

## QUANTIZED VORTEX DYNAMICS AND INTERACTION PATTERNS IN SUPERCONDUCTIVITY BASED ON THE REDUCED DYNAMICAL LAW

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**ABSTRACT.** We study analytically and numerically stability and interaction patterns of quantized vortex lattices governed by the reduced dynamical law – a system of ordinary differential equations (ODEs) – in superconductivity. By deriving several non-autonomous first integrals of the ODEs, we obtain qualitatively dynamical properties of a cluster of quantized vortices, including global existence, finite time collision, equilibrium solution and invariant solution manifolds. For a vortex lattice with 3 vortices, we establish orbital stability when they have the same winding number and find different collision patterns when they have different winding numbers. In addition, under several special initial setups, we can obtain analytical solutions for the nonlinear ODEs.

**1. Introduction.** In this paper, we study analytically and numerically stability and interaction patterns of the following system of ordinary differential equations (ODEs) describing the dynamics of  $N \geq 2$  quantized vortices in superconductivity based on the reduced dynamical law [23, 12, 17, 30, 31]

$$\dot{\mathbf{x}}_j(t) = 2m_j \sum_{k=1, k \neq j}^N m_k \frac{\mathbf{x}_j(t) - \mathbf{x}_k(t)}{|\mathbf{x}_j(t) - \mathbf{x}_k(t)|^2}, \quad 1 \leq j \leq N, \quad t > 0, \quad (1.1)$$

with initial data

$$\mathbf{x}_j(0) = \mathbf{x}_j^0 = (x_j^0, y_j^0)^T \in \mathbb{R}^2, \quad 1 \leq j \leq N. \quad (1.2)$$

Here  $t$  is time,  $\mathbf{x}_j(t) = (x_j(t), y_j(t))^T \in \mathbb{R}^2$  is the center of the  $j$ -th ( $1 \leq j \leq N$ ) quantized vortex at time  $t$ ,  $m_j = +1$  or  $-1$  is the winding number or index or circulation of the  $j$ -th ( $1 \leq j \leq N$ ) quantized vortex. We always assume that the initial data satisfies  $\mathbf{X}^0 := (\mathbf{x}_1^0, \dots, \mathbf{x}_N^0) \in \mathbb{R}_*^{2 \times N} := \{\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathbb{R}^{2 \times N} \mid \mathbf{x}_j \neq \mathbf{x}_k, j \neq k\}$ .

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$\mathbb{R}^{2 \times N} \mid \mathbf{x}_j \neq \mathbf{x}_l \in \mathbb{R}^2 \text{ for } 1 \leq j < l \leq N\}$  and denote its mass center as  $\bar{\mathbf{x}}^0 := \frac{1}{N} \sum_{j=1}^N \mathbf{x}_j^0$ . Throughout this paper, we assume that  $N \geq 2$ .

The ODEs (1.1) with (1.2) was derived asymptotically as a reduced dynamical law for the dynamics of  $N$  quantized vortices – particle-like or topological defects – in the Ginzburg-Landau equation [22, 12, 17]

$$\partial_t \psi(\mathbf{x}, t) = \nabla^2 \psi(\mathbf{x}, t) + \frac{1}{\varepsilon^2} (1 - |\psi(\mathbf{x}, t)|^2) \psi(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^2, \quad t > 0, \quad (1.3)$$

with initial condition

$$\psi(\mathbf{x}, 0) = \psi_0(\mathbf{x}) = \prod_{j=1}^N \phi_{m_j}(\mathbf{x} - \mathbf{x}_j^0), \quad \mathbf{x} \in \mathbb{R}^2, \quad (1.4)$$

for superconductivity when either  $\varepsilon = 1$  and  $d_{\min}^0 := \min_{1 \leq j < l \leq N} |\mathbf{x}_j^0 - \mathbf{x}_l^0| \rightarrow \infty$  [23] or for a given  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times N}$  and  $\varepsilon \rightarrow 0^+$  [12, 17]. Here  $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$  is the Cartesian coordinates in two dimensions (2D),  $\psi := \psi(\mathbf{x}, t)$  is a complex-valued order parameter,  $\varepsilon > 0$  is a constant, and  $\phi_m(\mathbf{x}) = f(r)e^{im\theta}$  ( $m = +1$  or  $-1$ ) with  $(r, \theta)$  the polar coordinates in 2D and  $f(r)$  satisfying [23, 12, 17, 30, 31]

$$\begin{aligned} \frac{1}{r} \frac{d}{dr} \left( r \frac{df(r)}{dr} \right) - \frac{1}{r^2} f(r) + \frac{1}{\varepsilon^2} (1 - f^2(r)) f(r) &= 0, \quad 0 < r < +\infty, \\ f(0) &= 0, \quad \lim_{r \rightarrow +\infty} f(r) = 1. \end{aligned}$$

Here  $\phi_m(\mathbf{x})$  is a typical quantized vortex in 2D, which is zero of the order parameter at the vortex center located at the origin and has localized phase singularity with integer  $m$  topological charge usually called also as winding number or index or circulation. In fact, quantized vortices have been widely observed in superconductor [11, 17, 4], liquid helium [21], Bose-Einstein condensates [26, 2, 13]; and they are key signatures of superconductivity and superfluidity. The study of quantized vortices and their dynamics is one of the most important and fundamental problems in superconductivity and superfluidity [23, 3, 20, 28, 6, 8, 9, 10, 16, 18, 5, 24, 25].

Based on the reduced dynamical law, i.e. (1.1), for the quantized vortex dynamics in superconductivity, when two quantized vortices have the same winding number (i.e. vortex pair), they undergo a repulsive interaction; and respectively, when they have opposite winding numbers (i.e. vortex dipole or vortex-antivortex), they undergo an attractive interaction [23, 30, 31]. For  $N \geq 2$  and  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times N}$ , it is straightforward to obtain local existence of the ODEs (1.1) with (1.2) by the standard theory of ODEs. Specifically, when  $N = 2$ , one can obtain explicitly the analytical solution of (1.1) with (1.2): when  $m_1 = m_2$  (i.e. vortex pair), the two vortices move outwards by repelling each other along the line passing through their initial locations  $\mathbf{x}_1^0 \neq \mathbf{x}_2^0$  and they never collide at finite time; and when  $m_1 = -m_2$  (i.e. vortex dipole or vortex-antivortex), the two vortices move towards each other along the line passing through their initial locations  $\mathbf{x}_1^0 \neq \mathbf{x}_2^0$  and they will collide at  $\frac{1}{2}(\mathbf{x}_1^0 + \mathbf{x}_2^0)$  in finite time [23, 30, 31]. For analytical solutions of the ODEs (1.1) with several special initial setups in (1.2), we refer to [30, 31] and references therein. In addition, define the *mass center* of the  $N$  vortices as

$$\bar{\mathbf{x}}(t) := \frac{1}{N} \sum_{j=1}^N \mathbf{x}_j(t), \quad t \geq 0, \quad (1.5)$$

then it was proven that the mass center is conserved under the dynamics of (1.1) with (1.2) [30, 31]

$$\bar{\mathbf{x}}(t) \equiv \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}^0, \quad t \geq 0. \quad (1.6)$$

Introduce

$$\begin{aligned} W(\mathbf{X}) &= - \sum_{1 \leq j \neq k \leq N} m_j m_k \ln |\mathbf{x}_j - \mathbf{x}_k| \\ &= - \ln \prod_{1 \leq j \neq k \leq N} |\mathbf{x}_j - \mathbf{x}_k|^{m_j m_k}, \quad \mathbf{X} \in \mathbb{R}_*^{2 \times N}, \end{aligned} \tag{1.7}$$

then (1.1) can be reformulated as

$$\dot{\mathbf{X}}(t) = -\nabla_{\mathbf{X}} W(\mathbf{X}), \quad t > 0, \tag{1.8}$$

which implies that

$$W(\mathbf{X}(t_2)) \leq W(\mathbf{X}(t_1)) \leq W(\mathbf{X}(0)) = W(\mathbf{X}^0), \quad 0 \leq t_1 \leq t_2. \tag{1.9}$$

In addition, let  $\mathbf{z}_j(t) := x_j(t) + iy_j(t) \in \mathbb{C}$  for  $1 \leq j \leq N$ , then (1.1) can be reformulated as

$$\begin{aligned} \dot{\mathbf{z}}_j(t) &= 2m_j \sum_{k=1, k \neq j}^N m_k \frac{\mathbf{z}_j(t) - \mathbf{z}_k(t)}{|\mathbf{z}_j(t) - \mathbf{z}_k(t)|^2} \\ &= 2m_j \sum_{k=1, k \neq j}^N \frac{m_k}{\bar{\mathbf{z}}_j(t) - \bar{\mathbf{z}}_k(t)}, \quad 1 \leq j \leq N, \quad t > 0, \end{aligned} \tag{1.10}$$

where  $\bar{\mathbf{z}}$  denotes the complex conjugate of  $\mathbf{z} \in \mathbb{C}$ .

For rigorous mathematical justification of the derivation of the above reduced dynamical law (1.1) with (1.2) for superconductivity, we refer to [12, 17] and references therein, and respectively, for numerical comparison of quantized vortex center dynamics under the Ginzburg-Landau equation (1.3) with (1.4) and its corresponding reduced dynamical law (1.1) with (1.2), we refer to [30, 31] and references therein. Based on the mathematical and numerical results [12, 17, 30, 31], the dynamics of the  $N$  quantized vortex centers under the reduced dynamical law agrees qualitatively (and quantitatively when they are well-separated) with that under the Ginzburg-Landau equation. The main aim of this paper is to study analytically and numerically the dynamics and interaction patterns of the reduced dynamical law (1.1) with (1.2), which will generate important insights about quantized vortex dynamics and interaction patterns in superconductivity and is much simpler than to solve the Ginzburg-Landau equation (1.3) with (1.4). We establish global existence of the ODEs (1.1) when the  $N$  quantized vortices have the same winding number and possible finite time collision when they have opposite winding numbers. For  $N = 3$ , we prove orbital stability when they have the same winding number and find different collision patterns when they have different winding numbers. Analytical solutions of the ODEs (1.1) are obtained under several initial setups with symmetry.

The paper is organized as follows. In section 2, we obtain some invariant solution manifolds and several non-autonomous first integrals of the ODEs (1.1) and establish its global existence when the  $N$  quantized vortices have the same winding number and possible finite time collision when they have opposite winding numbers. In section 3, we prove orbital stability when they have the same winding number and find different collision patterns when they have different winding numbers for the dynamics of  $N = 3$  vortices. Analytical solutions of the ODEs (1.1) are presented under several initial setups with symmetry in section 4. Finally, some conclusions are drawn in section 5.

**2. Dynamical properties of a cluster with  $N$  quantized vortices.** In this section, we establish dynamical properties of the system of ODEs (1.1) with the initial data (1.2) for describing the dynamics – reduced dynamical law – of a cluster with  $N$  quantized vortices in superconductivity.

For any two vortices  $\mathbf{x}_j(t)$  and  $\mathbf{x}_l(t)$  ( $1 \leq j < l \leq N$ ), if there exists a finite time  $0 < T_c < +\infty$  such that  $d_{jl}(t) := |\mathbf{x}_j(t) - \mathbf{x}_l(t)| > 0$  for  $0 \leq t < T_c$  and  $d_{jl}(T_c) = 0$ , then we say that they will be *finite time collision* or annihilation (cf. Fig. 2.1a); otherwise, i.e.  $d_{jl}(t) > 0$  for  $t \geq 0$ , then we say that they will not collide. When  $N \geq 2$  and let  $I \subseteq \{1, 2, \dots, N\}$  be a set with at least 2 elements, if there exists a finite time  $0 < T_c < +\infty$  such that  $\min_{1 \leq j < l \leq N} d_{jl}(t) > 0$  for  $0 \leq t < T_c$ ,  $\lim_{t \rightarrow T_c^-} \mathbf{x}_j(t) = \mathbf{x}^0 \in \mathbb{R}^2$  for  $j \in I$  with  $\mathbf{x}^0$  a fixed point and  $\min_{j \in I} d_{jl}(t) > 0$  for  $0 \leq t \leq T_c$  and  $l \in J := \{m \mid 1 \leq m \leq N, m \notin I\}$ , then we say that all vortices in the set  $I$  will form a (finite time) *collision cluster* among the  $N$  vortices (cf. Fig. 2.1b). Define

$$T_{\max} = \sup \{t \geq 0 \mid \mathbf{x}_j(t) \neq \mathbf{x}_l(t), \text{ for all } 1 \leq j \neq l \leq N\},$$

it is easy to see that  $0 < T_{\max} \leq +\infty$  by noting (1.2). If  $T_{\max} < +\infty$ , a finite time collision happens among at least two vortices in the  $N$  vortices (or the ODEs (1.1) with (1.2) will blow-up at finite time); otherwise, i.e.  $T_{\max} = +\infty$ , there is no collision among all the  $N$  quantized vortices (or the ODEs (1.1) with (1.2) is global well-posed in time).

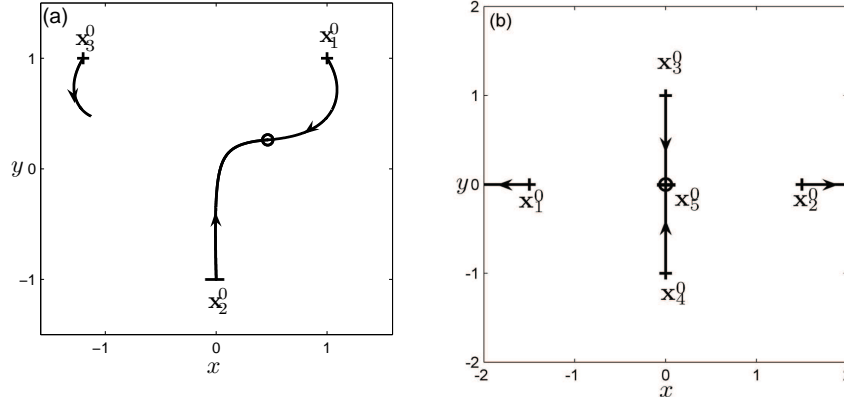


FIGURE 2.1. Illustrations of a finite time collision of a vortex dipole in a vortex cluster with 3 vortices (a) and a (finite time) collision cluster with 3 vortices in a vortex cluster with 5 vortices (b). Here and in the following figures, ‘+’ and ‘-’ denote the initial vortex centers with winding numbers  $m = +1$  and  $m = -1$ , respectively; and ‘o’ denotes the finite time collision position.

**2.1. Invariant solution manifolds.** Let  $\alpha > 0$  be a positive constant,  $0 \leq \theta_0 < 2\pi$  be a constant,  $\mathbf{x}^0 \in \mathbb{R}^2$  be a given point and  $Q(\theta)$  be the rotational matrix defined as

$$Q(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad 0 \leq \theta < 2\pi.$$

Then it is easy to see that the ODEs (1.1) with (1.2) is translational and rotational invariant with the proof omitted here for brevity.

**Lemma 2.1.** *Let  $\mathbf{X}(t) = (\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t)) \in \mathbb{R}_*^{2 \times N}$  be the solution of the ODEs (1.1) with (1.2), then we have*

- (i) *If  $\mathbf{x}_j^0 \rightarrow \mathbf{x}_j^0 + \mathbf{x}^0$  for  $1 \leq j \leq N$  in (1.2), then  $\mathbf{x}_j(t) \rightarrow \mathbf{x}_j(t) + \mathbf{x}^0$  for  $1 \leq j \leq N$ .*
- (ii) *If  $\mathbf{x}_j^0 \rightarrow \alpha \mathbf{x}_j^0$  for  $1 \leq j \leq N$  in (1.2), then  $\mathbf{x}_j(t) \rightarrow \alpha \mathbf{x}_j(t/\alpha^2)$  for  $1 \leq j \leq N$ .*
- (iii) *If  $\mathbf{x}_j^0 \rightarrow Q(\theta) \mathbf{x}_j^0$  for  $1 \leq j \leq N$  in (1.2), then  $\mathbf{x}_j(t) \rightarrow Q(\theta) \mathbf{x}_j(t)$  for  $1 \leq j \leq N$ .*

Denote  $S_e(\mathbf{x}^0) := \{ \mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathbb{R}_*^{2 \times N} \mid |(\mathbf{x}_j - \mathbf{x}^0) \cdot \mathbf{e}| = |\mathbf{x}_j - \mathbf{x}^0| \times |\mathbf{e}|, 1 \leq j \leq N \}$ , where  $\mathbf{e} \in \mathbb{R}^2$  is a given unit vector. In fact,  $S_e(\mathbf{x}^0)$  is a line in 2D passing the point  $\mathbf{x}^0$  and parallel to the unit vector  $\mathbf{e}$ . For  $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathbb{R}_*^{2 \times N}$ , if there exist  $\mathbf{x}^0 \in \mathbb{R}^2$  and a unit vector  $\mathbf{e} \in \mathbb{R}^2$  such that  $\mathbf{X} \in S_e(\mathbf{x}^0)$ , then we say that  $\mathbf{X}$  is collinear.

**Lemma 2.2.** *If the initial data  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times N}$  in (1.2) is collinear, i.e. there exist  $\mathbf{x}^0 \in \mathbb{R}^2$  and a unit vector  $\mathbf{e} \in \mathbb{R}^2$  such that  $\mathbf{X}^0 \in S_e(\mathbf{x}^0)$ , then the solution  $\mathbf{X}(t)$  of (1.1)-(1.2) is collinear, i.e.  $\mathbf{X}(t) \in S_e(\mathbf{x}^0)$  for  $0 \leq t < T_{\max}$ .*

*Proof.* From  $\mathbf{X}^0 \in S_e(\mathbf{x}^0)$ , there exist  $a_j^0 \in \mathbb{R}$  ( $1 \leq j \leq N$ ) satisfying  $a_j^0 \neq a_l^0$  for  $1 \leq j < l \leq N$  such that

$$\mathbf{x}_j^0 = \mathbf{x}^0 + a_j^0 \mathbf{e}, \quad 1 \leq j \leq N. \tag{2.1}$$

Noting the symmetric structure in (1.1) and (2.1), we can assume

$$\mathbf{x}_j(t) = \mathbf{x}^0 + a_j(t) \mathbf{e}, \quad 1 \leq j \leq N, \quad t \geq 0. \tag{2.2}$$

Plugging (2.2) into (1.1), we have

$$\dot{a}_j(t) = 2m_j \sum_{k=1, k \neq j}^N m_k \frac{a_j(t) - a_k(t)}{|a_j(t) - a_k(t)|^2}, \quad 1 \leq j \leq N, \quad t > 0, \tag{2.3}$$

with the initial data by noting (2.1)

$$a_j(0) = a_j^0, \quad 1 \leq j \leq N. \tag{2.4}$$

The ODEs (2.3) with (2.4) is locally well-posed. Thus  $\mathbf{X}(t) \in S_e(\mathbf{x}^0)$  for  $0 \leq t < T_{\max}$ . □

Let  $\mathbf{e} \in \mathbb{R}^2$  be a unit vector, denote  $\theta_N^j := \frac{2(j-1)\pi}{N}$  and  $\mathbf{x}_j^{(0)} = Q(\theta_N^j + \theta_0) \mathbf{e}$  for  $1 \leq j \leq N$  and define

$$S_e^N(\mathbf{x}^0, \theta_0) := \left\{ \mathbf{X}_r^0 = (\mathbf{x}^0 + r \mathbf{x}_1^{(0)}, \dots, \mathbf{x}^0 + r \mathbf{x}_N^{(0)}) \in \mathbb{R}_*^{2 \times N} \mid r > 0 \right\},$$

$$S_e^N(\mathbf{x}^0) := \bigcup_{0 \leq \theta_0 < 2\pi} S_e^N(\mathbf{x}^0, \theta_0).$$

**Lemma 2.3.** *Assume the  $N$  vortices have the same winding number, i.e.  $m_1 = \dots = m_N = \pm 1$ . If there exists a unit vector  $\mathbf{e} \in \mathbb{R}^2$ ,  $\mathbf{x}^0 \in \mathbb{R}^2$  and  $0 \leq \theta_0 < 2\pi$  such that the initial data  $\mathbf{X}^0 \in S_e^N(\mathbf{x}^0, \theta_0)$  in (1.2), then the solution  $\mathbf{X}(t)$  of (1.1) satisfies  $\mathbf{X}(t) \in S_e^N(\mathbf{x}^0, \theta_0)$  for  $t \geq 0$ .*

*Proof.* Since  $\mathbf{X}^0 \in S_e^N(\mathbf{x}^0, \theta_0)$ , there exists a  $r_0 > 0$  such that

$$\mathbf{x}_j^0 = \mathbf{x}^0 + r_0 Q(\theta_N^j + \theta_0) \mathbf{e}, \quad 1 \leq j \leq N. \tag{2.5}$$

Noting the symmetric structure in (1.1) and (2.5), we can assume

$$\mathbf{x}_j(t) = \mathbf{x}^0 + r(t)Q(\theta_N^j + \theta_0)\mathbf{e}, \quad 1 \leq j \leq N, \quad t \geq 0. \quad (2.6)$$

Plugging (2.6) into (1.1), noting  $m_1 = \dots = m_N$  and (2.5), we have [30, 31]

$$\dot{r}(t) = \frac{N-1}{r(t)}, \quad t > 0, \quad r(0) = r_0,$$

which implies  $r(t) = \sqrt{r_0^2 + 2(N-1)t}$  for  $t \geq 0$ . Thus  $\mathbf{X}(t) \in S_{\mathbf{e}}^N(\mathbf{x}^0, \theta_0)$  for  $t \geq 0$ .  $\square$

From the above two lemmas, for any  $\theta_0 \in \mathbb{R}$ ,  $\mathbf{x}^0 \in \mathbb{R}^2$  and a unit vector  $\mathbf{e} \in \mathbb{R}^2$ ,  $S_{\mathbf{e}}(\mathbf{x}^0)$  is an invariant solution manifold of the ODEs (1.1) with (1.2). In addition, when  $m_1 = \dots = m_N$ , then  $S_{\mathbf{e}}^N(\mathbf{x}^0, \theta_0)$  is also an invariant solution manifold of the ODEs (1.1) with (1.2). Specifically, when  $\mathbf{X}^0 \in S_{\mathbf{e}}^N(\mathbf{0}, \theta_0)$  and  $m_1 = \dots = m_N$ , then the ODEs (1.1) with (1.2) admits the self-similar solution  $\mathbf{X}(t) = \sqrt{r_0^2 + 2(N-1)t}\mathbf{X}^0$  with  $r_0 = |\mathbf{x}_1^0| = \dots = |\mathbf{x}_N^0|$  for  $t \geq 0$ . For more self-similar solutions of the ODEs (1.1) with special initial setups, we refer to [30, 31] and references therein.

**2.2. Non-autonomous first integrals.** In this subsection, we establish several non-autonomous first integrals of the ODEs (1.1). Here a nonconstant function  $H(\mathbf{X}, t)$  is called a non-autonomous first integral of the ODEs (1.1) if it is constant along any solution curve  $\mathbf{X} = \mathbf{X}(t)$  of (1.1). In fact, first integral is an important concept in the qualitative analysis of dynamical system. The classical applications of first integral can be traced back to H. Poincaré for studying the three body problem [27] and S. Kowalevski for studying the Euler top [14] and so on. For more recent applications, we refer to [15, 21, 29] and references therein.

Let  $N^+$  and  $N^-$  be the number of vortices with winding number  $m = 1$  and  $m = -1$ , respectively, then we have

$$0 \leq N^+ \leq N, \quad 0 \leq N^- \leq N, \quad N^+ + N^- = N.$$

In addition, it is easy to get

$$\begin{aligned} M_0 &= \sum_{1 \leq j < l \leq N} m_j m_l = \frac{1}{2} \sum_{j, l=1, j \neq l}^N m_j m_l \\ &= \frac{N^+(N^+ - 1)}{2} + \frac{N^-(N^- - 1)}{2} - N^+ N^- = \frac{(N^+ - N^-)^2 - N}{2}. \end{aligned} \quad (2.7)$$

Define

$$H_1(\mathbf{X}, t) = -4NM_0t + \sum_{1 \leq j < l \leq N} |\mathbf{x}_j - \mathbf{x}_l|^2, \quad (2.8)$$

$$H_2(\mathbf{X}, t) = -4M_0t + \sum_{j=1}^N |\mathbf{x}_j|^2, \quad \mathbf{X} := (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) \in \mathbb{R}^{2 \times N}, \quad t \geq 0, \quad (2.9)$$

$$H_3(\mathbf{X}, t) = -4(N-2)M_0t + \sum_{1 \leq j < l \leq N} |\mathbf{x}_j + \mathbf{x}_l|^2. \quad (2.10)$$

then we have

**Lemma 2.4.** *Let  $\mathbf{X}(t) = (\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t)) \in \mathbb{R}_*^{2 \times N}$  be the solution of the ODEs (1.1) with (1.2), then  $H_1(\mathbf{X}, t)$ ,  $H_2(\mathbf{X}, t)$  and  $H_3(\mathbf{X}, t)$  are non-autonomous first integrals of (1.1), i.e.*

$$H_1(\mathbf{X}(t), t) \equiv H_1^0 := \sum_{1 \leq j < l \leq N} |\mathbf{x}_j^0 - \mathbf{x}_l^0|^2, \tag{2.11}$$

$$H_2(\mathbf{X}(t), t) \equiv H_2^0 := \sum_{j=1}^N |\mathbf{x}_j^0|^2, \quad t \geq 0, \tag{2.12}$$

$$H_3(\mathbf{X}(t), t) \equiv H_3^0 := \sum_{1 \leq j < l \leq N} |\mathbf{x}_j^0 + \mathbf{x}_l^0|^2. \tag{2.13}$$

*Specifically, when  $M_0 = 0$ ,  $H_1(\mathbf{X}) := H_1(\mathbf{X}, t) = \sum_{1 \leq j < l \leq N} |\mathbf{x}_j - \mathbf{x}_l|^2$  and  $H_2(\mathbf{X}) := H_2(\mathbf{X}, t) = \sum_{j=1}^N |\mathbf{x}_j|^2$  are two autonomous first integrals of (1.1); and when either  $N = 2$  or  $M_0 = 0$ ,  $H_3(\mathbf{X}) := H_3(\mathbf{X}, t) = \sum_{1 \leq j < l \leq N} |\mathbf{x}_j + \mathbf{x}_l|^2$  is an autonomous first integral of (1.1).*

*Proof.* Differentiating the left equation in (2.8) (with  $\mathbf{X} = \mathbf{X}(t)$ ) with respect to  $t$ , we have for  $t \geq 0$

$$\begin{aligned} \frac{dH_1(\mathbf{X}(t), t)}{dt} &= \nabla_{\mathbf{X}} H_1(\mathbf{X}, t) \cdot \dot{\mathbf{X}} + \left. \frac{\partial H_1(\mathbf{X}, t)}{\partial t} \right|_{\mathbf{X}=\mathbf{X}(t)} \\ &= -4NM_0 + 2 \sum_{1 \leq j < l \leq N} (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot (\dot{\mathbf{x}}_j(t) - \dot{\mathbf{x}}_l(t)). \end{aligned} \tag{2.14}$$

Using summation by parts and noting (2.7) and (1.1), we obtain

$$\begin{aligned} I &:= 2 \sum_{1 \leq j < l \leq N} (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot (\dot{\mathbf{x}}_j(t) - \dot{\mathbf{x}}_l(t)) \\ &= \sum_{j,l=1, j \neq l}^N (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot (\dot{\mathbf{x}}_j(t) - \dot{\mathbf{x}}_l(t)) \\ &= 2 \sum_{j,l=1, j \neq l}^N (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot \\ &\quad \left[ \sum_{k=1, k \neq j}^N m_j m_k \frac{\mathbf{x}_j(t) - \mathbf{x}_k(t)}{|\mathbf{x}_j(t) - \mathbf{x}_k(t)|^2} - \sum_{k=1, k \neq l}^N m_l m_k \frac{\mathbf{x}_l(t) - \mathbf{x}_k(t)}{|\mathbf{x}_l(t) - \mathbf{x}_k(t)|^2} \right] \\ &= 2 \sum_{j,l=1, j \neq l}^N 2m_j m_l \frac{|\mathbf{x}_j(t) - \mathbf{x}_l(t)|^2}{|\mathbf{x}_j(t) - \mathbf{x}_l(t)|^2} \\ &\quad + 2 \sum_{j,l=1, j \neq l}^N (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot \sum_{k=1, k \neq j, l}^N m_j m_k \frac{\mathbf{x}_j(t) - \mathbf{x}_k(t)}{|\mathbf{x}_j(t) - \mathbf{x}_k(t)|^2} \\ &\quad - 2 \sum_{j,l=1, j \neq l}^N (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot \sum_{k=1, k \neq j, l}^N m_l m_k \frac{\mathbf{x}_l(t) - \mathbf{x}_k(t)}{|\mathbf{x}_l(t) - \mathbf{x}_k(t)|^2}. \end{aligned} \tag{2.15}$$

$$I = 4 \sum_{j,l=1, j \neq l}^N m_j m_l + 2 \sum_{j,k=1, j \neq k}^N m_j m_k \sum_{l=1, l \neq j, k}^N \frac{(\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot (\mathbf{x}_j(t) - \mathbf{x}_k(t))}{|\mathbf{x}_j(t) - \mathbf{x}_k(t)|^2}$$

$$\begin{aligned}
& -2 \sum_{l,k=1,l \neq k}^N m_l m_k \sum_{j=1,l \neq l,k}^N \frac{(\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot (\mathbf{x}_l(t) - \mathbf{x}_k(t))}{|\mathbf{x}_l(t) - \mathbf{x}_k(t)|^2} \\
= & 8M_0 + 2 \sum_{j,k=1,j \neq k}^N m_j m_k \sum_{l=1,l \neq j,k}^N \frac{(\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot (\mathbf{x}_j(t) - \mathbf{x}_k(t))}{|\mathbf{x}_j(t) - \mathbf{x}_k(t)|^2} \\
& -2 \sum_{j,k=1,j \neq k}^N m_j m_k \sum_{l=1,l \neq j,k}^N \frac{(\mathbf{x}_k(t) - \mathbf{x}_l(t)) \cdot (\mathbf{x}_j(t) - \mathbf{x}_k(t))}{|\mathbf{x}_j(t) - \mathbf{x}_k(t)|^2} \\
= & 2 \sum_{j,k=1,j \neq k}^N m_j m_k \sum_{l=1,l \neq j,k}^N \frac{(\mathbf{x}_j(t) - \mathbf{x}_k(t)) \cdot [(\mathbf{x}_j(t) - \mathbf{x}_l(t)) - (\mathbf{x}_k(t) - \mathbf{x}_l(t))]}{|\mathbf{x}_j(t) - \mathbf{x}_k(t)|^2} \\
& + 8M_0 \\
= & 8M_0 + 2 \sum_{j,k=1,j \neq k}^N (N-2)m_j m_k \\
= & 8M_0 + 4(N-2)M_0 = 4NM_0, \quad t \geq 0. \tag{2.16}
\end{aligned}$$

Plugging (2.15) and (2.16) into (2.14), we get

$$\frac{dH_1(\mathbf{X}(t), t)}{dt} = -4NM_0 + 4NM_0 = 0, \quad t \geq 0, \tag{2.17}$$

which immediately implies (2.11) by noting the initial condition (1.2).

Similarly, differentiating (2.10) (with  $\mathbf{X} = \mathbf{X}(t)$ ) with respect to  $t$ , we get

$$\begin{aligned}
\frac{dH_3(\mathbf{X}(t), t)}{dt} &= \nabla_{\mathbf{X}} H_3(\mathbf{X}, t) \cdot \dot{\mathbf{X}} + \left. \frac{\partial H_3(\mathbf{X}, t)}{\partial t} \right|_{\mathbf{X}=\mathbf{X}(t)} \\
&= -4(N-2)M_0 + 2 \sum_{1 \leq j < l \leq N} (\mathbf{x}_j(t) + \mathbf{x}_l(t)) \cdot (\dot{\mathbf{x}}_j(t) + \dot{\mathbf{x}}_l(t)). \tag{2.18}
\end{aligned}$$

Similar to (2.15), noting (1.1) and (2.7), we get

$$\begin{aligned}
II &:= 2 \sum_{1 \leq j < l \leq N} (\mathbf{x}_j(t) + \mathbf{x}_l(t)) \cdot (\dot{\mathbf{x}}_j(t) + \dot{\mathbf{x}}_l(t)) \\
&= \sum_{j,l=1,j \neq l}^N (\mathbf{x}_j(t) + \mathbf{x}_l(t)) \cdot (\dot{\mathbf{x}}_j(t) + \dot{\mathbf{x}}_l(t)) \\
&= 2 \sum_{j,l=1,j \neq l}^N (\mathbf{x}_j(t) + \mathbf{x}_l(t)) \cdot \\
&\quad \left[ \sum_{k=1,k \neq j}^N m_j m_k \frac{\mathbf{x}_j(t) - \mathbf{x}_k(t)}{|\mathbf{x}_j(t) - \mathbf{x}_k(t)|^2} + \sum_{k=1,k \neq l}^N m_l m_k \frac{\mathbf{x}_l(t) - \mathbf{x}_k(t)}{|\mathbf{x}_l(t) - \mathbf{x}_k(t)|^2} \right]. \tag{2.19}
\end{aligned}$$

$$\begin{aligned}
II &= 2 \sum_{j,k=1,j \neq k}^N m_j m_k \sum_{l=1,l \neq j,k}^N \frac{(\mathbf{x}_j(t) + \mathbf{x}_l(t)) \cdot (\mathbf{x}_j(t) - \mathbf{x}_k(t))}{|\mathbf{x}_j(t) - \mathbf{x}_k(t)|^2} \\
&\quad + 2 \sum_{l,k=1,l \neq k}^N m_l m_k \sum_{j=1,l \neq l,k}^N \frac{(\mathbf{x}_j(t) + \mathbf{x}_l(t)) \cdot (\mathbf{x}_l(t) - \mathbf{x}_k(t))}{|\mathbf{x}_l(t) - \mathbf{x}_k(t)|^2}
\end{aligned}$$



$$\begin{aligned}
 &= 2 \sum_{j,k=1, j \neq k}^N m_j m_k \sum_{l=1, l \neq j, k}^N \frac{(\mathbf{x}_j(t) + \mathbf{x}_l(t)) \cdot (\mathbf{x}_j(t) - \mathbf{x}_k(t))}{|\mathbf{x}_j(t) - \mathbf{x}_k(t)|^2} \\
 &\quad - 2 \sum_{j,k=1, j \neq k}^N m_j m_k \sum_{l=1, l \neq j, k}^N \frac{(\mathbf{x}_k(t) + \mathbf{x}_l(t)) \cdot (\mathbf{x}_j(t) - \mathbf{x}_k(t))}{|\mathbf{x}_j(t) - \mathbf{x}_k(t)|^2} \\
 &= 2 \sum_{j,k=1, j \neq k}^N m_j m_k \sum_{l=1, l \neq j, k}^N \frac{(\mathbf{x}_j(t) - \mathbf{x}_k(t)) \cdot [(\mathbf{x}_j(t) + \mathbf{x}_l(t)) - (\mathbf{x}_k(t) + \mathbf{x}_l(t))]}{|\mathbf{x}_j(t) - \mathbf{x}_k(t)|^2} \\
 &= 2 \sum_{j,k=1, j \neq k}^N (N-2)m_j m_k = 4(N-2)M_0, \quad t \geq 0. \tag{2.20}
 \end{aligned}$$

Plugging (2.19) and (2.20) into (2.18), we get

$$\frac{dH_3(\mathbf{X}(t), t)}{dt} = -4(N-2)M_0 + 4(N-2)M_0 = 0, \quad t \geq 0, \tag{2.21}$$

which immediately implies (2.13) by noting the initial condition (1.2).

From (2.8) and (2.10), it is easy to see that

$$H_2(\mathbf{X}(t), t) = \frac{1}{2(N-1)} [H_1(\mathbf{X}(t), t) + H_3(\mathbf{X}(t), t)], \quad t \geq 0. \tag{2.22}$$

Differentiating (2.22) with respect to  $t$ , noticing (2.17) and (2.21), we have

$$\frac{dH_2(\mathbf{X}(t), t)}{dt} = \frac{1}{2(N-1)} \left[ \frac{dH_1(\mathbf{X}(t), t)}{dt} + \frac{dH_3(\mathbf{X}(t), t)}{dt} \right] = 0, \quad t \geq 0,$$

which immediately implies (2.12) by noting the initial condition (1.2). Therefore  $H_1(\mathbf{X}, t)$ ,  $H_2(\mathbf{X}, t)$  and  $H_3(\mathbf{X}, t)$  are three non-autonomous first integrals of the ODEs (1.1). □

**2.3. Global existence in the case with the same winding number.** Let  $m_0 = +1$  or  $-1$  be fixed. When the  $N$  quantized vortices have the same winder number, e.g.  $m_0$ , we have

**Theorem 2.1.** *Suppose the  $N$  vortices have the same winding number, i.e.  $m_j = m_0$  for  $1 \leq j \leq N$  in (1.1), then  $T_{\max} = +\infty$ , i.e. there is no finite time collision among the  $N$  quantized vortices. In addition, at least two vortices move to infinity as  $t \rightarrow +\infty$ .*

*Proof.* The proof will be proceeded by the method of contradiction. Assume  $0 < T_{\max} < +\infty$ , i.e. there exist  $M$  ( $2 \leq M \leq N$ ) vortices (without loss of generality, we assume here that they are  $\mathbf{x}_1, \dots, \mathbf{x}_M$ ) that collide at a fixed point  $\mathbf{x}^0 \in \mathbb{R}^2$  and the rest  $N - M$  vortices are all away from this point. Taking  $t = T_{\max}$  in the left equation in (2.11), noting (2.7), (2.8) and  $|N^+ - N^-| = N$ , we get

$$\begin{aligned}
 0 &< H_1^0 = H_1(\mathbf{X}(T_{\max}), T_{\max}) \\
 &= -4NM_0T_{\max} + \sum_{1 \leq j < l \leq N} |\mathbf{x}_j(T_{\max}) - \mathbf{x}_l(T_{\max})|^2 \\
 &= -2N^2(N-1)T_{\max} + \sum_{j=M+1}^N |\mathbf{x}_j(T_{\max}) - \mathbf{x}^0|^2 \\
 &\quad + \sum_{M+1 \leq j < l \leq N} |\mathbf{x}_j(T_{\max}) - \mathbf{x}_l(T_{\max})|^2,
 \end{aligned}$$

which immediately implies that  $2 \leq M < N$ . Denote the non-empty sets  $I = \{1, \dots, M\}$  and  $J = \{M+1, \dots, N\}$ , and define

$$D_I(t) = \sum_{1 \leq j < l \leq M} d_{jl}^2(t), \quad d_{I,J}(t) = \min_{j \in I, l \in J} d_{jl}(t), \quad 0 \leq t \leq T_{\max}, \quad (2.23)$$

where

$$d_{jl}(t) = |\mathbf{x}_j(t) - \mathbf{x}_l(t)|, \quad t \geq 0, \quad 1 \leq j < l \leq N.$$

Then we have

$$\lim_{t \rightarrow T_{\max}^-} D_I(t) = 0, \quad \liminf_{t \rightarrow T_{\max}^-} d_{I,J}(t) > 0,$$

which yields

$$d_1 := \min_{0 \leq t \leq T_{\max}} d_{I,J}(t) > 0.$$

Choose  $\varepsilon = \frac{Md_1}{3(N-M)} > 0$ , then there exists a  $0 < T_1 < T_{\max}$  such that

$$0 \leq D_I(t) < \varepsilon, \quad T_1 \leq t \leq T_{\max}.$$

Differentiating (2.23) with respect to  $t$ , we obtain

$$\begin{aligned} \dot{D}_I(t) &= 2 \sum_{1 \leq j < l \leq M} (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot (\dot{\mathbf{x}}_j(t) - \dot{\mathbf{x}}_l(t)) \\ &= \sum_{j,l=1, j \neq l}^M (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot (\dot{\mathbf{x}}_j(t) - \dot{\mathbf{x}}_l(t)) \\ &= 2 \sum_{j,l=1, j \neq l}^M (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot \left[ \sum_{k=1, k \neq j}^N \frac{\mathbf{x}_j(t) - \mathbf{x}_k(t)}{d_{jk}^2(t)} - \sum_{k=1, k \neq l}^N \frac{\mathbf{x}_l(t) - \mathbf{x}_k(t)}{d_{lk}^2(t)} \right] \\ &= 2 \sum_{j,l=1, j \neq l}^M (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot \left[ \sum_{k=1, k \neq j}^M \frac{\mathbf{x}_j(t) - \mathbf{x}_k(t)}{d_{jk}^2(t)} - \sum_{k=1, k \neq l}^M \frac{\mathbf{x}_l(t) - \mathbf{x}_k(t)}{d_{lk}^2(t)} \right] \\ &\quad + 2 \sum_{j,l=1, j \neq l}^M (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot \sum_{k=M+1}^N \left[ \frac{\mathbf{x}_j(t) - \mathbf{x}_k(t)}{d_{jk}^2(t)} - \frac{\mathbf{x}_l(t) - \mathbf{x}_k(t)}{d_{lk}^2(t)} \right]. \end{aligned}$$

Similar to (2.14) via (2.15) (with details omitted here for brevity), we get

$$\begin{aligned} \dot{D}_I(t) &= 4 \sum_{1 \leq j < l \leq M} \left[ M + (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot \sum_{k=M+1}^N \left( \frac{\mathbf{x}_j(t) - \mathbf{x}_k(t)}{d_{jk}^2(t)} - \frac{\mathbf{x}_l(t) - \mathbf{x}_k(t)}{d_{lk}^2(t)} \right) \right] \\ &\geq 4 \sum_{1 \leq j < l \leq M} \left[ M - d_{jl}(t) \sum_{k=M+1}^N \left( \frac{1}{d_{jk}(t)} + \frac{1}{d_{lk}(t)} \right) \right] \\ &\geq 4 \sum_{1 \leq j < l \leq M} \left[ M - 2 \sum_{k=M+1}^N \frac{\varepsilon}{d_1} \right] \\ &= 2M(M-1) \left( M - 2(N-M) \frac{\varepsilon}{d_1} \right) > 0, \quad T_1 \leq t \leq T_{\max}, \quad (2.24) \end{aligned}$$

which immediately implies that  $0 = D_I(T_{\max}^-) \geq D_I(T_1) > 0$ . This is a contradiction, and thus  $T_{\max} = +\infty$ , i.e. there is no finite time collision among the  $N$  quantized vortices.

Noticing  $M_0 = \frac{1}{2}N(N-1) > 0$ , combining (2.8) and (2.11), we get  $\sum_{j=1}^N |\mathbf{x}_j(t)|^2 = \lim_{t \rightarrow +\infty} H_2^0 + 4M_0t \rightarrow +\infty$  when  $t \rightarrow +\infty$ . Hence there exists an  $1 \leq i_0 \leq N$  such that  $|\mathbf{x}_{i_0}(t)| \rightarrow +\infty$  when  $t \rightarrow +\infty$ . Due to the conservation of mass center, i.e.  $\bar{\mathbf{x}}(t) := \frac{1}{N} \sum_{j=1}^N \mathbf{x}_j(t) \equiv \bar{\mathbf{x}}(0)$ , there exists at least another  $1 \leq j_0 \neq i_0 \leq N$  such that  $|\mathbf{x}_{j_0}(t)| \rightarrow +\infty$  when  $t \rightarrow +\infty$ . Thus there exist at least two vortices move to infinity when  $t \rightarrow +\infty$ .  $\square$

Define

$$\begin{aligned} d_{\min}(t) &= \min_{1 \leq j < l \leq N} d_{jl}(t), \\ D_{\min}(t) &= d_{\min}^2(t), \quad 1 \leq j \neq l \leq N, \quad t \geq 0, \\ D_{jl}(t) &= d_{jl}^2(t). \end{aligned} \tag{2.25}$$

Then it is easy to see that  $d_{\min}(t)$  and  $D_{\min}(t)$  are continuous and piecewise smooth functions. In addition, for  $1 \leq j < l \leq N$ , noting (1.1), we have for  $t \geq 0$

$$\begin{aligned} \dot{D}_{jl}(t) &= 2(\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot (\dot{\mathbf{x}}_j(t) - \dot{\mathbf{x}}_l(t)) \\ &= 4(\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot \left[ \sum_{k=1, k \neq j}^N \frac{\mathbf{x}_j(t) - \mathbf{x}_k(t)}{d_{jk}^2(t)} - \sum_{k=1, k \neq l}^N \frac{\mathbf{x}_l(t) - \mathbf{x}_k(t)}{d_{lk}^2(t)} \right] \\ &= 4 \left[ 2 + (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot \sum_{k=1, k \neq j, l}^N \left( \frac{\mathbf{x}_j(t) - \mathbf{x}_k(t)}{d_{jk}^2(t)} - \frac{\mathbf{x}_l(t) - \mathbf{x}_k(t)}{d_{lk}^2(t)} \right) \right]. \end{aligned} \tag{2.26}$$

When the  $N$  vortices are initially collinear, we have

**Theorem 2.2.** *Suppose the  $N$  vortices have the same winding number, i.e.  $m_j = m_0$  for  $1 \leq j \leq N$  in (1.1), and the initial data  $\mathbf{X}^0$  in (1.2) is collinear, then  $d_{\min}(t)$  and  $D_{\min}(t)$  are monotonically increasing functions.*

*Proof.* Since  $\mathbf{X}^0$  is collinear, there exist  $\mathbf{x}^0 \in \mathbb{R}^2$  and a unit vector  $\mathbf{e} \in \mathbb{R}^2$  such that  $\mathbf{X}^0 \in S_{\mathbf{e}}(\mathbf{x}^0)$ , by Lemma 2.2, we know that  $\mathbf{X}(t) \in S_{\mathbf{e}}(\mathbf{x}^0)$  for  $t \geq 0$ . Thus there exist  $a_j(t)$  ( $1 \leq j \leq N$ ) satisfying  $a_j(t) \neq a_l(t)$  for  $1 \leq j < l \leq N$  such that

$$\mathbf{x}_j(t) = \mathbf{x}^0 + a_j(t)\mathbf{e}, \quad t \geq 0, \quad 1 \leq j \leq N. \tag{2.27}$$

Taking  $0 \leq t_0 < t_1$  such that  $d_{\min}(t)$  is smooth on  $[t_0, t_1]$ , without loss of generality, we assume that there exists  $1 \leq i_0 \leq N-1$  (otherwise by re-ordering) such that

$$\begin{aligned} a_1(t) &< a_2(t) < \dots < a_{i_0}(t) < a_{i_0+1}(t) < \dots < a_N(t), \\ d_{\min}(t) &= d_{i_0, i_0+1}(t), \quad t_0 \leq t < t_1. \end{aligned} \tag{2.28}$$

Plugging  $j = i_0$  and  $l = i_0 + 1$  into (2.26) and noting (2.28), (2.27) and (2.25), we gave

$$\dot{D}_{\min}(t) = \dot{D}_{i_0, i_0+1}(t) = 4 \left[ 2 + d_{\min}(t) \sum_{k=1, k \neq i_0, i_0+1}^N \left( \frac{1}{d_{i_0 k}(t)} - \frac{1}{d_{i_0+1, k}(t)} \right) \right]$$

$$\begin{aligned}
 &= 4 \left[ 2 - \sum_{k=1}^{i_0-1} \frac{d_{\min}^2(t)}{d_{i_0k}(t)d_{i_0+1,k}(t)} - \sum_{k=i_0+2}^N \frac{d_{\min}^2(t)}{d_{i_0k}(t)d_{i_0+1,k}(t)} \right] \\
 &\geq 4 \left[ 2 - \sum_{k=1}^{i_0-1} \frac{1}{(i_0-k)(i_0+1-k)} - \sum_{k=i_0+2}^N \frac{1}{(k-i_0)(k-i_0-1)} \right] \\
 &= 4 \left( \frac{1}{i_0} + \frac{1}{N-i_0} \right) > 0, \quad t_0 \leq t < t_1.
 \end{aligned}$$

Here we used  $\frac{d_{jl}(t)}{d_{\min}(t)} \geq |j-l|$  for  $1 \leq j < l \leq N$  by noting (2.28). Thus  $D_{\min}(t)$  (and  $d_{\min}(t)$ ) is a monotonically increasing function over  $[t_0, t_1]$ . Therefore,  $D_{\min}(t)$  (and  $d_{\min}(t)$ ) is a monotonically increasing function over its every piecewise smooth interval. Due to that it is a continuous function, thus  $D_{\min}(t)$  (and  $d_{\min}(t)$ ) is a monotonically increasing function for  $t \geq 0$ .  $\square$

Similarly, when  $2 \leq N \leq 4$  and  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times N}$ , we have

**Theorem 2.3.** *Suppose  $2 \leq N \leq 4$  and the  $N$  vortices have the same winding number, i.e.  $m_j = m_0$  for  $1 \leq j \leq N$  in (1.1), then  $d_{\min}(t)$  and  $D_{\min}(t)$  are monotonically increasing functions.*

*Proof.* Taking  $0 \leq t_0 < t_1$  such that  $d_{\min}(t)$  is smooth on  $[t_0, t_1]$ , without loss of generality, we assume that  $d_{\min}(t) = d_{12}(t)$  for  $t_0 \leq t < t_1$  (otherwise by re-ordering). Taking  $j = 1$  and  $l = 2$  in (2.26), we get for  $t_0 \leq t < t_1$

$$\begin{aligned}
 \dot{D}_{\min}(t) &= \dot{D}_{12}(t) \\
 &= 4 \left[ 2 + (\mathbf{x}_1(t) - \mathbf{x}_2(t)) \cdot \sum_{k=1, k \neq 1, 2}^N \left( \frac{\mathbf{x}_1(t) - \mathbf{x}_k(t)}{d_{1k}^2(t)} - \frac{\mathbf{x}_2(t) - \mathbf{x}_k(t)}{d_{2k}^2(t)} \right) \right].
 \end{aligned}$$

When  $N = 2$  or  $3$ , noting  $0 < d_{12}(t) \leq d_{jl}(t)$  for  $1 \leq j \neq l \leq N$ , we get

$$\begin{aligned}
 \dot{D}_{\min}(t) &> 4 \left[ 2 - \sum_{k=1, k \neq 1, 2}^N \left( \frac{d_{12}(t)}{d_{1k}(t)} + \frac{d_{12}(t)}{d_{2k}(t)} \right) \right] \\
 &\geq 4(2 - 2(N - 2)) \geq 0, \quad t_0 \leq t < t_1,
 \end{aligned}$$

which implies that  $D_{\min}(t)$  and  $d_{\min}(t)$  are monotonically increasing functions over  $t \in [t_0, t_1]$ . When  $N = 4$ , without loss of generality, we can assume

$$d_{12}(t) \leq d_{13}(t) \leq d_{23}(t), \quad d_{12}(t) \leq d_{14}(t) \leq d_{24}(t), \quad t_0 \leq t < t_1,$$

then we get

$$\dot{D}_{\min}(t) > 4 \left[ 2 - \left( \frac{d_{12}(t)}{d_{23}(t)} + \frac{d_{12}(t)}{d_{24}(t)} \right) \right] \geq 0, \quad t_0 \leq t < t_1,$$

which implies that  $D_{\min}(t)$  and  $d_{\min}(t)$  are monotonically increasing functions over  $t \in [t_0, t_1]$ .  $\square$

**Remark 2.1.** When  $N \geq 5$  and the initial data  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times N}$  is not collinear,  $d_{\min}(t)$  might not be a monotonically increasing function, especially when  $0 \leq t \ll 1$ . Based on our extensive numerical results, for any given  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times N}$ , there exists a constant  $T_0 > 0$  depending on  $\mathbf{X}^0$  such that  $d_{\min}(t)$  is a monotonically increasing function when  $t \geq T_0$ . Rigorous mathematical justification is ongoing.

**2.4. Finite time collision in the case with opposite winding numbers.**  
 When the  $N$  vortices have opposite winding numbers, we have

**Theorem 2.4.** *Suppose the  $N$  vortices have opposite winding numbers, i.e.  $|N^+ - N^-| < N$ , we have*

- (i) *If  $M_0 < 0$ , finite time collision happens, i.e.  $0 < T_{\max} < +\infty$ , and there exists a collision cluster among the  $N$  vortices. In addition,  $T_{\max} \leq T_a := -\frac{H_1^0}{4NM_0}$ .*
- (ii) *If  $M_0 = 0$ , then the solution of (1.1) is bounded, i.e.*

$$|\mathbf{x}_j(t)| \leq \sqrt{H_2^0} = \sqrt{\sum_{j=1}^N |\mathbf{x}_j^0|^2}, \quad t \geq 0, \quad 1 \leq j \leq N. \tag{2.29}$$

- (iii) *If  $M_0 > 0$  and there is no finite time collision, i.e.  $T_{\max} = +\infty$ , then at least two vortices move to infinity as  $t \rightarrow +\infty$ .*

(iv) *Let  $I \subseteq \{1, 2, \dots, N\}$  be a set with  $M$  ( $2 \leq M \leq N$ ) elements. If the collective winding number of  $I$  defined as  $M_1 := \frac{1}{2} \sum_{j, l \in I, j \neq l} m_j m_l \geq 0$ , then the set of vortices  $\{\mathbf{x}_j(t) \mid j \in I\}$  cannot be a collision cluster among the  $N$  vortices for  $0 \leq t \leq T_{\max}$ .*

*Proof.* (i) Combining (2.8) and (2.11), we get

$$0 \leq \sum_{1 \leq j < l \leq N} |\mathbf{x}_j(t) - \mathbf{x}_l(t)|^2 = 4NM_0t + H_1^0, \quad t \geq 0. \tag{2.30}$$

If  $M_0 < 0$ , when  $t \rightarrow T_a = -\frac{H_1^0}{4NM_0}$ , we have  $4NM_0t + H_1^0 \rightarrow 0$ . Thus finite time collision happens at  $t = T_{\max} \leq T_a < +\infty$ .

(ii) If  $M_0 = 0$ , combining (2.9) and (2.12), we get

$$0 \leq |\mathbf{x}_j(t)|^2 \leq \sum_{j=1}^N |\mathbf{x}_j(t)|^2 \equiv H_2^0 = \sum_{j=1}^N |\mathbf{x}_j^0|^2, \quad t \geq 0,$$

which immediately implies (2.29).

(iii) If  $M_0 > 0$  and  $T_{\max} = +\infty$ , then there exists no finite time collision cluster among the  $N$  vortices. The proof can be proceeded similarly as the last part in Theorem 2.1 and details are omitted here for brevity.

(iv) When  $M = N$ , for any given  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times N}$ , we get  $H_1^0 > 0$ . Noting (2.30) and  $M_0 = M_1 \geq 0$ , we have

$$\sum_{1 \leq j < l \leq N} |\mathbf{x}_j(t) - \mathbf{x}_l(t)|^2 = 4NM_0t + H_1^0 \geq H_1^0 > 0, \quad 0 \leq t \leq T_{\max},$$

which immediately implies that the  $N$  vortices cannot be a collision cluster when  $t \in [0, T_{\max}]$  for any given  $\mathbf{X}^0$ .

When  $2 \leq M < N$  and  $N \geq 3$ , without loss of generality, we assume  $I = \{1, \dots, M\}$  and denote  $J = \{M + 1, \dots, N\}$ . Thus  $M_1 = \sum_{1 \leq j < l \leq M} m_j m_l \geq 0$ . We will proceed the proof by the method of contradiction. Assume that the  $M$  vortices  $\mathbf{x}_1, \dots, \mathbf{x}_M$  collide at  $\mathbf{x}^0 \in \mathbb{R}^2$  when  $t \rightarrow T_c$  satisfying  $0 < T_c \leq T_{\max}$ , i.e.  $\mathbf{x}_j(t) \rightarrow \mathbf{x}^0$  when  $t \rightarrow T_c^-$  for  $1 \leq j \leq M$  and  $|\mathbf{x}_j(t) - \mathbf{x}^0| > 0$  when  $t \rightarrow T_c^-$  for  $M + 1 \leq j \leq N$ . Denote  $d_2 := \min_{j \in J} \min_{0 \leq t \leq T_c} |\mathbf{x}_j(t) - \mathbf{x}^0|^2$  and we have  $d_2 > 0$ . Since

$\lim_{t \rightarrow T_c} D_I(t) = 0$ , there exists  $0 < T_1 < T_c$ , such that  $D_I(t) < \frac{d_2}{2}$  and  $d_{I,J}(t) > \frac{d_2}{2}$  for

$t \in [T_1, T_c)$ . Choose  $T_2 \in [T_1, T_c)$ , such that

$$0 < T_c - T_2 < \frac{d_2}{8M(M-1)(N-M)}.$$

Since  $D_I(t)$  is a continuous function, there exists  $T_3 \in [T_2, T_c]$ , such that  $D_I(T_3) = \max_{t \in [T_2, T_c]} D_I(t) > 0$ . Similar to (2.24), we have

$$\begin{aligned} D_I(T_3) &= D_I(T_3) - D_I(T_c) = - \int_{T_3}^{T_c} \frac{d}{dt} D_I(t) dt \\ &= -4 \int_{T_3}^{T_c} \sum_{1 \leq j < l \leq M} (\mathbf{x}_j(t) - \mathbf{x}_l(t)) \cdot \\ &\quad \sum_{k=M+1}^N m_k \left[ m_j \frac{\mathbf{x}_j(t) - \mathbf{x}_k(t)}{d_{jk}^2(t)} - m_l \frac{\mathbf{x}_l(t) - \mathbf{x}_k(t)}{d_{lk}^2(t)} \right] dt - 4MM_1(T_c - T_3) \\ &\leq 4 \int_{T_3}^{T_c} \sum_{1 \leq j < l \leq M} \sum_{k=M+1}^N \left( \frac{d_{jl}(t)}{d_{jk}(t)} + \frac{d_{jl}(t)}{d_{lk}(t)} \right) dt \\ &\leq 16 \int_{T_3}^{T_c} \sum_{1 \leq j < l \leq M} \sum_{k=M+1}^N \frac{D_I(T_3)}{d_2} dt \\ &= \frac{8M(M-1)(N-M)(T_c - T_3)}{d_2} D_I(T_3) \\ &\leq \frac{8M(M-1)(N-M)(T_c - T_2)}{d_2} D_I(T_3) \\ &< D_I(T_3). \end{aligned}$$

This is a contradiction and thus the set of vortices  $\{\mathbf{x}_j(t) \mid j \in I\}$  cannot be a collision cluster among the  $N$  vortices for  $0 \leq t \leq T_{\max}$ .  $\square$

**Proposition 2.1.** *If the  $N$  vortices be a collision cluster at  $0 < T_{\max} < +\infty$  under a given initial data  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times N}$ , then we have*

$$M_0 < 0, \quad H_1^0 = NH_2^0, \quad H_3^0 = (N-2)H_2^0. \tag{2.31}$$

*Proof.* Due to the conservation of mass center and  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times N}$ , we get

$$\bar{\mathbf{x}}(t) \equiv \bar{\mathbf{x}}^0 \implies \lim_{t \rightarrow T_{\max}^-} \mathbf{x}_j(t) = \mathbf{x}_j(T_{\max}) = \bar{\mathbf{x}}^0, \quad 1 \leq j \leq N. \tag{2.32}$$

Plugging (2.32) into (2.8) and (2.11), we get

$$\begin{aligned} H_1(\mathbf{X}(T_{\max}), T_{\max}) &= \sum_{1 \leq j < l \leq N} |\mathbf{x}_j(T_{\max}) - \mathbf{x}_l(T_{\max})|^2 - 4NM_0T_{\max} \\ &= -4NM_0T_{\max} = H_1^0. \end{aligned} \tag{2.33}$$

Similarly, we have

$$\begin{aligned} H_2(\mathbf{X}(T_{\max}), T_{\max}) &= -4M_0T_{\max} = H_2^0, \\ H_3(\mathbf{X}(T_{\max}), T_{\max}) &= -4(N-2)M_0T_{\max} = H_3^0. \end{aligned} \tag{2.34}$$

Combining (2.33) and (2.34), we obtain (2.31).  $\square$

**Proposition 2.2.** *If the ODEs (1.1) admits an equilibrium solution, then  $N \geq 4$  is a square of an integer, i.e.  $N = (N^+ - N^-)^2$  and*

$$1 \leq N^+ = \frac{1}{2} (N \pm \sqrt{N}) < N, \quad 1 \leq N^- = N - N^+ < N. \tag{2.35}$$

*Proof.* Assume  $\mathbf{X}(t) \equiv \mathbf{X}^0 \in \mathbb{R}_*^{2 \times N}$  be an equilibrium solution of (1.1), noting (2.8) and (2.11), we get

$$H_1(\mathbf{X}(t), t) = H_1^0 - 4NM_0t \equiv H_1^0, \quad t \geq 0. \tag{2.36}$$

Thus  $M_0 = 0$ . Noting (2.7), we have

$$4 \leq N = (N^+ - N^-)^2 = (2N^+ - N)^2 = (2N^- - N)^2. \tag{2.37}$$

Thus  $N \geq 4$  is a square of an integer and we obtain (2.35) by solving (2.37).  $\square$

**Remark 2.2.** When  $N = 4$ , an equilibrium solution of (1.1) was constructed in [30, 31] by taking  $m_4 = -1$ ,  $\mathbf{x}_4^0 = (0, 0)^T$  and  $m_1 = m_2 = m_3 = 1$ ,  $\mathbf{x}_j^0$  located in the vertices of a right triangle centered at the origin. Here we want to remark that any equilibrium solution of (1.1) is dynamically unstable.

**3. Interaction patterns of a cluster with 3 quantized vortices.** In this section, we assume  $N = 3$  in (1.1) and (1.2).

**3.1. Structural/orbital stability in the case with the same winding number.** Assume that  $m_1 = m_2 = m_3$  and by Theorem 2.1, we know the ODEs (1.1) with (1.2) is globally well-posed, i.e.  $T_{\max} = +\infty$ .

**Lemma 3.1.** *If the initial data  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times 3}$  in (1.2) with  $N = 3$  is collinear, then one vortex moves to the mass center  $\bar{\mathbf{x}}^0$  and the other two vortices repel with each other and move outwards to far field along the line when  $t \rightarrow +\infty$ .*

*Proof.* Since  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times 3}$  is collinear, there exist  $\mathbf{x}^0 \in \mathbb{R}^2$  and a unit vector  $\mathbf{e} \in \mathbb{R}^2$  such that

$$\mathbf{x}_j^0 = \mathbf{x}^0 + a_j^0 \mathbf{e}, \quad j = 1, 2, 3.$$

Without loss of generality, we assume that

$$a_1^0 < a_2^0 < a_3^0, \quad a_1^0 < 0, \quad a_3^0 > 0, \quad a_1^0 + a_2^0 + a_3^0 = 0.$$

Based on the results in Lemma 2.2 and Theorem 2.1, we know that there exist  $a_j(t)$  ( $j = 1, 2, 3$ ) such that

$$\mathbf{x}_j(t) = \mathbf{x}^0 + a_j(t) \mathbf{e}, \quad j = 1, 2, 3, \tag{3.1}$$

satisfying

$$a_1(t) < a_2(t) < a_3(t), \quad a_1(t) + a_2(t) + a_3(t) \equiv 0, \quad t \geq 0. \tag{3.2}$$

Plugging (3.1) into (1.1) with  $N = 3$  and  $m_1 = m_2 = m_3$ , noting (3.2), we get

$$\begin{aligned} \dot{a}_1(t) &= -\frac{2}{a_2(t) - a_1(t)} - \frac{2}{a_3(t) - a_1(t)} = \frac{6a_1(t)}{[a_2(t) - a_1(t)][a_3(t) - a_1(t)]} < 0, \\ \dot{a}_2(t) &= \frac{2}{a_2(t) - a_1(t)} - \frac{2}{a_3(t) - a_2(t)} = \frac{-6a_2(t)}{[a_2(t) - a_1(t)][a_3(t) - a_2(t)]}, \quad t > 0, \\ \dot{a}_3(t) &= \frac{2}{a_3(t) - a_1(t)} + \frac{2}{a_3(t) - a_2(t)} = \frac{6a_3(t)}{[a_3(t) - a_1(t)][a_3(t) - a_2(t)]} > 0, \end{aligned}$$

with the initial data

$$a_j(0) = a_j^0, \quad j = 1, 2, 3. \tag{3.3}$$

Thus  $a_1(t)$  is a monotonically decreasing function and  $a_3(t)$  is a monotonically increasing function for  $t \geq 0$ . Let  $\rho_2(t) = a_2^2(t) \geq 0$ , then we have

$$\dot{\rho}_2(t) = \frac{-12\rho_2(t)}{[a_2(t) - a_1(t)][a_3(t) - a_2(t)]} < 0, \quad t > 0,$$

which immediately implies that  $\rho_2(t)$  is a monotonically decreasing function and  $\lim_{t \rightarrow +\infty} \rho_2(t) = 0$ . Thus we have

$$\lim_{t \rightarrow +\infty} a_2(t) = 0 \implies \lim_{t \rightarrow +\infty} \mathbf{x}_2(t) = \bar{\mathbf{x}}^0 = \frac{1}{3} \sum_{j=1}^3 \mathbf{x}_j^0.$$

Thus the vortex  $\mathbf{x}_2(t)$  moves towards  $\bar{\mathbf{x}}^0$  along the line  $S_e(\bar{\mathbf{x}}^0)$ . Based on the results in Theorem 2.1, we know that at least two vortices must move to infinity when  $t \rightarrow +\infty$ . Thus we have

$$a_1(t) \rightarrow -\infty, \quad a_3(t) \rightarrow +\infty \quad \text{when} \quad t \rightarrow +\infty.$$

Thus the other two vortices  $\mathbf{x}_1(t)$  and  $\mathbf{x}_3(t)$  repel with each other and move outwards to far field along the line  $S_e(\mathbf{x}^0)$  when  $t \rightarrow +\infty$ .  $\square$

**Theorem 3.1.** *Assume the initial data  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times 3}$  in (1.2) with  $N = 3$  is not collinear, then there exists a unit vector  $\mathbf{e} \in \mathbb{R}^2$  such that*

$$\lim_{t \rightarrow +\infty} d_S(t) := \inf_{\mathbf{X} \in S_e^3(\bar{\mathbf{x}}^0)} \|\mathbf{X}(t) - \mathbf{X}\|_2 = 0. \tag{3.4}$$

*Proof.* Without loss of generality, as shown in Fig. 3.1a, we assume  $\bar{\mathbf{x}}^0 = \mathbf{0}$  and  $d_{12}^0 := d_{12}(0) \leq d_{13}^0 := d_{13}(0) \leq d_{23}^0 := d_{23}(0)$ . Thus  $0 < \theta_3^0 := \theta_3(0) \leq \theta_2^0 := \theta_2(0) \leq \theta_1^0 := \theta_1(0) < \pi$  satisfying  $\theta_1^0 + \theta_2^0 + \theta_3^0 = \pi$  (cf. Fig. 3.1a). From (1.1) with  $N = 3$ , we get

$$\begin{aligned} \dot{d}_{12}(t) &= \frac{4}{d_{12}(t)} + \frac{2 \cos(\theta_1(t))}{d_{13}(t)} + \frac{2 \cos(\theta_2(t))}{d_{23}(t)}, \\ \dot{d}_{13}(t) &= \frac{4}{d_{13}(t)} + \frac{2 \cos(\theta_1(t))}{d_{12}(t)} + \frac{2 \cos(\theta_3(t))}{d_{23}(t)}, \quad t > 0, \\ \dot{d}_{23}(t) &= \frac{4}{d_{23}(t)} + \frac{2 \cos(\theta_2(t))}{d_{12}(t)} + \frac{2 \cos(\theta_3(t))}{d_{13}(t)}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \dot{\theta}_3(t) &= B(t) [d_{13}^2(t) + d_{23}^2(t) - 2d_{12}^2(t)], \\ \dot{\theta}_2(t) &= B(t) [d_{12}^2(t) + d_{23}^2(t) - 2d_{13}^2(t)], \quad t > 0, \\ \dot{\theta}_1(t) &= B(t) [d_{12}^2(t) + d_{13}^2(t) - 2d_{23}^2(t)], \end{aligned} \tag{3.6}$$

where  $B(t) := 4A(t)/(d_{12}(t)d_{13}(t)d_{23}(t))^2$  with  $A(t)$  denoting the area of the triangle with vertices  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$  and  $\mathbf{x}_3(t)$ . Denote

$$\rho_{12}(t) = d_{12}^2(t), \quad \rho_{13}(t) = d_{13}^2(t), \quad \rho_{23}(t) = d_{23}^2(t), \quad t \geq 0.$$

From (3.5)-(3.6) and noting the initial data, we get  $\frac{\pi}{3} \leq \theta_1(t) < \pi$  and  $0 < \theta_3(t) \leq \frac{\pi}{3}$  are monotonically decreasing and increasing functions, respectively, and

$$\begin{aligned} \rho_{12}(t) \leq \rho_{23}(t), \quad 0 < \theta_3(t) \leq \frac{\pi}{3} \leq \theta_1(t) < \pi, \quad t \geq 0; \\ \lim_{t \rightarrow +\infty} \theta_3(t) = \lim_{t \rightarrow +\infty} \theta_1(t) = \frac{\pi}{3}. \end{aligned} \tag{3.7}$$

Combining this with  $\theta_1(t) + \theta_2(t) + \theta_3(t) \equiv \pi$  for  $t \geq 0$ , we get  $\lim_{t \rightarrow +\infty} \theta_2(t) = \frac{\pi}{3}$ , which immediately implies (3.4).  $\square$

For  $\theta \in \mathbb{R}$  and  $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in S_e^N(\mathbf{0}, \theta_0)$ , define  $Q(\theta)\mathbf{X} := (Q(\theta)\mathbf{x}_1, \dots, Q(\theta)\mathbf{x}_N)$ .



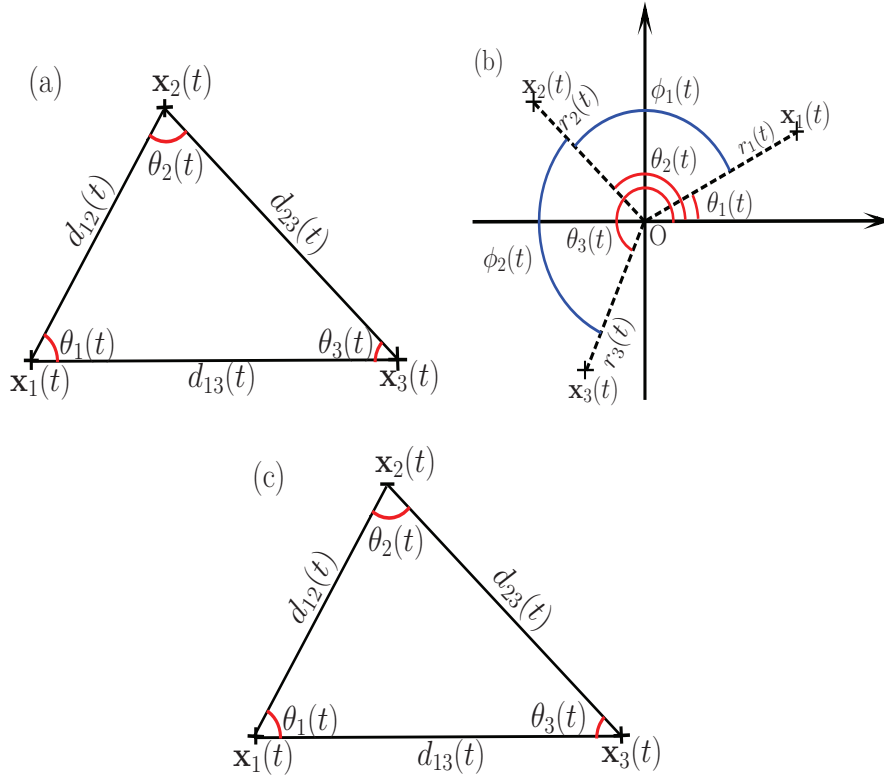


FIGURE 3.1. Interaction of 3 vortices with the same winding number (a and b) and opposite winding numbers (c).

**Definition 3.1.** For the self-similar solution  $\tilde{\mathbf{X}}(t) = \sqrt{r_0^2 + 2(N-1)t} \tilde{\mathbf{X}}^0$  with  $\tilde{\mathbf{X}}^0 = (\tilde{\mathbf{x}}_1^0, \dots, \tilde{\mathbf{x}}_N^0) \in S_e^N(\mathbf{0}, \theta_0)$  and  $r_0 = |\tilde{\mathbf{x}}_1^0|$  of the ODEs (1.1) with  $m_1 = \dots = m_N$ , if for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, when the initial data  $\mathbf{X}^0$  in (1.2) satisfies  $\|\mathbf{X}^0 - \tilde{\mathbf{X}}^0\|_2 < \delta$ , the solution  $\mathbf{X}(t)$  of the ODEs (1.1) with (1.2) satisfies

$$\sup_{t \geq 0} \inf_{r > 0, \theta \in [0, 2\pi)} \left\| \mathbf{X}(t) - \bar{\mathbf{x}}^0 - rQ(\theta)\tilde{\mathbf{X}}(t) \right\| < \varepsilon,$$

then the self-similar solution  $\tilde{\mathbf{X}}(t)$  is called as orbitally stable.

**Theorem 3.2.** For any  $\theta_0 \in \mathbb{R}$  and  $\tilde{\mathbf{X}}^0 = (\tilde{\mathbf{x}}_1^0, \tilde{\mathbf{x}}_2^0, \tilde{\mathbf{x}}_3^0) \in S_e^3(\mathbf{0}, \theta_0)$ , the solution  $\tilde{\mathbf{X}}(t) = \sqrt{4t + r_0^2} \tilde{\mathbf{X}}^0$  with  $r_0 = |\tilde{\mathbf{x}}_1^0|$  of the ODEs (1.1) with  $N = 3$  and  $m_1 = m_2 = m_3$  is orbitally stable.

*Proof.* By using Lemma 2.1, without loss of generality, we can assume that  $\theta_0 = 0$ ,  $r_0 = 1$  and  $\bar{\mathbf{x}}^0 = \mathbf{0}$ . In addition, as shown in Fig. 3.1b, we assume

$$\mathbf{x}_j(t) = r_j(t)(\cos(\theta_j(t)), \sin(\theta_j(t)))^T, \quad j = 1, 2, 3, \tag{3.8}$$

satisfying  $\theta_1^0 := \theta_1(0) < \theta_2^0 := \theta_2(0) < \theta_3^0 := \theta_3(0) < 2\pi$  and  $\|\mathbf{X}^0 - \tilde{\mathbf{X}}^0\|_2 \leq \delta$  with  $0 < \delta \leq \frac{1}{12}$  sufficiently small and to be determined later. In fact, from

$$\|\mathbf{X}^0 - \tilde{\mathbf{X}}^0\|_2^2 = \sum_{j=1}^3 (r_j(0) - 1)^2 + \sum_{j=1}^3 4r_j(0) \sin^2(\varphi_j^0), \tag{3.9}$$

with  $\varphi_j^0 = \frac{\theta_j^0}{2} - \frac{(j-1)\pi}{3}$  for  $j = 1, 2, 3$ , we can get

$$\begin{aligned} |r_1(0) - 1| + |r_2(0) - 1| + |r_3(0) - 1| &\leq 3\delta < \frac{1}{2}, \\ |\theta_1^0| + \left| \theta_2^0 - \frac{2\pi}{3} \right| + \left| \theta_3^0 - \frac{4\pi}{3} \right| &\leq 6\delta < \frac{\pi}{4}. \end{aligned} \tag{3.10}$$

Plugging (3.8) into (1.6), we get

$$\begin{aligned} r_1(t) \cos(\theta_1(t)) + r_2(t) \cos(\theta_2(t)) + r_3(t) \cos(\theta_3(t)) &\equiv 0, \\ r_1(t) \sin(\theta_1(t)) + r_2(t) \sin(\theta_2(t)) + r_3(t) \sin(\theta_3(t)) &\equiv 0, \quad t \geq 0. \end{aligned}$$

Solving the above equations, we obtain

$$r_2(t) = -r_1(t) \frac{\sin(\theta_3(t) - \theta_1(t))}{\sin(\theta_3(t) - \theta_2(t))} = -r_1(t) \frac{\sin(\phi_1(t) + \phi_2(t))}{\sin(\phi_2(t))}, \tag{3.11}$$

$$r_3(t) = r_1(t) \frac{\sin(\theta_2(t) - \theta_1(t))}{\sin(\theta_3(t) - \theta_2(t))} = r_1(t) \frac{\sin(\phi_1(t))}{\sin(\phi_2(t))}, \quad t \geq 0, \tag{3.12}$$

where (cf. Fig. 3.1b)

$$\phi_1(t) = \theta_2(t) - \theta_1(t), \quad \phi_2(t) = \theta_3(t) - \theta_2(t), \quad t \geq 0. \tag{3.13}$$

By Lemma 2.4, we have

$$r_1^2(t) + r_2^2(t) + r_3^2(t) = 12t + H_3^0, \quad t \geq 0, \tag{3.14}$$

with  $H_3^0 = r_1^2(0) + r_2^2(0) + r_3^2(0)$ . Substituting (3.11) and (3.12) into (3.14), we get

$$r_1(t) = \frac{(12t + H_3^0)^{1/2} \sin(\phi_2(t))}{D^{1/2}(t)}, \tag{3.15}$$

$$D(t) := \sin^2(\phi_1(t)) + \sin^2(\phi_2(t)) + \sin^2(\phi_1(t) + \phi_2(t)).$$

Plugging (3.8) into (1.1) with  $N = 3$ , noting (3.11)-(3.13) and (3.15), we have

$$\dot{\Phi}(t) = \frac{2}{12t + H_3^0} F(\phi_1(t), \phi_2(t)) = \frac{2}{12t + H_3^0} F(\Phi(t)), \quad t > 0, \tag{3.16}$$

where  $\Phi(t) := (\phi_1(t), \phi_2(t))^T$  and  $F(\Phi) = (f_1(\Phi), f_2(\Phi))^T$  is defined as

$$\begin{aligned} f_1(\Phi) &= \frac{\sin(\phi_1)}{\sin(\phi_2) \sin(\phi_1 + \phi_2)} \left[ \frac{\sin^2(\phi_1 + \phi_2)}{D_{13}(\Phi)} + \frac{\sin^2(\phi_2)}{D_{23}(\Phi)} \right. \\ &\quad \left. - \frac{\sin^2(\phi_2) + \sin^2(\phi_1 + \phi_2)}{D_{12}(\Phi)} \right], \\ f_2(\Phi) &= \frac{\sin(\phi_2)}{\sin(\phi_1) \sin(\phi_1 + \phi_2)} \left[ \frac{\sin^2(\phi_1 + \phi_2)}{D_{13}(\Phi)} + \frac{\sin^2(\phi_1)}{D_{12}(\Phi)} \right. \\ &\quad \left. - \frac{\sin^2(\phi_1) + \sin^2(\phi_1 + \phi_2)}{D_{23}(\Phi)} \right], \end{aligned}$$

with

$$\begin{aligned} D_{12}(\Phi) &= \frac{1}{D(\Phi)} (\sin^2(\phi_2) + \sin^2(\phi_1 + \phi_2) + 2 \sin(\phi_2) \sin(\phi_1 + \phi_2) \cos(\phi_1)), \\ D_{13}(\Phi) &= \frac{1}{D(\Phi)} (\sin^2(\phi_1) + \sin^2(\phi_2) - 2 \sin(\phi_1) \sin(\phi_2) \cos(\phi_1 + \phi_2)), \\ D_{23}(\Phi) &= \frac{1}{D(\Phi)} (\sin^2(\phi_1) + \sin^2(\phi_1 + \phi_2) + 2 \sin(\phi_1) \sin(\phi_1 + \phi_2) \cos(\phi_2)), \\ P(\Phi) &= \frac{1}{D(\Phi)} \left[ \sin(\phi_2) - \sin(\phi_1 + \phi_2) \cos\left(\phi_1 - \frac{2\pi}{3}\right) \right. \\ &\quad \left. + \sin(\phi_1) \cos\left(\phi_1 + \phi_2 - \frac{4\pi}{3}\right) \right]^2; \end{aligned}$$

and

$$\dot{\theta}_1(t) = \frac{2}{12t + H_3^0} g(\Phi(t)), \quad t > 0, \tag{3.17}$$

with

$$g(\Phi) = g(\phi_1, \phi_2) = \frac{\sin(\phi_1) \sin(\phi_1 + \phi_2) (D_{13}(\Phi) - D_{12}(\Phi))}{\sin(\phi_2) D_{12}(\Phi) D_{13}(\Phi)}.$$

Let

$$s = \frac{1}{4} \ln \left( \frac{12t + H_3^0}{H_3^0} \right), \quad \Psi(s) = \Phi(t) - (2\pi/3, 2\pi/3)^T, \quad s \geq 0, \tag{3.18}$$

then (3.16) can be re-written as

$$\dot{\Psi}(s) = F(\Psi(s) + (2\pi/3, 2\pi/3)^T) = -2\Psi(s) + G(\Psi(s)), \quad s > 0, \tag{3.19}$$

where

$$G(\Psi) = F(\Psi + (2\pi/3, 2\pi/3)^T) + 2\Psi.$$

It is easy to verify that  $\Psi(s) \equiv \mathbf{0}$  is an equilibrium solution of (3.19). By the variation-of-constant formula, we have

$$\Psi(s) = e^{-2s} \Psi(0) + \int_0^s e^{-2(s-\tau)} G(\Psi(\tau)) d\tau, \quad s \geq 0. \tag{3.20}$$

By using the Taylor expansion, there exist constants  $K_j > 0$  ( $j = 1, 2, 3$ ) and  $0 < \delta_1 < 1$  such that

$$\begin{aligned} \|G(\Psi)\|_2 &\leq \|\Psi\|_2, \quad \|G(\Psi)\|_2 \leq K_1 \|\Psi\|_2^2, \\ |g(\Phi)| &= |g(\Psi + (2\pi/3, 2\pi/3)^T)| \leq K_2 \|\Psi\|_2, \\ |3 - P(\Phi)| &= |3 - P(\Psi + (2\pi/3, 2\pi/3)^T)| \leq K_3 \|\Psi\|_2^2, \quad \text{when } \|\Psi\|_2 < \delta_1. \end{aligned} \tag{3.21}$$

For any  $0 < \delta_2 \leq \delta_1$ , when  $\|\Psi(0)\|_2 \leq \frac{\delta_2}{2}$  ( $\Leftrightarrow \|\Phi(0) - (2\pi/3, 2\pi/3)^T\|_2 \leq \frac{\delta_2}{2}$ ) and let  $S > 0$  such that  $\|\Psi(s)\|_2 \leq \delta_2$  for  $0 \leq s \leq S$ , noting (3.20) and (3.21) and using the triangle inequality, we have

$$\|\Psi(s)\|_2 \leq e^{-2s} \|\Psi(0)\|_2 + \int_0^s e^{-2(s-\tau)} \|\Psi(\tau)\|_2 d\tau, \quad 0 \leq s \leq S,$$

which is equivalent to

$$e^{2s} \|\Psi(s)\|_2 = \|\Psi^0\|_2 + \int_0^s e^{2\tau} \|\Psi(\tau)\|_2 d\tau, \quad 0 \leq s \leq S.$$

Using the Gronwall's inequality, we get

$$\|\Psi(s)\|_2 \leq \|\Psi(0)\|_2 e^{-s} \leq \|\Psi(0)\|_2 \leq \frac{\delta_2}{2}, \quad 0 \leq s \leq S. \quad (3.22)$$

From (3.22) and using the standard extension theorem for ODEs, we can obtain

$$\|\Psi(s)\|_2 \leq \|\Psi(0)\|_2 e^{-s}, \quad 0 \leq s < +\infty. \quad (3.23)$$

Combining (3.23) and (3.20), using the triangle inequality, we obtain

$$\begin{aligned} \|\Psi(s)\|_2 &\leq e^{-2s}\|\Psi(0)\|_2 + e^{-2s} \int_0^s e^{2\tau} \|G(\Psi(\tau))\|_2 d\tau \\ &\leq e^{-2s}\|\Psi(0)\|_2 + e^{-2s} \int_0^s e^{2\tau} K_1 \|\Psi(\tau)\|_2^2 d\tau \\ &\leq e^{-2s}\|\Psi(0)\|_2 + e^{-2s} \int_0^s e^{2\tau} K_1 \|\Psi(0)\|_2^2 e^{-2\tau} d\tau \\ &\leq [\|\Psi(0)\|_2 + K_1 \|\Psi(0)\|_2^2] e^{-2s} \\ &\leq (1 + K_1)\delta_2 e^{-2s}, \quad 0 \leq s < +\infty, \end{aligned}$$

which immediately implies

$$\|\Phi(t) - (2\pi/3, 2\pi/3)^T\|_2 < (1 + K_1)\delta_2 \sqrt{\frac{H_3^0}{12t + H_3^0}}, \quad 0 \leq t < +\infty.$$

Noting (3.18) and (3.21), we have

$$\begin{aligned} |g(\Phi)| &= |g(\Psi + (\pi/3, \pi/3)^T)| \leq K_2 \|\Psi\|_2 \\ &\leq K_2(1 + K_1)\delta_2 \sqrt{\frac{H_3^0}{12t + H_3^0}}, \quad 0 \leq t < +\infty. \end{aligned}$$

This implies that the ODE (3.17) is globally solvable, and the solution can be written as

$$\theta_1(t) = \theta_1(0) + \int_0^t \frac{3}{12s + H_3^0} g(\phi_1(s), \phi_2(s)) ds, \quad 0 \leq t < +\infty.$$

Denote  $\theta_1^\infty = \lim_{t \rightarrow \infty} \theta_1(t)$  and  $\theta(t) = \theta_1(t) - \theta_1^\infty$ , then we have

$$\begin{aligned} \inf_{r>0} \|\mathbf{X}(t) - rQ(\theta(t))\tilde{\mathbf{X}}(t)\|_2 &= \inf_{r>0} \{12t + H_3^0 + 3r^2 - 2r d(t)\} \\ &= 12t + H_3^0 - \frac{1}{3}d^2(t) \\ &= \frac{12t + H_3^0}{3} (3 - P(\Phi(t))), \quad t \geq 0, \end{aligned} \quad (3.24)$$

where

$$d(t) := r_1(t) + r_2(t) \cos(\phi_1(t) - 2\pi/3) + r_3(t) \cos(\phi_1(t) + \phi_2(t) - 4\pi/3).$$

Noting (3.21), we have

$$\begin{aligned} |3 - P(\Phi(t))| &= |3 - P(\Psi(t) + (2\pi/3, 2\pi/3)^T)| \leq K_3 \|\Psi(t)\|_2^2 \\ &\leq \frac{K_3(1 + K_1)^2 \delta_2 H_3^0}{12t + H_3^0}, \quad t \geq 0. \end{aligned} \quad (3.25)$$

For any  $\varepsilon > 0$ , taking  $\delta_3 = \frac{\varepsilon}{2K_3(1+K_1)^2H_3^0}$  and  $0 < \delta = \min\{\frac{1}{12}, \frac{\delta_1}{12}, \delta_3\}$ , when  $\|\mathbf{X}^0 - \tilde{\mathbf{X}}^0\|_2 < \delta$ , noting (3.25) and (3.24), we get

$$\sup_{t \geq 0} \inf_{r > 0} \|\mathbf{X}(t) - rQ(\theta(t))\tilde{\mathbf{X}}(t)\|_2 \leq K_3(1 + K_1)^2 H_3^0 \delta < \varepsilon, \tag{3.26}$$

which completes the proof by taking  $\delta_2 = \delta$  in the above proof. □

**3.2. Collision patterns in the case with opposite winding numbers.** Without loss of generality, we assume  $m_1 = m_3 = +1$  and  $m_2 = -1$  in (1.1) with  $N = 3$ . Then we have  $M_0 = \frac{1}{2}[(N^+ - N^-)^2 - N] = \frac{1}{2}(1^2 - 3) = -1 < 0$ , thus finite time collision must happen.

**Theorem 3.3.** *For any given initial data  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times 3}$  in (1.2) with  $N = 3$ , we have*

(i) *If  $|\mathbf{x}_1^0 - \mathbf{x}_2^0| = |\mathbf{x}_2^0 - \mathbf{x}_3^0|$ , then the three vortices be a collision cluster and they will collide at  $\bar{\mathbf{x}}^0$  when  $t \rightarrow T_{\max}^- = \frac{H_1^0}{12}$  with  $H_1^0 = \sum_{1 \leq j < l \leq 3} |\mathbf{x}_j^0 - \mathbf{x}_l^0|^2$ .*

(ii) *If  $|\mathbf{x}_1^0 - \mathbf{x}_2^0| < |\mathbf{x}_2^0 - \mathbf{x}_3^0|$ , then only  $\mathbf{x}_1$  and  $\mathbf{x}_2$  form a collision cluster, and respectively, if  $|\mathbf{x}_1^0 - \mathbf{x}_2^0| > |\mathbf{x}_2^0 - \mathbf{x}_3^0|$ , then only  $\mathbf{x}_2$  and  $\mathbf{x}_3$  form a collision cluster. Moreover, the collision time  $0 < T_{\max} < \frac{H_1^0}{12}$ .*

*Proof.* (i) If the initial data  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times 3}$  in (1.2) with  $N = 3$  is collinear, i.e. there exists a unit vector  $\mathbf{e} \in \mathbb{R}^2$  such that

$$\mathbf{x}_j^0 = \mathbf{x}_2^0 + a_j^0 \mathbf{e}, \quad j = 1, 2, 3,$$

satisfying  $a_2^0 = 0$ ,  $a_1^0 < a_3^0$  and  $0 < |a_1^0| \leq |a_3^0|$  (without loss of generality, otherwise we need only switch  $\mathbf{x}_1$  and  $\mathbf{x}_3$ ). Based on the results in Lemma 2.2 and Theorem 2.1, we know that there exist  $a_j(t)$  ( $j = 1, 2, 3$ ) such that

$$\mathbf{x}_j(t) = \mathbf{x}_2^0 + a_j(t) \mathbf{e}, \quad j = 1, 2, 3,$$

satisfying  $a_1(t) + a_2(t) + a_3(t) \equiv a_1^0 + a_2^0 + a_3^0$  and  $a_1(t) < a_3(t)$  for  $0 \leq t < T_{\max}$  and

$$\begin{aligned} \dot{a}_1(t) &= -\frac{2}{a_1(t) - a_2(t)} + \frac{2}{a_1(t) - a_3(t)} = \frac{2[a_3(t) - a_2(t)]}{[a_2(t) - a_1(t)][a_3(t) - a_1(t)]}, \\ \dot{a}_2(t) &= -\frac{2}{a_2(t) - a_1(t)} - \frac{2}{a_2(t) - a_3(t)} = \frac{2[3a_2(t) - (a_1^0 + a_2^0 + a_3^0)]}{[a_2(t) - a_1(t)][a_3(t) - a_2(t)]}, \quad t > 0, \\ \dot{a}_3(t) &= \frac{2}{a_3(t) - a_1(t)} - \frac{2}{a_3(t) - a_2(t)} = \frac{2[a_1(t) - a_2(t)]}{[a_3(t) - a_1(t)][a_3(t) - a_2(t)]}, \end{aligned}$$

with the initial data (3.3).

If  $|a_1^0| = |a_3^0|$ , i.e.  $a_3^0 = -a_1^0 > 0$ , then the above ODEs with (3.3) admits the unique solution as

$$\begin{aligned} a_1(t) &= -\sqrt{(a_1^0)^2 - 2t}, \\ a_2(t) &\equiv 0, \quad 0 \leq t < T_{\max} := \frac{1}{2}(a_3^0)^2, \\ a_3(t) &= \sqrt{(a_3^0)^2 - 2t}, \end{aligned} \tag{3.27}$$

which immediately implies that the three vortices be a collision cluster and they collide at  $\bar{\mathbf{x}}^0 = \mathbf{x}_2^0$  when  $t \rightarrow T_{\max}^- = \frac{H_1^0}{12}$  with  $H_1^0 = \sum_{1 \leq j < l \leq 3} |\mathbf{x}_j^0 - \mathbf{x}_l^0|^2 = 6(a_3^0)^2$ .

If  $|a_1^0| < |a_3^0|$ , then  $a_3^0 > 0$ . If  $0 = a_2^0 < a_1^0 < a_3^0$ , then we can show that  $a_2(t) < a_1(t) < a_3(t)$  for  $0 \leq t < T_{\max}$  and  $a_1(t)$ ,  $a_2(t)$  and  $a_3(t)$  are monotonically

decreasing, increasing and increasing functions over  $t \in [0, T_{\max})$ , respectively. Thus only  $\mathbf{x}_1$  and  $\mathbf{x}_2$  form a collision cluster among the 3 vortices. On the other hand, if  $a_1^0 < 0 = a_2^0 < a_3^0$ , then we can show that  $a_1(t) < a_2(t) < a_3(t)$  for  $0 \leq t < T_{\max}$  and  $a_1(t)$  and  $a_2(t)$  are monotonically increasing and decreasing functions over  $t \in [0, T_{\max})$ , respectively. In addition we have  $a_1(T_{\max}) \leq a_2(T_{\max}) < 0$  and  $a_3(T_{\max}) = a_3^0 + a_2^0 + a_1^0 - a_1(T_{\max}) - a_2(T_{\max}) > 0$ , therefore, again only  $\mathbf{x}_1$  and  $\mathbf{x}_2$  form a collision cluster among the 3 vortices.

(ii) If the initial data  $\mathbf{X}^0 \in \mathbb{R}_*^{2 \times 3}$  in (1.2) with  $N = 3$  is not collinear, i.e. the initial locations of the 3 vortices form a triangle. Without loss of generality, as shown in Fig. 3.1c, we assume  $\bar{\mathbf{x}}^0 = \mathbf{0}$  and  $d_{12}^0 := d_{12}(0) \leq d_{23}^0 := d_{23}(0)$ . Thus  $0 < \theta_3^0 := \theta_3(0) \leq \theta_1^0 := \theta_1(0) < \pi$  satisfying  $\theta_1^0 + \theta_2^0 + \theta_3^0 = \pi$  (cf. Fig. 3.1b). From (1.1) with  $N = 3$ , we get

$$\begin{aligned} \dot{d}_{12}(t) &= -\frac{4}{d_{12}(t)} + \frac{2 \cos(\theta_1(t))}{d_{13}(t)} - \frac{2 \cos(\theta_2(t))}{d_{23}(t)}, \\ \dot{d}_{13}(t) &= \frac{4}{d_{13}(t)} - \frac{2 \cos(\theta_1(t))}{d_{12}(t)} - \frac{2 \cos(\theta_3(t))}{d_{23}(t)}, \quad t > 0, \\ \dot{d}_{23}(t) &= -\frac{4}{d_{23}(t)} - \frac{2 \cos(\theta_2(t))}{d_{12}(t)} + \frac{2 \cos(\theta_3(t))}{d_{13}(t)}, \end{aligned} \quad (3.28)$$

and

$$\begin{aligned} \dot{\theta}_3(t) &= -B(t) [d_{13}^2(t) + d_{23}^2(t)] < 0, \\ \dot{\theta}_2(t) &= B(t) [d_{12}^2(t) + d_{23}^2(t) + 2d_{13}^2(t)] > 0, \quad t > 0, \\ \dot{\theta}_1(t) &= -B(t) [d_{12}^2(t) + d_{13}^2(t)] < 0. \end{aligned} \quad (3.29)$$

If  $d_{12}^0 = d_{23}^0$ , then  $0 < \theta_3^0 = \theta_1^0 < \frac{\pi}{2}$  (cf. Fig. 3.1c), this together with (3.28)-(3.29) yields

$$\begin{aligned} d_{12}(t) &= d_{23}(t), \quad 0 < \theta_3(t) = \theta_1(t) < \frac{\pi}{2}, \quad 0 \leq t < T_{\max}; \\ \lim_{t \rightarrow T_{\max}^-} \theta_3(t) &= \lim_{t \rightarrow T_{\max}^-} \theta_1(t) = 0, \end{aligned} \quad (3.30)$$

which immediately implies that the three vortices are forming a collision cluster. By using Theorem 2.4, we get  $T_{\max} = H_1^0/12$ .

If  $0 < d_{12}^0 < d_{23}^0$ , then  $0 < \theta_3^0 < \theta_1^0 < \frac{\pi}{2}$  (cf. Fig. 3.1b). From (3.28) and (3.29), we have

$$\begin{aligned} \dot{d}_{23}(t) - \dot{d}_{12}(t) &= \frac{[4 - 2 \cos(\theta_2(t))](d_{23}(t) - d_{12}(t))}{d_{12}(t)d_{23}(t)} \\ &\quad + \frac{2[\cos(\theta_3(t)) - \cos(\theta_1(t))]}{d_{13}(t)} > 0, \quad t > 0, \\ \dot{\theta}_1(t) - \dot{\theta}_3(t) &= B(t) [d_{23}^2(t) - d_{12}^2(t)] > 0. \end{aligned} \quad (3.31)$$

Then we have

$$d_{23}(t) \geq d_{12}(t) + d_{23}^0 - d_{12}^0, \quad \theta_1(t) \geq \theta_3(t) + \theta_1^0 - \theta_3^0, \quad 0 \leq t \leq T_{\max}.$$

This, together with that  $\theta_1(t)$  and  $\theta_3(t)$  are monotonically decreasing functions,  $0 < \theta_2(t) = \pi - \theta_1(t) - \theta_3(t) < \pi$  is a monotonically increasing functions and finite time collision must happen (cf. Fig. 3.1c), we get that  $\lim_{t \rightarrow T_{\max}^-} d_{12}(t) = 0$  and  $\lim_{t \rightarrow T_{\max}^-} \theta_3(t) = 0$ . Thus only the two vortices  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$  form a collision

cluster among the 3 vortices. By using Theorem 2.4, we get the collision time  $0 < T_{\max} < \frac{H_1^0}{12}$ .  $\square$

**4. Analytical solutions under special initial setups.** Let  $0 \leq \theta_0 < 2\pi$  be a constant,  $n \geq 2$  be an integer,  $0 < a_1 < a_2$  be two constants,  $C_1 := \frac{1}{2}(a_1^2 + a_2^2)$ ,  $C_2 := \frac{1}{2}(a_2^2 - a_1^2)$ , and  $m_0 = +1$  or  $-1$ . Denote

$$\theta_n^j = \frac{2(j-1)\pi}{n} + \theta_0, \quad \alpha_n^j = \frac{2(j-1)\pi}{n} + \frac{\pi}{n} + \theta_0, \quad 1 \leq j \leq n.$$

**4.1. For the interaction of two clusters.** Here we take  $N = 2n$  with  $n \geq 2$ .

**Proposition 4.1.** *Taking  $m_j = m_0$  for  $1 \leq j \leq N = 2n$  and the initial data  $\mathbf{X}^0$  in (1.2) as*

$$\mathbf{x}_j^0 = a_1 (\cos(\theta_n^j), \sin(\theta_n^j))^T, \quad \mathbf{x}_{n+j}^0 = a_2 (\cos(\theta_n^j), \sin(\theta_n^j))^T, \quad 1 \leq j \leq n, \quad (4.1)$$

then the solution of the ODEs (1.1) with (4.1) can be given as

$$\begin{aligned} \mathbf{x}_j(t) &= \sqrt{\rho_1(t)} (\cos(\theta_n^j), \sin(\theta_n^j))^T, \\ \mathbf{x}_{n+j}(t) &= \sqrt{\rho_2(t)} (\cos(\theta_n^j), \sin(\theta_n^j))^T, \quad 1 \leq j \leq n, \quad t \geq 0, \end{aligned} \quad (4.2)$$

where when  $n = 2$ ,

$$\begin{aligned} \rho_1(t) &= C_1 + 6t - \sqrt{C_2^2 + 8C_1t + 24t^2}, \\ \rho_2(t) &= C_1 + 6t + \sqrt{C_2^2 + 8C_1t + 24t^2}, \quad t \geq 0; \end{aligned} \quad (4.3)$$

and when  $n \geq 3$ ,

$$\rho_1(t) \sim \alpha_1 t, \quad \rho_2(t) \sim \beta_1 t, \quad t \gg 1, \quad (4.4)$$

with  $\alpha_1$  and  $\beta_1$  being two positive constants satisfying

$$0 < \alpha_1 < \beta_1, \quad \alpha_1 + \beta_1 = 8n - 4, \quad \beta_1 - \alpha_1 = 4n \frac{\beta_1^{n/2} + \alpha_1^{n/2}}{\beta_1^{n/2} - \alpha_1^{n/2}}. \quad (4.5)$$

Specifically, when  $n \gg 1$ , we have

$$\alpha_1 \approx 2n - 2, \quad \beta_1 \approx 6n - 2. \quad (4.6)$$

*Proof.* Noting the symmetry of the ODEs (1.1) with the initial data (4.1), we can take the solution ansatz (4.2). Substituting (4.2) into (1.1) and (1.2), we obtain

$$\begin{aligned} \dot{\rho}_1(t) &= 4 \sum_{j=2}^n \frac{\mathbf{n}(\theta_n^1) \cdot (\mathbf{n}(\theta_n^1) - \mathbf{n}(\theta_n^j))}{|\mathbf{n}(\theta_n^1) - \mathbf{n}(\theta_n^j)|^2} \\ &\quad + 4 \sum_{j=1}^n \frac{\mathbf{n}(\theta_n^1) \cdot (\rho_1(t)\mathbf{n}(\theta_n^1) - \sqrt{\rho_1(t)\rho_2(t)}\mathbf{n}(\theta_n^j))}{|\sqrt{\rho_1(t)}\mathbf{n}(\theta_n^1) - \sqrt{\rho_2(t)}\mathbf{n}(\theta_n^j)|^2} \\ &= 2n - 2 + 4 \sum_{j=1}^n \frac{\rho_1(t) - \sqrt{\rho_1(t)\rho_2(t)} \cos(\theta_n^1 - \theta_n^j)}{\rho_1(t) + \rho_2(t) - 2\sqrt{\rho_1(t)\rho_2(t)} \cos(\theta_n^1 - \theta_n^j)}, \quad t > 0, \end{aligned} \quad (4.7)$$

$$\begin{aligned} \dot{\rho}_2(t) &= 4 \sum_{j=1}^n \frac{\mathbf{n}(\theta_n^1) \cdot \left( \rho_2(t)\mathbf{n}(\theta_n^1) - \sqrt{\rho_1(t)\rho_2(t)}\mathbf{n}(\theta_n^j) \right)}{\left| \sqrt{\rho_2(t)}\mathbf{n}(\theta_n^1) - \sqrt{\rho_1(t)}\mathbf{n}(\theta_n^j) \right|^2} \\ &\quad + 4 \sum_{j=2}^n \frac{\mathbf{n}(\theta_n^1) \cdot (\mathbf{n}(\theta_n^1) - \mathbf{n}(\theta_n^j))}{\left| \mathbf{n}(\theta_n^1) - \mathbf{n}(\theta_n^j) \right|^2} \\ &= 2n - 2 + 4 \sum_{l=1}^n \frac{\rho_2(t) - \sqrt{\rho_1(t)\rho_2(t)} \cos(\theta_n^1 - \theta_n^j)}{\rho_1(t) + \rho_2(t) - 2\sqrt{\rho_1(t)\rho_2(t)} \cos(\theta_n^1 - \theta_n^j)}, \quad t > 0, \end{aligned} \tag{4.8}$$

where

$$\mathbf{n}(\theta) = (\cos(\theta), \sin(\theta))^T, \quad \theta \in \mathbb{R}. \tag{4.9}$$

Summing (4.7) and (4.8), we have

$$\dot{\rho}_1(t) + \dot{\rho}_2(t) = 8n - 4, \quad t > 0, \tag{4.10}$$

Subtracting (4.7) from (4.8), we get

$$\dot{\rho}_2(t) - \dot{\rho}_1(t) = 4n \frac{\rho_2^{n/2}(t) + \rho_1^{n/2}(t)}{\rho_2^{n/2}(t) - \rho_1^{n/2}(t)} = 4n + \frac{8n\rho_1^{n/2}(t)}{\rho_2^{n/2}(t) - \rho_1^{n/2}(t)}, \quad t > 0. \tag{4.11}$$

Here we use the equality

$$\sum_{j=1}^n \frac{x^2 - 1}{x^2 + 1 - 2x \cos(\theta_n^1 - \theta_n^j)} = n \frac{x^n + 1}{x^n - 1}, \quad 1 < x \in \mathbb{R}.$$

Combining (4.10) and (4.11), we obtain

$$\begin{aligned} \dot{\rho}_1(t) &= 2n - 2 - \frac{4n\rho_1^{n/2}(t)}{\rho_2^{n/2}(t) - \rho_1^{n/2}(t)}, \\ \dot{\rho}_2(t) &= 6n - 2 + \frac{4n\rho_1^{n/2}(t)}{\rho_2^{n/2}(t) - \rho_1^{n/2}(t)}, \quad t \geq 0, \end{aligned} \tag{4.12}$$

with the initial data

$$\rho_1(0) = \rho_1^0 := a_1^2 < \rho_2(0) = \rho_2^0 := a_2^2. \tag{4.13}$$

When  $n = 2$ , we can solve (4.12) with (4.13) analytically and obtain the solution (4.3) immediately. When  $n \geq 3$ , noting that all the vortices have the same winding number, i.e.  $T_{\max} = +\infty$  by using Theorem 2.1, we get  $\rho_1(t) < \rho_2(t)$  for  $t \geq 0$  and thus

$$\dot{\rho}_2(t) > 0, \quad \dot{\rho}_2(t) - \dot{\rho}_1(t) > 0, \quad t \geq 0.$$

Therefore, we conclude that  $\rho_2(t)$  and  $\rho_2(t) - \rho_1(t)$  are monotonically increasing functions when  $t \geq 0$  and  $\lim_{t \rightarrow +\infty} \rho_2(t) = +\infty$  by noting Theorem 2.4. From (4.12), we can conclude that there exist two positive constants  $0 < \alpha_1 < \beta_1$  such that (4.4) is valid. Plugging (4.4) into (4.10), we get (4.5) immediately. When  $n \gg 1$ , (4.5) yields

$$\alpha_1 + \beta_1 = 8n - 4, \quad \beta_1 - \alpha_1 \approx 4n,$$

which immediately implies (4.6). In addition, Figure 4.1 depicts the solution  $\rho_1(t)$  and  $\rho_2(t)$  of (4.12) obtained numerically with  $\rho_1(0) = 1$  and  $\rho_2(0) = 4$  for different  $n \geq 2$ . □



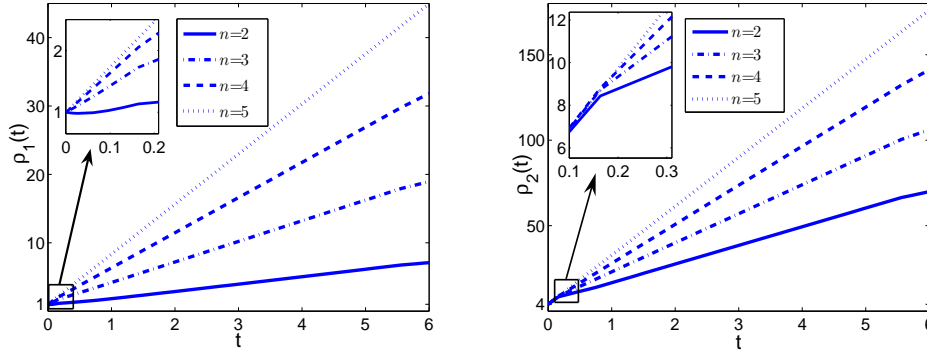


FIGURE 4.1. Time evolution of  $\rho_1(t)$  (left) and  $\rho_2(t)$  (right) of (4.12) with  $\rho_1^0 = 1$  and  $\rho_2^0 = 4$  for different  $n \geq 2$ .

**Proposition 4.2.** Taking  $m_j = m_0$  for  $1 \leq j \leq N = 2n$  and the initial data  $\mathbf{X}^0$  in (1.2) as

$$\mathbf{x}_j^0 = a_1 (\cos(\theta_n^j), \sin(\theta_n^j))^T, \quad \mathbf{x}_{n+j}^0 = a_2 (\cos(\alpha_n^j), \sin(\alpha_n^j))^T, \quad 1 \leq j \leq n, \quad (4.14)$$

then the solution of the ODEs (1.1) with (4.14) can be given as

$$\begin{aligned} \mathbf{x}_j(t) &= \sqrt{\rho_1(t)} (\cos(\theta_n^j), \sin(\theta_n^j))^T, \\ \mathbf{x}_{n+j}(t) &= \sqrt{\rho_2(t)} (\cos(\alpha_n^j), \sin(\alpha_n^j))^T, \quad 1 \leq j \leq n, \quad t \geq 0, \end{aligned} \quad (4.15)$$

where when  $n = 2$ ,

$$\begin{aligned} \rho_1(t) &= C_1 + 6t - C_2 \left(1 + \frac{6t}{C_1}\right)^{2/3}, \\ \rho_2(t) &= C_1 + 6t + C_2 \left(1 + \frac{6t}{C_1}\right)^{2/3}, \quad t \geq 0; \end{aligned}$$

and when  $n \geq 3$ ,

$$\rho_1(t) \sim \alpha_2 t, \quad \rho_2(t) \sim \beta_2 t, \quad t \gg 1,$$

with  $\alpha_2$  and  $\beta_2$  being two positive constants satisfying

$$0 < \alpha_2 < \beta_2, \quad \alpha_2 + \beta_2 = 8n - 4, \quad \beta_2 - \alpha_2 = 4n \frac{\beta_2^{n/2} - \alpha_2^{n/2}}{\beta_2^{n/2} + \alpha_2^{n/2}}.$$

Specifically, when  $n \gg 1$ ,  $\alpha_2 \approx 2n - 2$  and  $\beta_2 \approx 6n - 2$ .

*Proof.* The proof is analogue to that of Proposition 4.1 and thus it is omitted here for brevity.  $\square$

**Proposition 4.3.** Taking  $m_j = m_0$  and  $m_{n+j} = -m_0$  for  $1 \leq j \leq n$  and the initial data  $\mathbf{X}^0$  in (1.2) as (4.14), then the solution of the ODEs (1.1) with (4.14) can be given as (4.15), where

$$\rho_1(t) > 0, \quad \rho_2(t) > 0, \quad 0 \leq t < T_c := \frac{1}{4}(a_1^2 + a_2^2), \quad \lim_{t \rightarrow T_c^-} \rho_1(t) = \lim_{t \rightarrow T_c^-} \rho_2(t) = 0,$$

which implies that the  $N = 2n$  vortices will be a (finite time) collision cluster.

*Proof.* Similar to the proof of Proposition 4.1, noting the symmetry of the ODEs (1.1) with the initial data (4.14), we can take the solution ansatz (4.15). In addition, plugging (4.15) into (1.1) and (1.2), we get

$$\dot{\rho}_1(t) = 2n - 2 - 4 \sum_{l=1}^n \frac{\rho_1(t) - \sqrt{\rho_1(t)\rho_2(t)} \cos(\theta_n^1 - \alpha_n^l)}{\rho_1(t) + \rho_2(t) - 2\sqrt{\rho_1(t)\rho_2(t)} \cos(\theta_n^1 - \alpha_n^l)}, \quad (4.16)$$

$$\dot{\rho}_2(t) = 2n - 2 - 4 \sum_{l=1}^n \frac{\rho_2(t) - \sqrt{\rho_1(t)\rho_2(t)} \cos(\alpha_n^1 - \theta_n^l)}{\rho_1(t) + \rho_2(t) - 2\sqrt{\rho_1(t)\rho_2(t)} \cos(\alpha_n^1 - \theta_n^l)}, \quad (4.17)$$

with the initial data (4.13).

Summing (4.16) and (4.17), we obtain

$$\dot{\rho}_1(t) + \dot{\rho}_2(t) = 4n - 4 - 4 \sum_{l=1}^n 1 = 4n - 4 - 4n = -4, \quad t \geq 0. \quad (4.18)$$

Subtracting (4.16) from (4.17), we get

$$\dot{\rho}_2(t) - \dot{\rho}_1(t) = -4n \frac{\rho_2^{n/2}(t) - \rho_1^{n/2}(t)}{\rho_2^{n/2}(t) + \rho_1^{n/2}(t)} = -4n + \frac{8n\rho_1^{n/2}(t)}{\rho_2^{n/2}(t) + \rho_1^{n/2}(t)}, \quad t > 0. \quad (4.19)$$

Here we use the equality

$$\sum_{j=1}^n \frac{x^2 - 1}{x^2 + 1 - 2x \cos(\theta_n^1 - \theta_n^j + \frac{\pi}{n})} = n \frac{x^n - 1}{x^n + 1}, \quad 1 < x \in \mathbb{R}.$$

Combining (4.18) and (4.19), we obtain

$$\begin{aligned} \dot{\rho}_1(t) &= 2n - 2 - \frac{4n\rho_1^{n/2}(t)}{\rho_2^{n/2}(t) + \rho_1^{n/2}(t)}, \\ \dot{\rho}_2(t) &= -2n - 2 + \frac{4n\rho_1^{n/2}(t)}{\rho_2^{n/2}(t) + \rho_1^{n/2}(t)}, \quad t \geq 0, \end{aligned} \quad (4.20)$$

with the initial data (4.13).

Solving (4.18) by noting (4.13), we get

$$\rho_1(t) + \rho_2(t) = -4t + a_1^2 + a_2^2, \quad 0 \leq t < T_c := \frac{1}{4}(a_1^2 + a_2^2).$$

Noticing  $N^+ = N^- = n = \frac{N}{2}$ , thus  $M_0 = -\frac{N}{2} = -n < 0$  by noting (2.7). From Theorem 2.4, finite time collision must happen among the  $N = 2n$  vortices. Thus there exist  $1 \leq j_0 \leq n$  and  $1 \leq l_0 \leq n$  such that the vortex dipole  $\mathbf{x}_{j_0}$  and  $\mathbf{x}_{n+l_0}$  will collide at  $t = T_c$ , i.e.  $\rho_1(T_c) = \rho_2(T_c) = 0$ . Therefore, the  $N = 2n$  vortices will be a (finite time) collision cluster. In addition, Figure 4.2 depicts the solution  $\rho_1(t)$  and  $\rho_2(t)$  of (4.20) obtained numerically with  $\rho_1(0) = 1$  and  $\rho_2(0) = 4$  for different  $n \geq 2$ .  $\square$

**Remark 4.1.** When  $a_1 = a_2$ , i.e.  $\rho_1^0 = \rho_2^0$ , we can get

$$\rho_1(t) = \rho_2(t) = -2t + a_1^2,$$

which also implies the  $N = 2n$  vortices will be a (finite time) collision cluster.

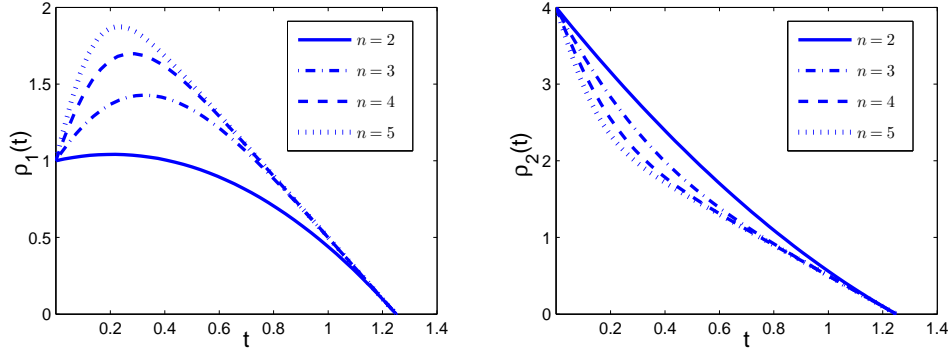


FIGURE 4.2. Time evolution of  $\rho_1(t)$  (left) and  $\rho_2(t)$  (right) of (4.20) with  $\rho_1^0 = 1$  and  $\rho_2^0 = 4$  for different  $n \geq 2$ .

4.2. **For the interaction of two clusters and a single vortex.** Here we take  $N = 2n + 1$  with  $n \geq 2$ .

**Proposition 4.4.** Taking  $m_j = m_0$  for  $1 \leq j \leq N = 2n + 1$  and the initial data  $\mathbf{X}^0$  in (1.2) as

$$\begin{aligned} \mathbf{x}_N^0 &= \mathbf{0}, & \mathbf{x}_j^0 &= a_1 (\cos(\theta_n^j), \sin(\theta_n^j))^T, \\ \mathbf{x}_{n+j}^0 &= a_2 (\cos(\theta_n^j), \sin(\theta_n^j))^T, & 1 \leq j \leq n, \end{aligned} \tag{4.21}$$

then the solution of the ODEs (1.1) with (4.21) can be given as

$$\begin{aligned} \mathbf{x}_N(t) &\equiv \mathbf{0}, & \mathbf{x}_j(t) &= \sqrt{\rho_1(t)} (\cos(\theta_n^j), \sin(\theta_n^j))^T, \\ \mathbf{x}_{n+j}(t) &= \sqrt{\rho_2(t)} (\cos(\theta_n^j), \sin(\theta_n^j))^T, & 1 \leq j \leq n, \end{aligned} \tag{4.22}$$

where when  $n = 2$ ,

$$\begin{aligned} \rho_1(t) &= C_1 + 10t - \sqrt{C_2^2 + 8C_1t + 40t^2}, \\ \rho_2(t) &= C_1 + 10t + \sqrt{C_2^2 + 8C_1t + 40t^2}, \quad t \geq 0; \end{aligned}$$

and when  $n \geq 3$ ,

$$\rho_1(t) \sim \alpha_3 t, \quad \rho_2(t) \sim \beta_3 t, \quad t \gg 1,$$

with  $\alpha_3$  and  $\beta_3$  being two positive constants satisfying

$$0 < \alpha_3 < \beta_3, \quad \alpha_3 + \beta_3 = 8n + 4, \quad \beta_3 - \alpha_3 = 4n \frac{\beta_3^{n/2} + \alpha_3^{n/2}}{\beta_3^{n/2} - \alpha_3^{n/2}}.$$

Specifically, when  $n \gg 1$ , we have  $\alpha_3 \approx 2n + 2$ , and  $\beta_3 \approx 6n + 2$ .

*Proof.* Due to symmetry, we get  $\mathbf{x}_N(t) \equiv \mathbf{0}$  for  $t \geq 0$ . The rest of the proof is analogue to that of Proposition 4.1 and thus it is omitted here for brevity.  $\square$

**Proposition 4.5.** Taking  $m_j = m_0$  for  $1 \leq j \leq N = 2n + 1$  and the initial data  $\mathbf{X}^0$  in (1.2) as

$$\begin{aligned} \mathbf{x}_N^0 &= \mathbf{0}, & \mathbf{x}_j^0 &= a_1 (\cos(\theta_n^j), \sin(\theta_n^j))^T, \\ \mathbf{x}_{n+j}^0 &= a_2 (\cos(\alpha_n^j), \sin(\alpha_n^j))^T, & 1 \leq j \leq n, \end{aligned} \tag{4.23}$$

then the solution of the ODEs (1.1) with (4.23) can be given as

$$\begin{aligned} \mathbf{x}_N(t) &\equiv \mathbf{0}, \quad \mathbf{x}_j(t) = \sqrt{\rho_1(t)} (\cos(\theta_n^j), \sin(\theta_n^j))^T, \\ \mathbf{x}_{n+j}(t) &= \sqrt{\rho_2(t)} (\cos(\alpha_n^j), \sin(\alpha_n^j))^T, \quad 1 \leq j \leq n, \end{aligned} \tag{4.24}$$

where when  $n = 2$ ,

$$\begin{aligned} \rho_1(t) &= C_1 + 10t - C_2 \left(1 + \frac{10t}{C_1}\right)^{2/5}, \\ \rho_2(t) &= C_1 + 10t + C_2 \left(1 + \frac{10t}{C_1}\right)^{2/5}, \quad t \geq 0; \end{aligned}$$

and when  $n \geq 3$ ,

$$\rho_1(t) \sim \alpha_4 t, \quad \rho_2(t) \sim \beta_4 t, \quad t \gg 1,$$

with  $\alpha_4$  and  $\beta_4$  being two positive constants satisfying

$$0 < \alpha_4 < \beta_4, \quad \alpha_4 + \beta_4 = 8n + 4, \quad \beta_4 - \alpha_4 = 4n \frac{\beta_4^{n/2} - \alpha_4^{n/2}}{\beta_4^{n/2} + \alpha_4^{n/2}}.$$

Specifically, when  $n \gg 1$ ,  $\alpha_4 \approx 2n + 2$  and  $\beta_4 \approx 6n + 2$ .

*Proof.* Due to symmetry, we get  $\mathbf{x}_N(t) \equiv \mathbf{0}$  for  $t \geq 0$ , and the rest of the proof is analogue to that of Proposition 4.1 and 4.3, thus it is omitted here for brevity.  $\square$

**Proposition 4.6.** Taking  $m_N = -m_0$ ,  $m_j = m_0$  for  $1 \leq j \leq 2n = N - 1$  and the initial data  $\mathbf{X}^0$  in (1.2) as (4.21), then the solution of the ODEs (1.1) with (4.21) can be given as (4.22), where

(i) when  $n = 2$ , then  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$  and  $\mathbf{x}_5(t)$  be a collision cluster among the 5 vortices and they will collide at the origin  $(0, 0)^T$  in finite time;

(ii) when  $n = 3$ , then

$$\rho_1(t) \sim \left(\frac{a_1 a_2}{a_1 + a_2}\right)^2, \quad \rho_2(t) \sim 12t, \quad t \gg 1; \tag{4.25}$$

(iii) when  $n \geq 4$ , then

$$\rho_1(t) \sim \alpha_5 t, \quad \rho_2(t) \sim \beta_5 t, \quad t \gg 1,$$

with  $\alpha_5$  and  $\beta_5$  being two positive constants satisfying

$$0 < \alpha_5 < \beta_5, \quad \alpha_5 + \beta_5 = 8n - 12, \quad \beta_5 - \alpha_5 = 4n \frac{\beta_5^{n/2} + \alpha_5^{n/2}}{\beta_5^{n/2} - \alpha_5^{n/2}}.$$

Specifically, when  $n \gg 1$ , we have  $\alpha_5 \approx 2n - 6$ , and  $\beta_5 \approx 6n - 6$ .

*Proof.* Similar to the proof of Proposition 4.1 and 4.3, the solution of the ODEs (1.1) with (4.21) can be given as (4.22), where

$$\dot{\rho}_1(t) + \dot{\rho}_2(t) = 8n - 12, \quad \dot{\rho}_2(t) - \dot{\rho}_1(t) = 4n + \frac{8n\rho_1^{n/2}(t)}{\rho_2^{n/2}(t) - \rho_1^{n/2}(t)}, \quad t > 0,$$

which implies

$$\begin{aligned} \dot{\rho}_1(t) &= 2n - 6 - \frac{4n\rho_1^{n/2}(t)}{\rho_2^{n/2}(t) - \rho_1^{n/2}(t)}, \\ \dot{\rho}_2(t) &= 6n - 6 + \frac{4n\rho_1^{n/2}(t)}{\rho_2^{n/2}(t) - \rho_1^{n/2}(t)}, \quad t > 0. \end{aligned} \tag{4.26}$$

1) When  $n = 2$ , (4.26) reduces to

$$\dot{\rho}_1(t) = -2 - \frac{8\rho_1(t)}{\rho_2(t) - \rho_1(t)}, \quad \dot{\rho}_2(t) = 6 + \frac{8\rho_1(t)}{\rho_2(t) - \rho_1(t)}, \quad t > 0. \quad (4.27)$$

Solving (4.27) with the initial data (4.13), we get

$$\begin{aligned} \rho_1(t) &= 2t + C_1 - \sqrt{8t^2 + 8C_1t + C_2^2}, \\ \rho_2(t) &= 2t + C_1 + \sqrt{8t^2 + 8C_1t + C_2^2}, \quad t \geq 0. \end{aligned}$$

Thus there exists a  $T_c := \frac{1}{2} \left[ -C_1 + \sqrt{C_1^2 + 2a_1^2 a_2^2} \right] > 0$ , such that

$$\rho_1(T_c) = 0, \quad \rho_2(T_c) > 0, \quad \rho_1(t) > 0, \quad \rho_2(t) > 0, \quad t \in [0, T_c),$$

which immediately implies that  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$  and  $\mathbf{x}_5(t)$  be a collision cluster among the 5 vortices and they will collide at the origin  $(0, 0)^T$  when  $t \rightarrow T_c^-$ .

2) When  $n = 3$ , (4.26) reduces to

$$\dot{\rho}_1(t) = -\frac{12\rho_1^{3/2}(t)}{\rho_2^{3/2}(t) - \rho_1^{3/2}(t)}, \quad \dot{\rho}_2(t) = \frac{12\rho_2^{3/2}(t)}{\rho_2^{3/2}(t) - \rho_1^{3/2}(t)}, \quad t > 0, \quad (4.28)$$

which immediately implies

$$\frac{d}{dt} \left[ \frac{1}{\sqrt{\rho_1(t)}} + \frac{1}{\sqrt{\rho_2(t)}} \right] = 0 \implies \frac{1}{\sqrt{\rho_1(t)}} + \frac{1}{\sqrt{\rho_2(t)}} \equiv \frac{a_1 + a_2}{a_1 a_2}, \quad t \geq 0. \quad (4.29)$$

Since  $0 < a_1 < a_2$ , then  $\rho_1(t)$  and  $\rho_2(t)$  are monotonically decreasing and increasing functions, respectively. From (4.29), we know that  $0 < \rho_1(t) < \rho_2(t)$  for  $t \geq 0$  and thus  $T_{\max} = +\infty$ , i.e. there is no finite time collision. Noting that  $M_0 > 0$ , by Theorem 2.4, we have

$$\lim_{t \rightarrow +\infty} \rho_2(t) = +\infty. \quad (4.30)$$

Combining (4.30), (4.29) and (4.28), we obtain (4.25) immediately.

3) When  $n \geq 4$ , the proof is analogue to that of Proposition 4.1 and thus it is omitted here for brevity.

In addition, Figure 4.3 depicts the solution  $\rho_1(t)$  and  $\rho_2(t)$  of (4.26) obtained numerically with  $\rho_1^0 = 1$  and  $\rho_2^0 = 4$  for different  $n \geq 2$ .  $\square$

**Proposition 4.7.** *Taking  $m_N = -m_0$ ,  $m_j = m_0$  for  $1 \leq j \leq 2n = N - 1$  and the initial data  $\mathbf{X}^0$  in (1.2) as (4.23), then the solution of the ODEs (1.1) with (4.23) can be given as (4.24), where*

(i) *when  $n = 2$ , only the three vortices  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$  and  $\mathbf{x}_5(t)$  be a collision cluster among the 5 vortices and they will collide at the origin  $(0, 0)^T$  in finite time;*

(ii) *when  $n = 3$ , then*

$$\rho_1(t) \sim \left( \frac{a_1 a_2}{a_2 - a_1} \right)^2, \quad \rho_2(t) \sim 12t, \quad t \gg 1;$$

(iii) *when  $n \geq 4$ , then*

$$\rho_1(t) \sim \alpha_6 t, \quad \rho_2(t) \sim \beta_6 t, \quad t \gg 1,$$

with  $\alpha_6$  and  $\beta_6$  being two positive constants satisfying

$$0 < \alpha_6 < \beta_6, \quad \alpha_6 + \beta_6 = 8n - 12, \quad \beta_6 - \alpha_6 = 4n \frac{\beta_6^{n/2} - \alpha_6^{n/2}}{\beta_6^{n/2} + \alpha_6^{n/2}}.$$

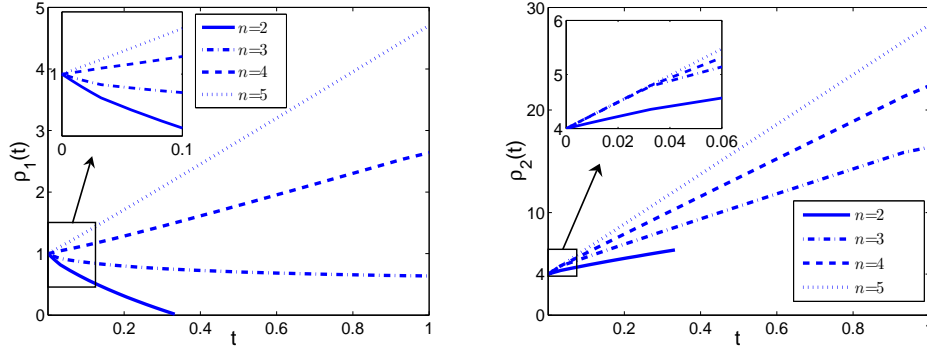


FIGURE 4.3. Time evolution of  $\rho_1(t)$  (left) and  $\rho_2(t)$  (right) of (4.26) with  $\rho_1^0 = 1$  and  $\rho_2^0 = 4$  for different  $n \geq 2$ .

Specifically, when  $n \gg 1$ , we have  $\alpha_6 \approx 2n - 6$ , and  $\beta_6 \approx 6n - 6$ .

*Proof.* The proof is analogue to that of Proposition 4.6 and thus it is omitted here for brevity.  $\square$

**Proposition 4.8.** Taking  $m_N = -m_0$ ,  $m_j = m_0$  and  $m_{n+j} = -m_0$  for  $1 \leq j \leq n$  and the initial data  $\mathbf{X}^0$  in (1.2) as (4.23), then the solution of the ODEs (1.1) with (4.23) can be given as (4.24), where

(i) when  $n = 2$ , only the three vortices  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$  and  $\mathbf{x}_5(t)$  be a collision cluster among the 5 vortices and they will collide at the origin  $(0, 0)^T$  in finite time;

(ii) When  $n \geq 3$ , all the  $N = 2n + 1$  vortices be a collision cluster and they will collide at the origin  $(0, 0)^T$  when  $t \rightarrow T_c := \frac{1}{4}[a_1^2 + a_2^2]$ .

*Proof.* Similar to the proof of Proposition 4.1 and 4.3, the solution of the ODEs (1.1) with (4.23) can be given as (4.24), where

$$\dot{\rho}_1(t) + \dot{\rho}_2(t) = -4, \quad \dot{\rho}_2(t) - \dot{\rho}_1(t) = 8 - 4n + \frac{8n\rho_1^{n/2}(t)}{\rho_2^{n/2}(t) + \rho_1^{n/2}(t)}, \quad t > 0, \quad (4.31)$$

which implies

$$\begin{aligned} \dot{\rho}_1(t) &= 2n - 6 - \frac{4n\rho_1^{n/2}(t)}{\rho_2^{n/2}(t) + \rho_1^{n/2}(t)}, \\ \dot{\rho}_2(t) &= 2 - 2n + \frac{4n\rho_1^{n/2}(t)}{\rho_2^{n/2}(t) + \rho_1^{n/2}(t)}, \quad t > 0. \end{aligned} \quad (4.32)$$

Solving (4.31) with initial data (4.13), we get

$$\rho_1(t) + \rho_2(t) = -4t + a_1^2 + a_2^2, \quad t \geq 0,$$

which implies that a finite time collision must happen and  $0 < T_{\max} \leq T_c := \frac{1}{4}(a_1^2 + a_2^2)$ .

1) When  $n = 2$ , from (4.32), we obtain

$$\dot{\rho}_1(t) = -\frac{10\rho_1(t) + 2\rho_2(t)}{\rho_1(t) + \rho_2(t)} < 0, \quad \dot{\rho}_2(t) = \frac{6\rho_1(t) - 2\rho_2(t)}{\rho_1(t) + \rho_2(t)}, \quad t > 0. \quad (4.33)$$

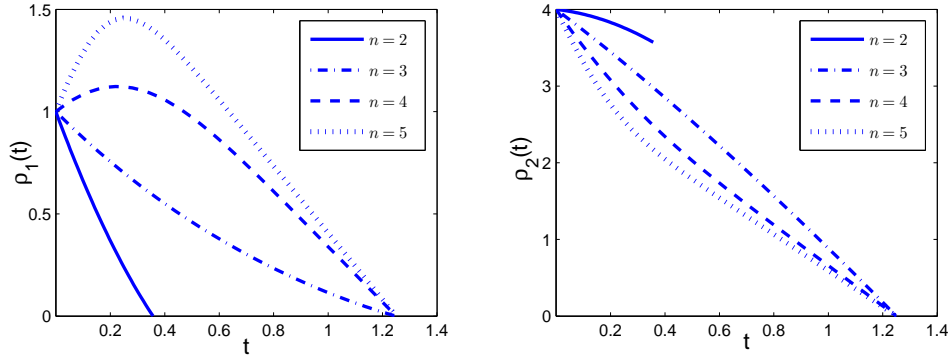


FIGURE 4.4. Time evolution of  $\rho_1(t)$  (left) and  $\rho_2(t)$  (right) of (4.32) with  $\rho_1^0 = 1$  and  $\rho_2^0 = 4$  for different  $n \geq 2$ .

Solving (4.33) with the initial data (4.13), we have

$$\begin{aligned} \rho_1(t) &= (-2t + C_1)(2C_3(-2t + C_1) - 1), \\ \rho_2(t) &= (-2t + C_1)(3 - 2C_3(-2t + C_1)), \quad t \geq 0. \end{aligned}$$

where  $C_3 = \frac{3a_1^2 + a_2^2}{(a_1^2 + a_2^2)^2} > 0$ . Denote

$$0 < T_{\max} = \frac{2C_1C_3 - 1}{4C_3} = \frac{a_1^2(a_1^2 + a_2^2)}{6a_1^2 + 2a_2^2} = \frac{2a_1^2}{3a_1^2 + a_2^2}T_c < T_c,$$

then we have

$$\rho_1(T_{\max}) = 0, \quad \rho_2(T_{\max}) > 0, \quad \rho_1(t) > 0, \quad \rho_2(t) > 0, \quad t \in [0, T_{\max}),$$

which implies that only the three vortices  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$  and  $\mathbf{x}_5(t)$  be a collision cluster among the 5 vortices and they will collide at the origin  $(0, 0)^T$  when  $t \rightarrow T_{\max}^-$ .

2) When  $n \geq 3$ , by Theorem 2.4, only the  $n + 1$  vortices with  $\mathbf{x}_{n+1}(t), \dots, \mathbf{x}_{2n}(t)$  and  $\mathbf{x}_N(t)$  cannot be a collision cluster among the  $N$  vortices since they have the same winding number; and similarly, only the  $n + 1$  vortices with  $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$  and  $\mathbf{x}_N(t)$  cannot be a collision cluster among the  $N$  vortices since their collective winding number defined as  $M_1 := \sum_{1 \leq j < l \leq n} m_j m_l + \sum_{j=1}^n m_j m_N = \frac{1}{2}[(n - 1)^2 - n - 1] = \frac{1}{2}n(n - 3) \geq 0$ . Thus, in order to have a finite time collision, there exist  $1 \leq j_0 \leq n$  and  $1 \leq l_0 \leq n$  such that the vortex dipole  $\mathbf{x}_{j_0}(t)$  and  $\mathbf{x}_{n+l_0}(t)$  will collide at  $t = T_c$ , i.e.  $\rho_1(T_c) = \rho_2(T_c) = 0$ . Therefore, the  $N = 2n + 1$  vortices will be a (finite time) collision cluster.

In addition, Figure 4.4 depicts the solution  $\rho_1(t)$  and  $\rho_2(t)$  of (4.32) obtained numerically with  $\rho_1(0) = 1$  and  $\rho_2(0) = 4$  for different  $n \geq 2$ .  $\square$

**5. Conclusion.** Based on the reduced dynamical law of a system of ordinary differential equations (ODEs) for the dynamics of  $N$  vortex centers, we have obtained stability and interaction patterns of quantized vortices in superconductivity. By deriving several non-autonomous first integrals of the ODEs system, we proved global well-posedness of the  $N$  vortices when they have the same winding number and demonstrated that finite time collision might happen when they have different winding numbers. When  $N = 3$ , we established rigorously orbital stability when they have the same winding number and classified their collision patterns when

they have different winding numbers. Finally, under several special initial setups including interaction of two clusters, we obtained explicitly the analytical solutions of the ODEs system. The analytical and numerical results demonstrated the rich dynamics and interaction patterns of  $N$  vortices in superconductivity.

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