate of the dimension reduction from 3D to 2D and 1D. In order to do so, without loss of generality, we assume the potential $V_d(\eta)\ge 0$ for $\eta\in\mathbb{R}^d$ $(d=1,2,3)$. It is natural to consider the energy space in d dimensions $(d=1,2,3)$ defined as

$$X_d=\left\{u\in H^1(\mathbb{R}^d)\ \Big|\ \|u\|_{X_d}^2=\|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\int_{\mathbb{R}^d}V_d(\eta)|u(\eta)|^2\,d\eta<\infty\right\},$$

and the unit sphere of X_d defined as

$$S_d=X_d\bigcap\{u\in L^2(\mathbb{R}^d)\big|\ \|u\|_{L^2(\mathbb{R}^d)}=1\}.$$

This paper is organized as follows. In sections 2, 3, and 4, we study quasi-2D equations I (1.11) and II (1.15) and quasi-1D equation (1.17), respectively. In each section, we first establish existence and uniqueness as well as nonexistence of the ground state under different parameter regimes, and then study the well-posedness of the corresponding Cauchy problem. In section 5, we rigorously prove the validity of dimension reduction from 3D GPPS (1.4)–(1.5) to 2D and 1D in the weak interaction regimes. Our approach is based on a priori estimates from the energy and mass conservation together with the Strichartz estimates.

Throughout the paper, we adopt the standard notation of Sobolev space and use $\|f\|_p^p := \int_{\mathbb{R}^d} |f(\eta)|^p \, d\eta$ for $p \in (0,\infty)$ when there is no confusion about the space $\mathbb{R}^d$, denote C as a generic constant which is independent of ε, let X^* be the dual space of X, and adopt the Fourier transform of a function $f(\eta) \in L^1(\mathbb{R}^d)$ as

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(\eta) e^{-i\xi\cdot\eta} \, d\eta, \qquad \xi \in \mathbb{R}^d. \tag{1.19}$$

2. Results for the quasi-2D equation I.

In this section, we prove existence and uniqueness as well as nonexistence of ground states for the quasi-2D equation I under different parameter regimes and local (global) existence for the Cauchy problem. For considering the ground state in 2D, let C_b be the best constant from the Gagliardo–Nirenberg inequality [29], i.e.,

$$C_b := \inf_{0\neq f\in H^1(\mathbb{R}^2)} \frac{\|\nabla f\|^2_{L^2(\mathbb{R}^2)} \cdot \|f\|^2_{L^2(\mathbb{R}^2)}}{\|f\|^4_{L^4(\mathbb{R}^2)}}. \tag{2.1}$$

2.1. Existence and uniqueness of ground state.

Associated to the quasi-2D equation I (1.11)–(1.12), the energy is

$$E_{2D}(\Phi) = \int_{\mathbb{R}^2} \left[\frac{1}{2}|\nabla\Phi|^2 + V_2(\mathbf{x})|\Phi|^2 + \frac{\beta - \lambda + 3n_3^2\lambda}{2\sqrt{2\pi}\,\varepsilon}|\Phi|^4 - \frac{3\lambda}{4}|\Phi|^2 \widetilde{\varphi^{2D}} \right] d\mathbf{x}, \quad \Phi \in X_2, \tag{2.2}$$

where

$$\widetilde{\varphi^{2D}} = \left(\partial_{\mathbf{n}_\perp \mathbf{n}_\perp} - n_3^2 \Delta\right) \varphi^{2D}, \qquad \varphi^{2D} = U_\varepsilon^{2D} * |\Phi|^2. \tag{2.3}$$

The ground state $\Phi_g \in S_2$ of (1.11) is the minimizer of the following nonconvex minimization problem:

$$\text{Find } \Phi_g \in S_2 \quad \text{such that } E_{2D}(\Phi_g) = \min_{\Phi\in S_2} E_{2D}(\Phi). \tag{2.4}$$

For the ground state, we have the following results.

THEOREM 2.1 (existence and uniqueness of ground state). *Assume* $0 \le V_2(\mathbf{x}) \in L^\infty_{loc}(\mathbb{R}^2)$ *and* $\lim_{|\mathbf{x}|\to\infty} V_2(\mathbf{x}) = \infty$*; then we have the following:*

(i) *There exists a ground state* $\Phi_g \in S_2$ *of the system* (1.11)–(1.12) *if one of the following conditions holds:*
  (A1) $\lambda \ge 0$ *and* $\beta - \lambda > -\sqrt{2\pi} C_b\, \varepsilon$*;*
  (A2) $\lambda < 0$ *and* $\beta + \frac{1}{2}(1 + 3|2n_3^2 - 1|)\lambda > -\sqrt{2\pi} C_b\, \varepsilon$.

(ii) *The positive ground state* $|\Phi_g|$ *is unique under one of the following conditions:*
  (A1′) $\lambda \ge 0$ *and* $\beta - \lambda \ge 0$*;*
  (A2′) $\lambda < 0$ *and* $\beta + \frac{1}{2}(1 + 3|2n_3^2 - 1|)\lambda \ge 0$.
  Moreover, any ground state is of the form $\Phi_g = e^{i\theta_0}|\Phi_g|$ *for some constant* $\theta_0 \in \mathbb{R}$.

(iii) *If* $\beta + \frac{1}{2}\lambda(1 - 3n_3^2) < -\sqrt{2\pi} C_b\, \varepsilon$*, there exists no ground state of* (1.11).

In order to prove this theorem, we first study the property of the nonlocal term.

LEMMA 2.2 (kernel U_ε^{2D} in (1.12)). *For any real function $f(\mathbf{x})$ in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$, we have*

$$\widehat{U_\varepsilon^{2D} * f}(\xi) = \hat{f}(\xi)\,\widehat{U_\varepsilon^{2D}}(\xi) = \frac{\hat{f}(\xi)}{\pi}\int_{\mathbb{R}} \frac{e^{-\varepsilon^2 s^2/2}}{|\xi|^2+s^2}ds, \qquad f \in \mathcal{S}(\mathbb{R}^2). \tag{2.5}$$

Moreover, define the operator

$$T_{\alpha\alpha'}(f) = \partial_{\alpha\alpha'}(U_\varepsilon^{2D} * f), \qquad \alpha, \alpha' = x, y;$$

then we have

$$\|T_{\alpha\alpha'} f\|_2 \le \frac{\sqrt{2}}{\sqrt{\pi}\,\varepsilon}\|f\|_2, \qquad \|T_{\alpha\alpha'} f\|_2 \le \|\nabla f\|_2, \tag{2.6}$$

and hence $T_{\alpha\alpha'}$ can be extended to a bounded linear operator from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.

Proof. From (1.12), we have

$$|U_\varepsilon^{2D}(\mathbf{x})| = \left|\frac{1}{2\sqrt{2}\pi^{3/2}}\int_{\mathbb{R}} \frac{e^{-s^2/2}}{\sqrt{|\mathbf{x}|^2+\varepsilon^2 s^2}}\,ds\right| \le \frac{1}{2\pi|\mathbf{x}|}, \quad \mathbf{0} \ne \mathbf{x} \in \mathbb{R}^2. \tag{2.7}$$

This immediately implies that $U_\varepsilon^{2D} * g$ is well defined for any $g \in L^1(\mathbb{R}^2)\bigcap L^2(\mathbb{R}^2)$ since the right-hand side in the above inequality is the singular kernel of the Riesz potential. Rewrite $U_\varepsilon^{2D}(\mathbf{x})$ as [11]

$$U_\varepsilon^{2D}(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{w_\varepsilon^2(z)w_\varepsilon^2(z')}{\sqrt{|\mathbf{x}|^2+(z-z')^2}}dzdz', \qquad \mathbf{x} \in \mathbb{R}^2.$$

Using the Plancherel formula, we get

$$\widehat{U_\varepsilon^{2D}}(\xi_1,\xi_2) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\widehat{w_\varepsilon^2}(\xi_3)\overline{\widehat{w_\varepsilon^2}}(\xi_3)}{\xi_1^2+\xi_2^2+\xi_3^2}d\xi_3 = \frac{1}{\pi}\int_{\mathbb{R}} \frac{e^{-\varepsilon^2 s^2/2}}{|\xi|^2+s^2}ds, \qquad \xi = (\xi_1,\xi_2) \in \mathbb{R}^2, \tag{2.8}$$

which immediately implies (2.5). Here $\bar{c}$ denotes the complex conjugate of c. Concerning $T_{\alpha\alpha'}$, we need only prove the results for T_{xx} since others are similar. Applying the Fourier transform, we have

$$\left|\widehat{T_{xx}f}(\xi)\right| = \left|\frac{\hat{f}(\xi)}{\pi}\int_{\mathbb{R}} \frac{e^{-\varepsilon^2 s^2/2}\xi_1^2}{|\xi|^2+s^2}ds\right| \le \frac{|\hat{f}(\xi)|}{\pi}\int_{\mathbb{R}} e^{-\varepsilon^2 s^2/2}ds = \frac{\sqrt{2}}{\sqrt{\pi}\,\varepsilon}|\hat{f}(\xi)|, \quad \xi \in \mathbb{R}^2. \tag{2.9}$$

Thus we can get the first inequality in (2.6) and know that $T_{xx}: L^2 \to L^2$ is bounded. Moreover, from

$$\left|\widehat{T_{xx}f}(\xi)\right| = \left|\frac{\hat{f}(\xi)}{\pi}\int_{\mathbb{R}} \frac{e^{-\varepsilon^2 s^2/2}\xi_1^2}{|\xi|^2+s^2}ds\right| \le \frac{|\hat{f}(\xi)|\,|\xi_1|^2}{\pi}\int_{\mathbb{R}} \frac{1}{|\xi|^2+s^2}ds \le |\xi|\,|\hat{f}(\xi)|, \tag{2.10}$$

we obtain the second inequality in (2.6) and know that $T_{xx}: H^1 \to L^2$ is bounded, too. ∎

Remark 2.1. In fact, $T_{\alpha\alpha'}$ is bounded from $L^p \to L^p$; i.e., there exists $C_p > 0$ independent of ε, such that

$$\|T_{\alpha\alpha'}(f)\|_p \le \frac{C_p}{\varepsilon}\|f\|_p, \quad p \in (1,\infty). \tag{2.11}$$

This can be obtained by the Minkowski inequality and L^p estimates for the Poisson equation.

LEMMA 2.3. *For the energy $E_{2D}(\cdot)$ in (2.2), we have the following:*

(i) *For any $\Phi \in S_2$, denote $\rho(\mathbf{x}) = |\Phi(\mathbf{x})|^2$; then we have*

$$E_{2D}(\Phi) \geq E_{2D}(|\Phi|) = E_{2D}\left(\sqrt{\rho}\right) \qquad \forall \Phi \in S_2, \tag{2.12}$$

so the ground state Φ_g of (2.2) is of the form $e^{i\theta_0}|\Phi_g|$ for some constant $\theta_0 \in \mathbb{R}$.

(ii) *Under condition* (A1) *or* (A2) *in Theorem* 2.1*, $E_{2D}(\sqrt{\rho})$ is bounded below.*

(iii) *Under condition* (A1′) *or* (A2′) *in Theorem* 2.1*, $E_{2D}(\sqrt{\rho})$ is strictly convex.*

Proof. (i) For any $\Phi \in S_2$, we have $|\Phi| \in S_2$, and a simple calculation shows

$$E_{2D}(\Phi) - E_{2D}(|\Phi|) = \frac{1}{2}\|\nabla\Phi\|_2^2 - \frac{1}{2}\|\nabla|\Phi|\,\|_2^2 \geq 0, \qquad \Phi \in S_2, \tag{2.13}$$

where the equality holds iff [20]

$$|\nabla\Phi(\mathbf{x})| = \nabla|\Phi(\mathbf{x})| \quad \text{a.e. } \mathbf{x} \in \mathbb{R}^2, \tag{2.14}$$

which is equivalent to

$$\Phi(\mathbf{x}) = e^{i\theta_0}|\Phi(\mathbf{x})| \quad \text{ for some } \ \theta_0 \in \mathbb{R}. \tag{2.15}$$

Then the conclusion follows.

(ii) For $\sqrt{\rho} = \Phi \in S_2$, we split the energy E_{2D} into two parts, i.e.,

$$E_{2D}(\Phi) = E_1(\Phi) + E_2(\Phi) = E_1(\sqrt{\rho}) + E_2(\sqrt{\rho}), \tag{2.16}$$

where

$$E_1(\sqrt{\rho}) = \int_{\mathbb{R}^2}\left[\frac{1}{2}|\nabla\sqrt{\rho}|^2 + V_2(\mathbf{x})\rho\right]d\mathbf{x}, \tag{2.17}$$

$$E_2(\sqrt{\rho}) = \int_{\mathbb{R}^2}\left[\frac{\beta - \lambda + 3n_3^2\lambda}{2\sqrt{2\pi}\,\varepsilon}|\rho|^2 - \frac{3\lambda}{4}\rho\widetilde{\varphi^{2D}}\right]d\mathbf{x}, \tag{2.18}$$

with

$$\widetilde{\varphi^{2D}} = \left(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta\right)U_\varepsilon^{2D} * \rho. \tag{2.19}$$

Applying the Plancherel formula and Lemma 2.2, there holds

$$\begin{aligned}\int_{\mathbb{R}^2}\widetilde{\varphi^{2D}}(\mathbf{x})\rho(\mathbf{x})\,d\mathbf{x} &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\widehat{\widetilde{\varphi^{2D}}}(\xi)\bar{\hat{\rho}}(\xi)d\xi \\ &= \frac{-1}{4\pi^3}\int_{\mathbb{R}^3}\frac{\left((n_1\xi_1 + n_2\xi_2)^2 - n_3^2|\xi|^2\right)e^{-\varepsilon^2 s^2/2}}{|\xi|^2 + s^2}|\hat{\rho}|^2 ds d\xi.\end{aligned} \tag{2.20}$$

Recalling the Cauchy inequality and $n_1^2 + n_2^2 + n_3^2 = 1$, we have

$$-n_3^2|\xi|^2 \leq (n_1\xi_1 + n_2\xi_2)^2 - n_3^2|\xi|^2 \leq (1 - 2n_3^2)|\xi|^2, \qquad \xi \in \mathbb{R}^2. \tag{2.21}$$

Denoting $C_0 = \max\{|n_3^2|, |1 - 2n_3^2|\}$, we can derive that

$$\left|\int_{\mathbb{R}^2}\widetilde{\varphi^{2D}}(\mathbf{x})\rho(\mathbf{x})\,d\mathbf{x}\right| \leq \frac{C_0}{4\pi^3}\int_{\mathbb{R}^3}e^{-\varepsilon^2 s^2/2}|\hat{\rho}|^2 ds d\xi = \frac{\sqrt{2}C_0}{\sqrt{\pi}\,\varepsilon}\|\rho\|_2^2. \tag{2.22}$$

Hence, $E_2(\sqrt{\rho})$ can be bounded below by $\|\rho\|_2^2$. In fact, under condition (A1), i.e., $\lambda \geq 0$ and $\beta - \lambda > -\sqrt{2\pi}C_b\,\varepsilon$, we have

$$E_2(\sqrt{\rho}) \geq \frac{\beta - \lambda + 3n_3^2\lambda}{2\sqrt{2\pi}\,\varepsilon}\|\rho\|_2^2 - \frac{3\sqrt{2}n_3^2\lambda}{4\sqrt{\pi}\,\varepsilon}\|\rho\|_2^2 > -\frac{C_b}{2}\|\rho\|_2^2. \tag{2.23}$$

Similarly, under condition (A2), if $\lambda < 0$ and $n_3^2 \geq \frac{1}{2}$, then

$$E_2(\sqrt{\rho}) \geq \frac{\beta - \lambda + 3n_3^2\lambda}{2\sqrt{2\pi}\,\varepsilon}\|\rho\|_2^2 > -\frac{C_b}{2}\|\rho\|_2^2, \tag{2.24}$$

and if $\lambda < 0$ and $n_3^2 < \frac{1}{2}$, then

$$E_2(\sqrt{\rho}) \geq \frac{\beta - \lambda + 3n_3^2\lambda}{2\sqrt{2\pi}\,\varepsilon}\|\rho\|_2^2 + \frac{3\sqrt{2}(1-2n_3^2)\lambda}{4\sqrt{\pi}\,\varepsilon}\|\rho\|_2^2 > -\frac{C_b}{2}\|\rho\|_2^2. \tag{2.25}$$

Recalling the choice of the best constant C_b, under either condition (A1) or (A2), the energy can be written as

$$E_{2D}(\sqrt{\rho}) = E_1(\sqrt{\rho}) + E_2(\sqrt{\rho}) > \frac{1}{2}\|\nabla\sqrt{\rho}\|_2^2 - \frac{C_b}{2}\|\rho\|_2^2 \geq 0. \tag{2.26}$$

(iii) Again, we split the energy as (2.16). It is well known that $E_1(\sqrt{\rho})$ is strictly convex in ρ [20]. It remains to show that $E_2(\sqrt{\rho})$ is convex in ρ. For any real function $u \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, let

$$H(u) = \int_{\mathbb{R}^2}\left[\frac{\beta - \lambda + 3n_3^2\lambda}{2\sqrt{2\pi}\,\varepsilon}|u|^2 - \frac{3\lambda}{4}u\left(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta_\perp\right)\left(U_\varepsilon^{2D} * u\right)\right]d\mathbf{x}. \tag{2.27}$$

Then $E_2(\sqrt{\rho}) = H(\rho)$. It suffices to show that $H(\rho)$ is convex in ρ. For this purpose, let $\sqrt{\rho_1} = \Phi_1 \in S_2$ and $\sqrt{\rho_2} = \Phi_2 \in S_2$. For any $\theta \in [0, 1]$, consider $\rho_\theta = \theta\rho_1 + (1-\theta)\rho_2$ and $\sqrt{\rho_\theta} \in S_2$; then we compute directly and get

$$\theta H(\rho_1) + (1-\theta)H(\rho_2) - H(\rho_\theta) = \theta(1-\theta)H(\rho_1 - \rho_2). \tag{2.28}$$

Similar to (2.20), looking at the Fourier domain, we can obtain the lower bounds for $H(\rho_1 - \rho_2)$ under condition (A1′) or (A2′), while replacing C_b with 0 in the above proof of (ii), i.e.,

$$H(\rho_1 - \rho_2) \geq 0. \tag{2.29}$$

This shows that $H(\rho)$, i.e., $E_2(\sqrt{\rho})$, is convex in ρ. Thus $E_{2D}(\sqrt{\rho})$ is strictly convex in ρ. ☐

Proof of Theorem 2.1. (i) We first prove the existence results. Lemma 2.3 ensures that there exists a minimizing sequence of nonnegative function $\{\Phi^n\}_{n=0}^\infty \subset S_2$, such that $\lim_{n\to\infty} E_{2D}(\Phi^n) = \inf_{\Phi\in S_2} E(\Phi)$. Then, under condition (A1) or (A2), there exists a constant C such that

$$\|\nabla\Phi^n\|_2 + \|\Phi^n\|_4 + \int_{\mathbb{R}^2} V_2(\mathbf{x})|\Phi^n(\mathbf{x})|^2 d\mathbf{x} \leq C, \qquad n \geq 0. \tag{2.30}$$

Therefore Φ^n belongs to a weakly compact set in $L^4(\mathbb{R}^2)$, $H^1(\mathbb{R}^2)$, and $L^2_{V_2}(\mathbb{R}^2)$ with a weighted L^2-norm given by $\|\Phi\|_{L_{V_2}} = [\int_{\mathbb{R}^2}|\Phi(\mathbf{x})|^2 V_2(\mathbf{x})d\mathbf{x}]^{1/2}$. Thus, there exist a

$\Phi^\infty \in W := H^1(\mathbb{R}^2)\bigcap L^2_{V_2}(\mathbb{R}^2)\bigcap L^4(\mathbb{R}^2)$ and a subsequence of $\{\Phi^n\}_{n=0}^\infty$ (which we denote as the original sequence for simplicity) such that

$$\Phi^n \rightharpoonup \Phi^\infty \quad \text{in } W, \qquad \nabla\Phi^n \rightharpoonup \nabla\Phi^\infty \quad \text{in } L^2. \tag{2.31}$$

The confining condition $\lim_{|\mathbf{x}|\to\infty} V_2(\mathbf{x}) = \infty$ will give that $\|\Phi^\infty\|_2 = 1$ [19, 2, 1]. Hence $\Phi^\infty \in S_2$ and $\Phi^n \to \Phi^\infty$ in $L^2(\mathbb{R}^2)$ due to the L^2-norm convergence and weak convergence of $\{\Phi^n\}_{n=0}^\infty$. By the lower semicontinuity of the H^1- and $L^2_{V_2}$-norms, for E_1 in (2.17), we know that

$$E_1(\Phi^\infty) \le \liminf_{n\to\infty} E_1(\Phi^n). \tag{2.32}$$

By the Sobolev inequality, there exists $C(p) > 0$ depending on $p \ge 2$, such that $\|\Phi^n\|_p \le C(p)(\|\nabla\Phi^n\|_2 + \|\Phi^n\|_2) \le C(p)(1+C)$, uniformly for $n \ge 0$. Applying Hölder's inequality, we have

$$\|(\Phi^n)^2 - (\Phi^\infty)^2\|_2^2 \le C_1(\|\Phi^n\|_6^3 + \|\Phi^n\|_6^3)\|\Phi^n - \Phi^\infty\|_2, \tag{2.33}$$

which shows that $\rho^n = (\Phi^n)^2 \to \rho^\infty = (\Phi^\infty)^2$ in $L^2(\mathbb{R}^2)$. Using the Fourier transform of U_ε^{2D} in Lemma 2.2 and (2.22), it is easy to derive the convergence for E_2 in (2.18), i.e.,

$$E_2(\Phi^\infty) = \lim_{n\to\infty} E_2(\Phi^n). \tag{2.34}$$

Hence

$$E_{2D}(\Phi^\infty) = E_1(\Phi^\infty) + E_2(\Phi^\infty) \le \liminf_{n\to\infty} E_{2D}(\Phi^n). \tag{2.35}$$

Now, we see that Φ^∞ is indeed a minimizer. For the uniqueness part, it is straightforward by the strict convexity of $E_{2D}(\sqrt{\rho})$ shown in Lemma 2.3.

(ii) Since the nonlinear term in the equation behaves as a cubic nonlinearity, it is natural to consider the following. Let $\Phi \in S_2$ be a real function that attains the best constant C_b [29]; then $\Phi(\mathbf{x})$ is radially symmetric. Choose $\Phi_\delta(\mathbf{x}) = \delta^{-1}\Phi(\delta^{-1}\mathbf{x})$ with $\delta > 0$; then $\Phi_\delta \in S_2$. Denote $\varphi_\delta = \left(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta_\perp\right)(U_\varepsilon^{2D} * |\Phi_\delta|^2)$; by the same computation as in Lemma 2.3, there holds

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi_\delta|\Phi_\delta|^2\,d\mathbf{x} &= \frac{-1}{4\pi^3}\int_{\mathbb{R}^3} \frac{(n_1\xi_1 + n_2\xi_2)^2 - n_3^2|\xi|^2}{|\xi|^2 + s^2} e^{-\varepsilon^2 s^2/2}\left|\widehat{|\Phi|^2}(\delta\xi)\right|^2 ds d\xi \\ &= \frac{-1}{4\pi^3\delta^2}\int_{\mathbb{R}^3} \frac{(n_1\xi_1 + n_2\xi_2)^2 - n_3^2|\xi|^2}{|\xi|^2 + \delta^2 s^2} e^{-\varepsilon^2 s^2/2}\left|\widehat{|\Phi|^2}(\xi)\right|^2 ds d\xi.\end{aligned}$$

Using the fact that $\Phi(\mathbf{x})$ is radially symmetric, $\widehat{|\Phi|^2}(\xi)$ is also radially symmetric. Then we obtain

$$\int_{\mathbb{R}^2} \varphi_\delta|\Phi_\delta|^2\,d\mathbf{x} = -\frac{(n_1^2 + n_2^2 - 2n_3^2) + o(1)}{\sqrt{2\pi}\,\varepsilon\delta^2}\|\Phi\|_4^4 \qquad \text{as } \delta \to 0^+. \tag{2.36}$$

Hence, as $\delta \to 0^+$, we get

$$E_{2D}(\Phi_\delta) = \frac{1}{2\delta^2}\left(\|\nabla\Phi\|_2^2 + \frac{\beta + \frac{1}{2}\lambda(1-3n_3^2) + o(1)}{\sqrt{2\pi}\,\varepsilon}\|\Phi\|_4^4\right) + \int_{\mathbb{R}^2} V_2(\delta\mathbf{x})|\Phi|^2(\mathbf{x})d\mathbf{x}.$$

Recalling that $\|\nabla\Phi\|_2^2 = C_b\|\Phi\|_4^4$, we know that $\lim_{\delta\to0^+} E_{2D}(\Phi_\delta) = -\infty$ if $\beta + \frac{1}{2}\lambda(1 - 3n_3^2) < -\sqrt{2\pi}C_b\,\varepsilon$; i.e., there is no ground state in this case. ∎

2.2. Well-posedness for the Cauchy problem. Here, we study the well-posedness of the Cauchy problem corresponding to the quasi-2D equation I (1.11)–(1.12). Using the Fourier transform of the kernel U_ε^{2D} in Lemma 2.2, it is straightforward to see that the nonlinear term introduced by U_ε^{2D} behaves like a cubic term. Thus, those methods for classic cubic nonlinear Schrödinger equations would apply [13, 29, 27]. In particular, we have the following theorem concerning the Cauchy problem of (1.11)–(1.12).

THEOREM 2.4 (well-posedness of Cauchy problem). *Suppose the real-valued trap potential satisfies* $V_2(\mathbf{x}) \geq 0$ *for* $\mathbf{x} \in \mathbb{R}^2$ *and*

$$V_2(\mathbf{x}) \in C^\infty(\mathbb{R}^2) \text{ and } D^{\mathbf{k}} V_2(\mathbf{x}) \in L^\infty(\mathbb{R}^2) \qquad \forall\, \mathbf{k} \in \mathbb{N}_0^2 \text{ with } |\mathbf{k}| \geq 2. \tag{2.37}$$

Then we have the following:

(i) *For any initial data* $\phi(\mathbf{x}, t = 0) = \phi_0(\mathbf{x}) \in X_2$, *there exists a* $T_{\max} \in (0, +\infty]$ *such that the problem* (1.11)–(1.12) *has a unique maximal solution* $\phi \in C([0, T_{\max}), X_2)$. *It is maximal in the sense that if* $T_{\max} < \infty$, *then* $\|\phi(\cdot, t)\|_{X_2} \to \infty$ *when* $t \to T_{\max}^-$.

(ii) *As long as the solution* $\phi(\mathbf{x}, t)$ *remains in the energy space* X_2, *the* L^2-norm $\|\phi(\cdot, t)\|_2$ *and* energy $E_{2D}(\phi(\cdot, t))$ *in* (2.2) *are conserved for* $t \in [0, T_{\max})$.

(iii) *Under either condition* (A1) *or* (A2) *in Theorem* 2.1 *with constant* C_b *being replaced by* $C_b/\|\phi_0\|_2^2$, *the solution of* (1.11)–(1.12) *is global in time, i.e.,* $T_{\max} = \infty$.

Proof. The proof is standard. We shall use the known results for semilinear Schrödinger equations [13]. For $\phi \in X_2$, denote $\rho = |\phi|^2$ and consider the following:

$$G(\phi, \bar{\phi}) := G(\rho) = \frac{1}{2} \int_{\mathbb{R}^2} |\phi|^2 \left(\partial_{\mathbf{n}_\perp \mathbf{n}_\perp} - n_3^2 \Delta\right) (U_\varepsilon^{2D} * |\phi|^2)\, d\mathbf{x},$$

$$g(\phi) = \frac{\delta G(\phi, \bar{\phi})}{\delta \bar{\phi}} = \phi \left(\partial_{\mathbf{n}_\perp \mathbf{n}_\perp} - n_3^2 \Delta\right) (U_\varepsilon^{2D} * |\phi|^2), \qquad \phi \in X_2.$$

Then (1.11)–(1.12) read as

$$i\partial_t \phi = \left[-\frac{1}{2}\Delta + V_2(\mathbf{x})\right] \phi + \beta_0 |\phi|^2 \phi - 3\lambda g(\phi), \qquad \mathbf{x} \in \mathbb{R}^2, \quad t > 0, \tag{2.38}$$

where $\beta_0 = \frac{\beta - \lambda + 3n_3^2 \lambda}{\sqrt{2\pi}\,\varepsilon}$. Using the L^p boundedness of T_{jk} (cf. Lemma 2.2 and Remark 2.1) and the Sobolev inequality, for $\|u\|_{X_2} + \|v\|_{X_2} \leq M$, it is easy to prove the following:

$$\|g(u) - g(v)\|_{4/3} \leq C(M) \|u - v\|_4. \tag{2.39}$$

In view of the standard theorems, Theorems 9.2.1, 4.12.1, and 5.7.1, in [13], and [27] for the well-posedness of the nonlinear Schrödinger equation, we can obtain results (i) and (ii) immediately. The global existence (iii) comes from the uniform bound for $\|\phi(\cdot, t)\|_{X_2}$, which can be derived from energy and L^2-norm conservation. ∎

When the initial data is small, there also exist global solutions [13, 12]. Otherwise, blow-up can happen in finite time, and we would have the following results.

THEOREM 2.5 (finite time blow-up). *For any initial data* $\phi(\mathbf{x}, t = 0) = \phi_0(\mathbf{x}) \in X_2$ *with* $\int_{\mathbb{R}^2} |\mathbf{x}|^2 |\phi_0(\mathbf{x})|^2\, d\mathbf{x} < \infty$, *let* $\phi := \phi(\mathbf{x}, t)$ *be the solution of the problem* (1.11). *Assume that* $V_2(\mathbf{x})$ *satisfies* $2V_2(\mathbf{x}) + \mathbf{x} \cdot \nabla V_2(\mathbf{x}) \geq 0$, *and that conditions* (A1) *and* (A2) *with constant* C_b *being replaced by* $C_b/\|\phi_0\|_2^2$ *are not satisfied; then there exists finite time blow-up, i.e.,* $T_{\max} < \infty$, *if* $\lambda = 0$, *or* $\lambda > 0$ *and* $n_3^2 \geq \frac{1}{2}$, *and one of the following holds:*

(i) $E_{2D}(\phi_0) < 0$;
(ii) $E_{2D}(\phi_0) = 0$ *and* $\mathrm{Im}\left(\int_{\mathbb{R}^2} \bar{\phi}_0(\mathbf{x})\ (\mathbf{x}\cdot\nabla\phi_0(\mathbf{x}))\, d\mathbf{x}\right) < 0$;
(iii) $E_{2D}(\phi_0) > 0$ *and* $\mathrm{Im}\left(\int_{\mathbb{R}^2} \bar{\phi}_0(\mathbf{x})\ (\mathbf{x}\cdot\nabla\phi_0(\mathbf{x}))\, d\mathbf{x}\right) < -\sqrt{2E_{2D}(\phi_0)}\|\mathbf{x}\phi_0\|_2$,

where $\mathrm{Im}(f)$ *denotes the imaginary part of* f.

Proof. Define the variance

$$\sigma_V(t) := \sigma_V(\phi(\cdot,t)) = \int_{\mathbb{R}^2} |\mathbf{x}|^2|\phi(\mathbf{x},t)|^2\, d\mathbf{x} = \sigma_x(t) + \sigma_y(t), \qquad t \geq 0, \tag{2.40}$$

where

$$\sigma_\alpha(t) := \sigma_\alpha(\phi(\cdot,t)) = \int_{\mathbb{R}^2} \alpha^2|\phi(\mathbf{x},t)|^2\, d\mathbf{x}, \qquad \alpha = x, y. \tag{2.41}$$

For $\alpha = x$, or y, differentiating (2.41) with respect to t and integrating by parts, we get

$$\frac{d}{dt}\sigma_\alpha(t) = -i\int_{\mathbb{R}^2} \left[\alpha\bar{\phi}(\mathbf{x},t)\partial_\alpha\phi(\mathbf{x},t) - \alpha\phi(\mathbf{x},t)\partial_\alpha\bar{\phi}(\mathbf{x},t)\right]\, d\mathbf{x}, \qquad t \geq 0. \tag{2.42}$$

Similarly, we have

(2.43)
$$\frac{d^2}{dt^2}\sigma_\alpha(t) = \int_{\mathbb{R}^2} \left[2|\partial_\alpha\phi|^2 + \beta_0|\phi|^4 + 3\lambda|\phi|^2\alpha\partial_\alpha(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta)\varphi - 2\alpha|\phi|^2\partial_\alpha V_2(\mathbf{x})\right]\, d\mathbf{x},$$

where $\beta_0 = \frac{\beta-\lambda+3\lambda n_3^2}{\sqrt{2\pi}\,\varepsilon}$, $\varphi = U_\varepsilon^{2D} * |\phi|^2$. Writing $\rho = |\phi|^2$, $\tilde{\varphi} = (\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta)\varphi$, and $n_\xi(\xi) = (n_1\xi_1+n_2\xi_2)^2 - n_3^2|\xi|^2$ and noticing that ρ is a real function, by the Plancherel formula, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} |\phi|^2\, (\mathbf{x}\cdot\nabla\tilde{\varphi})\ d\mathbf{x} &= \frac{-1}{4\pi^2}\int_{\mathbb{R}^2} \hat{\rho}(\xi)\,\nabla\cdot\left(\xi\overline{\hat{\tilde{\varphi}}}\right)\, d\xi = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \hat{\rho}(\xi)\,\nabla\cdot\left(\xi n_\xi\widehat{U_\varepsilon^{2D}}\overline{\hat{\rho}}\right)\, d\xi \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \hat{\rho}\,\left(\overline{\hat{\rho}}\nabla(\xi n_\xi\widehat{U_\varepsilon^{2D}}) + n_\xi\widehat{U_\varepsilon^{2D}}\xi\cdot\nabla\overline{\hat{\rho}}\right)\, d\xi \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \left(|\hat{\rho}|^2\nabla(\xi n_\xi\widehat{U_\varepsilon^{2D}}) + n_\xi\widehat{U_\varepsilon^{2D}}\xi\cdot\frac{1}{2}\nabla|\hat{\rho}|^2\right)\, d\xi \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \left(n_\xi\widehat{U_\varepsilon^{2D}} + \frac{1}{2}\xi\cdot\nabla(n_\xi\widehat{U_\varepsilon^{2D}})\right)|\hat{\rho}|^2\, d\xi \\
&= -\int_{\mathbb{R}^2} |\phi|^2\tilde{\varphi}\, d\mathbf{x} + \frac{1}{4\pi^3}\int_{\mathbb{R}^3} \frac{n_\xi s^2 e^{-\varepsilon^2 s^2/2}|\hat{\rho}|^2}{(|\xi|^2+s^2)^2}\, ds d\xi.
\end{aligned}$$

Denote

$$I(t) := I(\phi(\cdot,t)) = \frac{1}{4\pi^3}\int_{\mathbb{R}^3} \frac{n_\xi s^2 e^{-\varepsilon^2 s^2/2}|\hat{\rho}|^2}{(|\xi|^2+s^2)^2}\, ds d\xi, \qquad t \geq 0. \tag{2.44}$$

Using $n_\xi \in [-n_3^2|\xi|^2, (1-2n_3^2)|\xi|^2]$, we obtain

$$\frac{-\sqrt{2}n_3^2}{\sqrt{\pi}\,\varepsilon}\|\phi(t)\|_4^4 \leq I(t) \leq \frac{\sqrt{2}(1-2n_3^2)}{\sqrt{\pi}\,\varepsilon}\|\phi(t)\|_4^4, \qquad t \geq 0. \tag{2.45}$$

If $\lambda = 0$, or $\lambda > 0$ and $n_3 \geq \frac{1}{2}$, noting $\lambda I(t) \leq 0$ in these cases, summing (2.43) for $\alpha = x, y$, and using the energy conservation, we have

$$\begin{aligned}\frac{d^2}{dt^2}\sigma_V(t) &= 2\int_{\mathbb{R}^2}\left[|\nabla\phi|^2 + \beta_0|\phi|^4 + \frac{3}{2}\lambda|\phi|^2\left(\mathbf{x}\cdot\nabla\tilde{\varphi}\right) - |\phi|^2\mathbf{x}\cdot\nabla V_2(\mathbf{x})\right]d\mathbf{x}\\ &= 4E_{2D}(\phi(\cdot,t)) + 3\lambda I(t) - 2\int_{\mathbb{R}^2}|\phi|^2(2V_2(\mathbf{x}) + \mathbf{x}\cdot\nabla V_2(\mathbf{x}))\,d\mathbf{x}\\ &\leq 4E_{2D}(\phi(\cdot,t)) \equiv 4E_{2D}(\phi_0), \qquad t \geq 0.\end{aligned}$$

Thus,

$$\sigma_V(t) \leq 2E_{2D}(\phi_0)t^2 + \sigma_V'(0)t + \sigma_V(0), \qquad t \geq 0,$$

and the conclusion follows in the same manner as those in [27, 13] for the standard nonlinear Schrödinger equation. ∎

3. Results for the quasi-2D equation II. In this section, we investigate the existence and uniqueness as well as nonexistence of the ground state of the quasi-2D equation II (1.15), and the well-posedness of the corresponding Cauchy problem.

3.1. Existence and uniqueness of ground state. Associated to the quasi-2D equation II (1.15), the energy is

(3.1)
$$\tilde{E}_{2D}(\Phi) = \int_{\mathbb{R}^2}\left[\frac{1}{2}|\nabla\Phi|^2 + V_2(\mathbf{x})|\Phi|^2 + \frac{\beta - \lambda + 3n_3^2\lambda}{2\sqrt{2\pi}\,\varepsilon}|\Phi|^4 - \frac{3\lambda}{4}|\Phi|^2\varphi\right]d\mathbf{x}, \quad \Phi \in X_2,$$

where

$$\varphi(\mathbf{x}) = \left(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta\right)\left((-\Delta)^{-1/2}|\Phi|^2\right). \tag{3.2}$$

The ground state $\Phi_g \in S_2$ of (1.15) is defined as the minimizer of the nonconvex minimization problem:

$$\text{Find } \Phi_g \in S_2 \quad \text{such that } \tilde{E}_{2D}(\Phi_g) = \min_{\Phi\in S_2}\tilde{E}_{2D}(\Phi). \tag{3.3}$$

For the above ground state, we have the following results.

THEOREM 3.1 (existence and uniqueness of ground state). *Assume* $0 \leq V_2(\mathbf{x}) \in L^\infty_{loc}(\mathbb{R}^2)$ *and* $\lim_{|\mathbf{x}|\to\infty} V_2(\mathbf{x}) = \infty$*; then we have the following:*

(i) *There exists a ground state* $\Phi_g \in S_2$ *of* (1.15) *if one of the following conditions holds:*
 (B1) $\lambda = 0$ *and* $\beta > -\sqrt{2\pi}C_b\,\varepsilon$;
 (B2) $\lambda > 0$, $n_3 = 0$, *and* $\beta - \lambda > -\sqrt{2\pi}C_b\,\varepsilon$;
 (B3) $\lambda < 0$, $n_3^2 \geq \frac{1}{2}$, *and* $\beta - (1 - 3n_3^2)\lambda > -\sqrt{2\pi}C_b\,\varepsilon$.

(ii) *The positive ground state* $|\Phi_g|$ *is unique under one of the following conditions:*
 (B1′) $\lambda = 0$ *and* $\beta \geq 0$;
 (B2′) $\lambda > 0$, $n_3 = 0$, *and* $\beta \geq \lambda$;
 (B3′) $\lambda < 0$, $n_3^2 \geq \frac{1}{2}$, *and* $\beta - (1 - 3n_3^2)\lambda \geq 0$.
 Moreover, any ground state $\Phi_g = e^{i\theta_0}|\Phi_g|$ *for some constant* $\theta_0 \in \mathbb{R}$.

(iii) *There exists no ground state of* (1.15) *if one of the following conditions holds:*
 (B1″) $\lambda > 0$ *and* $n_3 \neq 0$;
 (B2″) $\lambda < 0$ *and* $n_3^2 < \frac{1}{2}$;
 (B3″) $\lambda = 0$ *and* $\beta < -\sqrt{2\pi}C_b\,\varepsilon$.

Again, in order to prove this theorem, we first analyze the nonlocal part in (1.15). In fact, following the standard proof in [25], we can get the following lemma.

LEMMA 3.2 (property of fractional Poisson equation (1.13)). *Assume $f(\mathbf{x})$ is a real-valued function good enough for the fractional Poisson equation*

$$(-\Delta)^{-1/2}\varphi(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \qquad \lim_{|\mathbf{x}|\to\infty} \varphi(\mathbf{x}) = 0.$$

We have

$$\varphi(\mathbf{x}) = \int_{\mathbb{R}^2} \frac{f(\mathbf{x}')}{2\pi|\mathbf{x}-\mathbf{x}'|} d\mathbf{x}' = \left(\frac{1}{2\pi|\mathbf{x}|}\right) * f, \qquad \mathbf{x} \in \mathbb{R}^2,$$

and the Hardy–Littlewood–Sobolev inequality implies

$$\|\varphi\|_{p^*} \le C_p \|f\|_p, \qquad p^* = \frac{2p}{2-p}, \qquad p \in (1,2). \tag{3.4}$$

Moreover, the first order derivatives of φ are the Riesz transforms of f and satisfy

$$\|\partial_\alpha \varphi\|_q \le C_q \|f\|_q, \qquad q \in (1,\infty), \quad \alpha = x, y, \tag{3.5}$$

and the second order derivatives satisfy

$$\|\partial_{\alpha\alpha'}\varphi\|_q = \|\partial_\alpha \left((-\Delta)^{-1/2}\partial_{\alpha'} f\right)\|_q \le C_q \|\partial_{\alpha'} f\|_q, \qquad q \in (1,\infty), \quad \alpha, \alpha' = x, y. \tag{3.6}$$

Remark 3.1. Similar results hold for $T_{\alpha\alpha'}$ defined in Lemma 2.2, i.e.,

$$\|T_{\alpha\alpha'} f\|_p \le C_p \|\nabla f\|_p \qquad \text{for } p \in (1,\infty). \tag{3.7}$$

Since the fractional Poisson operator $(-\Delta)^{-1/2}$ is taken as an approximation of U_ε^{2D} (1.12), we consider the convergence regarding the first order derivatives of $(-\Delta)^{-1/2}$ and U_ε^{2D}.

LEMMA 3.3. *For any real-valued function $f \in L^p(\mathbb{R}^2)$, let*

$$T_\alpha^\varepsilon(f) = \partial_\alpha(U_\varepsilon^{2D} * f), \qquad R_\alpha(f) = \partial_\alpha(-\Delta)^{-1/2} f, \qquad \alpha = x, y; \tag{3.8}$$

then T_α^ε is bounded from L^p to L^p for $1 < p < \infty$ with the bounds independent of ε. In particular, for any fixed $f \in L^p(\mathbb{R}^2)$ with $p \in (1,\infty)$, we have

$$\lim_{\varepsilon\to 0^+} \|T_\alpha^\varepsilon(f) - R_\alpha(f)\|_p = 0, \qquad p \in (1,\infty). \tag{3.9}$$

Proof. We can write R_α and T_α^ε as

$$R_\alpha(f) = K_\alpha * f, \qquad T_\alpha^\varepsilon(f) = K_\alpha^\varepsilon * f, \tag{3.10}$$

where R_α is the Riesz transform and

$$K_\alpha(\mathbf{x}) = \frac{\alpha}{2\pi|\mathbf{x}|^3}, \quad K_\alpha^\varepsilon(\mathbf{x}) = \frac{1}{2\sqrt{2}\pi^{3/2}} \int_{\mathbb{R}} \frac{\alpha e^{-s^2/2}}{(|\mathbf{x}|^2 + \varepsilon^2 s^2)^{3/2}} \, ds, \quad \mathbf{x} \in \mathbb{R}^2, \quad \alpha = x, y. \tag{3.11}$$

It is easy to check that K_α^ε satisfies

$$|K_\alpha^\varepsilon(\mathbf{x})| \le B|\mathbf{x}|^{-2}, \qquad |\nabla K_\alpha^\varepsilon(\mathbf{x})| \le B|\mathbf{x}|^{-3}, \qquad |\mathbf{x}|>0,$$
$$\int_{R_1<|\mathbf{x}|<R_2} K_j^\varepsilon(\mathbf{x})d\mathbf{x} = 0, \qquad 0<R_1<R_2<\infty,$$

for some ε-independent constant B. Then the standard theorem on singular integrals [25] implies that T_α^ε is well defined for L^p functions and is bounded from L^p to L^p with an ε-independent bound.

Thus, we need only prove the convergence in L^2; other cases can be derived by an approximation argument and interpolation. For the L^2 convergence, looking at the Fourier domain, we find that

$$\begin{aligned}\|T_\alpha^\varepsilon(f)-R_\alpha(f)\|_2^2 &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\left|\hat f(\xi)\right|^2\left[\frac{\alpha}{|\xi|}-\frac{\alpha}{\pi}\int_{\mathbb{R}}\frac{e^{-\varepsilon^2 s^2/2}}{|\xi|^2+s^2}ds\right]^2 d\xi\\ &\le \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\left|\hat f(\xi)\right|^2\left[\frac{1}{\pi}\int_{\mathbb{R}}\frac{(1-e^{-\varepsilon^2 s^2/2})|\xi|}{|\xi|^2+s^2}ds\right]^2 d\xi.\end{aligned}$$

Note that for fixed $0\ne\xi\in\mathbb{R}^2$, the dominated convergence theorem suggests that

$$\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\left|\int_{\mathbb{R}}\frac{(1-e^{-\varepsilon^2 s^2/2})|\xi|}{|\xi|^2+s^2}ds\right| = 0; \tag{3.12}$$

hence, the conclusion in the L^2 case is obvious by using the dominated convergence theorem again. Using approximation and the fact that $L^2\cap L^q$ is dense in L^p for $q\in(1,\infty)$, noting the uniform bound on $T_\alpha^\varepsilon : L^p\to L^p$ for $p\in(1,\infty)$, we can complete the proof. □

LEMMA 3.4. *For the energy $\tilde E_{2D}(\cdot)$ in* (3.1), *the following properties hold:*

(i) *For any $\Phi\in S_2$, denote $\rho(\mathbf{x})=|\Phi(\mathbf{x})|^2$; then we have*

$$\tilde E_{2D}(\Phi)\ge \tilde E_{2D}(|\Phi|)=\tilde E_{2D}\left(\sqrt{\rho}\right) \qquad \forall \Phi\in S_2, \tag{3.13}$$

so the ground state Φ_g of (3.1) *is of the form $e^{i\theta_0}|\Phi_g|$ for some constant $\theta_0\in\mathbb{R}$.*

(ii) *If condition* (B1), (B2), *or* (B3) *in Theorem* 3.1 *holds, then $\tilde E_{2D}$ is bounded below.*

(iii) *If condition* (B1′), (B2′), *or* (B3′) *in Theorem* 3.1 *holds, then $\tilde E_{2D}(\sqrt{\rho})$ is strictly convex.*

Proof. (i) The proof is similar to that of Lemma 2.3.

(ii) Similar to Lemma 2.3, for $\Phi\in S_2$, denote $\rho=|\Phi|^2$; we need only consider the lower bound of the functional

$$\tilde H(\rho) = -\lambda\int_{\mathbb{R}^2}\rho\left(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp}-n_3^2\Delta\right)[(-\Delta)^{-1/2}\rho]\,d\mathbf{x}. \tag{3.14}$$

Using the Plancherel formula and the Cauchy inequality, for $\lambda<0$ and $n_3^2\ge\frac12$, we have

$$\begin{aligned}&\tilde H(\rho)\\ &=\frac{\lambda}{4\pi^2}\int_{\mathbb{R}^2}\frac{(n_1\xi_1+n_2\xi_2)^2-n_3^2|\xi_3|^2}{|\xi|}\left|\hat\rho(\xi)\right|^2 d\xi \ge \frac{\lambda}{4\pi^2}\int_{\mathbb{R}^2}(1-2n_3^2)|\xi|\ \left|\hat\rho(\xi)\right|^2 d\xi\ge 0.\end{aligned} \tag{3.15}$$

For $\lambda > 0$ and $n_3 = 0$, it is easy to see that $\tilde{H}(\rho) \geq 0$. Hence, assertion (ii) is proved.

(iii) Similar to Lemma 2.3, it is sufficient to prove the convexity of $\tilde{H}(\rho)$ in ρ. For $\sqrt{\rho_1} \in S_2$, $\sqrt{\rho_2} \in S_2$, and any $\theta \in [0,1]$, denote $\rho_\theta = \theta\rho_1 + (1-\theta)\rho_2$. We have

$$\theta\tilde{H}(\rho_1) + (1-\theta)\tilde{H}(\rho_2) - \tilde{H}(\rho_\theta) = \theta(1-\theta)\tilde{H}(\rho_1 - \rho_2), \tag{3.16}$$

where the right-hand side is nonnegative under the given condition, i.e., $\tilde{H}(\rho)$ is convex. □

Proof of Theorem 3.1. (i) We need only consider the existence since the uniqueness is a consequence of the convexity of $\tilde{E}_{2D}(\sqrt{\rho})$ in Lemma 3.4. For existence, we may apply the same arguments as in Theorem 2.1, except that here, for the sequence $\rho^n = (\Phi^n)^2$, we have to show

$$\liminf_{n\to\infty} \tilde{H}(\rho^n) \geq \tilde{H}(\rho^\infty), \quad \text{with} \quad \rho^\infty = |\Phi^\infty|^2. \tag{3.17}$$

Denote

$$\varphi^n = \left(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta\right)[(-\Delta)^{-1/2}\rho^n], \qquad n = 0,1,\ldots \text{ or } n = \infty.$$

Using $\Phi^n \to \Phi^\infty$ in $L^2(\mathbb{R}^2)$ and $\Phi^n \rightharpoonup \Phi^\infty$ in $H^1(\mathbb{R}^2)$, we have $\rho^n \to \rho^\infty$ in $L^p(\mathbb{R}^2)$ $p > 1$, and Lemma 3.2 shows that $\varphi^n \to \varphi^\infty$ in $W^{-1,p'}(\mathbb{R}^2)$, which is the dual space of $W^{1,p}$ with $p' = p/(p-1)$. Thus (3.17) is true, and the existence of the ground state follows.

(ii) To prove the nonexistence results, we try to find the case where $\tilde{E}_{2D}$ does not have a lower bound. For any $\Phi \in S_2$, denote $\rho(\mathbf{x}) = |\Phi(\mathbf{x})|^2$ and let $\theta \in \mathbb{R}$ such that $(\cos\theta, \sin\theta) = \frac{1}{\sqrt{n_1^2+n_2^2}}(n_1, n_2)$ when $n_1^2 + n_2^2 \neq 0$ and $\theta = 0$ if $n_1 = n_2 = 0$. For any $\varepsilon_1, \varepsilon_2 > 0$, consider the following function:

$$\Phi_{\varepsilon_1,\varepsilon_2}(x,y) = \varepsilon_1^{-1/2}\varepsilon_2^{-1/2}\Phi(\varepsilon_1^{-1}(x\cos\theta + y\sin\theta), \varepsilon_2^{-1}(-x\sin\theta + y\cos\theta)). \tag{3.18}$$

Denoting $\rho_{\varepsilon_1,\varepsilon_2} = |\Phi_{\varepsilon_1,\varepsilon_2}|^2$, we have

$$\widehat{\rho_{\varepsilon_1,\varepsilon_2}}(\xi_1,\xi_2) = \hat{\rho}(\varepsilon_1(\xi_1\cos\theta + \xi_2\sin\theta), \varepsilon_2(-\xi_1\sin\theta + \xi_2\cos\theta)). \tag{3.19}$$

By the Plancherel formula and changing variables, we get

$$\begin{aligned}
\tilde{H}(\rho_{\varepsilon_1,\varepsilon_2}) &= \frac{\lambda}{4\pi^2}\int_{\mathbb{R}^2} \frac{(n_1\xi_1 + n_2\xi_2)^2 - n_3^2|\xi^2|}{|\xi|}|\widehat{\rho_{\varepsilon_1,\varepsilon_2}}|^2 d\xi \\
&= \frac{\lambda}{4\pi^2}\int_{\mathbb{R}^2} \frac{(n_1^2+n_2^2)\eta_1^2 - n_3^2|\eta|^2}{|\eta|}|\widehat{\rho}|^2(\varepsilon_1\eta_1, \varepsilon_2\eta_2)d\eta \\
&= \frac{\lambda}{4\varepsilon_1^2\varepsilon_2\pi^2}\int_{\mathbb{R}^2} \frac{(1-n_3^2)\eta_1^2 - n_3^2(\eta_1^2 + \frac{\varepsilon_2^2}{\varepsilon_1^2}\eta_2^2)}{\sqrt{\eta_1^2 + \frac{\varepsilon_2^2}{\varepsilon_1^2}\eta_2^2}}|\widehat{\rho}|^2(\eta_1,\eta_2)d\eta.
\end{aligned}$$

Let $\kappa = \frac{\varepsilon_2}{\varepsilon_1}$; then the dominated convergence theorem implies that

$$\tilde{H}(\rho_{\varepsilon_1,\varepsilon_2}) = \begin{cases} \frac{1-2n_3^2+o(1)}{4\varepsilon_1^2\varepsilon_2}\lambda\int_{\mathbb{R}^2}|\eta_1| \ \left|\widehat{\rho}(\eta_1,\eta_2)\right|^2 \, d\eta, & \kappa \to 0^+, \\ \frac{-n_3^2+o(1)}{4\varepsilon_1^2\varepsilon_2}\lambda\int_{\mathbb{R}^2}|\eta_1| \ \left|\widehat{\rho}(\eta_1,\eta_2)\right|^2 \, d\eta, & \kappa \to +\infty. \end{cases} \tag{3.20}$$

For fixed $\kappa > 0$ and letting $\varepsilon_1 \to 0^+$, we have $\int_{\mathbb{R}^2} V_2(\mathbf{x})|\Phi_{\varepsilon_1,\varepsilon_2}|^2\, d\mathbf{x} = O(1)$ and

$$\|\nabla\Phi_{\varepsilon_1,\varepsilon_2}\|_2^2 = \frac{1}{\varepsilon_1^2}\|\partial_x\Phi\|_2^2 + \frac{1}{\varepsilon_2^2}\|\partial_y\Phi\|_2^2, \qquad \|\Phi_{\varepsilon_1,\varepsilon_2}\|_4^4 = \frac{1}{\varepsilon_1\varepsilon_2}\|\Phi\|_4^4. \tag{3.21}$$

Thus under condition (B1″), i.e., $n_3 \neq 0$ and $\lambda > 0$, choosing κ large enough, we get

$$\tilde{E}_{2D}(\Phi_{\varepsilon_1,\varepsilon_2}) = \frac{C_1}{\varepsilon_1^2} + \frac{C_2}{\kappa^2\varepsilon_1^2} + \frac{C_3}{\kappa\varepsilon_1^2} + C_4\lambda\frac{-n_3^2 + o(1)}{\kappa\varepsilon_1^3} + O(1), \tag{3.22}$$

where C_k $(k = 1, 2, 3, 4)$ are constants independent of κ, ε_1, and $C_4 > 0$. Since $n_3 \neq 0$, the last term is negative for κ large; sending $\varepsilon_1 \to 0^+$, one immediately finds that

$$\lim_{\varepsilon_1\to 0^+,\varepsilon_2=\kappa\varepsilon_1} \tilde{E}_{2D}(\Phi_{\varepsilon_1,\varepsilon_2}) = -\infty,$$

which justifies the nonexistence. Under condition (B2″), i.e., $n_3^2 < \frac{1}{2}$ and $\lambda < 0$, by choosing κ small enough in (3.20) and sending ε_1 to 0^+, we will have the same results. Case (B3″) will reduce to Theorem 2.1. ☐

3.2. Existence results for the Cauchy problem. Let us consider the Cauchy problem of (1.15); noting that the nonlinearity $\phi(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta)((-\Delta)^{-1/2}|\phi|^2)$ is actually a derivative nonlinearity, it would bring significant difficulty in analyzing the dynamic behavior. The common approach to solving the Schrödinger equation is to try to solve the corresponding integral equation by the fixed point theorem. However, the loss of an order 1 derivative due to the nonlocal term will cause trouble. This can be overcome by the smoothing effect of the inhomogeneous problem $iu_t + \Delta u = g(x,t)$, which provides a gain of an order 1 derivative [10, 17]. When $V_2(\mathbf{x}) = 0$, i.e., without an external trapping potential which corresponds to the free expansion of a dipolar BEC after turning off the confinement, the above approach can be extended in a straightforward manner. However, when $V_2(\mathbf{x}) \neq 0$, i.e., with an external trapping potential, especially a confinement trapping potential with $\lim_{|\mathbf{x}|\to\infty} V_2(\mathbf{x}) = \infty$, the approach in [10, 17] has some difficulties. By configuring that $(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta)((-\Delta)^{-1/2}|\phi|^2)$ is almost a first order derivative, we are able to establish the well-posedness of (1.15) with a general external potential $V_2(\mathbf{x})$ by a different approach.

The Cauchy problem of the Schrödinger equation with derivative nonlinearity has been investigated extensively in the literature [16, 18]. Here, we present an existence result in the energy space with the special structure of our nonlinearity, which will show that the approximation (1.15) of (1.11) is reasonable in an appropriate function space with the weak convergence. We are interested in the case of $\lambda \neq 0$.

THEOREM 3.5 (existence for the Cauchy problem). *Suppose that the real-valued potential* $V_2(\mathbf{x})$ *satisfies* (2.37) *and* $\lim_{|\mathbf{x}|\to\infty} V_2(\mathbf{x}) = \infty$*, and initial value* $\phi_0(\mathbf{x}) \in X_2$*, and that either condition* (B2) *or* (B3) *in Theorem* 3.1 *holds with constant* C_b *being replaced by* $C_b/\|\phi_0\|_2^2$*; then there exists a solution* $\phi \in L^\infty([0,\infty); X_2) \cap W^{1,\infty}([0,\infty); X_2^*)$ *for the Cauchy problem of* (1.15)*. Moreover, there hold* L^2*-norm and energy* $\tilde{E}_{2D}$ (3.1) *conservations, i.e.,*

$$\|\phi(\cdot,t)\|_{L^2(\mathbb{R}^2)} = \|\phi_0\|_{L^2(\mathbb{R}^2)}, \quad \tilde{E}_{2D}(\phi(\cdot,t)) \leq \tilde{E}_{2D}(\phi_0) \quad \forall t \geq 0. \tag{3.23}$$

Proof. We first consider the Cauchy problem for the following equation:

$$i\partial_t\phi^\delta(\mathbf{x},t) = \mathbf{H}_\mathbf{x}^V\phi^\delta + g_1(\phi^\delta) + g_2(\phi^\delta), \qquad \mathbf{x} \in \mathbb{R}^2, \quad t > 0, \tag{3.24}$$

with the initial data $\phi^\delta(\mathbf{x},0)=\phi_0(\mathbf{x})$, $\beta_0=\frac{\beta-\lambda+3\lambda n_3^2}{\sqrt{2\pi}\,\varepsilon}$, $\varphi^\delta=U_\delta^{2D}*|\phi^\delta|^2$, where U_δ^{2D} is given in (1.12) as $U_\delta^{2D}(\mathbf{x})=\frac{1}{2\sqrt{2}\pi^{3/2}}\int_{\mathbb{R}}\frac{e^{-s^2/2}}{\sqrt{|\mathbf{x}|^2+\delta^2 s^2}}\,ds$ $(\delta>0)$, and

(3.25)
$$\mathbf{H}_{\mathbf{x}}^V=-\frac{1}{2}\Delta+V_2(\mathbf{x}),\quad g_1(\phi^\delta)=\beta_0|\phi^\delta|^2\phi^\delta,\quad g_2(\phi^\delta)=-\frac{3\lambda}{2}\phi^\delta(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp}-n_3^2\Delta)\varphi^\delta.$$

Then our quasi-2D equation II (1.15) can be written as

$$i\partial_t\phi=\mathbf{H}_{\mathbf{x}}^V\phi+g_1(\phi)+\tilde{g}_2(\phi), \tag{3.26}$$

where

$$\tilde{g}_2(\phi)=-\frac{3\lambda}{2}\phi(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp}-n_3^2\Delta)(-\Delta)^{-1/2}(|\phi|^2). \tag{3.27}$$

We denote the pairing of X_2 and its dual X_2^* by $\langle,\rangle_{X_2,X_2^*}$ as

$$\langle f_1,f_2\rangle_{X_2,X_2^*}=\mathrm{Re}\int_{\mathbb{R}^2}f_1(\mathbf{x})\bar{f}_2(\mathbf{x})\,d\mathbf{x}, \tag{3.28}$$

where Re(f) denotes the real part of f. Using the results in Theorem 2.4 and [13], we see that there exists a unique maximal solution $\varphi^\delta\in C([-T_{\min}^\delta,T_{\max}^\delta],X_2)\cap C^1([-T_{\min}^\delta,T_{\max}^\delta],X_2^*)$. Here maximal means that if either $t\uparrow T_{\max}^\delta$ or $t\downarrow -T_{\min}^\delta$, $\|\phi^\delta(t)\|_{X_2}\to\infty$. We want to show that as $\delta\to 0^+$, ϕ^δ will converge to a solution of (1.15).

First, we show that $T_{\min}^\delta=+\infty$ and $T_{\max}^\delta=+\infty$. The energy conservation for (3.24) is

(3.29)
$$E_\delta(t):=\frac{1}{2}\|\nabla\phi^\delta\|_2^2+\frac{1}{2}\beta_0\|\phi^\delta\|_4^4+\int_{\mathbb{R}^2}V_2(\mathbf{x})|\phi^\delta|^2d\mathbf{x}+E_{\mathrm{dip}}^\delta(t)\equiv E_\delta(0),\qquad t\ge 0,$$

where

$$E_{\mathrm{dip}}^\delta(t)=-\frac{3\lambda}{4}\int_{\mathbb{R}^2}|\phi^\delta|^2(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp}-n_3^2\Delta)\varphi^\delta d\mathbf{x},\qquad t\ge 0. \tag{3.30}$$

Computation similar to that in Lemma 3.4 confirms that $E_{\mathrm{dip}}^\delta\ge 0$. Hence energy conservation implies that $\|\phi^\delta(t)\|_{X_2}<\infty$ for all t, i.e., $T_{\max}^\delta=T_{\min}^\delta=+\infty$.

We note that

$$X_2\hookrightarrow H^1\hookrightarrow L^2\hookrightarrow H^{-1}\hookrightarrow X_2^*, \tag{3.31}$$

where H^{-1} is viewed as the dual of H^1. Consider a bounded time interval $I=[-T,T]$; it follows from energy conservation that there exists a constant $C_1(\phi_0)>0$ such that

$$\|\phi^\delta\|_{C([-T,T];X_2)}\le C_1(\phi_0). \tag{3.32}$$

Moreover, Lemma 2.2 and Remark 3.1 imply that

$$\|\phi^\delta(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp}-n_3^2\Delta)\varphi^\delta\|_q\le C\|\phi^\delta\|_{q^*}\|\nabla|\phi^\delta|^2\|_p\le C\|\phi^\delta\|_{q^*}\|\phi^\delta\|_{2p/(2-p)}\|\nabla\phi^\delta\|_2 \tag{3.33}$$

for $q,p\in(1,2)$, $\frac{1}{q^*}+\frac{1}{p}=\frac{1}{q}$. Then we have

$$\|\phi^\delta\|_{C^1([-T,T];X_2^*)}\le C_2(\phi_0). \tag{3.34}$$

Thus, from (3.32) and (3.34), there exist a sequence $\delta_n \to 0^+$ $(n = 1, 2, \ldots)$ and a function $\phi \in L^\infty([-T,T]; X_2) \cap W^{1,\infty}([-T,T]; X_2^*)$ [13], such that

$$\phi^{\delta_n}(t) \rightharpoonup \phi(t) \quad \text{in } X_2 \quad \forall t \in [-T,T]. \tag{3.35}$$

For each $t \in [-T,T]$, due to the mass conservation of (3.24), we know $\|\phi^{\delta_n}(t)\|_2 = \|\phi_0\|_2$. By a similar proof in Theorem 2.1, the weak convergence of $\phi^{\delta_n}(t)$ in X_2 would imply that $\phi^{\delta_n}(t)$ converges strongly in L^2, which is a consequence of the fact that $V_2(\mathbf{x})$ is a confining potential. So, $\lim_{n\to\infty} \|\phi^{\delta_n}(t)\|_2 = \|\phi(t)\|_2$, and it turns out that [13]

$$\phi^{\delta_n} \to \phi \quad \text{in } C([-T,T]; L^2(\mathbb{R}^2)). \tag{3.36}$$

In view of (3.35), (3.36), and the Gagliardo–Nirenberg inequality, we obtain

$$\phi^{\delta_n} \to \phi \quad \text{in } C([-T,T]; L^p(\mathbb{R}^2)) \quad \forall p \in [2,\infty). \tag{3.37}$$

We now try to say that ϕ actually solves (1.15). For any function $\psi(\mathbf{x}) \in X_2$ and $f(t) \in C_c^\infty([-T,T])$, from (3.24), we have

$$\int_{-T}^{T} \left[\langle i\phi^{\delta_n}, \psi\rangle_{X_2,X_2^*} f'(t) + \langle \mathbf{H}_{\mathbf{x}}^V \phi^{\delta_n} + g_1(\phi^{\delta_n}) + g_2(\phi^{\delta_n}), \psi\rangle_{X_2,X_2^*} f(t)\right] dt = 0. \tag{3.38}$$

Recalling $|g_1(u) - g_1(v)| \le C(|u|^2 + |v|^2)|u - v|$, (3.37) implies that for all $t \in [-T,T]$ [13]

$$g_1(\phi^{\delta_n}(t)) \to g_1(\phi(t)) \quad \text{in } L^\rho(\mathbb{R}^2) \text{ for some } \rho \in [1,\infty), \tag{3.39}$$

$$\langle g_1(\phi^{\delta_n}(t)), \psi(t)\rangle_{X_2,X_2^*} \to \langle g_1(\phi(t)), \psi(t)\rangle_{X_2,X_2^*}. \tag{3.40}$$

For $g_2(\phi^{\delta_n})$, considering $\varphi^{\delta_n}(\mathbf{x},t)$ and noting $\partial_\alpha \varphi^{\delta_n} = T_\alpha^{\delta_n}(|\phi^{\delta_n}|^2)$ $(\alpha = x, y)$ (defined in Lemma 3.3), we have proved in Lemma 3.3 that $T_\alpha^{\delta_n}$ is uniformly bounded from L^p to L^p and

$$T_\alpha^{\delta_n}(|\phi(t)|^2) \to R_\alpha(|\phi(t)|^2) = \partial_\alpha(-\Delta)^{-1/2}(|\phi(t)|^2) \quad \text{in } L^p(\mathbb{R}^2),\ p \in (1,\infty),\ \delta_n \to 0^+. \tag{3.41}$$

Rewriting

$$T_\alpha^{\delta_n}(|\phi^{\delta_n}(t)|^2) = T_\alpha^{\delta_n}(|\phi^{\delta_n}(t)|^2 - |\phi(t)|^2) + T_\alpha^{\delta_n}(|\phi(t)|^2), \tag{3.42}$$

recalling the fact (3.37), we immediately have

$$T_\alpha^{\delta_n}(|\phi^{\delta_n}(t)|^2) \to R_\alpha(|\phi(t)|^2) \quad \text{in } L^p(\mathbb{R}^2) \quad \text{for some } p \in (1,\infty), \tag{3.43}$$

which is actually

$$\partial_\alpha \varphi^{\delta_n}(t) \to \partial_\alpha \left((-\Delta)^{-1/2}|\phi(t)|^2\right) \quad \text{in } L^p(\mathbb{R}^2) \quad \text{for some } p \in (1,\infty). \tag{3.44}$$

Hence, by integration by parts, for $\alpha' = x, y$,

$$\begin{aligned}\langle \phi^{\delta_n}(t)\partial_{\alpha\alpha'}\varphi^{\delta_n}(t), \psi(t)\rangle_{X_2,X_2^*} &= \mathrm{Re}\int_{\mathbb{R}^2} \phi^{\delta_n}(t)\partial_{\alpha\alpha'}\varphi^{\delta_n}(t)\bar{\psi}(t)d\mathbf{x}\\ &= -\mathrm{Re}\int_{\mathbb{R}^2} \partial_\alpha\varphi^{\delta_n}(t)(\partial_{\alpha'}\phi^{\delta_n}(t)\bar{\psi}(t) + \phi^{\delta_n}(t)\overline{\partial_{\alpha'}\psi}(t))d\mathbf{x};\end{aligned}$$

passing to the limit as $n \to \infty$,

$$\lim_{n\to\infty} \langle \phi^{\delta_n}(t)\partial_{\alpha\alpha'}\varphi^{\delta_n}(t), \psi(t)\rangle_{X_2,X_2^*} = -\mathrm{Re}\int_{\mathbb{R}^2} R_\alpha(|\phi(t)|^2)(\partial_{\alpha'}\phi(t)\bar{\psi}(t) + \phi(t)\overline{\partial_{\alpha'}\psi}(t))d\mathbf{x}$$
$$= \langle \phi(t)\partial_{\alpha\alpha'}(-\Delta)^{-1/2}(|\phi(t)|^2), \psi(t)\rangle_{X_2,X_2^*};$$

and in view of (3.44) and (3.35), we obtain

$$\lim_{n\to\infty} \langle g_2(\varphi^{\delta_n}(t)), \psi(t)\rangle_{X_2,X_2^*} = \langle \tilde{g}_2(\phi(t)), \psi(t)\rangle_{X_2,X_2^*}. \tag{3.45}$$

Combining the above results and (3.40) and sending $n \to \infty$, the dominated convergence theorem will yield

$$\int_{-T}^{T} \left[\langle i\phi, \psi\rangle_{X_2,X_2^*} f'(t) + \langle \mathbf{H}_{\mathbf{x}}^V \phi + g_1(\phi) + \tilde{g}_2(\phi), \psi\rangle_{X_2,X_2^*} f(t)\right] dt = 0,$$

which proves that

$$i\partial_t\phi = \mathbf{H}_{\mathbf{x}}^V\phi + g_1(\phi) + \tilde{g}_2(\phi) \quad \text{ in } X_2^* \quad \text{ a.e. } t \in [-T, T], \tag{3.46}$$

with $\phi(t = 0) = \phi_0$ and $\phi \in L^\infty([-T, T]; X_2)\cap W^{1,\infty}([-T, T]; X_2^*)$. Moreover, by lower semicontinuity of the X_2-norm, (3.39), and (3.45), the energy $\tilde{E}_{2D}$ (3.1) satisfies

$$\tilde{E}_{2D}(\phi(t)) \le \tilde{E}_{2D}(\phi_0). \tag{3.47}$$

It is easy to see that we can choose $T = \infty$. ∎

If the uniqueness of the $L^\infty([-T, T]; X_2) \cap W^{1,\infty}([-T, T]; X_2^*)$ solution to the quasi-2D equation II (1.15) is known, we can prove that the solution constructed above in Theorem 3.5 is actually $C([-T, T]; X_2) \cap C^1([-T, T]; X_2^*)$ and conserves the energy.

Next, we discuss possible finite time blow-up for the continuous solutions of the quasi-2D equation II (1.15). To this purpose, the following assumptions are introduced:

(A) Assumption on the trap and coefficient of the cubic term, i.e., $V_2(\mathbf{x})$ satisfies $3V_2(\mathbf{x}) + \mathbf{x}\cdot\nabla V_2(\mathbf{x}) \ge 0$, $\frac{\beta-\lambda+3\lambda n_3^2}{\sqrt{2\pi}\,\varepsilon} \ge -\frac{C_b}{\|\phi_0\|_2^2}$, with ϕ_0 being the initial data of (1.15).

(B) Assumption on the trap and coefficient of the nonlocal term, i.e., $V_2(\mathbf{x})$ satisfies $2V_2(\mathbf{x}) + \mathbf{x}\cdot\nabla V_2(\mathbf{x}) \ge 0$, $\lambda = 0$, or $\lambda > 0$ and $n_3^2 \ge \frac{1}{2}$.

THEOREM 3.6 (finite time blow-up). *For any initial data $\phi(\mathbf{x}, t = 0) = \phi_0(\mathbf{x}) \in X_2$ with $\int_{\mathbb{R}^2} |\mathbf{x}|^2|\phi_0(\mathbf{x})|^2\, d\mathbf{x} < \infty$, if conditions* (B1), (B2), *and* (B3) *with constant C_b being replaced by $C_b/\|\phi_0\|_2^2$ are not satisfied, and if $\phi := \phi(\mathbf{x}, t)$ is a $C([0, T_{\max}), X_2)$ solution of problem* (1.15) *with L^2-norm and energy conservation, then there exists finite time blow-up, i.e., $T_{\max} < \infty$, if one of the following conditions holds:*

(i) $\tilde{E}_{2D}(\phi_0) < 0$, *and either assumption* (A) *or* (B) *holds;*

(ii) $\tilde{E}_{2D}(\phi_0) = 0$ *and* $\mathrm{Im}\left(\int_{\mathbb{R}^2} \bar{\phi}_0(\mathbf{x})\ (\mathbf{x}\cdot\nabla\phi_0(\mathbf{x}))\, d\mathbf{x}\right) < 0$, *and either assumption* (A) *or* (B) *holds;*

(iii) $\tilde{E}_{2D}(\phi_0) > 0$, *and* $\mathrm{Im}(\int_{\mathbb{R}^2} \bar{\phi}_0(\mathbf{x})\ (\mathbf{x}\cdot\nabla\phi_0(\mathbf{x}))\, d\mathbf{x}) < -\sqrt{3\tilde{E}_{2D}(\phi_0)}\|\mathbf{x}\phi_0\|_2$ *if assumption* (A) *holds, or* $\mathrm{Im}\left(\int_{\mathbb{R}^2} \bar{\phi}_0(\mathbf{x})\ (\mathbf{x}\cdot\nabla\phi_0(\mathbf{x}))\, d\mathbf{x}\right) < -\sqrt{2\tilde{E}_{2D}(\phi_0)}\|\mathbf{x}\phi_0\|_2$ *if assumption* (B) *holds.*

Proof. Calculating derivatives of the variance defined in (2.40), for $\alpha = x, y$, we have

$$\frac{d}{dt}\sigma_\alpha(t) = 2\,\mathrm{Im}\left(\int_{\mathbb{R}^2} \bar{\phi}\ \alpha\partial_\alpha\phi\, d\mathbf{x}\right), \qquad t \ge 0, \tag{3.48}$$

and

(3.49)

$$\frac{d^2}{dt^2}\sigma_\alpha(t) = \int_{\mathbb{R}^2} \left[2|\partial_\alpha\phi|^2 + \beta_0|\phi|^4 + 3\lambda|\phi|^2\alpha\partial_\alpha(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta)\varphi - 2\alpha|\phi|^2\partial_\alpha V_2(\mathbf{x})\right] d\mathbf{x},$$

where $\beta_0 = \frac{\beta-\lambda+3\lambda n_3^2}{\sqrt{2\pi}\varepsilon}$, $(-\Delta)^{1/2}\varphi = |\phi|^2$. Writing $\rho = |\phi|^2$, $\tilde{\varphi} = (\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta)\varphi$ and noting that ρ is a real function, by the Plancherel formula, as for Theorem 2.5, we get

$$\int_{\mathbb{R}^2} |\phi|^2 \left(\mathbf{x}\cdot\nabla\tilde{\varphi}\right)\, d\mathbf{x} = -\frac{3}{2}\int_{\mathbb{R}^2} |\phi|^2\tilde{\varphi}\, d\mathbf{x}. \tag{3.50}$$

Hence, summing (3.49) for $\alpha = x, y$ and using the energy conservation, if assumption (A) holds, we have

$$\begin{aligned}\frac{d^2}{dt^2}\sigma_V(t) &= 2\int_{\mathbb{R}^2}\left(|\nabla\phi|^2 + \beta_0|\phi|^4 - \frac{9}{4}\lambda|\phi|^2\left(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta\right)\varphi - |\phi|^2(\mathbf{x}\cdot\nabla V_2(\mathbf{x}))\right) d\mathbf{x}\\ &= 6\tilde{E}_{2D}(\phi) - \int_{\mathbb{R}^2}(|\nabla\phi|^2 + \beta_0|\phi|^4)\, d\mathbf{x} - 2\int_{\mathbb{R}^2}|\phi|^2\left(3V_2(\mathbf{x}) + \mathbf{x}\cdot\nabla V_2(\mathbf{x})\right) d\mathbf{x}\\ &\le 6\tilde{E}_{2D}(\phi(\cdot,t)) \le 6\tilde{E}_{2D}(\phi_0), \qquad t\ge 0. \end{aligned}\tag{3.51}$$

Thus,

$$\sigma_V(t) \le 3\tilde{E}_{2D}(\phi_0)t^2 + \sigma_V'(0)t + \sigma_V(0), \qquad t \ge 0,$$

and the conclusion follows as in Theorem 2.5. If assumption (B) holds, the energy contribution of the nonlocal part is nonpositive, and we have

$$\begin{aligned}\frac{d^2}{dt^2}\sigma_V(t) &= 2\int_{\mathbb{R}^2}\left(|\nabla\phi|^2 + \beta_0|\phi|^4 - \frac{9}{4}\lambda|\phi|^2\left(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta\right)\varphi - |\phi|^2(\mathbf{x}\cdot\nabla V_2(\mathbf{x}))\right) d\mathbf{x}\\ &= 4\tilde{E}_{2D}(\phi) - \frac{3\lambda}{2}\int_{\mathbb{R}^2}|\phi|^2\tilde{\varphi}\, d\mathbf{x} - 2\int_{\mathbb{R}^2}|\phi|^2\left(2V_2(\mathbf{x}) + \mathbf{x}\cdot\nabla V_2(\mathbf{x})\right) d\mathbf{x}\\ &\le 4\tilde{E}_{2D}(\phi(\cdot,t)) \le 4\tilde{E}_{2D}(\phi_0), \qquad t\ge 0, \end{aligned}\tag{3.52}$$

and the conclusion follows in a way similar to that of the assumption (A) case. ◻

4. Results for quasi-1D equation. In this section, we prove the existence and uniqueness of the ground state for quasi-1D equation (1.17) and establish the well-posedness for dynamics.

4.1. Existence and uniqueness of ground state. Associated to the quasi-1D equation (1.17), the energy is

$$E_{1D}(\Phi) = \int_{\mathbb{R}}\left[\frac{1}{2}|\partial_z\Phi|^2 + V_1(z)|\Phi|^2 + \frac{1}{2}\beta_{1D}|\Phi|^4 + \frac{3\lambda(1-3n_3^2)}{16\sqrt{2\pi}\,\varepsilon}|\Phi|^2\varphi\right] dz, \tag{4.1}$$

where $\beta_{1D} = \frac{\beta+\frac{1}{2}\lambda(1-3n_3^2)}{2\pi\varepsilon^2}$ and

$$\varphi(z) = \partial_{zz}(U_\varepsilon^{1D} * |\Phi|^2), \quad U_\varepsilon^{1D}(z) = \frac{2e^{\frac{z^2}{2\varepsilon^2}}}{\sqrt{\pi}}\int_{|z|}^{\infty} e^{-\frac{s^2}{2\varepsilon^2}}\, ds. \tag{4.2}$$

Again, the ground state $\Phi_g \in S_1$ of (1.17) is defined as the minimizer of the nonconvex minimization problem:

$$\text{Find } \Phi_g \in S_1 \quad \text{such that } E_{1D}(\Phi_g) = \min_{\Phi\in S_1} E_{1D}(\Phi). \tag{4.3}$$

For the above ground state, we have the following results.

THEOREM 4.1 (existence and uniqueness of ground state). *Assuming that* $0 \le V_1(z) \in L^\infty_{\text{loc}}(\mathbb{R})$ *and* $\lim_{|z|\to\infty} V_1(z) = \infty$, *for any parameters* β, λ, *and* ε, *there exists a ground state* $\Phi_g \in S_1$ *of the quasi-*1*D problem* (1.17)–(1.18), *and the positive ground state* $|\Phi_g|$ *is unique under one of the following conditions:*

(C1) $\lambda(1-3n_3^2) \ge 0$ *and* $\beta - (1-3n_3^2)\lambda \ge 0$;

(C2) $\lambda(1-3n_3^2) < 0$ *and* $\beta + \frac{\lambda}{2}(1-3n_3^2) \ge 0$.

Moreover, $\Phi_g = e^{i\theta_0}|\Phi_g|$ *for some constant* $\theta_0 \in \mathbb{R}$.

To complete the proof, we first study the property of the convolution kernel U^{1D}_ε (1.18).

LEMMA 4.2 (kernel U^{1D}_ε (1.18)). *For any* $f(z)$ *in the Schwartz space* $\mathcal{S}(\mathbb{R})$, *we have*

$$\widehat{U^{1D}_\varepsilon * f}(\xi) = \hat{f}(\xi)\widehat{U^{1D}_\varepsilon}(\xi) = \frac{\sqrt{2}\,\varepsilon \hat{f}(\xi)}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\varepsilon^2 s/2}}{|\xi|^2+s} ds, \quad \xi \in \mathbb{R}. \tag{4.4}$$

Hence

$$\|\partial_{zz}(U^{1D}_\varepsilon * f)\|_2 \le \frac{2\sqrt{2}}{\sqrt{\pi}\,\varepsilon}\|f\|_2. \tag{4.5}$$

Proof. For any $f(z) \in \mathcal{S}(\mathbb{R})$, rewrite the kernel as [11]

$$U^{1D}_\varepsilon(z) = \frac{\sqrt{2}\,\varepsilon}{\sqrt{\pi}} \int_{\mathbb{R}^4} \frac{w^2_\varepsilon(x,y) w^2_\varepsilon(x',y')}{\sqrt{z^2+(x-x')^2+(y-y')^2}}\, dxdydx'dy'.$$

Applying Fourier transform to both sides and using the Plancherel formula as in Lemma 2.2, we obtain

$$\widehat{U^{1D}_\varepsilon}(\xi) = \frac{\sqrt{2}\,\varepsilon}{\pi^{3/2}} \int_{\mathbb{R}^2} \frac{|\widehat{w^2_\varepsilon}(\xi_1,\xi_2)|^2}{\xi^2+\xi_1^2+\xi_2^2}\, d\xi_1\, d\xi_2, \qquad \xi \in \mathbb{R}. \tag{4.6}$$

Then a direct computation yields the conclusion. ☐

LEMMA 4.3. *For the energy* $E_{1D}(\cdot)$ *in* (4.1), *we have the following:*

(i) *For any* $\Phi \in S_1$, *denote* $\rho(z) = |\Phi(z)|^2$; *then*

$$E_{1D}(\Phi) \ge E_{1D}(|\Phi|) = E\left(\sqrt{\rho}\right) \qquad \forall \Phi \in S_1, \tag{4.7}$$

so the ground state Φ_g *of* (4.1) *is of the form* $e^{i\theta_0}|\Phi_g|$ *for some constant* $\theta_0 \in \mathbb{R}$.

(ii) E_{1D} *is bounded below.*

(iii) *If condition* (C1) *or* (C2) *in Theorem* 4.1 *holds,* $E_{1D}(\sqrt{\rho})$ *is strictly convex in* ρ.

Proof. Part (i) is similar to the proof of Lemma 2.2. Part (ii) is well known, once we note the property of kernel U^{1D}_ε (cf. Lemma 4.2) and the Sobolev inequality in one dimension,

$$\|f\|^2_\infty \le \|f\|_2\|f'\|_2. \tag{4.8}$$

(iii) We come to the convexity of $E_{1D}(\sqrt{\rho})$. Following Lemma 2.3, we need only consider the functional

$$H_{1D}(\rho) = \int_{\mathbb{R}} \left[\frac{\beta + \lambda(1-3n_3^2)/2}{4\pi\varepsilon^2}\rho^2 + \frac{3\lambda(1-3n_3^2)}{16\sqrt{2\pi\varepsilon^2}}\rho\partial_{zz}(U_\varepsilon^{1D} * \rho)\right] dz. \tag{4.9}$$

Then under condition (C1) or (C2), using the Plancherel formula and Lemma 4.2, after similar computation as in Lemma 2.2, we would have $H_{1D}(\rho) \geq 0$. For arbitrary $\sqrt{\rho_1}, \sqrt{\rho_2} \in S_1$ and $\theta \in [0,1]$, denote $\rho_\theta = \theta\rho_1 + (1-\theta)\rho_2$; then $\sqrt{\rho_\theta} \in S_1$ and

$$\theta H_{1D}(\rho_1) + (1-\theta)H_{1D}(\rho_2) - H_{1D}(\rho_\theta) = \theta(1-\theta)H_{1D}(\rho_1 - \rho_2) \geq 0, \tag{4.10}$$

which proves the convexity. □

Proof of Theorem 4.1. The uniqueness follows from the strict convexity in Lemma 4.3. The existence part is similar to the proof of Theorem 2.1, and we omit it here for brevity. □

4.2. Well-posedness for the Cauchy problem. Concerning the Cauchy problem, Lemma 4.2 shows that the nonlinearity in quasi-1D equation (1.17) is almost a cubic nonlinearity, while the same property has been observed for quasi-2D equation I (1.11)–(1.12). Hence results similar to those in Theorem 2.4 can be obtained for (1.17), and we omit the proof here.

THEOREM 4.4 (well-posedness for Cauchy problem). *Suppose the real-valued trap potential satisfies $V_1(z) \geq 0$ for $z \in \mathbb{R}$ and $V_1(z) \in C^\infty(\mathbb{R})$, $D^k V_1(z) \in L^\infty(\mathbb{R})$ for all integers $k \geq 2$; then there exists a unique solution $\phi \in C([0,\infty), X_1) \cap C^1([0,\infty), X_1^*)$ to the Cauchy problem of* (1.17) *for any initial data $\phi(z, t=0) = \phi_0(z) \in X_1$.*

5. Convergence rate of dimension reduction. In this section, we discuss the dimension reduction of the 3D GPPS to lower dimensions. Inspired by the previous work of Ben Abdallah and colleagues (see [9, 7]) for GPE without the dipolar term (i.e., $\lambda = 0$) and [6, 8] for Schrödinger–Poisson systems, we are going to find a limiting ε-independent equation as $\varepsilon \to 0^+$. Thus in quasi-2D equations I (1.11) and II (1.15) and quasi-1D equation (1.17), we have to consider the coefficients to be $O(1)$. The existence of the global solution for the full 3D system (1.4)–(1.5) has been proved in [12, 2] when $\beta \geq 0$ and $\lambda \in [-\frac{1}{2}\beta, \beta]$; hence we would expect the limiting equation in lower dimensions to be valid in a similar regime. Thus in lower dimensions, we require that in the quasi-2D case, $\beta = O(\varepsilon)$, $\lambda = O(\varepsilon)$, and in the quasi-1D case, $\beta = O(\varepsilon^2)$, $\lambda = O(\varepsilon^2)$, i.e., we are considering the weak interaction regime; then we would get an ε-independent limiting equation. In this regime, we will see that the GPPS will reduce to a regular GPE in lower dimensions.

5.1. Reduction to 2D. We consider the weak interaction regime, i.e., $\beta \to \varepsilon\beta$, $\lambda \to \varepsilon\lambda$. In Case I (1.8), for full 3D GPPS (1.4)–(1.5), we introduce the rescaling $z \to \varepsilon z$, $\psi \to \varepsilon^{-1/2}\psi^\varepsilon$ which preserves the normalization; then

(5.1)

$$i\partial_t\psi^\varepsilon(\mathbf{x}, z, t) = \left[\mathbf{H}_\mathbf{x}^V + \frac{1}{\varepsilon^2}\mathbf{H}_z + (\beta - \lambda)|\psi^\varepsilon|^2 - 3\varepsilon\lambda\partial_{\mathbf{n}_\varepsilon\mathbf{n}_\varepsilon}\varphi^\varepsilon\right]\psi^\varepsilon, \quad (\mathbf{x}, z) \in \mathbb{R}^3,\ t > 0,$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and

$$\text{(5.2)} \qquad \mathbf{H}_{\mathbf{x}}^V = -\frac{1}{2}(\partial_{xx} + \partial_{yy}) + V_2(x, y), \qquad \mathbf{H}_z = -\frac{1}{2}\partial_{zz} + \frac{z^2}{2},$$

$$\text{(5.3)} \qquad \mathbf{n}_\varepsilon = (n_1, n_2, n_3/\varepsilon), \qquad \partial_{\mathbf{n}_\varepsilon} = \mathbf{n}_\varepsilon \cdot \nabla, \qquad \partial_{\mathbf{n}_\varepsilon \mathbf{n}_\varepsilon} = \partial_{\mathbf{n}_\varepsilon}(\partial_{\mathbf{n}_\varepsilon}),$$

$$\text{(5.4)} \qquad \left(-\partial_{xx} - \partial_{yy} - \frac{1}{\varepsilon^2}\partial_{zz}\right)\varphi^\varepsilon = \frac{1}{\varepsilon}|\psi^\varepsilon|^2, \qquad \lim_{|\mathbf{x}|\to\infty} \varphi^\varepsilon(\mathbf{x}) = 0.$$

It is well known that $\mathbf{H}_z$ has eigenvalues $\mu_k = k+1/2$ with corresponding eigenfunction $w_k(z)$ $(k = 0, 1, \ldots)$, where $\{w_k\}_{k=0}^\infty$ forms an orthonormal basis of $L^2(\mathbb{R})$ [14, 28], specifically, $w_0(z) = \frac{1}{\pi^{1/4}} e^{-z^2/2}$. Following [9], it is convenient to consider the initial data concentrated on the ground mode of $\mathbf{H}_z$, i.e.,

$$\text{(5.5)} \qquad \psi^\varepsilon(\mathbf{x}, z, 0) = \phi_0(\mathbf{x}) w_0(z), \quad \phi_0 \in X_2 \text{ and } \|\phi_0\|_{L^2(\mathbb{R}^2)} = 1.$$

In Case I (1.8), when $\varepsilon \to 0^+$, quasi-2D equations I (1.11) and II (1.15) will yield an ε-independent equation in the weak interaction regime,

$$\text{(5.6)} \qquad i\partial_t \phi(\mathbf{x}, , t) = \mathbf{H}_{\mathbf{x}}^V \phi + \frac{\beta - (1 - 3n_3^2)\lambda}{\sqrt{2\pi}}|\phi|^2\phi, \quad \mathbf{x} = (x, y) \in \mathbb{R}^2,$$

with initial condition $\phi(\mathbf{x}, 0) = \phi_0(\mathbf{x})$. We follow the ideas in [6, 9, 7] to show the convergence from the 3D GPPS to the 2D approximation. First, let us state the main result.

THEOREM 5.1 (dimension reduction to 2D). *Suppose that V_2 satisfies condition* (2.37), *and that $-\frac{\beta}{2} \leq \lambda \leq \beta$ and $\beta \geq 0$. Let $\psi^\varepsilon \in C([0, \infty); X_3)$ and $\phi \in C([0, \infty); X_2)$ be the unique solutions of equations* (5.1)–(5.5) *and* (5.6), *respectively; then for any $T > 0$, there exists $C_T > 0$ such that*

$$\text{(5.7)} \qquad \left\|\psi^\varepsilon(\mathbf{x}, z, t) - e^{-i\frac{\mu_0 t}{\varepsilon^2}}\phi(\mathbf{x}, t) w_0(z)\right\|_{L^2(\mathbb{R}^3)} \leq C_T\, \varepsilon \qquad \forall t \in [0, T].$$

Under the assumption, we have the global existence of ψ^ε [2, 12] as well as ϕ [9, 13]. Define the projection operator to the ground mode $\mathbf{H}_z$ by

$$\text{(5.8)} \qquad \Pi\psi^\varepsilon(\mathbf{x}, z, t) = e^{-i\mu_0 t/\varepsilon^2}\phi^\varepsilon(\mathbf{x}, t) w_0(z), \qquad (\mathbf{x}, z) \in \mathbb{R}^3, \quad t \geq 0,$$

where

$$\text{(5.9)} \qquad \phi^\varepsilon(\mathbf{x}, t) = e^{i\mu_0 t/\varepsilon^2}\int_{\mathbb{R}} \psi^\varepsilon(\mathbf{x}, z, t) w_0(z) dz, \qquad \mathbf{x} = (x, y) \in \mathbb{R}^2, \quad t \geq 0.$$

Since the space $(\mathbf{x}, z) \in \mathbb{R}^3$ is anisotropic, we introduce the $L_z^p L_{\mathbf{x}}^q$ space by the norm

$$\text{(5.10)} \qquad \|f\|_{(p,q)} := \|f\|_{L_z^p L_{\mathbf{x}}^q} = \|\, \|f(\cdot, z)\|_{L_{\mathbf{x}}^q}\|_{L_z^p}, \quad p, q \in [1, \infty].$$

The corresponding anisotropic Sobolev inequalities are available [9].

LEMMA 5.2 (uniform bound). *Let ψ^ε and ϕ be the solutions of* (5.1) *and* (5.6), *respectively, ϕ^ε be defined in* (5.9), *$\lambda \in [-\frac{\beta}{2}, \beta]$, and $\beta \geq 0$; we have*

$$\text{(5.11)} \qquad \psi^\varepsilon \in L^\infty((0, \infty), H^1(\mathbb{R}^3)), \quad \phi, \phi^\varepsilon \in L^\infty((0, T), H^1(\mathbb{R}^2)),$$

with uniform bound in ε. Moreover, for $p \in [2, \infty]$,

$$\text{(5.12)} \quad \|\psi^\varepsilon - \Pi\psi^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_z(\psi^\varepsilon - \Pi\psi^\varepsilon)\|_{L^2(\mathbb{R}^3)}^2 \leq C\varepsilon^2, \quad \|\psi^\varepsilon - \Pi\psi^\varepsilon\|_{(p,2)} \leq C\varepsilon,$$

with C depending on $\|\phi_0\|_{X_2}$, uniform in time t.

Proof. From energy conservation of (5.1), we know that the following energy is constant:

$$E(t) := (\mathbf{H}_{\mathbf{x}}^V \psi^\varepsilon(t), \psi^\varepsilon(t)) + \frac{1}{\varepsilon^2}(\mathbf{H}_z \psi^\varepsilon(t), \psi^\varepsilon(t)) + \frac{\beta-\lambda}{2}\|\psi^\varepsilon\|_{L^4(\mathbb{R}^3)}^4 + \frac{3\lambda\varepsilon^2}{2}\|\partial_{\mathbf{n}_\varepsilon}\nabla\varphi^\varepsilon(t)\|_{L^2(\mathbb{R}^3)}^2,$$

where $(\cdot,\cdot)$ denotes the standard L^2 inner product in $\mathbb{R}^3$. Using standard L^p estimates for Poisson equation (5.4), we have $\|\partial_{\mathbf{n}_\varepsilon}\nabla\varphi^\varepsilon(t)\|_{L^2(\mathbb{R}^3)} \le \frac{1}{\varepsilon}\|\psi^\varepsilon(t)\|_{L^4(\mathbb{R}^3)}^2$, which implies

$$\frac{\beta-\lambda}{2}\|\psi^\varepsilon\|_{L^4(\mathbb{R}^3)}^4 + \frac{3\lambda\varepsilon^2}{2}\|\partial_{\mathbf{n}_\varepsilon}\nabla\varphi^\varepsilon(t)\|_{L^2(\mathbb{R}^3)}^2 \ge 0, \tag{5.13}$$

and $E(0) = \frac{\mu_0}{\varepsilon^2} + C_0$, where C_0 depends on $\|\phi_0\|_{X_2}$. Writing $\psi^\varepsilon(t) = \sum_{k=0}^\infty \phi_k(\mathbf{x}, t) w_k(z)$, and using the L^2 conservation $\sum_{k=0}^\infty \|\phi_k(t)\|_{L^2(\mathbb{R}^2)}^2 = 1$, we can deduce from the energy conservation that

$$\begin{aligned}\frac{\mu_0}{\varepsilon^2} + C_0 &\ge (\mathbf{H}_{\mathbf{x}}^V \psi^\varepsilon(t), \psi^\varepsilon(t)) + \frac{1}{\varepsilon^2}(\mathbf{H}_z \psi^\varepsilon(t), \psi^\varepsilon(t)) \\ &= (\mathbf{H}_{\mathbf{x}}^V \psi^\varepsilon(t), \psi^\varepsilon(t)) + \frac{1}{\varepsilon^2}\sum_{k=0}^\infty \mu_k \|\phi_k(t)\|_{L^2(\mathbb{R}^2)}^2 \\ &= (\mathbf{H}_{\mathbf{x}}^V \psi^\varepsilon(t), \psi^\varepsilon(t)) + \frac{1}{\varepsilon^2}\sum_{k=1}^\infty (\mu_k - \mu_0)\|\phi_k(t)\|_{L^2(\mathbb{R}^2)}^2 + \frac{\mu_0}{\varepsilon^2}.\end{aligned}$$

Hence,

$$\|\partial_x \psi^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_y \psi^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \le (\mathbf{H}_{\mathbf{x}}^V \psi^\varepsilon(t), \psi^\varepsilon(t)) \le C_0, \tag{5.14}$$

$$\|\partial_z \psi^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \le (\mathbf{H}_z \psi^\varepsilon, \psi^\varepsilon) \le \mu_0 + C_0\varepsilon^2, \tag{5.15}$$

$$\|\psi^\varepsilon - \Pi\psi^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \le \frac{1}{\mu_1 - \mu_0}\sum_{k=1}^\infty (\mu_k - \mu_0)\|\phi_k(t)\|_{L^2(\mathbb{R}^2)}^2 \le 2C_0\varepsilon^2, \tag{5.16}$$

$$\|\partial_z(\psi^\varepsilon - \Pi\psi^\varepsilon)\|_{L^2(\mathbb{R}^3)}^2 \le \sum_{k=1}^\infty \frac{\mu_k}{\mu_k - \mu_0}(\mu_k - \mu_0)\|\phi_k(t)\|_{L^2(\mathbb{R}^2)}^2 \le \frac{3}{2}C_0\varepsilon^2. \tag{5.17}$$

The estimate on $\|\psi^\varepsilon - \Pi\psi^\varepsilon\|_{(p,2)}$ follows from Sobolev embedding. ∎

We also need the following Strichartz estimates for the unitary group $e^{-it\mathbf{H}_{\mathbf{x}}^V}$, which is valid when V_2 satisfies condition (2.37) [13].

DEFINITION 5.3. *In 2D, let q', r' be the conjugate index of q and r ($1 \le q, r \le \infty$), respectively; i.e., $1 = 1/q' + 1/q = 1/r' + 1/r$. We call the pair (q, r) admissible and (q', r') conjugate admissible if*

$$\frac{2}{q} = 2\left(\frac{1}{2} - \frac{1}{r}\right), \quad 2 \le r < \infty. \tag{5.18}$$

The following estimates are established in [13, 12, 26].

LEMMA 5.4 (Strichartz estimates). *Let (q, r) be an admissible pair and (γ, ρ) be a conjugate admissible pair, and let $I \subset \mathbb{R}$ be a bounded interval satisfying $0 \in I$. Then we have the following:*

(i) *There exists a constant* C *depending on* I *and* q *such that*

$$\left\| e^{-it\mathbf{H}_{\mathbf{x}}^V}\varphi \right\|_{L^q(I,L^r(\mathbb{R}^2))} \le C(I,q)\|\varphi\|_{L^2(\mathbb{R}^2)}. \tag{5.19}$$

(ii) *If* $f \in L^\gamma(I, L^\rho(\mathbb{R}^2))$, *there exists a constant* C *depending on* I, q, *and* ρ *such that*

$$\left\| \int_{I\bigcap s\le t} e^{-i(t-s)\mathbf{H}_{\mathbf{x}}^V} f(s)\,ds \right\|_{L^q(I,L^r(\mathbb{R}^2))} \le C(I,q,\rho)\|f\|_{L^\gamma(I,L^\rho(\mathbb{R}^2))}. \tag{5.20}$$

Now we are able to prove the theorem.

Proof of Theorem 5.1. In view of Lemma 5.2, we can derive

$$\begin{aligned} \|\psi^\varepsilon - e^{-i\frac{t\mu_0}{\varepsilon^2}}\phi w_0(z)\|_{L^2(\mathbb{R}^3)} &\le \|\psi^\varepsilon - \Pi\psi^\varepsilon\|_{L^2(\mathbb{R}^3)} + \|\Pi\psi^\varepsilon - e^{-i\frac{t\mu_0}{\varepsilon^2}}\phi w_0(z)\|_{L^2(\mathbb{R}^3)} \\ &\le C\varepsilon + \|\phi^\varepsilon(t) - \phi(t)\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{5.21}$$

Hence, we need to estimate the difference between ϕ^ε and ϕ. By (5.1) and (5.4), we know $\phi^\varepsilon(\mathbf{x},t)$ in (5.9) solves the following equation:

$$\begin{aligned} &i\partial_t\phi^\varepsilon = \mathbf{H}_{\mathbf{x}}^V\phi^\varepsilon + (\beta - \lambda + 3n_3^3\lambda)e^{i\mu_0 t/\varepsilon^2}\int_{\mathbb{R}} |\psi^\varepsilon|^2\psi^\varepsilon w_0(z)dz + \varepsilon g^\varepsilon, \\ &g^\varepsilon = e^{i\mu_0 t/\varepsilon^2}\int_{\mathbb{R}} P_\varepsilon(\varphi^\varepsilon)\psi^\varepsilon w_0(z)dz, \end{aligned}$$

where the differential operator P_ε is defined as

$$P_\varepsilon(\varphi^\varepsilon) = -3\lambda\left((n_1^2 - n_3^2)\partial_{xx} + (n_2^2 - n_3^2)\partial_{yy} + 2n_1n_2\partial_{xy} + \frac{2}{\varepsilon}(n_1n_3\partial_{xz} + n_2n_3\partial_{yz})\right)\varphi^\varepsilon. \tag{5.22}$$

Denote $\chi^\varepsilon(\mathbf{x},t) = \phi^\varepsilon - \phi$, noting that $\|w_0\|_4^4 = 1/\sqrt{2\pi}$; then χ^ε satisfies the following equation:

$$\begin{aligned} &i\partial_t\chi^\varepsilon = \mathbf{H}_{\mathbf{x}}^V\chi^\varepsilon + f_1^\varepsilon + f_2^\varepsilon + \varepsilon g^\varepsilon, \quad \chi^\varepsilon(t=0) = 0, \\ &f_1^\varepsilon = \frac{\beta - \lambda + 3n_3^3\lambda}{\sqrt{2\pi}}(|\phi^\varepsilon|^2\phi^\varepsilon - |\phi|^2\phi), \\ &f_2^\varepsilon = (\beta - \lambda + 3n_3^3\lambda)e^{i\mu_0 t/\varepsilon^2}\int_{\mathbb{R}}\left(|\psi^\varepsilon|^2\psi^\varepsilon - e^{-i\mu_0 t/\varepsilon^2}|\phi^\varepsilon w_0|^2\phi^\varepsilon w_0\right)w_0(z)dz. \end{aligned}$$

Applying Strichartz estimates on bounded interval $[0,T]$ and recalling that $(\infty, 2)$ is an admissible pair, we can obtain

$$\begin{aligned} &\|\chi^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^2))} \\ &\quad \le C\left[\|f_1^\varepsilon\|_{L^{\rho^*}([0,T];L^\rho(\mathbb{R}^2))} + \|f_2^\varepsilon\|_{L^{\gamma^*}([0,T];L^\gamma(\mathbb{R}^2))} + \varepsilon\|g^\varepsilon\|_{L^{q^*}([0,T];L^q(\mathbb{R}^2))}\right], \end{aligned}$$

where (ρ^*, ρ), (γ^*, γ), and (q^*, q) are some conjugate admissible pairs. By an argument similar to one in [9], we have the estimates for f_1^ε and f_2^ε which come from the cubic nonlinearity, i.e., for appropriate $\rho \in (1,2)$ and $\gamma \in (1,2)$,

$$\|f_1^\varepsilon\|_{L^{\rho^*}([0,T];L^\rho(\mathbb{R}^2))} \le C\|\chi^\varepsilon\|_{L^{\rho^*}([0,T];L^2(\mathbb{R}^2))}, \quad \|f_2^\varepsilon\|_{L^{\gamma^*}([0,T];L^\gamma(\mathbb{R}^2))} \le C\varepsilon. \tag{5.23}$$

The basic tools involved are the Hölder inequality, Sobolev inequalities, and the estimates in Lemma 5.2, and we omit the proof of this part for brevity. Thus,

$$\|\chi^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^2))} \le C(\|\chi^\varepsilon\|_{L^{\rho^*}([0,T];L^\rho(\mathbb{R}^2))} + \varepsilon\|g^\varepsilon\|_{L^{q^*}([0,T];L^q(\mathbb{R}^2))} + \varepsilon). \tag{5.24}$$

Next, we shall estimate g^ε. Let φ_1^ε and φ_2^ε be the solution of the rescaled Poisson equation (5.4) with $|\psi^\varepsilon|^2$ replaced by $|\Pi\psi^\varepsilon|^2$ and $|\psi^\varepsilon|^2 - |\Pi\psi^\varepsilon|^2$, respectively; then $\varphi^\varepsilon = \varphi_1^\varepsilon + \varphi_2^\varepsilon$, and we can rewrite

$$g^\varepsilon = J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon, \tag{5.25}$$

where

$$J_1^\varepsilon = \int_{\mathbb{R}} P_\varepsilon(\varphi_1^\varepsilon)\phi^\varepsilon w_0^2 dz, \qquad J_2^\varepsilon = e^{\frac{i\mu_0 t}{\varepsilon^2}} \int_{\mathbb{R}} P_\varepsilon(\varphi^\varepsilon)(\psi^\varepsilon - \Pi\psi^\varepsilon) w_0 dz,$$
$$J_3^\varepsilon = e^{\frac{i\mu_0 t}{\varepsilon^2}} \int_{\mathbb{R}} P_\varepsilon(\varphi_2^\varepsilon)\Pi\psi^\varepsilon w_0 dz.$$

For J_1^ε, this reduces to the quasi-2D equation I (1.11), where we have that

$$J_1^\varepsilon = -3\lambda(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\Delta)\varphi_{2D}^\varepsilon\phi^\varepsilon \quad \text{and} \quad \varphi_{2D}^\varepsilon = U_\varepsilon^{2D} * |\phi^\varepsilon|^2, \tag{5.26}$$

with U_ε^{2D} given in (1.12). In view of the property of U_ε^{2D} in Lemma 2.2 and Remark 3.1, recalling $\phi^\varepsilon \in L^\infty([0,\infty); H^1(\mathbb{R}^2))$ and using the Hölder inequality and Sobolev inequality, we obtain

$$\|J_1^\varepsilon\|_p \le \|P_\varepsilon(\varphi_{2D}^\varepsilon)\|_{p_1}\|\phi^\varepsilon\|_{p_2} \le C\|\nabla|\phi^\varepsilon|^2\|_{p_1}\|\phi^\varepsilon\|_{p_2} \le C, \tag{5.27}$$

where $1 < p < p_1 < 2$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

For J_2^ε, applying the Minkowski inequality, Hölder inequality, and Sobolev inequality, as well as estimates for the Poisson equation, and noting $\psi^\varepsilon \in L^\infty([0,\infty); H^1(\mathbb{R}^3))$ and Lemma 5.2, we estimate

$$\begin{aligned}\|J_2^\varepsilon\|_p &\le \|P_\varepsilon(\varphi^\varepsilon)(\psi^\varepsilon - \Pi\psi^\varepsilon)w_0\|_{(1,p)} \le C\|P_\varepsilon(\varphi^\varepsilon)\|_{L^{p^*}(\mathbb{R}^3)}\|\psi^\varepsilon - \Pi\psi^\varepsilon\|_{(\infty,2)} \\ &\le C\varepsilon\left\|\frac{|\psi^\varepsilon|^2}{\varepsilon}\right\|_{L^{p^*}(\mathbb{R}^3)} \le C,\end{aligned}$$

where $p^* = 2p/(2-p) \le 3$.

For J_3^ε, similarly to J_ε^1, J_ε^2, we have

$$\begin{aligned}\|J_3^\varepsilon\|_p &\le \|P_\varepsilon(\varphi_2^\varepsilon)\Pi\psi^\varepsilon w_0\|_{(1,p)} \le C\|P_\varepsilon(\varphi_2^\varepsilon)\|_{L^{p_1}(\mathbb{R}^3)}\|\phi^\varepsilon\|_{L^{p_2}(\mathbb{R}^2)} \\ &\le \frac{C}{\varepsilon}\|\,|\psi^\varepsilon|^2 - |\Pi\psi^\varepsilon|^2\,\|_{L^{p_1}(\mathbb{R}^3)} \\ &\le \frac{C}{\varepsilon}\|\psi^\varepsilon - \Pi\psi^\varepsilon\|_{L^2(\mathbb{R}^3)}(\|\psi^\varepsilon\|_{L^{p_3}(\mathbb{R}^3)} + \|\Pi\psi^\varepsilon\|_{L^{p_3}(\mathbb{R}^3)}) \le C,\end{aligned}$$

where $p_3 = 2p_1^2/(2-p_1) \le 6$. Hence, by choosing $p = 6/5$ and $p_1 = 4/3$, p, p_1, p^*, and p_3 would satisfy all the conditions for J_k^ε $(k = 1,2,3)$, where we shall derive that uniformly in t,

$$\|g^\varepsilon\|_{L^p(\mathbb{R}^2)} \le \|J_1^\varepsilon\|_{L^p(\mathbb{R}^2)} + \|J_2^\varepsilon\|_{L^p(\mathbb{R}^2)} + \|J_3^\varepsilon\|_{L^p(\mathbb{R}^2)} \le C. \tag{5.28}$$

Then choosing $q = p$ in (5.24), we have

$$\|\chi^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^2))} \le C\left[\|\chi^\varepsilon\|_{L^{\rho^*}([0,T];L^2(\mathbb{R}^2))} + \varepsilon\right]. \tag{5.29}$$

Applying the results for all $t \in [0,T]$, we find

$$\|\chi^\varepsilon(t)\|_2^{\rho^*} \le C\left[\int_0^t \|\chi^\varepsilon(s)\|_2^{\rho^*}\, ds + \varepsilon^{\rho^*}\right], \quad t \in [0,T], \tag{5.30}$$

and Gronwall's inequality will give that $\|\chi^\varepsilon(t)\|_2 \le C\varepsilon$ for all $t \in [0,T]$. Combining (5.30) with (5.21), we can draw the desired conclusion. □

5.2. Reduction to 1D. In this case, we again consider the weak interaction regime $\beta \to \varepsilon^2\beta$, $\lambda \to \varepsilon^2\lambda$. In Case II (1.9), for the full 3D GPPS (1.4)–(1.5), we introduce the rescaling $x \to \varepsilon x$, $y \to \varepsilon y$, $\psi \to \varepsilon^{-1}\psi^\varepsilon$ which preserves the normalization. Then

(5.31)

$$i\partial_t\psi^\varepsilon(\mathbf{x},z,t) = \left[\mathbf{H}_z^V + \frac{1}{\varepsilon^2}\mathbf{H}_\mathbf{x} + (\beta-\lambda)|\psi^\varepsilon|^2 - 3\varepsilon\lambda\partial_{\tilde{\mathbf{n}}_\varepsilon\tilde{\mathbf{n}}_\varepsilon}\varphi^\varepsilon\right]\psi^\varepsilon, \quad (\mathbf{x},z) \in \mathbb{R}^3,$$

where $\mathbf{x} = (x,y) \in \mathbb{R}^2$ and

$$\mathbf{H}_z^V = -\frac{1}{2}\partial_{zz} + V_1(z), \quad \mathbf{H}_\mathbf{x} = -\frac{1}{2}(\partial_{xx} + \partial_{yy} + x^2 + y^2), \tag{5.32}$$

$$\tilde{\mathbf{n}}_\varepsilon = (n_1/\varepsilon, n_2/\varepsilon, n_3), \quad \partial_{\tilde{\mathbf{n}}_\varepsilon} = \tilde{\mathbf{n}}_\varepsilon \cdot \nabla, \quad \partial_{\tilde{\mathbf{n}}_\varepsilon\tilde{\mathbf{n}}_\varepsilon} = \partial_{\tilde{\mathbf{n}}_\varepsilon}(\partial_{\tilde{\mathbf{n}}_\varepsilon}), \tag{5.33}$$

$$\left(-\frac{1}{\varepsilon^2}\partial_{xx} - \frac{1}{\varepsilon^2}\partial_{yy} - \partial_{zz}\right)\varphi^\varepsilon = \frac{1}{\varepsilon^2}|\psi^\varepsilon|^2, \qquad \lim_{|\mathbf{x}|\to\infty}\varphi^\varepsilon(\mathbf{x}) = 0. \tag{5.34}$$

Note that the ground state mode of $\mathbf{H}_\mathbf{x}$ would be given by $w_0(x)w_0(y)$ with eigenvalue 1, and the initial data is then assumed to be

$$\psi^\varepsilon(\mathbf{x},z,0) = \phi_0(z)w_0(x)w_0(y), \quad \phi_0 \in X_1 \text{ and } \|\phi_0\|_{L^2(\mathbb{R})} = 1. \tag{5.35}$$

In Case II (1.9) in the introduction, when $\varepsilon \to 0^+$, the quasi-1D equation (1.17) will lead to an ε-independent equation in the weak interaction regime:

$$i\partial_t\phi(z,t) = \mathbf{H}_z^V\phi + \frac{\beta + \frac{1}{2}\lambda(1-3n_3^2)}{2\pi}|\phi|^2\phi, \qquad z \in \mathbb{R}, \quad t > 0, \tag{5.36}$$

with the initial condition $\phi(z,0) = \phi_0(z)$.

Following the steps in subsection 5.1, we can prove the following results.

Theorem 5.5 (dimension reduction to 1D). *Suppose the real-valued trap potential satisfies $V_1(z) \ge 0$ for $z \in \mathbb{R}$ and $V_1(z) \in C^\infty(\mathbb{R})$, $D^kV_1(z) \in L^\infty(\mathbb{R})$ for all $k \ge 2$. Assume $-\frac{\beta}{2} \le \lambda \le \beta$ and $\beta \ge 0$, and let $\psi^\varepsilon \in C([0,\infty); X_3)$ and $\phi \in C([0,\infty); X_1)$ be the unique solutions of* (5.31)–(5.35) *and* (5.36)*, respectively. Then for any $T > 0$, there exists $C_T > 0$ such that*

$$\left\|\psi^\varepsilon(\mathbf{x},z,t) - e^{-it/\varepsilon^2}\phi(z,t)w_0(x)w_0(y)\right\|_{L^2(\mathbb{R}^3)} \le C_T\,\varepsilon \qquad \forall t \in [0,T]. \tag{5.37}$$

6. Conclusion. We have analyzed the effective lower dimensional models for the 3D Gross–Pitaevskii–Poisson system (GPPS) describing dipolar Bose–Einstein condensates (BECs) in anisotropic confinement. The quasi-2D approximate equations I (1.11) and II (1.15) are introduced in the case where the trap is strongly confined in the vertical z-direction, and the quasi-1D approximate equation (1.17) is presented in the case where the trap is strongly confined in the x- and y-directions. Properties of ground states for all equations, such as existence and uniqueness as well as nonuniqueness results, are studied. Well-posedness of the Cauchy problem for both equations and possible finite time blow-up in the 2D case are discussed. Finally, we rigorously prove the linear convergence rate of the dimension reduction from the 3D GPPS to its quasi-2D and quasi-1D approximations in the weak interaction regime, i.e., $\beta = \lambda = O(\varepsilon^{3-d})$, in lower d ($d = 1, 2$) dimensions. In such a situation, all the nonlocal terms in the effective equations (1.11), (1.15), and (1.17) vanish, resulting in a regular GPE in lower dimensions. We remark that the results in this paper hold true for a larger class of confinements rather than the harmonic ones. In fact, effective 2D models have been derived and analyzed recently for a multilayer stack of dipolar BECs formed by a strong lattice potential [22].

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